Quantum Trajectories for Measurement of Entangled States

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It describes a ray in the Hilbert space.
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For pure states, $\rho^2 = \rho$ and $\det(\rho) = 0$.
Any power-series expandable function $f(\rho)$ becomes a linear combination of ρ and I. So generic basis-independent functions of ρ, e.g. $\mathrm{Tr}(f(\rho)O)$, reduce to conventional expectation values.

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For a particle on a line (infinite dimensional Hilbert space):

$$W(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dy \;\rho(x-\tfrac{y}{2}, x+\tfrac{y}{2}) e^{ipy/\hbar} \;,$$

$$\rho(x-\tfrac{y}{2}, x+\tfrac{y}{2}) = \int_{-\infty}^{\infty} dp \; W(x,p) e^{-ipy/\hbar} \;,$$

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For a state in an odd finite dimensional Hilbert space:

$$W(n,k) = \frac{1}{d} \sum_{m=0}^{d-1} \rho_{n-m,n+m} e^{4\pi i k m/d} \ .$$

Here the indices are defined modulo d, i.e. $n, k, m \in Z_d = \{0, 1, ..., d-1\}$.
With odd d, all indices are covered in two cycles of Z_d.

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This construction does not work in even dimensions.
A "quantum square-root" is needed. Given the construction for $d=2$, tensor products can reach any d, as an odd number times a power of 2.

Wigner Function (contd.)

For a qubit, the Wigner function is a 2×2 matrix.
Eigendirections of σ_z and σ_x can be chosen as the conjugate coordinates.
(σ_x is the translation generator for the σ_z eigenstates.)

The components $\{1, \sigma_x, \sigma_y, \sigma_z\}$ respectively give the contributions:

$$\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \ \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

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The Wigner function for $\rho = \frac{1}{2}(I + \vec{n}\cdot\vec{\sigma})$ is then positive within the octahedron $\pm x \pm y \pm z = 1$ embedded in the Bloch sphere.

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The Wigner function for the two-qubit singlet state becomes:

$$W_{\mathrm{singlet}} = \frac{1}{8}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Marginals for anticorrelated components are equal, while those for correlated components vanish.

The negative contributions are enough to give $\langle (\vec{\sigma} \cdot \vec{n}_1)(\vec{\sigma} \cdot \vec{n}_2) \rangle = -\vec{n}_1 \cdot \vec{n}_2$.

Weak Measurements

Information about the measured observable is extracted from the system at a slow rate (e.g. by weak coupling). Stretching out the time scale can allow one to monitor collapse of the system to a measurement eigenstate.
Note: A measurement interaction is the one where the apparatus does not, for whatever reasons, remain in a superposition of pointer states.

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Such a dynamical process exists! Gisin (1984)

Salient Features

A precise ratio of evolution towards the measurement eigenstates and unbiased white noise is needed to reproduce the Born rule as a constant of evolution.

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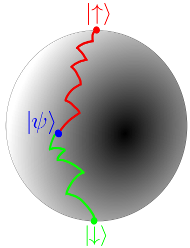
Technological advances allow us to monitor the quantum evolution during weak measurements. That can test the validity of the stochastic measurement formalism, and then help us figure out what may lie beyond.



Measurement ≡ An effective process of a more fundamental theory.

Ensemble of Quantum Geodesic Trajectories

Leave out $i[\rho, H]$ from the evolution description for simplicity.
The pointer basis $\{P_i\}$ is fixed by the system-apparatus interaction.
Unitary interpolation between ρ and $\{P_i\}$ gives the geodesic evolution:

$$\frac{d}{dt}\rho = \sum_i w_i \; g[\rho P_i + P_i\rho - 2\rho Tr(P_i\rho)] \; , \quad \sum_i w_i = 1 \; .$$

g is the system-apparatus coupling, and t is the "measurement time".

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- Diagonal projections of ρ fully determine the evolution:

$$\frac{2}{P_j\rho P_k}\frac{d}{dt}(P_j\rho P_k) = \frac{1}{P_j\rho P_j}\frac{d}{dt}(P_j\rho P_j) + \frac{1}{P_k\rho P_k}\frac{d}{dt}(P_k\rho P_k)$$

There are $n-1$ independent variables (diagonal projections $Tr(P_i\rho)$).
The evolution is totally decoupled from the decoherence process.

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- For the evolution satisfying the Born rule, free reparametrisation of the "measurement time" is allowed, but no other freedom. This choice governs the collapse time scale, and is fully local for each system-apparatus pair.

Quantum Diffusion: Single Qubit Measurement

The evolution equations simplify considerably for a qubit.
Let $|0\rangle$ and $|1\rangle$ be the measurement eigenstates.

$$\frac{d}{dt}\rho_{00} = 2g\ (w_0 - w_1)\rho_{00}\rho_{11}\ ,$$

$$\rho_{01}(t) = \rho_{01}(0)\left[\frac{\rho_{00}(t)\rho_{11}(t)}{\rho_{00}(0)\rho_{11}(0)}\right]^{1/2} .$$

With $\rho_{11}(t) = 1 - \rho_{00}(t)$ and $w_1(t) = 1 - w_0(t)$, only one independent variable describes evolution of the system.

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Evolution obeys Langevin dynamics, when unbiased white noise with spectral density S_ξ is added to w_i^{IB}. The trajectory weights become:

$$w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\ \xi\ .$$

$$\langle\langle \xi(t) \rangle\rangle = 0\ ,\quad \langle\langle \xi(t)\xi(t') \rangle\rangle = \delta(t - t')\ .$$

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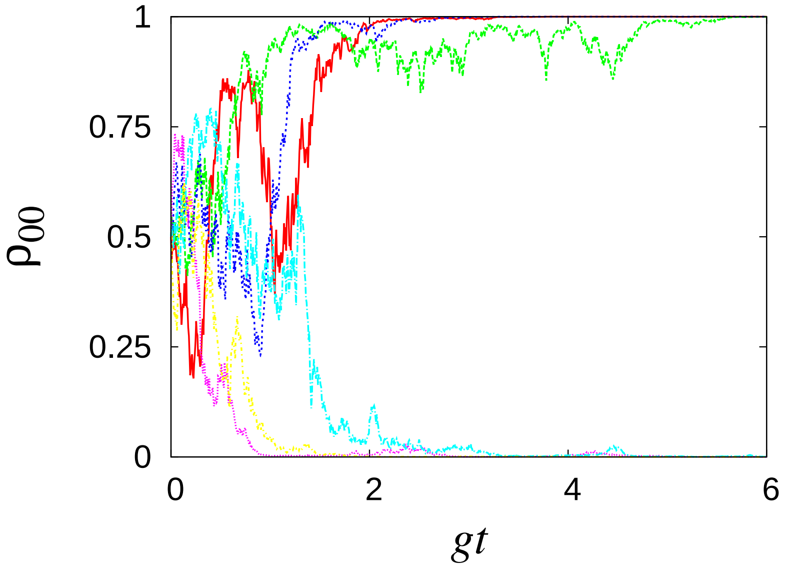
This is a stochastic differential process on the interval $[0,1]$.
The fixed points at $\rho_{00} = 0, 1$ are perfectly absorbing boundaries.
A quantum trajectory would zig-zag through the interval before ending at one of the two boundary points.



Individual quantum evolution trajectories for the initial state $\rho_{00} = 0.5$, with measurement eigenstates $\rho_{00} = 0, 1$, and in presence of measurement noise satisfying $gS_\xi = 1$.

Single Qubit Measurement (contd.)

It is instructive to convert the stochastic evolution equation from the differential Stratonovich form to the Itô form that specifies forward evolutionary increments:

$$d\rho_{00} = 2g\ \rho_{00}\rho_{11}(\rho_{00} - \rho_{11})(1 - gS_\xi)dt + 2g\sqrt{S_\xi}\ \rho_{00}\rho_{11}\ dW\ ,$$
$$\langle\langle dW(t)\rangle\rangle = 0\ ,\quad \langle\langle (dW(t))^2\rangle\rangle = dt\ .$$

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The first term produces drift in the evolution, while the second gives rise to diffusion. The evolution with no drift, i.e. the pure Wiener process with $gS_\xi = 1$, is rather special:

$\langle\langle d\rho_{00} \rangle\rangle = 0 \Longleftrightarrow$ Born rule is a constant of evolution.

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The constraint $gS_\xi = 1$ also gives the coupling-free relation:

$$\langle\langle (d\rho_{00} - d\rho_{11})^2\rangle\rangle = 4\rho_{00}\rho_{11}\frac{(d\rho_{00} - d\rho_{11})_{\rm geo}}{\rho_{00} - \rho_{11}}\ .$$

The fact that both vanishing drift and fluctuation-dissipation relation lead to the Born rule is an exceptional feature of quantum trajectory dynamics.

Implication: The environment can influence the measurement process only via the apparatus.

Ensemble Evolution Dynamics

During measurement, the probability distribution $p(\rho_{00}, t)$ of the set of quantum trajectories evolves according to the Fokker-Planck equation:

$$\frac{\partial p(\rho_{00},t)}{\partial t} = 2g \frac{\partial^2}{\partial^2 \rho_{00}} \left(\rho_{00}^2 (1-\rho_{00})^2 p(\rho_{00},t)\right) \ , \quad \text{with } gS_\xi = 1 \ .$$

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Its exact solution corresponding to initial $p(\rho_{00}, 0) = \delta(x)$ has two non-interfering components with areas x and $1-x$, monotonically travelling to the boundaries at $\rho_{00} = 1$ and 0 respectively.

Let $\tanh(z) = \rho_{00} - \rho_{11}$ map $\rho_{00} \in [0,1]$ to $z \in (-\infty, \infty)$. Then the two components are Gaussians centred at $z_\pm = z_0 \pm gt$, $z_0 = \tanh^{-1}(2x-1)$:

$$p(z,t) = \frac{1}{\sqrt{2\pi gt}} \left(x \exp\left[-\frac{(z-z_+)^2}{2gt}\right] + (1-x) \exp\left[-\frac{(z-z_-)^2}{2gt}\right] \right) .$$

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Parametric freedom: With the Born rule as a constant of evolution, g can be time-dependent, and gt is replaced by $\tau \equiv \int_0^t g(t')dt'$.
The white noise distribution remains unspecified beyond the mean and the variance. Suitable choice can be made, e.g. Gaussian noise or Z_2 noise.

Experimental Setup

The system is a superconducting 3D transmon qubit (nonlinear oscillator).

It consists of two Josephson junctions in a closed loop (SQUID) shunted by a capacitor.

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With a phase-sensitive amplifier, the scattering phase-shifts are Gaussians peaked at the two eigenvalues. Weak measurements result when the probe magnitude is small, making the two Gaussians closely overlap.

Experimental Results

A quantum state initially polarised in XZ-plane is measured in the Z-basis.
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The quantum state is infered from the integrated signal measurement, according to the Bayesian formalism (I_0, I_1, σ are known):

$$\frac{\rho_{00}(t+\Delta t)}{\rho_{11}(t+\Delta t)} = \frac{\rho_{00}(t)}{\rho_{11}(t)} \frac{\exp[-(I_m(\Delta t)-I_0)^2/2\sigma^2]}{\exp[-(I_m(\Delta t)-I_1)^2/2\sigma^2]} , \quad I_m(\Delta t) = \frac{1}{\Delta t} \int_0^{\Delta t} I(t')\, dt' .$$

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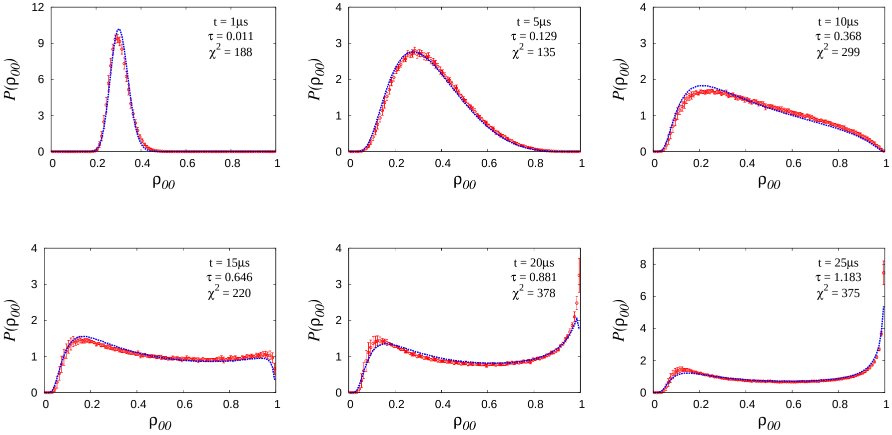
Quantum diffusion is not monotonic in time (unlike spontaneous collapse). Quantum trajectories stochastically diffuse along the meridians of the Bloch sphere (the phase of ρ_{01} remains unchanged).

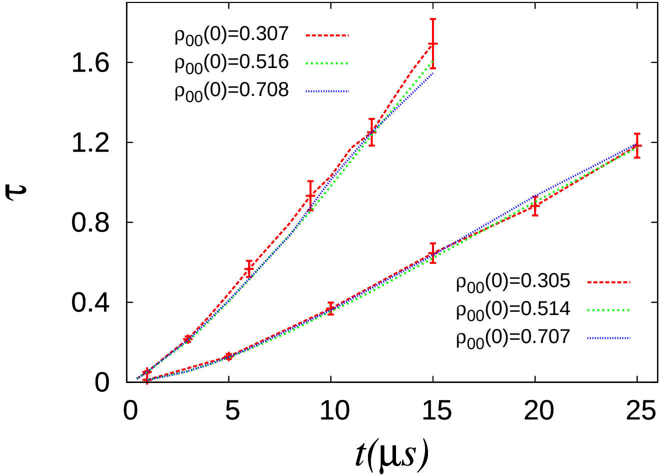
Experimental Results (contd.)

Relaxation time T_1 is determined from the decay rate of the ensemble averaged current, after preparing the qubit in the excited state.

The experimentally observed trajectory distribution fits the quantum diffusion prediction very well, in terms of the single dimensionless evolution parameter $\tau \equiv \int_0^t g(t')dt'$, and excited state relaxation T_1.



Evolution of the quantum trajectory distribution for weak Z-measurement of a superconducting transmon qubit with the initial state $\rho_{00} = 0.305(3)$. The histograms with bin width 0.01 (red) represent the experimental data for an ensemble of 10^6 trajectories. The curves (blue) are the best fits to the quantum diffusion model distribution, with the single dimensionless evolution parameter $\tau \in [0, 1.2]$. The trajectory parameters (with errors) were $T_1 = 45(4)\mu s$, $\Delta t = 0.5\mu s$, $I_0 = 128.44(2)$, $I_1 = 127.68(3)$, $\sigma = 5.50(1)$.



The best fit values of the time integrated measurement coupling τ for two values of the system-apparatus coupling, when experimental data for weak Z-measurement of a transmon with different initial states $\rho_{00}(0)$ are compared to the theoretical predictions. It is obvious that τ is essentially independent of the initial state, and varies almost linearly with time after a slower initial build-up. The error bars correspond to changes in τ that would change the χ^2-values for the trajectory distribution fits by 100.

Experimental Results (contd.)

• With a large ensemble of trajectories, the systematic errors dominate over the statistical ones.

• With 100 data points and only one fit parameter, χ^2 values less than a few hundred indicate good fits. The fits work well for $\tau < 2$.

$\tau > 10$ is essentially projective measurement.

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Systematic errors:

- Uncertainty in the initial state $\rho_{00}(0)$.
- Uncertainties in I_0, I_1.
  - Leftover heralding photons, after the initial state preparation pulse.
  - Thermal mixing with the higher excited transmon states.

Detector inefficiency is absorbed in the value of τ. (Formally, $g\Delta t = (\Delta I)^2/(4\sigma^2)$.)

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Understanding the quantum state collapse reduces to understanding why large amplitude coherent states are not observed in superposition.

Questions about origin of irreversibility remain open.

Nature of Noise

Where are the quantum properties (no coherence or entanglement left)?

Is the noise classical or quantum?

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The geodesic trajectory weights w_i are real, but are not confined to $[0, 1]$.
They cannot be interpreted as classical probabilities.
For a qubit, classical trajectory weights would have $w_0 - w_1 \in [-1, 1]$.
Instead, quantum diffusion has $w_0 - w_1 = \rho_{00} - \rho_{11} + \sqrt{S_\xi}\ \xi$.

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For the singlet state correlation determination at angle θ, there are two binary measurements. All four outcomes have probability $1/4$.

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ P_0^{(1)} \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ I \otimes P_\theta^{(2)} = \begin{pmatrix} c^2 & cs & 0 & 0 \\ cs & s^2 & 0 & 0 \\ 0 & 0 & c^2 & cs \\ 0 & 0 & cs & s^2 \end{pmatrix}.$$

Excursions of the trajectory weights outside $[0, 1]$ produce nontrivial correlations. (Time ordering of the two measurements does not matter).

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Beyond Quantum Mechanics

Physical:

(1) Hidden variables with novel dynamics may produce quantum mechanics as an effective theory, e.g. the GRW spontaneous collapse mechanism.

(2) Ignored (but known) interactions can produce effects that modify quantum dynamics at macroscopic scales, e.g. effects of CMBR or gravity.

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Bypass:

Many worlds interpretation—each evolutionary branch is a different world, and we only observe the measurement outcome corresponding to the world we live in (anthropic principle).

None of these have progressed to the level where they can be connected to verifiable experimental consequences.