BOSE 2018 Celebrating 125th Birth Anniversary

S N Bose Centre for Basic Sciences, Kolkata March 26-28 2018

Ideal and Imperfect Bose Fluids without and with Confinements

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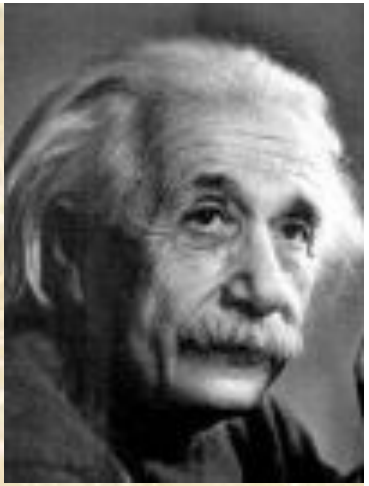
Plan of the talks

- ❑ Bose and Bosons
- ❑ Bose Einstein Condensation:

  An unusual concept of Phase Transition in 1-body physics
- ❑ Role of interaction to the Physics of Boson Condensation
- ❑ Effect of confinement to an interaction-free Bose gas
- ❑ Confined and interacting Bosons

  Many Body Aspect: Microscopic Theory of excitations

  Collective Aspect

  Superfluidity, Vortex matter, Dirty Bosons
- ❑ Summary

Publications on Bosonic Condensation:

1. Bose Einstein Condensation: from Atomic Physics to Quantum Liquids**,** Craig M. Savage and **Mukunda P Das**, World Scientific (2000)
2. Density Functional Theory of **Super-Phenomena** in Condensed Systems, **Mukunda P Das** in Electronic Density Functional Theory: Recent Progress and New Directions, J. F. Dobson, G. Vignale and M. P. Das **(Ed)**. Plenum (NY) pp 373 (1998).
3. Bose-Einstein Condensation of Trapped Alkali Atoms with Attractive Interaction, Nail Akhmediev, **Mukunda P Das** and Alexei Vagov in 9th International Conference in Many-Body Theories (Ed.) David Neilson and Ray F Bishop, World Scientific, pp 287.
4. Bose Einstein Condensation of Atoms with Attractive Interaction in a Harmonic Trap**,** Nail Akhmediev**, Mukunda P Das and** Alexei Vagov**,** Australian J Physics **53** 157 (2000)
5. Depletion of Harmonically Confined, Interacting Bose Atoms from the Ground State, G Gnanapragasam, Sang-Hoon Kim and **Mukunda P Das**, Modern Phys Lett B **20** 1839 (2006)
6. Interacting Bose Gas Confined by an External Potential G Gnanapragasam and **Mukunda P Das** International J Mod Phys A **22** 4923 (2007)
7. First-Order Quantum Correction to the Ground-State Energy Density of Two-Dimensional Hard-Sphere Bose Atoms, S. Kim & **M P Das,** International Journal of Modern Physics B, **24** 1007 (2010)
8. Collective Modes of Trapped Interacting Bosons, G. Gnanapragasam & **M. P. Das,** International Journal of Modern Physics B **22** 4349 (2008)
9. Ground-State Energy Density of a Dilute Bose Gas in the Canonical Transformation, S. Kim, C Kim & **M P Das**, International Journal of Modern Physics B **21** 5309. (2007)

A Brief History

Satyendra Nath BOSE

S N Bose The Man and His Work, C. K. Majumdar et al. Ed Vol I and II, SNBNCBS, Kolkata (1994), See particularly **P. Ghose** p.35 and **ECG Sudarshan** p. 7 in Vol I.

Satyendra Nath Bose– His Life and Times: Selected Works (with Commentaries) by **Kameshwar Wali**, World Scientific (2009). See also **The Man behind Bose Statistics K Wali** in Physics Today **59**, 46 (2006).

Satyendra Nath Bose by **Santimay and Minati Chattejee** in S N Bose Archive, SNBNCBS, Kolkata.

Satyendra Nath Bose: Co-founder of Quantum Statistics W A Blanpied, Am. J. Phys. **40**, 1212 (1972).

A World of Bose Particles, **E C G Sudarshan**, Am J Phys **43,** 69 (1975).

Bose and his Statistics, **G. Venkatraman**, Universities Press (1992).

Satyendra Nath BOSE

Born in Kolkata on 1 Jan 1894
Joined Presidency college in 1909 studied under J. C. Bose and P. C. Ray.
Was class mate of Meghnad Saha, both of them were self-taught physicists, collaborators and became Lectures in Applied Mathematics, Calcutta Univ.
In 1921 SNB became Reader in Physics at Dacca to teach B.Sc. (Hons) and M. Sc.
Spent nearly two years in Paris and Berlin with Langevin, Madame Curie, Einstein and others returned to Dacca in 1926, became Professor.
In 1945 returned to Calcutta as Khaira Professr of Physics
1958-FRS
1959-National Professor

Died in Kolkata on 4 February 1974 at the age of 80.

Bose Einstein Condensation

- 1924 Satyendra Nath Bose gave a novel derivation of photon statistics and explained the Planck distribution of Black Body Radiation.
- 1925 Based on this work Albert Einstein developed statistical mechanics of monoatomic Bosonic atoms, what is now known as BEC.
- It is an unusual concept in 1-body physics in the sense that one encounters a 'phase transition'.
- Uhlenbeck's criticism ???
- Effect of inter-particle interaction
- Superfluidity and ^4He (1938)

Q: Is Bose Condensate relevant to Superfluidity? (Ref: Russell Donnelly Sebastian Balibar))

Blackbody Radiation (Thermo-electrodynamics)

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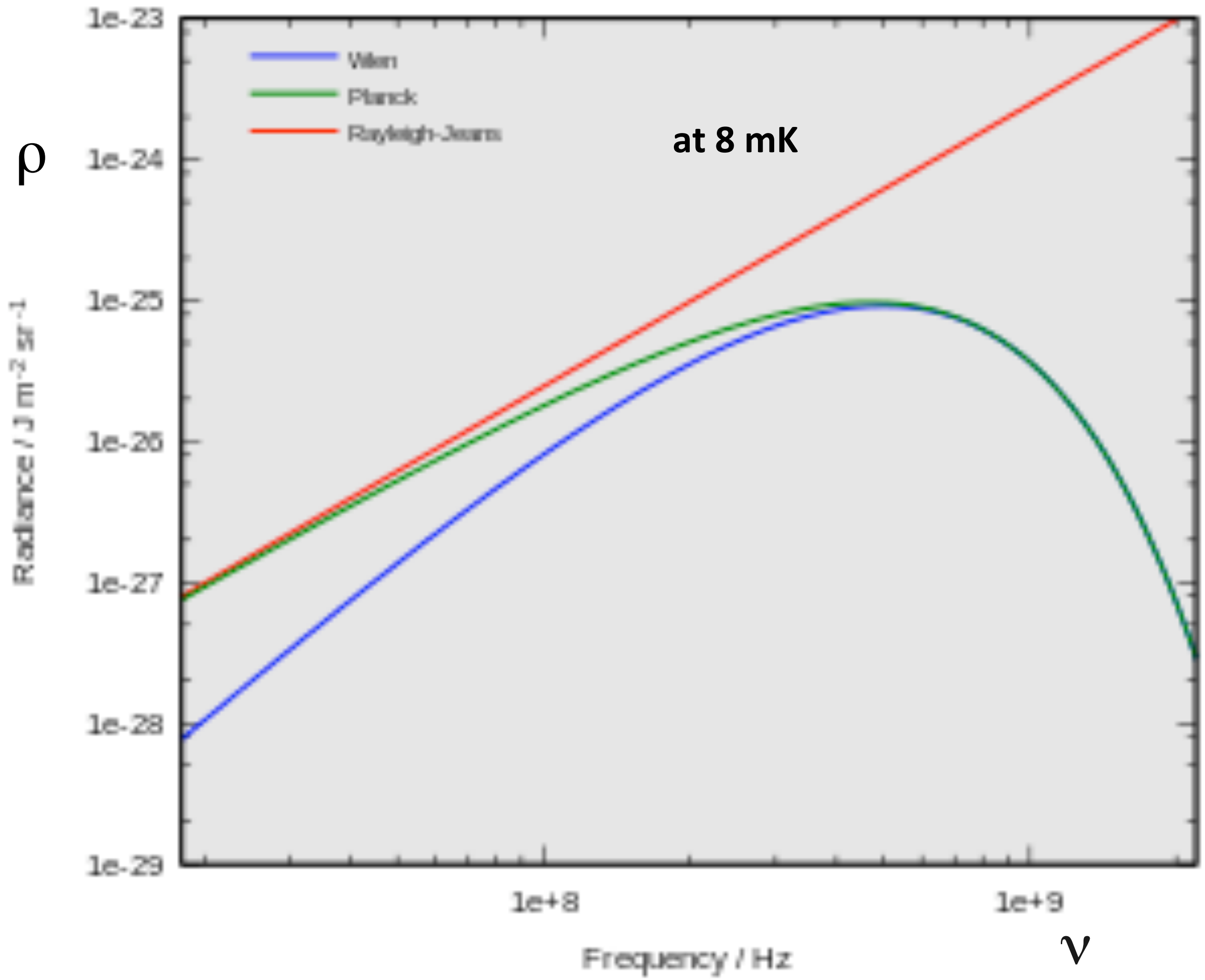

Rayleigh-Jeans Law

Wien's Law

Planck's Law

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

Fitted with h.

The paper of Bose on derivation of Blackbody Radiation and Bose Statistics was rejected by Phil. Mag. Bose sent the paper in English to Einstein with a request as given below.

Dated the 4th June, 1924

Respected Sir,

I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. You will see that I have tried to deduce the coefficient piv^2/c^3 in Planck's laws, independent of the classical electrodynamics, only assuming that the ultimate elementary regions in the phase space had the content h^3. I do not know sufficient German to translate the paper. If you think the paper worth publication, I shall be grateful if you arrange for its publication in *Zeitschrift fur Physik*. Though a complete stranger to you, I do not feel any hesitation in making such a request. Because we are all your pupils though profiting only by your teachings through your writings. I do not know whether you still remember that somebody from Calcutta asked your permission to translate your papers on relativity in English. You acceded to the request. The book has since been published. I was the one who translated your paper 'Generalised Relativity'.

Yours faithfully,

S.N. Bose

A bit of history !

Plancks Gesetz und Lichtquantenhypothese.

Von **Bose** (Dacca-University, Indien).

(Eingegangen am 2. Juli 1924.)

Translated by A. Einstein.

An important change in the paper (without Bose's consent) by a factor of 2 for the polarization of photon state.

Translator's remarks :

In my opinion Bose's derivation signifies an important advance. The method used here gives the quantum theory of an ideal gas as I will work out elsewhere.

*Z Physik **26**, pp. 178-181,1924 (Springer - Verlag, Heidelberg).*

Note: Einstein was the translator, certainly not the author of the paper. Therefore the statistics is "Bose Statistics".

Back to Bose (On Bose's first paper)

Some comments by Max Delbrück "Was Bose-Einstein statistics arrived by serendipity?"

There is evidence of haste in Einstein's handling of the paper in that Bose is not credited with his initials. Also the paper is astoundingly brief and abrupt. It has no literature references. I have a strong suspicion that Einstein cut it short, perhaps even rewrote it, but the original of the manuscript is not available

J. Chem Education **67** 467 (1980).

Back to Bose

Bose sent his 2nd Paper to Einstein soon after the first paper on
Thermal Equilibrium in the Radiation Field in the Presence of Matter

Not hearing from Einstein, Bose enquired about this paper after arrival in Paris (26.10.24)

Dear Master,
My heartfelt gratitude for taking the trouble of translating the paper yourself and publishing it. I just saw it in print before I left India. I have sent you about the middle of June a second paper entitled 'Thermal Equilibrium in Radiation Field in Presence of Matter'.
I am rather anxious to know your opinion about it as I think it to be rather important. I don't know whether it will be possible to have this paper published in Zeitschrift fur Physik.

Back to Bose

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Einstein replied in 3 Nov 24.

Most esteemed colleague!
Friendly thanks for your letter of 26.10. I am happy to have the opportunity to meet you personally.Your papers have appeared already some time ago; unfortunately the offprints were sent to me instead of you. You can have them any time.
I am not in agreement with your elementary law of probability for the interaction between radiation and matter, and have given the reason in a note that appeared together with your article. To wit, your law is not compatible
with the following two conditions:
1. The coefficient of absorption is independent of the density of the radiation.
2. The behavior of an oscillator in a radiation field must result from the statistical laws as a limiting case.
We can discuss this in more detail when you come here.
With friendly greetings.
A.E.

Back to Bose

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In fact Bose's 2nd Paper published after German translation by Einstein.

Thermal Equilibrium in the Radiation Field in the Presence of Matter

S. N. Bose in Ramna (India)

(Received 7 July, 1924

*Z. Physik **27** 384-393, 1924 (Springer - Verlag, Heidelberg).*

Bose was not informed about this publication.

*This time AE is also a translator but he added a **critical** comment (almost one page) without consultation with Bose which appeared in continuation of Bose's paper.*

The comments may be due to the following statement of Bose !!!

Back to Bose (his 2nd paper)

Thermal Equilibrium in the Radiation Field in the Presence of Matter- Bose wrote

" *However the form assumed by Pauli (Z. Phys.* ***18*** *272, 1923)and generalized by Einstein and Ehrenfest (Z. Phys.* ***19*** *301, 1923) appears to be completely arbitrary because one can not easily see how such an expression can be derived. The form suggested here is quite simple and can be justified on the basis of elementary considerations. The necessity of assuming relations between the coefficients themselves is also avoided.*

It was assumed in deriving the probability coefficients for the interaction (or coupling, as Bohr says) that even in a collision no interaction is as probable as the occurence of any special interaction. This assumption is a fundamental point in the derivation given here. "

Back to Bose **(his 2nd paper)**

Read with care:

*In his analysis George Sudarshan (Am J Phys **43** 69,1975) wrote on this controversy-*

*"Bose (Z. Phys. **27** 384,1924) propose to take up the general case and showed that both the Pauli processes and Einstein-Ehrenfest Processes are included as special cases. This by itself should have met with general acceptance and acclaim. But, Bose did point out that instead of Einstein's assumption of a stimulated and a spontaneous transition rate for absorption it is possible to consider only the spontaneous transition rate for emission provided the absorption rate is taken to be not proportional to the number of quanta per phase cell but this number divided by this number plus one. Now, as far as the radiative equilibrium is concerned this ansatz is as good as the Einstein ansatz (Z Phys **18** 121,1917), since only these ratios do come in! So it would have been quite possible for Einstein to add such a footnote to Bose's paper and call attention to the positive general features of Bose's formulation. But he (Einstein) chose otherwise."*

Back to Bose (his 3rd paper)

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Bose wrote to Einstein on 27 Jan 25.
A third paper to meet AE's comments was sent under separate cover. Unfortunately this paper is untraceable. See Partha Ghose.

Not available in AE Archives.

Back to Einstein

Following Bose, Einstein wrote three papers on Statistical Mechanics of Atomic gas within weeks (1924-25) presented in the Prussian Acad meetings. His 2nd paper is the one written clearly on the BEC. The important thing there is unlike photons, there is introduction of a chemical potential, since number of atoms are conserved.

Back to Bose

Ideal, Noninteracting, Free Bose gas

Nonideal, Interacting, Imperfect Bose Gas

Dirty Bose gas

Where does the Quantum Idea creep in?

- Identical Particles

  Degeneracy

- Indistinguishability =

  Identity of the Particles +

  Symmetry of the State

These two important aspects are in the choice of Probability function 'W' by S N Bose.

Derivation of Various Statistics

(See Text Book of Kerson Huang and/or Raj Pathria)

by (i) Combinatorics

(ii) Kinetic Eqn. Method

Combinatorics: Probability distribution W expressed as follows.

Maxwell- Boltzmann Statistics (1860,1868)

$$\mathbb{W}_{\text{MB}} = N! \prod_{i=1}^{s} \frac{g_i^{n_i}}{n_i!} = N! \prod_{i=1}^{s} \frac{(\alpha_i N)^{p_i N}}{(p_i N)!},$$

Bose-Einstein Statistics (1924)

$$W_{BE} = \prod_{i=1}^{s} \frac{(g_i + n_i - 1)!}{(g_i - 1)!\,n_i!} = \prod_{i=1}^{s} \frac{(\alpha_i N + p_i N - 1)!}{(\alpha_i N - 1)!\,(p_i N)!},$$

g_i is the degeneracy (multiplicity) of each state i,
$p_i = n_i/N$, is the probability of an entity being in state i, and
$\alpha_i = g_i/N$ is the relative degeneracy of state i.

Fermi-Dirac Statistics (1926)

$$W_{FD} = \prod_{i=1}^{s} \frac{g_i!}{n_i!(g_i - n_i)!} = \prod_{i=1}^{s} \frac{(\alpha_i N)!}{(p_i N)!(\alpha_i N - p_i N)!},$$

Method and Solutions:

S = k ln W

Maximise entropy subject to constraints of Energy and Probability by Lagrange method to get MB, Bose-Einstein and Fermi-Dirac statistics (0 for MB and ± 1 for Fermi and Bose in the denominator).

$$\overline{n}_s = \frac{1}{e^{\beta(\varepsilon_s - \mu)} - 1} \quad \text{bosons}$$

$$\overline{n}_s = \frac{1}{e^{\beta(\varepsilon_s - \mu)} + 1} \quad \text{fermions}$$

Alternately

All Statistics can be derived from kinetic Eqns.

Maxwell-Boltzmann

Bose-Einstein

Fermi-Dirac

See for example**: E A Uehling and G E Uhlenbeck Phys. Rev. 43 (1933) 552.**

$$\frac{\partial f}{\partial t}+D(f)=\int d\phi_1\int gw(\vartheta g)d\Omega\{f'f_1'(1+\theta f)(1+\theta f_1)-ff_1(1+\theta f')(1+\theta f_1')\},$$

$\theta = 0$ for Maxwell-Boltzmann

= +1 for Bose-Einstein

= -1 for Fermi-Dirac

Essential points of difference in this term in the collision term of the right hand side of the eqn.

(1)Use of the appropriate "Stoszzahlansatz".
(2) the necessity of determining the transition probability function , quantum mechanically, and of taking into account the identity of the molecules in this determination. (See Mott (1930) and Nordheim and Kikuchi(1930)).

Back to Bose-Einstein

Einstein extended Bose's work on light quanta to Bosonic Atoms. (from Wave to Particles)

$E_{\mathbf{k}} = c\mathbf{k}$ (for light quanta)

$E_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m$ (for atoms)

(In case of atoms the number conservation is required for which the chemical potential appears in the distribution, for light quanta chemical potential is zero).

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IDEAL QUANTUM GAS

- The two-particle wavefunction $\psi(x_1, x_2)$ only makes sense if

$$|\psi(x_1,x_2)|^2 = |\psi(x_2,x_1)|^2 \Rightarrow \psi(x_1,x_2) = e^{i\alpha}\psi(x_2,x_1)$$

- If we introduce exchange operator $\hat{P}_{\text{ex}}\psi(x_1,x_2) = \psi(x_2,x_1)$, since $\hat{P}_{\text{ex}}^2 = \mathbb{I}$, $e^{2i\alpha} = 1$ showing that $\alpha = 0$ or π, i.e.

$$\begin{array}{ll} \psi(x_1,x_2) = \psi(x_2,x_1) & \text{bosons} \\ \psi(x_1,x_2) = -\psi(x_2,x_1) & \text{fermions} \end{array}$$

IDEAL QUANTUM GAS (Bosons)

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- In bosonic systems, wavefunction must be symmetric under particle exchange.

- Such a wavefunction can be obtained by expanding all of terms contributing to Slater determinant and setting all signs positive.

  i.e. bosonic wave function describes uniform (equal phase) superposition of all possible permutations of product states.

IDEAL QUANTUM GAS (Bosons)

- In a system of N spinless non-interacting bosons, ground state of many-body system involves wavefunction in which all particles occupy lowest single-particle state, $\psi_{\rm B}(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \prod_{i=1}^{N} \phi_{\mathbf{k}=0}(\mathbf{r}_i)$.

- At non-zero temperature, partition function given by

$$\mathcal{Z} = \sum_{\{n_{\mathbf{k}}=0,1,2,\cdots\}} \exp\left[-\sum_{\mathbf{k}} \frac{(\epsilon_k - \mu)n_{\mathbf{k}}}{k_{\rm B}T}\right] = \prod_{\mathbf{k}} \frac{1}{1 - e^{-(\epsilon_k - \mu)/k_{\rm B}T}}$$

- The average state occupancy is given by the Bose-Einstein distribution,

$$\bar{n}(\epsilon_{\mathbf{q}}) = \frac{1}{e^{(\epsilon_k - \mu)/k_{\rm B}T} - 1}$$

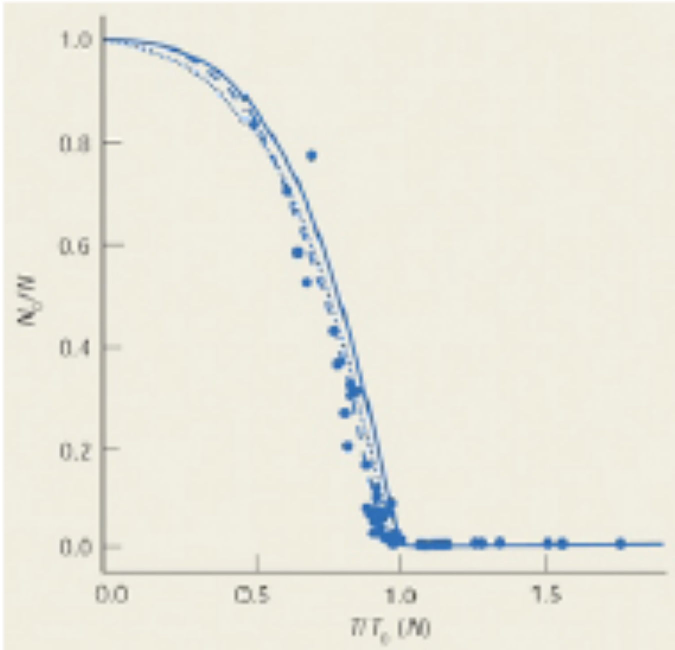
Note that Statistical mechanics of Free Bose gas in 3D with energy dispersion $E_k \sim k^2$

$$T_c = \frac{1}{2\pi mk}\left(\frac{\hbar^3}{g_{\frac{3}{2}}(1)}\right)^{\frac{2}{3}} n^{\frac{2}{3}} \qquad (1)$$

Dimensions - The required dimensionality for BEC to occur in a homogeneous Bose gas is D >2. That is, there can be NO condensation into the zero-momentum state for D$\leq$ 2 in a homogeneous Bose gas.

IDEAL QUANTUM GAS (?) Dilute and Ultracold

Rb^{87}



$$f(T) = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

J.R. Ensher, et al., Phys. Rev. Lett. 77 (1996) 4984.

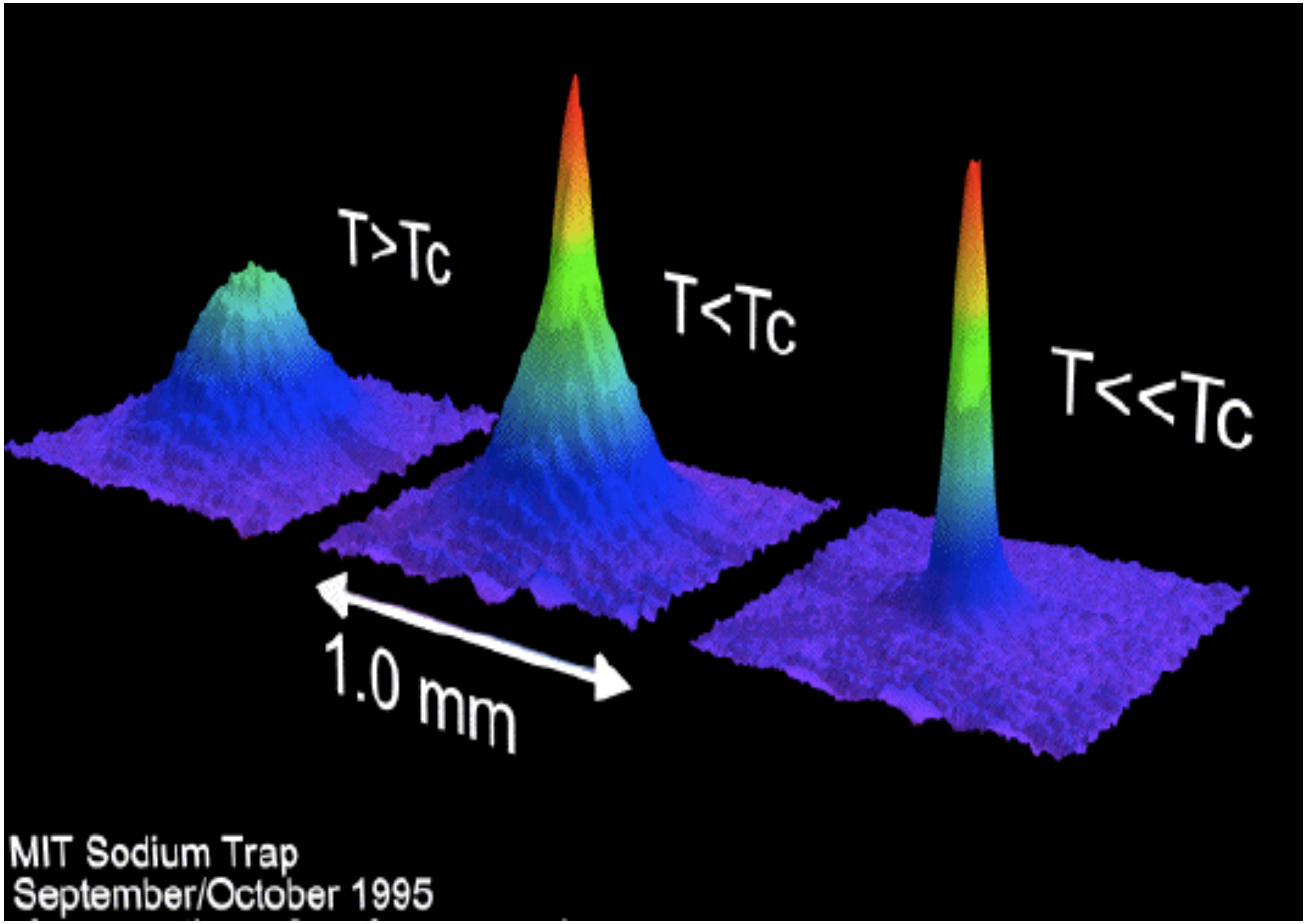
Questions?

Are the particles in the condensate assemble in a single state? Why not share among several states close in energy- will make no difference in the thermodynamic limit !!!
Or will there be a fragmentation of the condensate?

These questions beg the answers to –-

- ✧ **Is the BEC an ideal gas effect?**
- ✧ **What is the Role of Exchange?**
- ✧ **How real a Phase Transition in an ideal gas?**

Phase Transition (Ketterle Lecture)



EXCHANGE IN IDEAL CASE

G. E. Uhlenbeck and L. Gropper

Phys Rev **41** (1932) 79. See also Mullin and Blaylock (2003)

$P = nk_BT\,(1+nB(T))$,

B(T) is the second Virial Coeff. given by

$B(T) = -\eta\,\lambda^3/2^{5/2}$, where η =+ for Bosons,

- for Fermions

B(T) in the interacting case

$B(T) = \frac{1}{2}\int dr\,[1-\exp(-\beta U(r))]$

G. E. Uhlenbeck and L. Gropper

Phys Rev **41** (1932) 79.

U(r) → g(r)

$$g(r) = (1+\eta \exp(2\pi r^2/\lambda^2)$$

$$U_{eff}(r) = -k_B T \ln (1+\eta \exp(2\pi r^2/\lambda^2)$$

$$B(T) = 1/2V \int\int dr_1 dr_2 [1-g(1,2)]$$

Statistical Interaction (Exchange)

$$U_{\text{eff}}(r) = -k_B T \ln(1 + \eta e^{-2\pi r_{12}^2/\lambda^2}).$$

η = +1 for Bosons and
-1 for Fermions

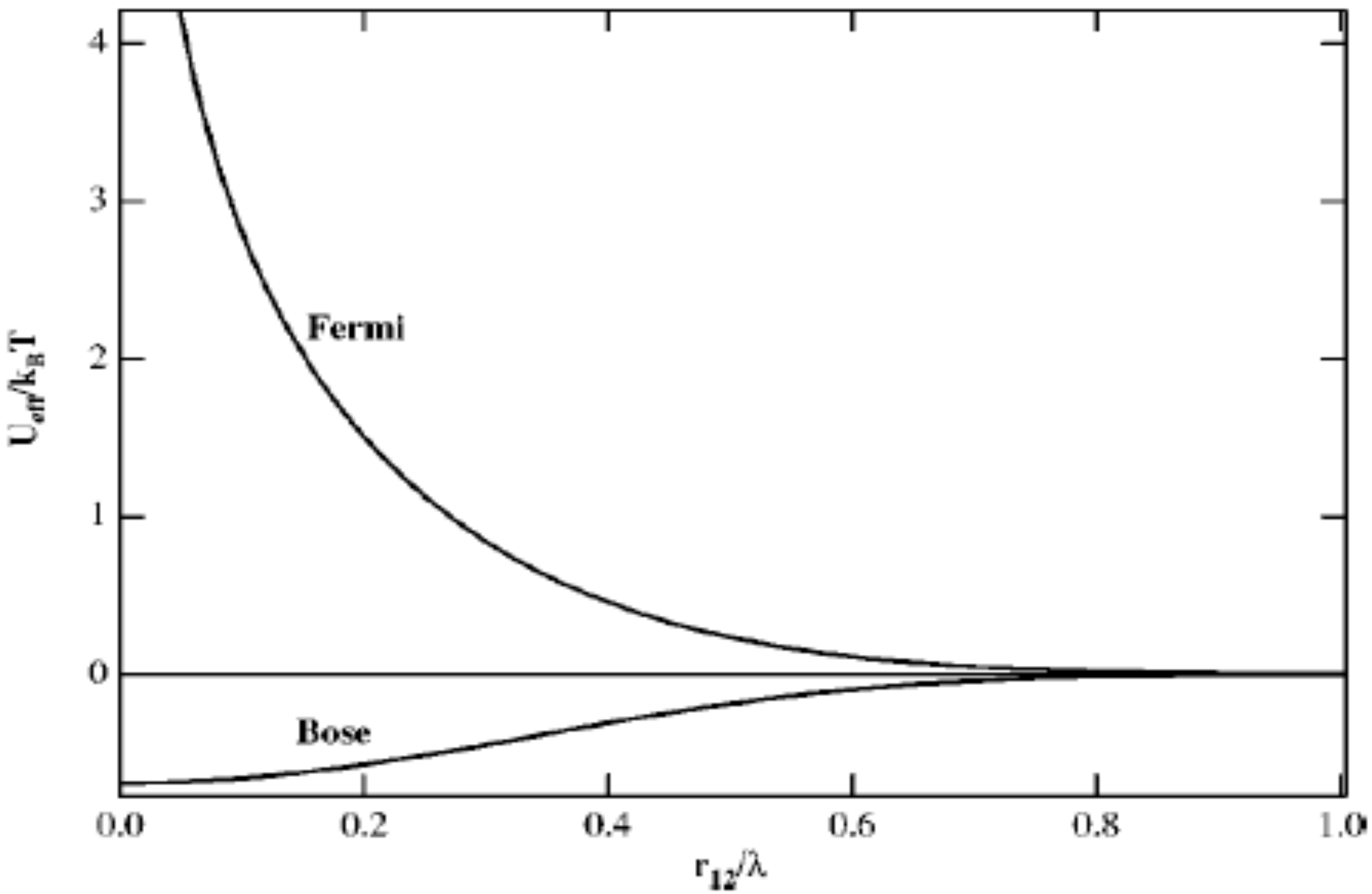


Fig. 1. Plot of the effective statistical interaction versus position. For bosons this function is attractive; for fermions it is repulsive.

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Interaction between particles are absolutely crucial- one may say that genuine condensation is an effect of *exchange coupling.*

Phillip Nozieres (1985)

Phillip Nozieres' argument

$$\hat{\mathcal{H}}_I = \frac{1}{2V} \sum_{p,p',q} V_q \hat{a}_p^+ \hat{a}_{p'}^+ \hat{a}_{p'-q} \hat{a}_{p+q}.$$

Case 1: Single state condensation

If all N particles are condensed in the lowest energy state

$$|\psi_0\rangle = \frac{1}{\sqrt{N!}} \left(\hat{a}_0^+\right)^N |0\rangle = |N\rangle,$$

the corresponding interaction energy is

$$\begin{aligned} E_0 \equiv \left\langle \psi_0 | \hat{\mathcal{H}}_I | \psi_0 \right\rangle &= \frac{V_0}{2V} \langle N | \hat{a}_0^{+2} \hat{a}_0^2 | N \rangle \\ &= \frac{V_0}{2V} N(N-1) \\ &\simeq \frac{V_0}{2V} N^2. \end{aligned}$$

Since Interaction energy adds to the total energy, $V_0 > 0$ (Repulsive)

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$$\hat{\mathcal{H}}_I = \frac{1}{2V}\sum_{p,p',q} V_q \hat{a}_p^+ \hat{a}_{p'}^+ \hat{a}_{p'-q}\hat{a}_{p+q}.$$

Case2 Two state Condensation: $N=N_1+N_2$

$$|\psi_{12}\rangle = \frac{1}{\sqrt{N_1!N_2!}}\left(\hat{a}_1^+\right)^{N_1}\left(\hat{a}_2^+\right)^{N_2}|0\rangle = |N_1\rangle_1|N_2\rangle_2,$$

$$E_{12} \equiv \left\langle \psi_{12}|\hat{\mathcal{H}}_I|\psi_{12}\right\rangle = \left(\underbrace{\frac{1}{2}V_0N_1^2 + \frac{1}{2}V_0N_2^2 + V_0N_1N_2}_{\text{Hartree term}} + \underbrace{V_qN_1N_2}_{\text{Fock term}}\right)/V$$

$$\simeq \frac{1}{2V}V_0N^2 + \frac{1}{V}V_qN_1N_2.$$

Since V_0 ~ *Vq > 0 (repulsive interaction), condensate fragmentation costs a macroscopic* exchange energy. **Genuine Bose-Einstein condensation is not an ideal gas effect but is due to exchange interaction (Fock energy).**

See for details P. Nozieres, in Bose-Einstein Condensation, Ed A. Griffin, D. W. Snoke and S. Stringari (Cambridge University Press, 1995).

Spontaneous Symmetry Breaking -

U(1) Global, Elitzur Theorem, Goldstone Theorem

$$\psi(\mathbf{r}) \to \psi(\mathbf{r})e^{i\alpha} \qquad H[\psi e^{i\alpha}] = H[\psi]$$

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}), \quad \psi_0(\mathbf{r}) \equiv a_0\varphi_0(\mathbf{r})$$

$$\psi_1(\mathbf{r}) = \sum_{k\neq 0} a_k\varphi_k(\mathbf{r}) \; .$$

$$\hat{N}_0 \equiv \int \psi_0^\dagger(\mathbf{r})\psi_0(\mathbf{r}) \; d\mathbf{r} = a_0^\dagger a_0 \; .$$

$$\hat{N}_1 \equiv \int \psi_1^\dagger(\mathbf{r})\psi_1(\mathbf{r}) \; d\mathbf{r} = \sum_{k\neq 0} a_k^\dagger a_k \qquad \hat{N} = \hat{N}_0 + \hat{N}_1 \; .$$

$$N_0 \equiv \langle \hat{N}_0 \rangle = \langle a_0^\dagger a_0 \rangle$$

BEC appears when

$$\lim_{N\to\infty} \frac{\langle a_0^\dagger a_0 \rangle}{N} > 0 \; .$$

Spontaneous Symmetry Breaking - U(1) Global

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(1)Landau's method of Order parameter

(2)Bogoliubov's symmetry breaking term in the H:

$H_\varepsilon[\psi] \equiv H[\psi] + \varepsilon\Gamma[\psi]$, where ε → 0 in the TL.

Spontaneous breakdown of gauge symmetry occurs, when

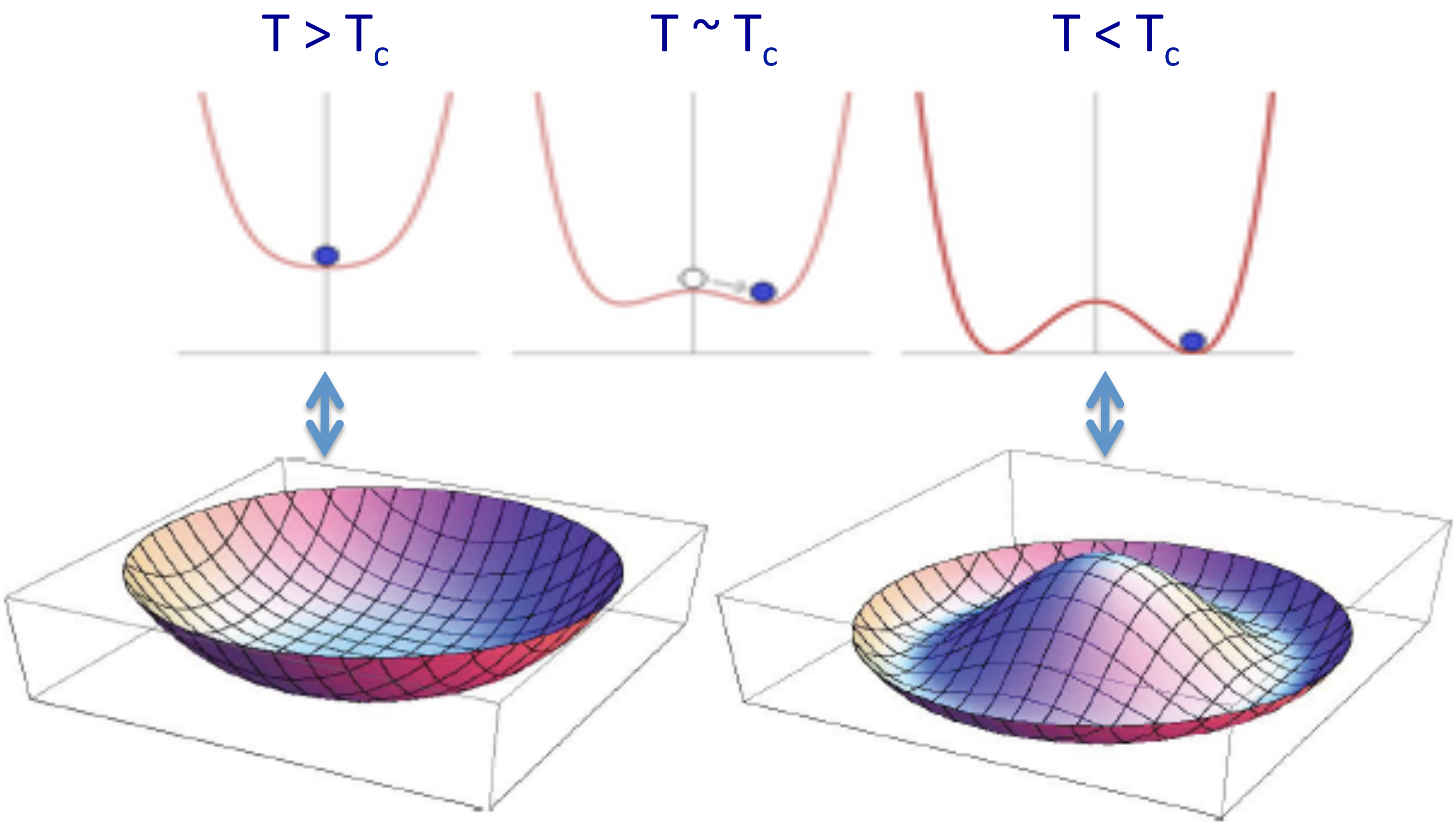
$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\int |\langle\psi_0(\mathbf{r})\rangle_\varepsilon|^2 d\mathbf{r} > 0\,.$$

or,
$$\lim_{N\to\infty}\frac{\langle a_0^\dagger a_0\rangle}{N} \le \lim_{\varepsilon\to 0}\lim_{N\to\infty}\frac{|\langle a_0\rangle_\varepsilon|^2}{N}$$

SBGS is necessary and sufficient condition of occurrence of BEC.

Spontaneous Symmetry Breaking - U(1) Global

Free Energy vs Order Parameter



SBGS is necessary and sufficient condition of occurrence of BEC.

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Mermin-Wagner Theorem

(Hohenberg, Coleman)

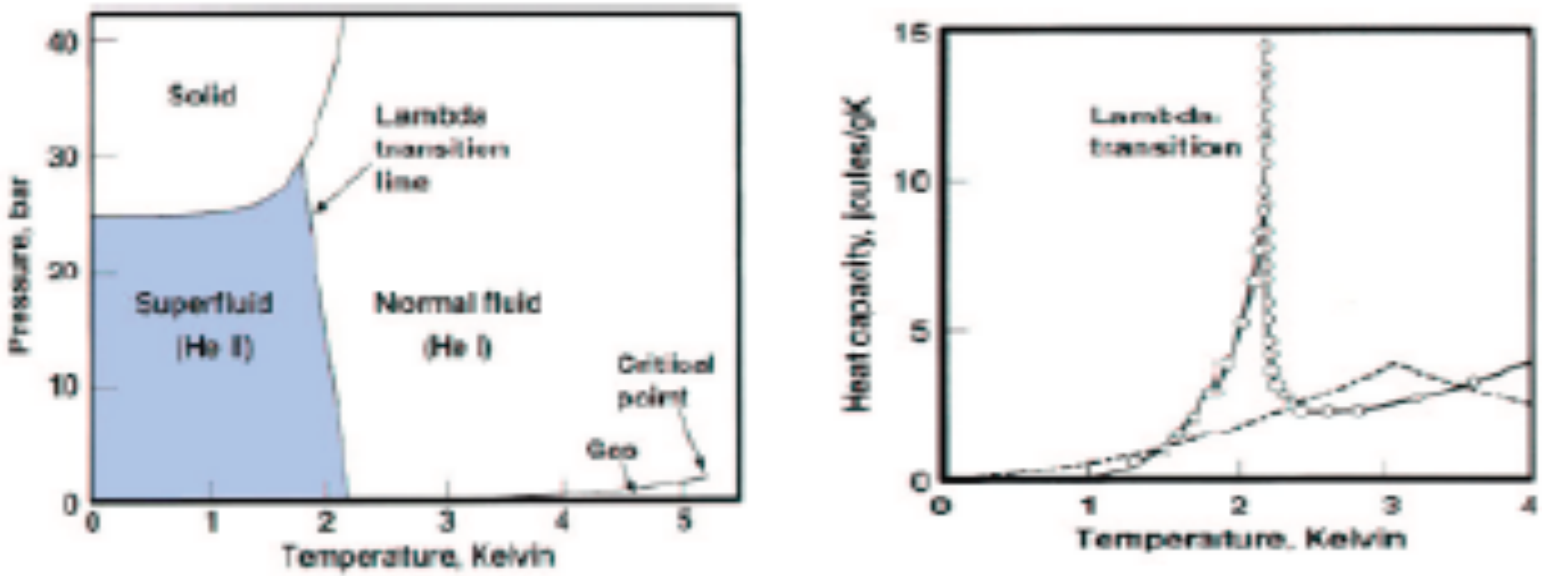
for Phase transition at $T \neq 0$ in $d > 2$

Chester-Penrose Theorem

in $d \geq 2$

Story of ^{4}He

1938 Kapitza and Allen and Misener discovered superfluidity



- Watch the Sp Ht: noninteracting theory vs expt.
- While superfluidity is total, condensate is <10%.
- ^{4}He atoms have strong two body interactions.

Figure 3. Shown are the famous papers of Kapitza (left) and Allen and Misener (right), which appeared back-to-back in *Nature* on January 8, 1938. (Reprinted by permission of Macmillan Publishers Ltd: *Nature* [16, 17], copyright 1938.)

BEC and Superfluidity

- 1924 Einstein: Theoretical prediction of BEC in an ideal gas
- 1938 Kapitza; Allen & Misner: Superfluidity in liquid He^4
- 1938 London: Suggests connection to BEC, Tisza
- 1941 Landau: Two fluid model; elementary excitations. No mention of BEC
- 1946 Bogoliubov: Seminal work on microscopic theory of Bose gases and superfluidity; assumes BEC

Off-Diagonal Long Range Order

One body density matrix is defined by $\rho(r,r') = <\psi^*(r)\,\psi(r')>$

ODLRO $\text{Lim}_{|r-r'|\rightarrow \text{inf}}\ \rho(r,r') \neq 0$ in the Bosonic state.

U(1) gauge symmetry is then spontaneously broken.

Condensation, Superfluidity and ODLRO

Equilibrium ⟶ Bose Condensation (Statistical Property)

↕

Dynamics ⟶ Superfluidity (Response)

↕

ODLRO
(Correlations)

BC is neither necessary nor sufficient for Superfluidity.

BC requires COHERENCE while

Supefluidity requires strong pair correlation

Superfluidity

Landau's critereon

General transformation of energy E and momentum **P** under Galilean boost:
(prime is the moving frame with respect to unprimed reference frame)

$\mathbf{P}' = \mathbf{P} - M\mathbf{V}$

$E' = |\mathbf{P}'|^2/2M = 1/2M\,|\mathbf{P} - M\mathbf{V}|^2 = E - \mathbf{P}.\mathbf{V} + 1/2\,M|\mathbf{V}|^2$

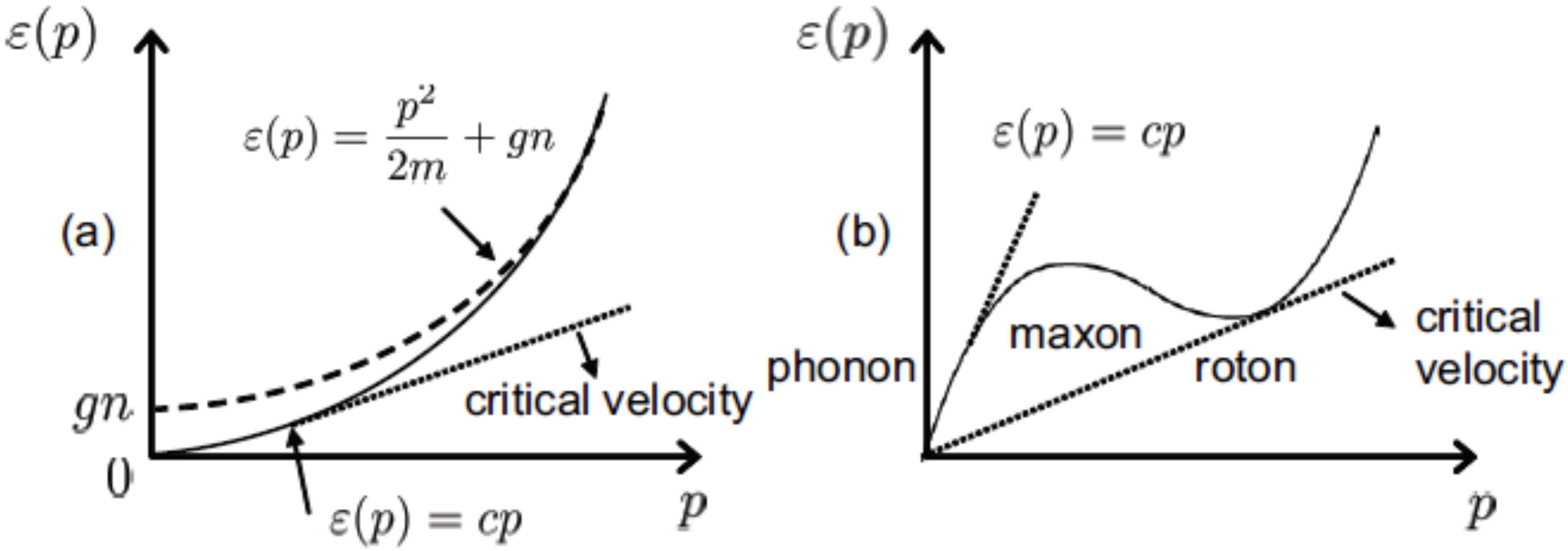
Let us move to the moving frame , in which the fluid moves with a velocity **v** but the container is at rest. In the moving frame, which moves with velocity –**v** with respect to the fluid, the energy and momentum of the fluid are now

$\mathbf{P}' = \mathbf{p} + M\mathbf{v}$ and $\mathbf{E}' = E_0 + \varepsilon(\mathbf{p}) + \mathbf{p}.\mathbf{v} + 1/2\,M|\mathbf{v}|^2$

From this result the changes in the energy and momentum are caused by the appearance of elementary excitation $\varepsilon(\mathbf{p}) + \mathbf{p}.\mathbf{v}$ and **p** respectively.
At thermal equilibrium the condition is $\varepsilon(\mathbf{p}) + \mathbf{p}.\mathbf{v} < 0$, implying nonzero dissipation. From this the critical velocity is $v_c = \min_{\mathbf{p}} [\varepsilon(\mathbf{p})/|p|]$.
Below v_c there will be no elementary excitation.

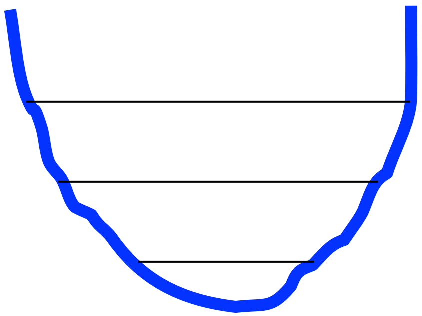
Superfluidity

Elementary Excitations of Bosons in weakly interacting and strongly interacting cases as in ^{4}He.



Critical velocity is smaller than the sound velocity, $v_c < v$.

Confinement brings a difference



Observation of BEC

1995 ^{87}Rb NIST
1995 ^{23}Na MIT
1995 ^{7}Li Rice
1997 ^{7}Li Rice
1997 ^{87}Rb Texas, Stanford, Konstanz, ^{23}Na (Rowland Inst)
1998 ^{87}Rb Münich, Hannover, Sussex, Kyoto, Paris, Otago, ^{3}H MIT, ^{23}Na NIST
1999 ^{87}Rb NIST Florence, Oxford, Pisa, Amsterdam
2000 ^{87}Rb Tokyo
2001 ^{87}Rb Tübingen, Wiezmann, ^{7}Li Paris, ^{41}K Florence, He^{*} Paris

Many others….

^{85}Rb, Cs, Yb, Cr, Dy, Er, Li_2, Na_2,Cs_2 and other mixtures,
Fermion Condensates

Alkalies

^{87}Rb

^{23}Na

^{7}Li ←

---

^{2}H

Physics in a Hurry

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VOLUME 75, NUMBER 9 PHYSICAL REVIEW LETTERS 28 AUGUST 1995

Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions

C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet

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(Received 25 July 1995)

Evidence for Bose-Einstein condensation of a gas of spin-polarized ^{7}Li atoms is reported. Atoms confined to a permanent-magnet trap are laser cooled to 200 μK and are then evaporatively cooled to lower temperatures. Phase-space densities consistent with quantum degeneracy are measured for temperatures in the range of 100 to 400 nK. At these high phase-space densities, diffraction of a probe laser beam is observed. Modeling shows that this diffraction is a sensitive indicator of the presence of a spatially localized condensate. Although measurements of the number of condensate atoms have not been performed, the measured phase-space densities are consistent with a majority of the atoms being in the condensate, for total trap numbers as high as 2×10^5 atoms. For ^{7}Li, the spin-triplet s-wave scattering length is known to be negative, corresponding to an attractive interatomic interaction. Previously, Bose-Einstein condensation was predicted not to occur in such a system.

ERRATA

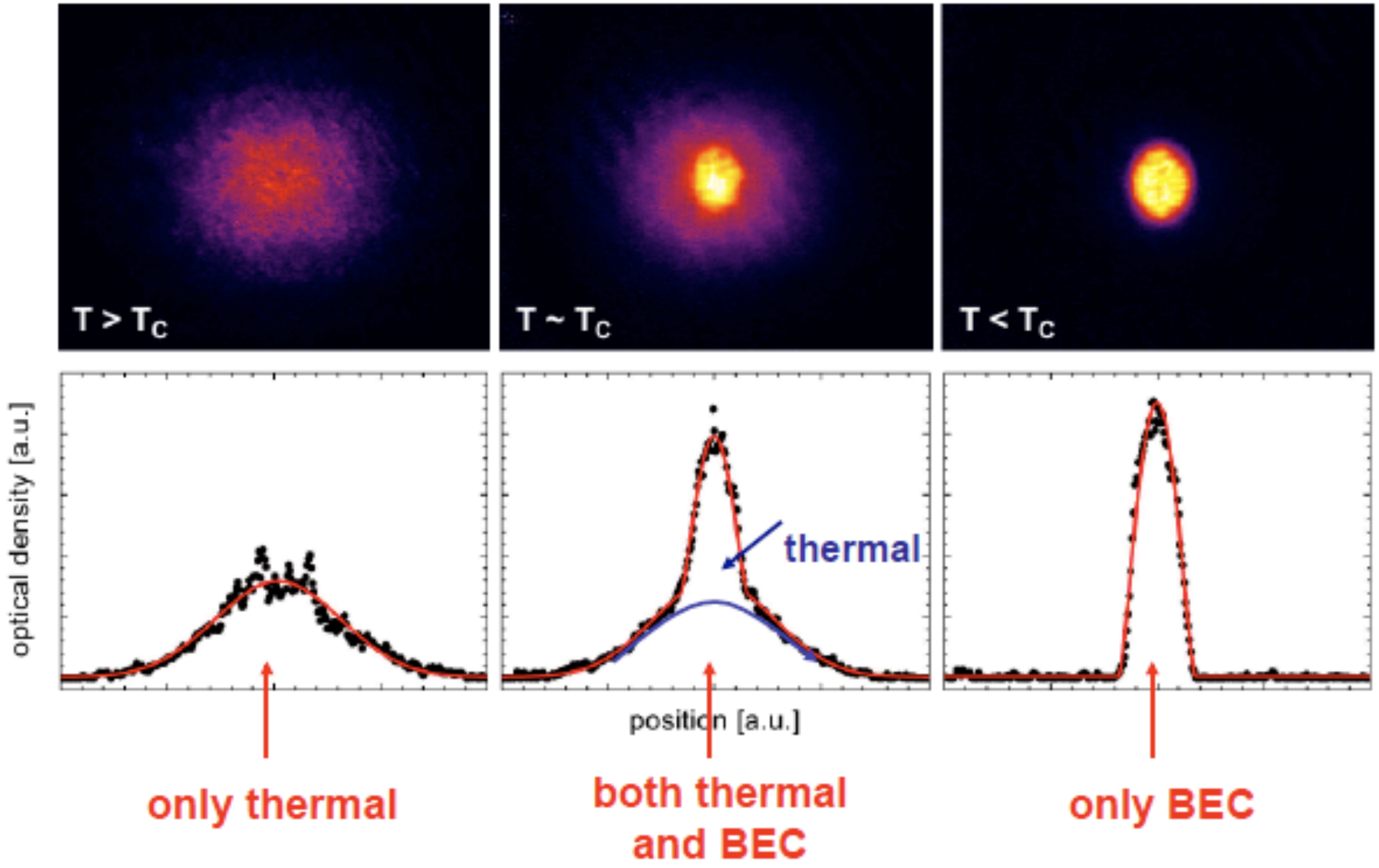
Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions [Phys. Rev. Lett. 75, 1687 (1995)]

C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet

[S0031-9007(97)03808-8]

In our Letter, we interpreted our observation of halolike distortions in absorption images of ultracold ^{7}Li clouds as evidence for Bose-Einstein condensation (BEC). The lens used to produce the images was thought to be nearly diffraction limited. We subsequently determined that the lens actually suffered from substantial spherical aberration, which played a significant role in the formation of the halos. A new analysis of the original data is presented in Ref. [1], and later experiments with an improved imaging system are described in Ref. [2]. Because of the aberration, the estimate of the number of condensate atoms in the Letter was inaccurate. While we stated that the images were consistent with as many as 2×10^5 condensate atoms, it is now clear that only about 10^3 condensate atoms were present. Nevertheless, the conclusion of the Letter that the halos were indicative of BEC does not change in light of these later results.

Delfovo Slide



Temperature is measured by fitting the tails of the thermal component !

?

- The 3D isotropic harmonic potential is given by :

$$V(r) = \frac{1}{2} m\omega^2 r^2 \tag{2}$$

- The energy levels are $E_n = \hbar\omega\left(n + \frac{3}{2}\right)$, where $n = 0,1,2...$
- The total number of particles N is

$$N = \sum_{n=0}^{\infty} \frac{1}{e^{\beta[\hbar\omega(n+\frac{3}{2})-\mu]} - 1} \tag{3}$$

- The condensate fraction for $T < T_c$ is given by:

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^3, \tag{4}$$

and the critical temperature is

$$T_c = \frac{\hbar\omega}{k}\left(\frac{N}{\zeta(3)}\right)^{1/3} \tag{5}$$

3D Non-interacting Bose Gas confined by a Harmonic Potential:

Gnanapragasam, Kim and Das, MPLB 20, 1839 (2006)

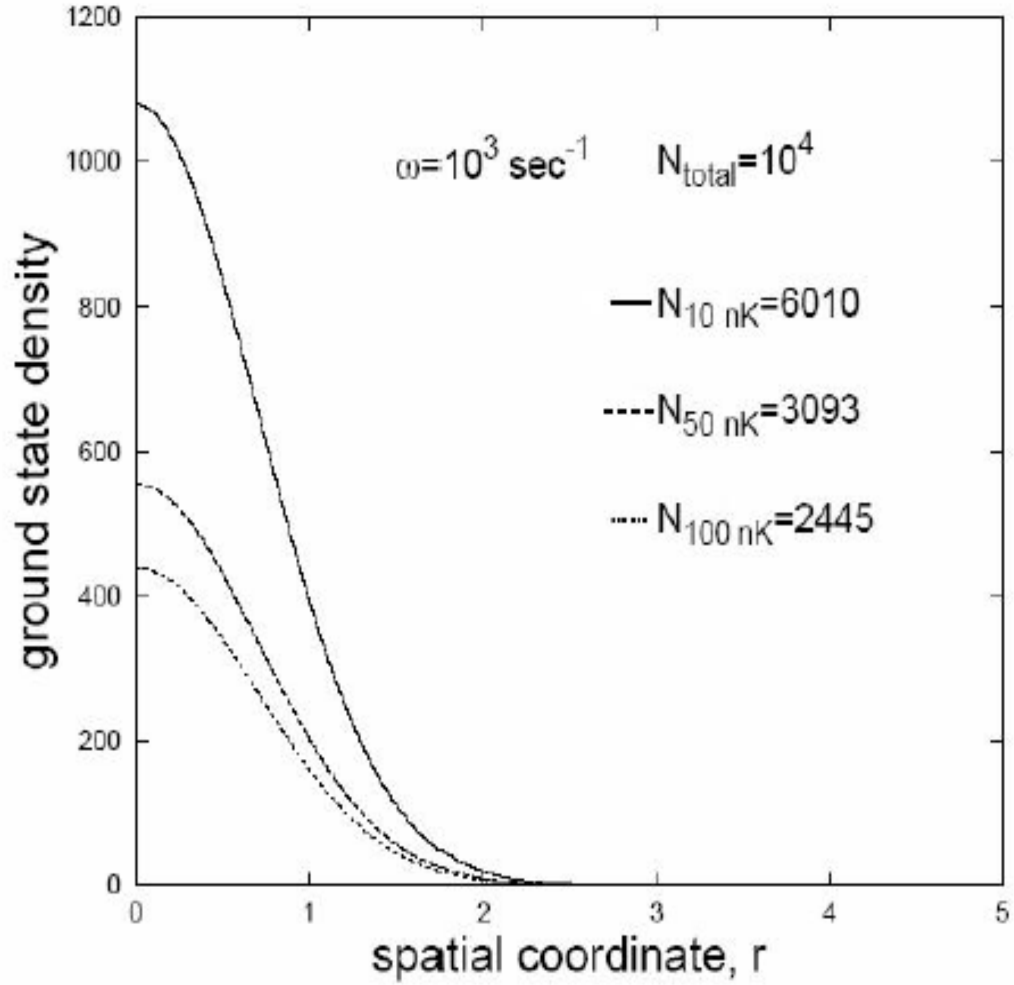


Figure: Shows the depletion of the condensate due to temperature

Part Summary of Lecture 1

→BEC is a Quantum State of accumulated Bosons. Take care of TL if one claims there is a phase transition.

→ Genuine Bose-Einstein condensation is not an ideal gas effect but is due to QM exchange interaction (Fock energy). Phillip Nozieres

→ Bose Statistics, Einstein's application to Bosonic atoms (BEC)

→ BEC and Superfluidity (Broken Symmetry, ODLRO)

→ **? A new state of matter of ultracold atoms ?**

Do we know about its material properties as usually understood in stastistical mech sense?- mechanical
thermal
electromagnetic ...

→ **ULTRACOLD ATOMS IN LIMBO/UNDER CONSTRAINT**

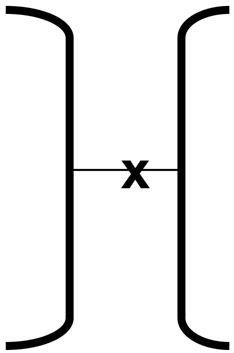
Lecture 2

One is born free, but …
where is the freedom? ……..

Everyone ought to be interacting.

Role of Interaction

$$V_{ijkl} = \iint d\mathbf{r}\, d\mathbf{r}' \;\psi_i^*(\mathbf{r})\psi_j^*(\mathbf{r}')\, V^{I}(r,r')\, \psi_k(\mathbf{r}')\psi_l(\mathbf{r})$$



Microscopic Theory in a Many-Body Paradigm: A reminder

- Homogeneous Interacting Boson Condensation
  - **Bogoliubov**
  - Beliaev
  - Lee, Huang and Yang
  - Hugenholtz and Pines
  - **Luban**
  - **Ter Haar**
- To go beyond Hartree-mean field theory (Gross Pitaevskii)
- Correlations among condensed and noncondensed atoms

Microscopic theory (Bogoliubov 1947, Beliaev 1958)

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$$H_\Lambda = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k,k',q} v(q) a_{k+q}^* a_{k'-q}^* a_{k'} a_k$$

This Hamiltonian in the general form is insoluble.
Bogoliubov chose the interaction to have this particular choice, direct (Hartree), Exchange and Pairing terms.

$$H_\Lambda^B = \sum_k \epsilon_k a_k^* a_k + \frac{1}{V} v(0) a_0^* a_0 \sum_{k\neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k\neq 0} v(k) [a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_{-k} a_k] + \frac{1}{2V} v(0) a_0^{*2} a_0^2 \, .$$

Microscopic Theory (Bogoliubov 1947, Beliaev 1958)

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- Bogoliubov Hamiltonian is solved by his Linear transformation method (discussed later)
- GF Eqn. of motion method by Zubarev
- Diagrammatic method by Beliaev.

Microscopic Theory (N N Bogoliubov) by CT

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See Pethick and Smith Chapter 8 for details-

$$H = \int d\mathbf{r} \left[-\hat{\psi}^\dagger(\mathbf{r}) \frac{\hbar^2}{2m} \nabla^2 \hat{\psi}(\mathbf{r}) + V(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) + \frac{U_0}{2} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}) \right].$$

$$\hat{\psi}(\mathbf{r}) = \psi(\mathbf{r}) + \delta\hat{\psi}(\mathbf{r}).$$

Excitations in a uniform gas:

$$H = \sum_{\mathbf{p}} \epsilon_p^0 a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{U_0}{2V} \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^\dagger a_{\mathbf{p}'-\mathbf{q}}^\dagger a_{\mathbf{p}'} a_{\mathbf{p}}$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = \delta_{\mathbf{p},\mathbf{p}'}, \quad [a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0, \quad \text{and} \quad [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0.$$

In the unperturbed state

$$a_0^\dagger |N_0\rangle = \sqrt{N_0+1}|N_0+1\rangle \;\; \text{and} \;\; a_0|N_0\rangle = \sqrt{N_0}|N_0-1\rangle$$

Bogoliubov's quasi-average $a_p = \sqrt{N_0}$, $\psi = \sqrt{N_0}\phi_0$, where $\phi_0 = V^{-1/2}$

Elementary Excitations:

Make the transformation

$$a_{\mathbf{p}} = u_p \alpha_{\mathbf{p}} - v_p \alpha^{\dagger}_{-\mathbf{p}}, \quad a_{-\mathbf{p}} = u_p \alpha_{-\mathbf{p}} - v_p \alpha^{\dagger}_{\mathbf{p}}.$$

Substitute back into the Hamiltonian

$$H = \frac{N^2 U_0}{2V} + \sum_{\mathbf{p}(\mathbf{p}\neq 0)} \epsilon_p \alpha^{\dagger}_{\mathbf{p}} \alpha_{\mathbf{p}} - \frac{1}{2} \sum_{\mathbf{p}(\mathbf{p}\neq 0)} (\epsilon^0_p + n_0 U_0 - \epsilon_p)$$

with

$$\epsilon_p = \sqrt{(\epsilon^0_p + n_0 U_0)^2 - (n_0 U_0)^2} = \sqrt{(\epsilon^0_p)^2 + 2\epsilon^0_p n_0 U_0}.$$

For small p, dispersion ε_p = sp , where $s^2 = \frac{n_0 U_0}{m}.$

Operators which create and destroy the excitations $\alpha^{\dagger}_{\mathbf{p}} = u_p a^{\dagger}_{\mathbf{p}} + v_p a_{-\mathbf{p}}.$

with normalisation $u_p^2 - v_p^2 = 1$

Explicitly $u_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} + 1\right)$ and $v_p^2 = \frac{1}{2}\left(\frac{\xi_p}{\epsilon_p} - 1\right),$

where $\xi_p = \epsilon^0_p + n_0 U_0$, difference between HF energy of a particle and μ

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Depletion

$$\hat{N} = N_0 + \sum_{\mathbf{p}(\mathbf{p}\neq 0)} v_p^2 + \sum_{\mathbf{p}(\mathbf{p}\neq 0)} (u_p^2 + v_p^2)\alpha_{\mathbf{p}}^\dagger \alpha_{\mathbf{p}} - \sum_{\mathbf{p}(\mathbf{p}\neq 0)} u_p v_p (\alpha_{\mathbf{p}}^\dagger \alpha_{-\mathbf{p}}^\dagger + \alpha_{-\mathbf{p}}\alpha_{\mathbf{p}}).$$

$$n_{\mathrm{ex}} = \frac{1}{V} \sum_{\mathbf{p}(\mathbf{p}\neq 0)} v_p^2 = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} v_p^2 = \frac{1}{3\pi^2}\left(\frac{ms}{\hbar}\right)^3$$

$$\frac{n_{\mathrm{ex}}}{n} = \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}.$$

Groundstate Energy

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Use p → p_c as a cut-off of p, the groundstate energy is

$$E_0 = \frac{N^2 U(p_c)}{2V} - \frac{1}{2} \sum_{\mathbf{p}(p<p_c)} \left(\epsilon_p^0 + n_0 U_0 - \epsilon_p\right).$$

where

$$U(p_c) = U_0 + \frac{U_0^2}{V} \sum_{\mathbf{p}(p<p_c)} \frac{1}{2\epsilon_p^0}.$$

$$E_0 = \frac{N^2 U_0}{2V} - \frac{1}{2} \sum_{\mathbf{p}(p<p_c)} \left[\epsilon_p^0 + n_0 U_0 - \epsilon_p - \frac{(nU_0)^2}{2\epsilon_p^0}\right].$$

$$\begin{aligned} \frac{E_0}{V} &= \frac{n^2 U_0}{2} + \frac{8}{15\pi^2}\left(\frac{ms}{\hbar}\right)^3 ms^2 \\ &= \frac{n^2 U_0}{2}\left[1 + \frac{128}{15\pi^{1/2}}(na^3)^{1/2}\right]. \end{aligned}$$

← Lee and Yang correction

Hartree-Fock Approximation

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = c_N \sum_{\text{sym}} \phi_1(\mathbf{r}_1)\phi_2(\mathbf{r}_2)\ldots\phi_N(\mathbf{r}_N).$$

$$c_N = \left(\frac{\prod_i N_i!}{N!}\right)^{1/2}$$

$$U_{ij}^{\text{Hartree}} = \int d\mathbf{r}d\mathbf{r}' U(\mathbf{r}-\mathbf{r}')|\phi_i(\mathbf{r})|^2|\phi_j(\mathbf{r}')|^2,$$

$$U_{ij}^{\text{Fock}} = \int d\mathbf{r}d\mathbf{r}' U(\mathbf{r}-\mathbf{r}')\phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')\phi_i(\mathbf{r}')\phi_j(\mathbf{r}).$$

Using second quantisation the interaction energy is

$$U = \frac{1}{2}\sum_{ijkl}\langle ij|U|kl\rangle a_i^\dagger a_j^\dagger a_l a_k,$$

where $\langle ij|U|kl\rangle = \int d\mathbf{r}d\mathbf{r}' U(\mathbf{r}-\mathbf{r}')\phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')\phi_l(\mathbf{r}')\phi_k(\mathbf{r})$

$$
\begin{aligned}
U &= \frac{1}{2}\sum_{ij}\langle ij|U|ij\rangle N_i(N_j-\delta_{ij}) + \frac{1}{2}\sum_{ij(i\neq j)}\langle ij|U|ji\rangle N_iN_j \\
&= \frac{1}{2}\sum_{i}\langle ii|U|ii\rangle N_i(N_i-1) + \sum_{i<j}(\langle ij|U|ij\rangle + \langle ij|U|ji\rangle)N_iN_j.
\end{aligned}
$$

For contact interaction $U = \frac{1}{2}\sum_{i}\langle ii|U|ii\rangle N_i(N_i-1) + 2\sum_{i<j}\langle ij|U|ij\rangle N_iN_j.$

In the uniform case the total energy

$$
\begin{aligned}
E &= \sum_{\mathbf{p}}\epsilon_p^0 N_{\mathbf{p}} + \frac{U_0}{2V}\sum_{\mathbf{p},\mathbf{p}'}N_{\mathbf{p}}(N_{\mathbf{p}'}-\delta_{\mathbf{p},\mathbf{p}'}) + \frac{U_0}{2V}\sum_{\mathbf{p},\mathbf{p}'(\mathbf{p}\neq\mathbf{p}')}N_{\mathbf{p}}N_{\mathbf{p}'} \\
&= \sum_{\mathbf{p}}\epsilon_p^0 N_{\mathbf{p}} + \frac{U_0}{2V}N(N-1) + \frac{U_0}{2V}\sum_{\mathbf{p},\mathbf{p}'(\mathbf{p}\neq\mathbf{p}')}N_{\mathbf{p}}N_{\mathbf{p}'} \\
&= \sum_{\mathbf{p}}\epsilon_p^0 N_{\mathbf{p}} + \frac{U_0}{V}\left(N^2 - \frac{1}{2}\sum_{\mathbf{p}}N_{\mathbf{p}}^2 - \frac{N}{2}\right).
\end{aligned}
$$

Excitation energy $\epsilon_{\mathbf{p}} = \epsilon_p^0 + \frac{N}{V}U_0 + \frac{N-N_{\mathbf{p}}}{V}U_0.$ (Last two terms are H and F terms)

Popov Approximation

To go beyond the Hartree–Fock approximation,
allow for the mixing of particle-like and hole-like excitations due to the interaction, which is reflected in the coupling of the equations for *u and v.*

Self energy of interacting bosons- See details in Pethick and Smith.

Dilute weakly interacting Bosons (Uniform Gas)

Hugenholtz and Pines(1959)

$$\frac{E_g}{V} = \frac{2\pi\hbar^2 a n^2}{m} \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} C_{ij} (na^3)^{i/2} \{\ln(na^3)\}^j$$

After elaborate calculations

$$\frac{E_g}{V} = \frac{2\pi\hbar^2 a n^2}{m} \left[1 + \frac{128}{15\sqrt{\pi}} (na^3)^{1/2} + \left\{ \frac{8(4\pi - 3\sqrt{3})}{3} \ln(na^3) + \kappa \right\} na^3 + \ldots \right]$$

$C_{00}=1$, $C_{10}=128/15\sqrt{\pi}$, $C_{21}=8(4\pi-3\sqrt{3})/3$, $\kappa = C_{20}$

The quantity C_{20} has never been determined

(See Fetter and Walecka p.221-222).

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Kim, Kim and Das (IJMPB, 21, 5309 (2007))

Calculations by canonical transformation method

$$\frac{N - N_0}{N} = \frac{1}{N}\sum_{q\neq 0} n_q$$

$$= \frac{8}{3}\sqrt{\frac{na^3}{\pi}} + 2\left(\pi - \frac{8}{3}\right) na^3 + \ldots .$$

$$\frac{E_2}{E_c} = 16\pi\left(\pi - \frac{8}{3}\right)p^2 + 32\pi\left(\pi - \frac{8}{3}\right)\left(\frac{10}{3} - \pi\right)p^3 + \mathcal{O}(p^4).$$

Here $p = \sqrt{na^3/\pi}$

$C_{20} = 16\pi(\pi - 8/3)$

Order parameter in microscopic theory (1958)

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(Bogoliubov-Beliaev) Normal and Anomalous Gren Functions

$G \sim <| a\, a^{+} |>$

$F \sim <| a\, a\, |>$

and

$F^{+} \sim <| a^{+} a^{+} |>$

Bogoliubov-Beliaev's gap eqn.

$$\Delta(\mathbf{k}) = -1/2 \sum_{\mathbf{k}'} \Delta(\mathbf{k}') \, v(\mathbf{k}-\mathbf{k}')/\sqrt{[\varepsilon^2(\mathbf{k}')-\Delta^2(\mathbf{k}')]}$$

Bogoliubov-Beliaev's gap eqn.

$$\Delta(\mathbf{k}) = -1/2 \sum_{\mathbf{k}'} \Delta(\mathbf{k}') \, v(\mathbf{k}-\mathbf{k}') / \sqrt{[\varepsilon^2(\mathbf{k}') - \Delta^2(\mathbf{k}')]}$$

Luban (1962) showed for solubility of the gap eqn., the pair potential, v(**r**) has to be repulsive-definite.

see also ter Haar (1977)

For a nonconfined interacting Bose gas
condensation CAN NOT occur without repulsion.

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Gross Pitaevskii Theory for Bosons

- similar to Ginzburg-Landau Theory of Superconductivity

$F = F[\Psi]$ - expanded in even powers of Ψ.

One of the most popular theories in BEC physics.

Gross Pitaevskii Free energy Functional expanded as even powers of Ψ

F = KE + Pot Energy + Interactions

* Contact interactions
* Hartree Mean-field (No fluctuations)

Mean Field Approximation

- In the mean field approximation the Boson field operator $\psi(\mathbf{r}, t)$ is given by :

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r}, t) + \psi'(\mathbf{r}, t) \tag{6}$$

where $\psi'(\mathbf{r}, t)$ is a small perturbation; $<\psi'>_g = 0$

- The function $\phi(\mathbf{r}, t)$ is a 'classical' field having the meaning of an order parameter and is often called the "macroscopic wave function of the condensate". The eqn for the order parameter is the GP eqn which is :

$$i\hbar\frac{\partial\phi(\mathbf{r}, t)}{\partial t} = \left(\frac{-\hbar^2\nabla^2}{2m} + V_{trap}(\mathbf{r}) + g|\phi(\mathbf{r}, t)|^2\right) \phi(\mathbf{r}, t) \tag{7}$$

where 'g' is the coupling constant $= \frac{4\pi\hbar^2 a}{m}$

:Repulsive pair interaction for BEC to occur

Gross-Pitaevskii Eqn.

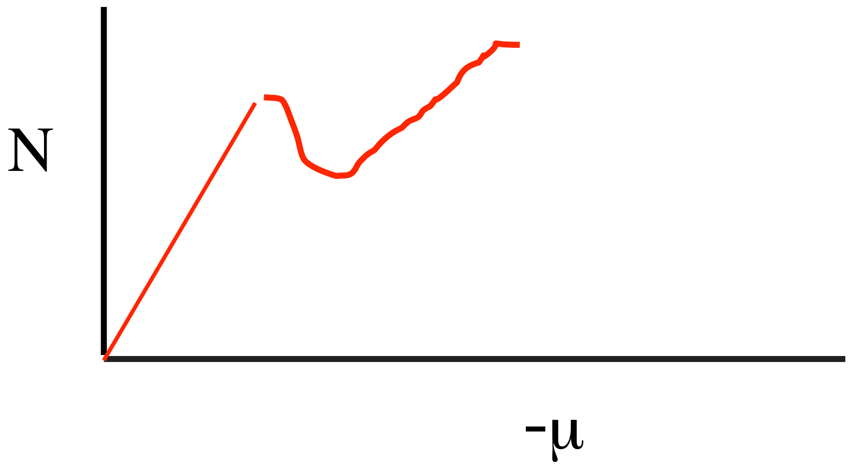
Akhmediev, Das and Vagov. Aust J Phys 53, 157 (2000)

(Ref to Randy Hulet's work as mentioned before)

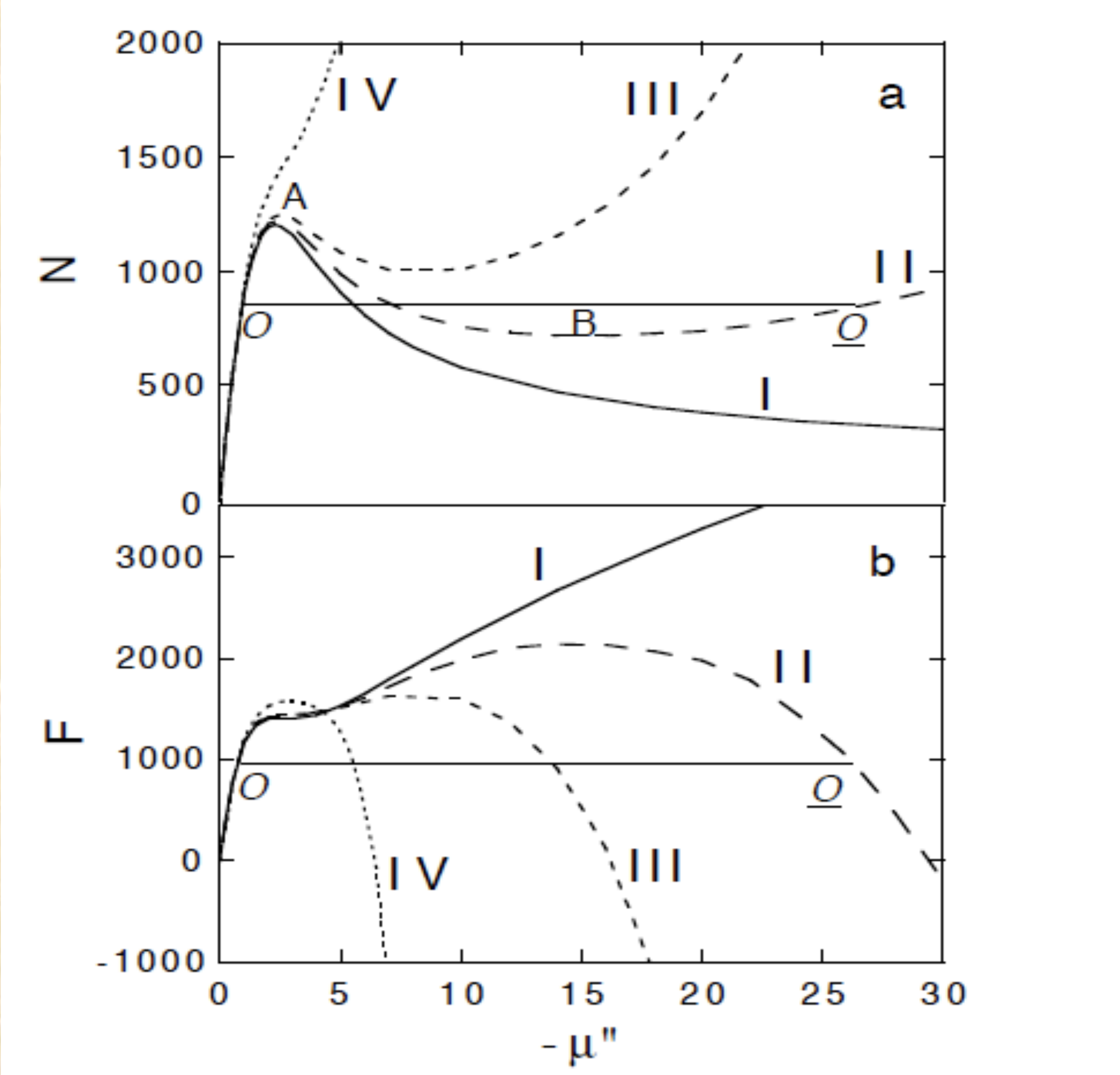
For attractive interaction, i.e the scattering length is negative, e.g. ^{7}Li, $N < 1400$ for stable solution.

(I) Higher order terms $|\phi|^5$ in GP eqn bring back stability.

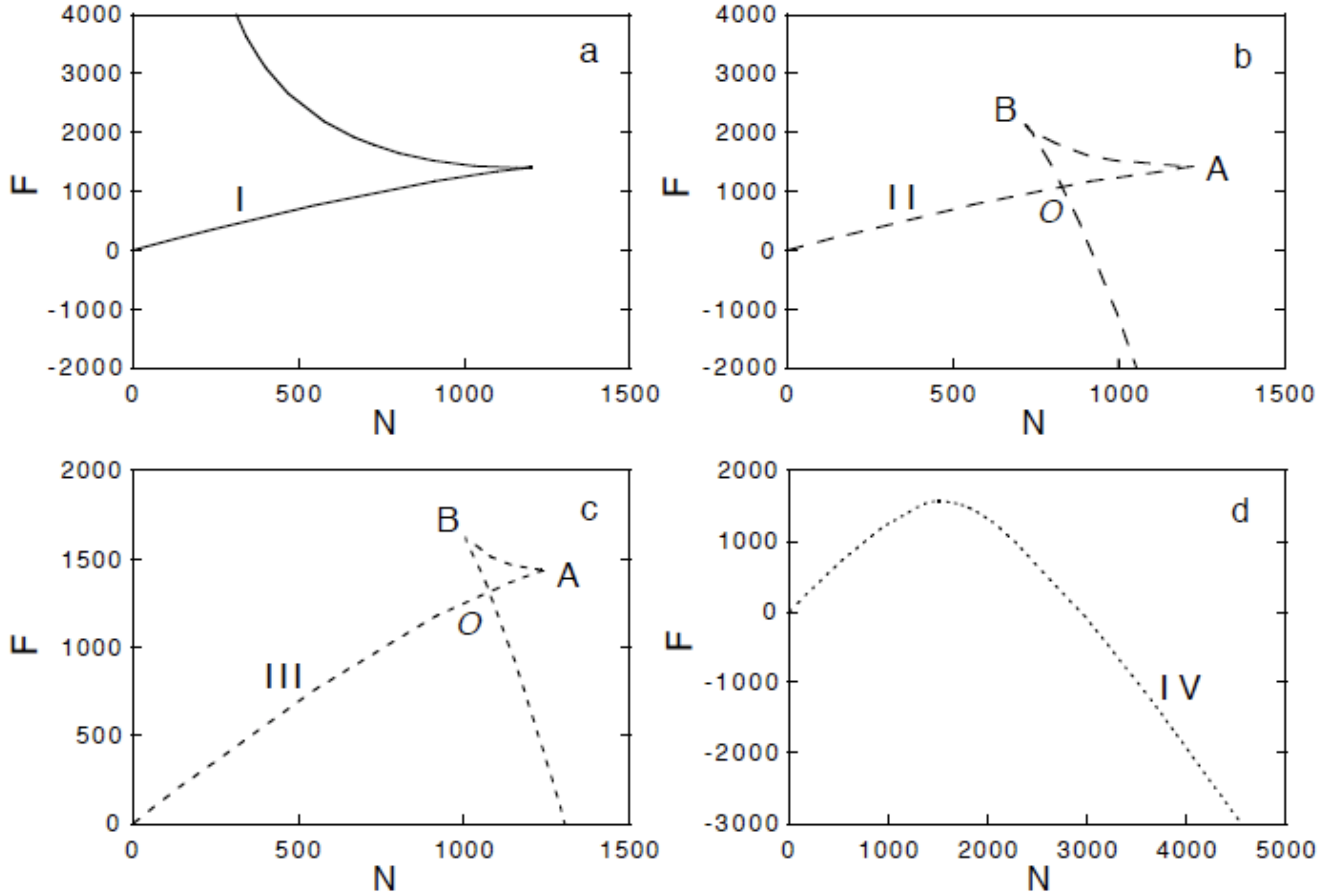
(II) Metastability will introduce vortices.



Akhmediev, Das and Vagov. Aust J Phys 53, 157 (2000)



Akhmediev, Das and Vagov. Aust J Phys 53, 157 (2000)



Comments on GP Theory

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1. It is a mean-field theory at the Hartree level.

2. No information on excited states.

3. What happens to interactions of atoms in the condensate and excited states?

Therefore, there is a need for a microscopic theory, that we described here.

Hamiltonian for trapped interacting Bosons

Gnanapragasam and Das, IJMPA 22, 4923 (2007).

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The total Hamiltonian for the system, in the second quantized form is:

$$H = \sum_{ij} H_{ij}^{(0)} a_i^\dagger a_j + \frac{1}{2} \sum_{ijkl} < ij|V^I|kl > a_i^\dagger a_j^\dagger a_k a_l \quad (8)$$

where

- $H_{ij}^{(0)}$, is the one-body Hamiltonian
- $a^\dagger, a$ are the boson creation and annihilation operators respectively

$$a_i^\dagger \, |n_0, n_1, ..., n_i, ... > = \sqrt{n_i + 1} \, |n_0, n_1, ..., n_i + 1, ... > \quad (9)$$

$$a_i \, |n_0, n_1, ..., n_i, ... > = \sqrt{n_i} \, |n_0, n_1, ..., n_i - 1, ... > \quad (10)$$

They obey the usual commutation rules:

$$[a_i, a_j^\dagger] = \delta_{i,j}, \quad [a_i, a_j] = 0 = [a_i^\dagger, a_j^\dagger]$$

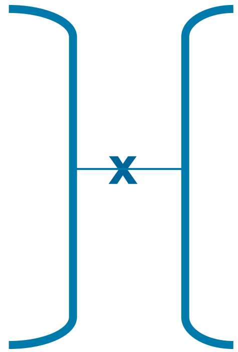
Hamiltonian for trapped Bosons

- V^I is the two-body interaction potential
- The matrix elements

$$H_{ij}^{(0)} = \int d\mathbf{r} \left[\phi_i^*(\mathbf{r}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{trap}(\mathbf{r}) \right) \phi_j(\mathbf{r}) \right] \tag{11}$$

and

$$< ij|V^I|kl > = \iint d\mathbf{r} d\mathbf{r}' \phi_i^*(\mathbf{r}) \phi_j^*(\mathbf{r}') V^I \phi_k(\mathbf{r}') \phi_l(\mathbf{r}) \tag{12}$$



Hamiltonian for trapped Bosons

$$H = \epsilon_0 a_0^\dagger a_0 + \frac{1}{2} V_0 a_0^\dagger a_0^\dagger a_0 a_0 + \sum_i{}' \epsilon_{i0} a_i^\dagger a_0 + \sum_i{}' \epsilon_{0i} a_0^\dagger a_i$$
$$+ \sum_{ij}{}' \epsilon_{ij} (a_i^\dagger a_j + a_j^\dagger a_i) + \sum_i{}' V_{i000} a_i^\dagger a_0^\dagger a_0 a_0 + \sum_i{}' V_{00i0} a_0^\dagger a_0^\dagger a_i a_0$$
$$+ \frac{1}{2} \sum_{ij}{}' V_{ij00} a_i^\dagger a_j^\dagger a_0 a_0 + \frac{1}{2} \sum_{ij}{}' V_{00ij} a_0^\dagger a_0^\dagger a_i a_j + 2 \sum_{ij}{}' V_{i0j0} a_i^\dagger a_0^\dagger a_j a_0$$
$$+ \sum_{ijk}{}' V_{ijk0} a_i^\dagger a_j^\dagger a_k a_0 + \sum_{ijk}{}' V_{k0ij} a_k^\dagger a_0^\dagger a_i a_j + \frac{1}{2} \sum_{ijkl}{}' V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \qquad (13)$$

where we have used the short hand notation:
$\epsilon_0 = H_{00}^{(0)}$, $\epsilon_{0j} = H_{0j}^{(0)}$, $V_0 =< 00|V^I|00 >$, $V_{i000} =< i0|V^I|00 >$ etc.

Double Time Thermodynamic Green Functions

The double-time temperature Green functions are defined as :

$$G_c(t,t') = -i\theta(t-t') < A(t)B(t') > -i\eta\theta(t'-t) < B(t')A(t) >$$
$$\equiv \ll A(t); B(t') \gg_c \quad (14)$$

$$G_r(t,t') = -i\theta(t-t') < [A(t), B(t')] > \equiv \ll A(t); B(t') \gg_r \quad (15)$$

$$G_a(t,t') = i\theta(t'-t) < [A(t), B(t')] > \equiv \ll A(t); B(t') \gg_a \quad (16)$$

where G_c G_r G_a are the causal, retarded and advanced Green functions, A and B are the Bose operators.

$$\theta = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

For applications in many-body physics one often uses the retarded and advanced Green functions.

Green Functions

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Here we define the normal and the anomalous Green functions:
$G_{mn} \equiv \ll a_m; a_n^\dagger \gg$ and $F_{mn} \equiv \ll a_m^\dagger; a_n^\dagger \gg$
The eqn of motion for G_{mn} is given by:

$$i\frac{d}{dt}G_{mn} = \delta(t-t') < [a_m; a_n^\dagger] > + \ll i\frac{da_m(t)}{dt}; a_n^\dagger \gg \tag{17}$$

where a_m satisfies the equation of motion:

$$i\frac{da_m(t)}{dt} = a_m H - H a_m \tag{18}$$

Thus we get:

$$i\frac{d}{dt}G_{mn} = \delta(t-t') < [a_m; a_n^\dagger] > + \ll [a_m, H]; a_n^\dagger \gg \tag{19}$$

Similarly the eqn of motion for F_{mn} is given by :

$$i\frac{d}{dt}F_{mn} = \delta(t-t') < [a_m^\dagger; a_n^\dagger] > + \ll [a_m^\dagger, H]; a_n^\dagger \gg \tag{20}$$

Higher order GF and decoupling

- Eqn of motion for $\ll a; a^\dagger \gg$ will produce a higher order Green function of the type $\ll a^\dagger aa; a^\dagger \gg$
- Hierarchy of never ending eqns $\Rightarrow$ No exact solution
- Wick's Decoupling

$$\ll a_0^+ a_0 a_0; a_0^+ \gg = N_0 G_{00},$$

$$\ll a_0^+ a_0^+ a_0; a_0^+ \gg = N_0 F_{00},$$

$$\ll a_i^+ a_0 a_0; a_0^+ \gg = A_{00} F_{i0},$$

$$\ll a_0^+ a_0^+ a_i; a_0^+ \gg = A_{00} G_{i0}$$

where $A_{00} = < a_0 a_0 > = < a_0^+ a_0^+ >^*$ and $N_0 = < a_0^+ a_0 >$

Green Functions for the Ground State

$$\left(E-\epsilon_0-V_0N_0-2\sum_i{}' V_{i0i0}N_i\right)G_{00}=\frac{1}{2\pi}+\left(\sum_i{}' V_{00ii}A_{ii}+\sum_{i,j\neq i}{}' V_{00ij}A_{ij}\right)F_{00}$$

$$+\left(\sum_i{}' \epsilon_{0i}+2\sum_i{}' V_{00i0}N_0+\sum_i{}' V_{i0ii}N_i+\sum_{i,j\neq i}{}' V_{j0ij}N_j\right)G_{i0}$$

$$+\left(\sum_i{}' V_{i000}A_{00}+2\sum_{i,j\neq i}{}' V_{i0j0}A_{j0}\right)F_{i0}+\sum_{i,j\neq i}{}' V_{i0ij}N_iG_{j0}$$

$$+\left(\sum_{i,jk\neq i}{}' V_{k0ij}A_{ij}+\sum_{i,k\neq i}{}' V_{k0ii}A_{ii}\right)F_{k0}$$

where $A_{ij} = <a_ia_j> = <a_i^\dagger a_j^\dagger>^*$ and $N_i = <a_i^\dagger a_i>$

Green Functions for the Ground State

$$\left(E+\epsilon_0+V_0N_0+2\sum_i{}' V_{i0i0}N_i\right)F_{00} = -\left(\sum_i{}' V_{ii00}A_{ii}+\sum_{i,j\neq i}{}' V_{ij00}A_{ij}\right)G_{00}$$

$$-\left(\sum_i{}' \epsilon_{i0}+2\sum_i{}' V_{i000}N_0+\sum_i{}' V_{iii0}N_i+\sum_{i,j\neq i}{}' V_{ijj0}N_j\right)F_{i0}$$

$$-\left(\sum_i{}' V_{00i0}A_{00}+2\sum_{i,j\neq i}{}' V_{j0i0}A_{j0}\right)G_{i0}-\sum_{i,j\neq i} V_{ijj0}N_iF_{j0}$$

$$-\left(\sum_{i,jk\neq i}{}' V_{ijk0}A_{ij}+\sum_{i,k\neq i}{}' V_{iik0}A_{ii}\right)G_{k0}$$

where $A_{ij} = <a_ia_j> = <a_i^\dagger a_i^\dagger>^*$ and $N_i = <a_i^\dagger a_i>$

Green Functions for the Ground State

$$(E - \tilde{\epsilon_0})\, G^0_{00} \;=\; \frac{1}{2\pi} + X_1 F^0_{00} \tag{23}$$

$$(E + \tilde{\epsilon_0})\, F^0_{00} \;=\; -X_1 G^0_{00} \tag{24}$$

where

$$X_1 = \sum_{i,j}{}' V_{ij00} A_{ij},$$

$$\tilde{\epsilon_0} = \epsilon_0 + V_0 N_0 + 2 \sum_i{}' V_{i0i0} N_i$$

Solving the above two eqns we get:

$$G^0_{00} = \frac{1}{2\pi} \frac{E + \tilde{\epsilon_0}}{(E^2 - \tilde{\epsilon_0}^2 + X_1^2)} \tag{25}$$

Condensate Number

The average occupation number of the condensed state N_0 is given by:

$$N_0 = \int_{-\infty}^{\infty} J_{00}(\omega) d\omega \qquad (26)$$

which gives:

$$N_0 = \frac{N}{2}\left[\frac{\tilde{\epsilon_0}}{\overline{\epsilon}} \coth\frac{\beta\overline{\epsilon}}{2} - 1\right] \qquad (27)$$

where

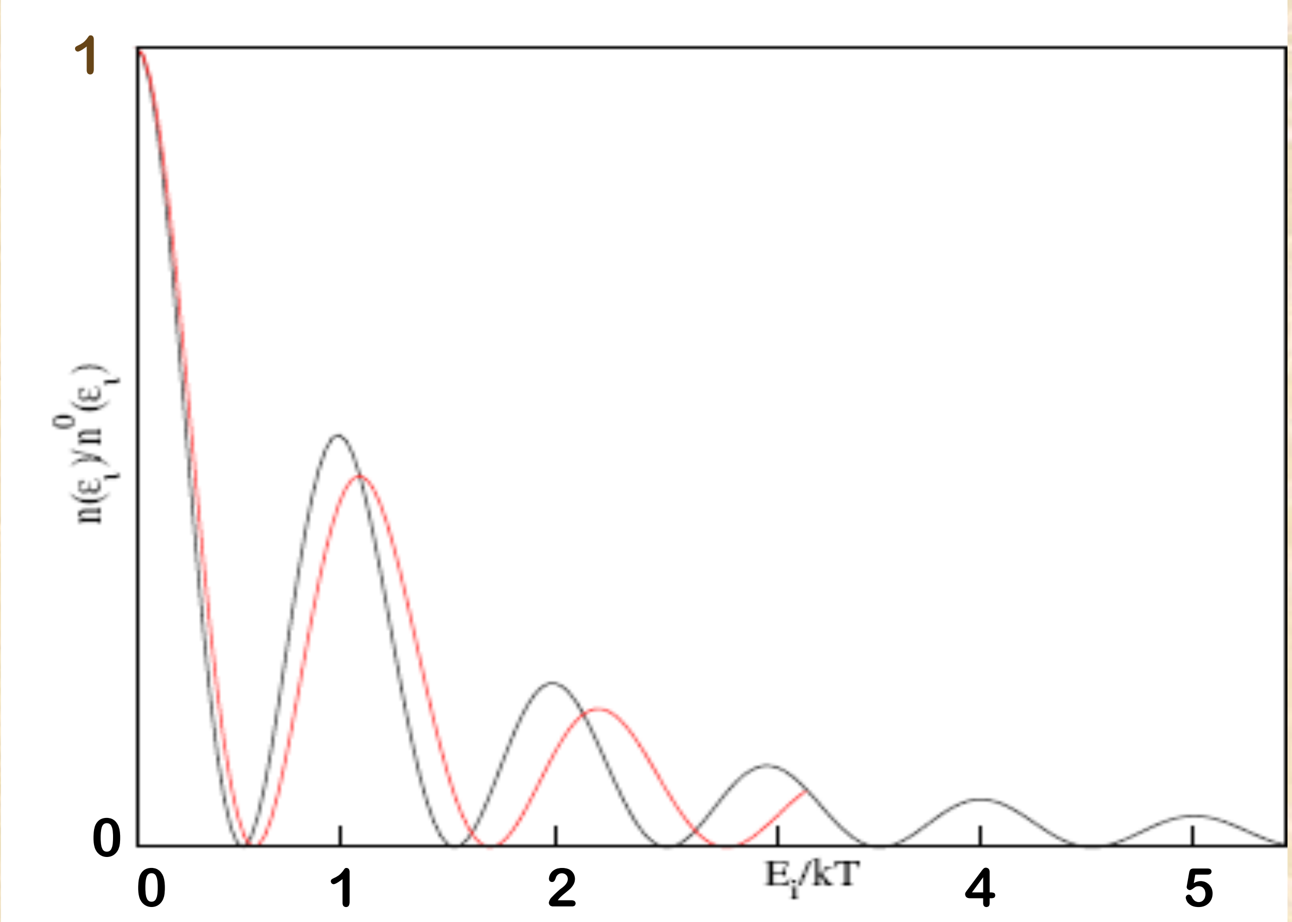
$$J_{00}(\omega) = \frac{1}{2}\left[\left(1 + \frac{\tilde{\epsilon_0}}{\overline{\epsilon}}\right)\frac{\delta(\omega - \overline{\epsilon})}{e^{\beta\omega} - 1} + \left(1 - \frac{\tilde{\epsilon_0}}{\overline{\epsilon}}\right)\frac{\delta(\omega + \overline{\epsilon})}{e^{\beta\omega} - 1}\right] \qquad (28)$$

- J_{00} is the Spectral Distribution Function
- $\beta = 1/kT$.
- $\overline{\epsilon} = (\tilde{\epsilon_0}^2 - X_1^2)^{1/2}$

Estimation of Condensate Number

- Rubidium atoms
- Isotropic harmonic potential of frequency $\omega = 10^3 s^{-1}$ and
- Total number of bosons $N = 10^4$
- the transition temperature T_c is ≈ 150 nK.
- $N_0 = 95\% N, N_1 = 5\% N, V_{1010} = \frac{1}{5} V_0, V_{1100} = \frac{1}{10} V_0, A_{11} = 1\% N_0$
- using eqn(27) we find that as we move away from the mean field regime, $N_0 \approx 500$, for a temperature of 125 nK.

Occupation as a function of energy



Green Functions for the Ground State

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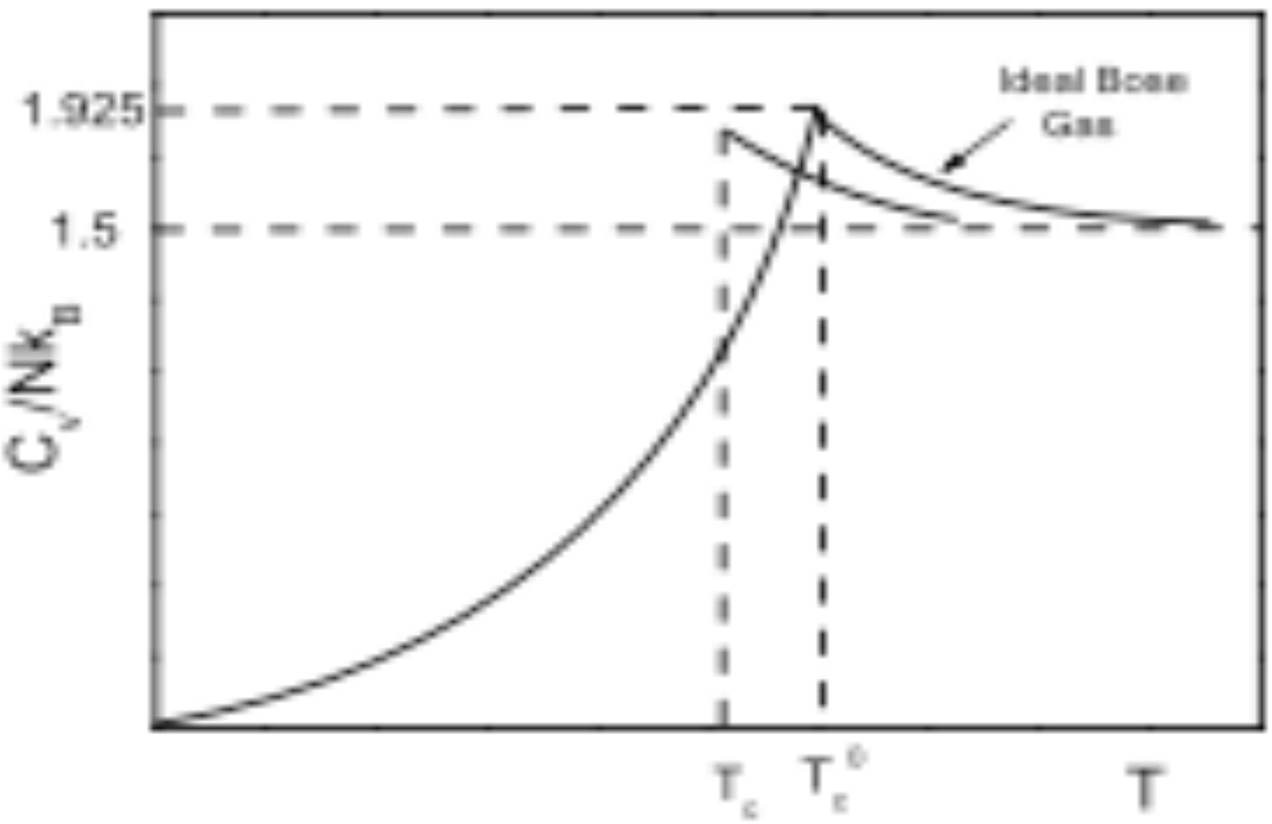
- Incorporating Green fns G_{10} and F_{10} that connect the ground state to the first excited state and solving,
- Ground state Green function involving higher orders is:

$$G_{00}^{1} = \frac{1}{2\pi}\left(\frac{\left(E+\tilde{\epsilon_0}\right)\left(E^2-\tilde{\epsilon_{11}}^2+X_4^2\right)}{(E^2-\tilde{\epsilon_0}^2+X_1^2)(E^2-\tilde{\epsilon_{11}}^2+X_4^2)+(X_2^2-\tilde{\epsilon_{10}}^2)\left(X_3^2-{\epsilon'_{10}}^2\right)}\right) \tag{29}$$

which is :

$$\frac{1}{G_{00}^{1}} = \frac{1}{G_{00}^{0}} + 2\pi\frac{\left(X_2^2-\tilde{\epsilon_{10}}^2\right)\left(X_3^2-{\epsilon'_{10}}^2\right)}{\left(E+\tilde{\epsilon_0}\right)\left(E^2-\tilde{\epsilon_{11}}^2+X_4^2\right)} \tag{30}$$

Specific Heat



Gnanapragasam Ph D Thesis, ANU (2008)

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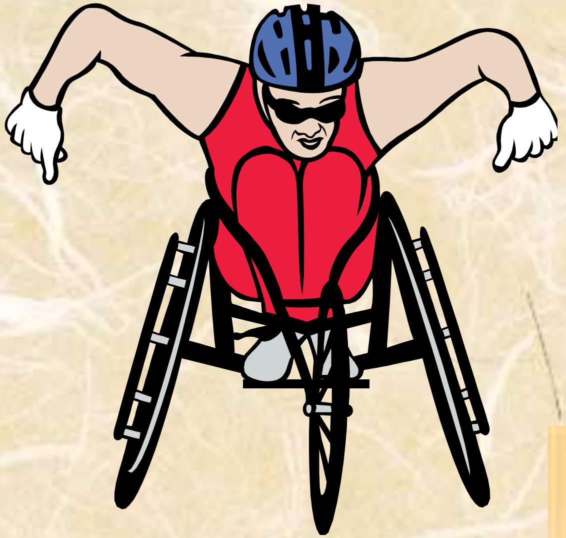

One body vs Collectiveness

Collective Modes Gnanapragasam and Das IJMPB 22, 4349 (2008)

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$$\frac{\partial \hat{n}}{\partial t} + \nabla \cdot \hat{\mathbf{j}} = 0$$

$$\hat{\mathbf{j}}(\mathbf{x},t) = \frac{\hbar}{2mi}\left[\hat{\psi}^\dagger(\mathbf{x},t)\nabla\hat{\psi}(\mathbf{x},t) - (\nabla\hat{\psi}^\dagger(\mathbf{x},t))\hat{\psi}(\mathbf{x},t)\right]$$

$$\frac{\partial^2 \hat{n}}{\partial t^2} + \nabla \cdot \frac{\partial \hat{\mathbf{j}}}{\partial t} = 0$$

The equation of motion for $\hat{\mathbf{j}}$ in the Heisenberg picture is

$$\frac{\partial \hat{\mathbf{j}}(\mathbf{x},t)}{\partial t} = \frac{1}{i\hbar}[\hat{\mathbf{j}}(\mathbf{x},t), \hat{H}]$$

or

$$\frac{\partial \hat{\mathbf{j}}(\mathbf{x},t)}{\partial t} = \frac{1}{i\hbar}e^{iEt/\hbar}[\hat{\mathbf{j}}(\mathbf{x}), \hat{H}]e^{-iEt/\hbar}$$

It can be shown that:

$$[\hat{\mathbf{j}}(\mathbf{x}), \hat{H}] = \frac{-i\hbar}{m}\Big(\hat{\psi}^\dagger(\mathbf{x})\nabla\left(T(\mathbf{x}) + V_t(\mathbf{x})\right)\hat{\psi}(\mathbf{x}) + \int d\mathbf{x}'\ \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}^\dagger(\mathbf{x}')\nabla(V^I(\mathbf{x},\mathbf{x}'))\hat{\psi}(\mathbf{x}')\hat{\psi}(\mathbf{x})\Big)$$

Collective Modes

$$\frac{\partial^2 \hat{n}(\mathbf{x},t)}{\partial t^2} = \frac{1}{m}\boldsymbol{\nabla}\cdot\Big(\hat{\psi}^\dagger(\mathbf{x},t)\boldsymbol{\nabla}\left(T(\mathbf{x})+V_t(\mathbf{x})\right)\hat{\psi}(\mathbf{x},t) + \int d\mathbf{x}'\ \hat{\psi}^\dagger(\mathbf{x},t)\hat{\psi}^\dagger(\mathbf{x}',t)\boldsymbol{\nabla}(V^I(\mathbf{x},\mathbf{x}'))\hat{\psi}(\mathbf{x}',t)\hat{\psi}(\mathbf{x},t)\Big)$$

$$\frac{\partial^2 n_0(\mathbf{x})}{\partial t^2} - \frac{1}{m}\boldsymbol{\nabla}\cdot\left[n_0(\mathbf{x})\left(\boldsymbol{\nabla}V_t(\mathbf{x}) + n_0\boldsymbol{\nabla}V_0(\mathbf{x}) + n_1\boldsymbol{\nabla}V_{0110}(\mathbf{x})\right)\right] = 0 \quad (18)$$

which could be written as

$$\frac{\partial^2 n_0(\mathbf{x})}{\partial t^2} - \frac{1}{m}\boldsymbol{\nabla}\cdot\left[n_0(\mathbf{x})\boldsymbol{\nabla}(V_t(\mathbf{x}) + n_0V_0(\mathbf{x}) + n_1V_{0110}(\mathbf{x}))\right] = 0 \quad (19)$$

With $n_0(\mathbf{x}) = n_0^{eq}(\mathbf{x}) + \delta n_0(\mathbf{x},t)$, where $n_0(\mathbf{x})$ is the equilibrium density, henceforth referred as $n_0(\mathbf{x})$ and $\delta n_0(\mathbf{x},t)$ the density variations, the above equation for a small variation in energy, becomes

$$m\frac{\partial^2 \delta n_0(\mathbf{x},t)}{\partial t^2} = \boldsymbol{\nabla}n_0(\mathbf{x})\cdot\boldsymbol{\nabla}\delta(V_t(\mathbf{x}) + n_0V_0(\mathbf{x}) + n_1V_{0110}(\mathbf{x})) + n_0(\mathbf{x})\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\delta(V_t(\mathbf{x}) + n_0V_0(\mathbf{x}) + n_1V_{0110}(\mathbf{x})) \quad (20)$$

Collective Modes

The total energy $\epsilon \approx \frac{1}{2}m\omega_0^2 R^2$, where R is the range of the trapping potential and thus $\nabla(\frac{1}{2}m\omega_0^2 R^2) = 0$. Again as an approximation we drop the term $\nabla V_{0110}(\mathbf{x})$. Thus the equation for density oscillations is

$$\begin{aligned} m\omega_c^2 \delta n_0(\mathbf{x}) &= \nabla V_t(\mathbf{x}) \cdot \nabla \delta n_0(\mathbf{x}) \\ &\quad -[\epsilon - (V_t(\mathbf{x}) + n_1 V_{0110}(\mathbf{x}))]\nabla^2 \delta n_0(\mathbf{x}) \end{aligned} \tag{23}$$

We consider two-body interaction of the form $C_6/|\mathbf{x}-\mathbf{x}'|^6$ and the integrated value of V_{0110} is $-\frac{1}{2}(kR_0C_6)m\omega_0^2$, where $k = \frac{24\alpha^3}{\sqrt{\pi}\hbar^2}$, R_0 is the range of the two-body interaction potential and $\alpha = \sqrt{\frac{m\omega_0}{\hbar}}$ is the harmonic oscillator length inverse. When the harmonic trapping is isotropic then Eq. (23) in spherical polar coordinates, becomes

$$\omega_c^2 \delta n_0(r) = \omega_0^2 r \frac{\partial}{\partial r}\delta n_0(r) - \frac{1}{2}\omega_0^2 \left[R^2 - r^2 + kR_0C_6n_1\right]\nabla^2 \delta n_0(r) \tag{24}$$

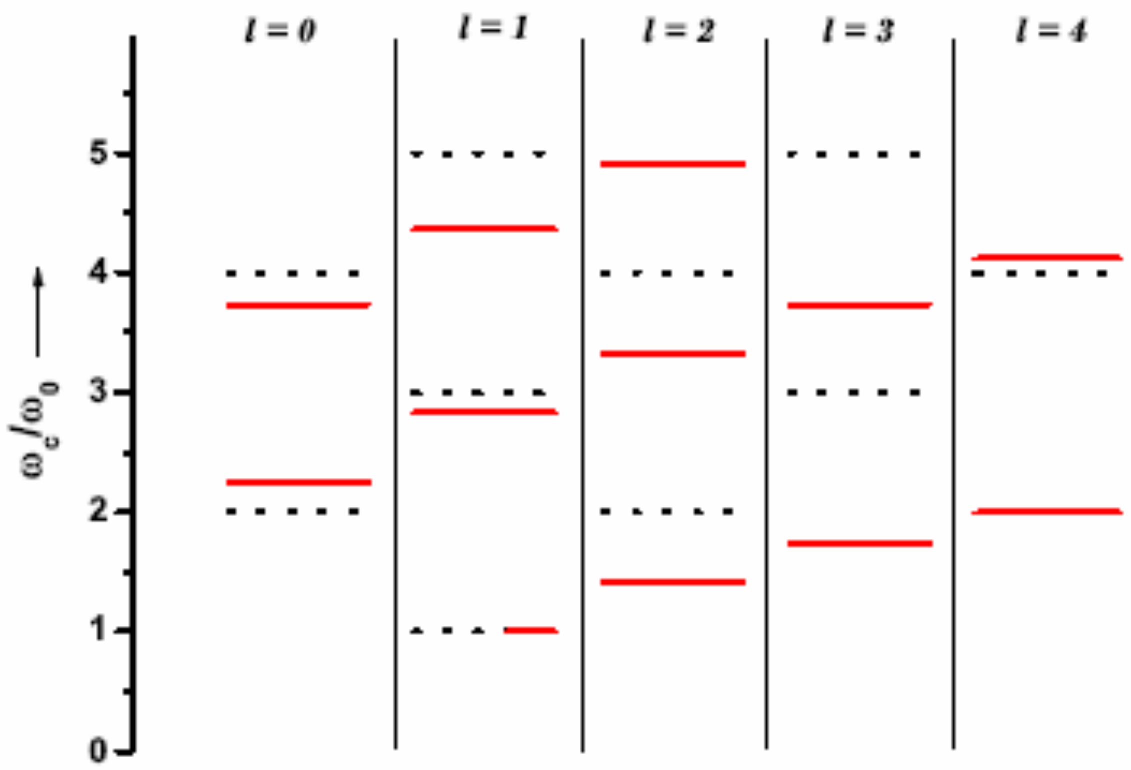
or

$$\eta\, \delta n_0(r) = r\frac{\partial}{\partial r}\delta n_0(r) - \frac{1}{2}\left[R^2 - r^2 + kR_0C_6n_1\right]\nabla^2 \delta n_0(r) \tag{25}$$

where $\eta = \omega_c^2/\omega_0^2$

Collective Modes

$$\omega_c^2 = \omega_0^2(l + 3n + 2nl + 2n^2)$$



Excitation frequencies of an interacting condensate in an isotropic harmonic trap (red)
Dotted lines represent in absence of interaction.

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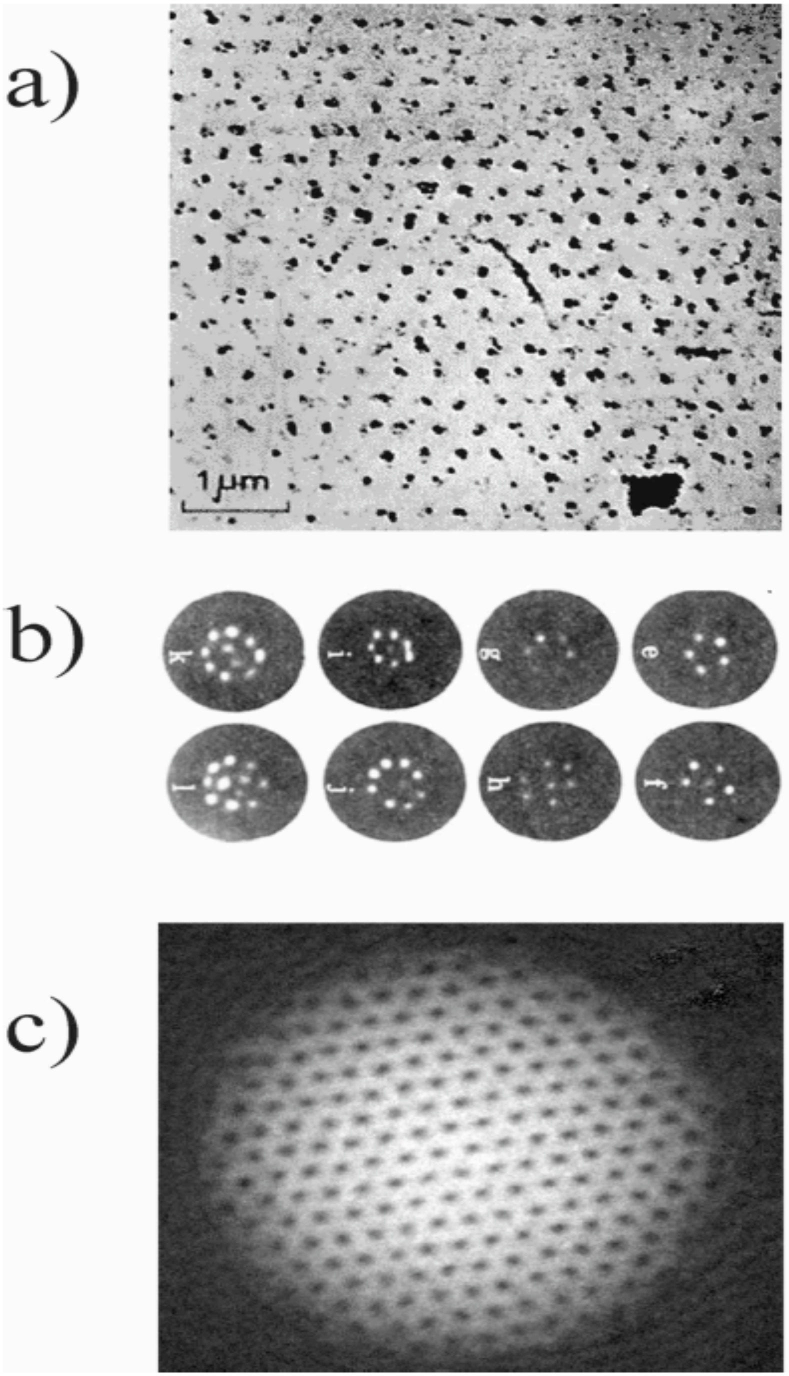
Vortex Structures

and Tkachenko modes

(Sonin)



Abrikosov vortices
In superconductors
Essmann and Trauble
(1967)

Vortices in ^{4}He
Packard's group (1979)

Vortices in cold atoms
(Cornell's group, JILA
2003) see also Ketterle's
group, MIT expt.

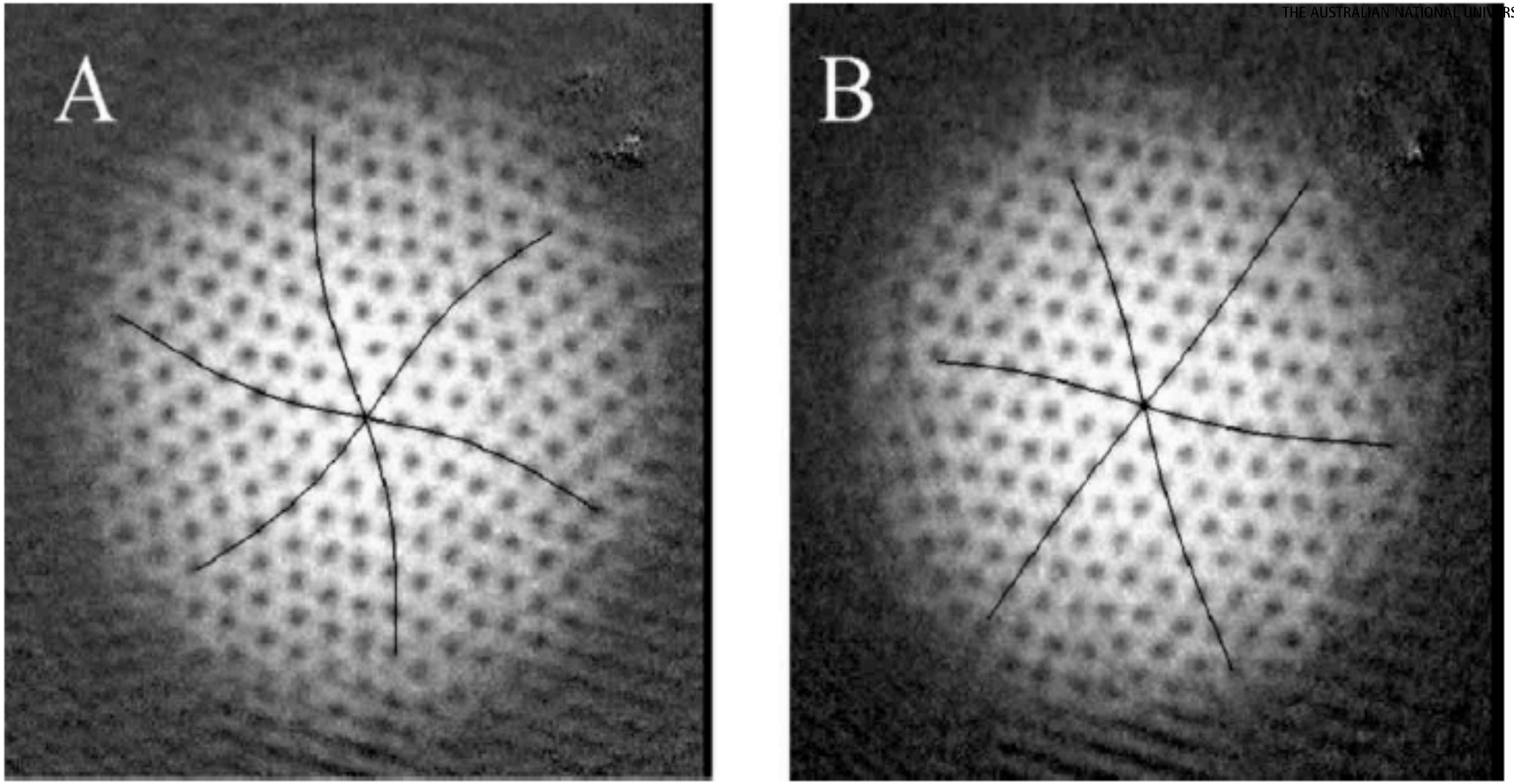


Figure 4.1: (1,0) Tkachenko mode excited by atom removal (a) taken 500 ms after the end of the blasting pulse (b) taken 1650 ms after the end of the blasting pulse. BEC rotation is counterclockwise. Lines are sine fits to the vortex lattice planes.

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Some useful References:

- Bose-Einstein Condensation and Superfluidity, Lev Pitaevskii and Sandro Stringari, Oxford Sc. Publications (2016)
- Bose-Einstein Condensation in dilute gases, C. J. Pethick and H. Smith Cambridge University Press, (2008)
- Fundamentals and New Frontiers on Bose Einstein Condensation, M. Ueda World Sci. (2010)
- Excitations in a Bose Condensed Liquid, A. Griffin, Cambridge Univ Press (1999)

Many Review Articles.

A Brief **SUMMARY**

Bose and Bosons

Role of confinement

Effect of temperature

Effect of interaction

Beyond the GP mean field-
correlation of atoms in the condensed and noncondensed states
Collective Excitations, Vortices etc.

Thank you for your attention.