Entanglement sharing via noisy channels: One-shot optimal singlet fraction

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Entanglement is a resource

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- Since entanglement cannot be created by LOCC, sharing of entanglement requires sending quantum systems through quantum channels along with LOCC.

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- **Preparation and Transmission:** Alice prepares $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ (the input state) and sends half of it down a d-dimensional quantum channel Λ to Bob. This gives rise to the following mixed state (output):

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- How useful such a state is (for QIP) can be quantified by its **singlet fraction** [Bennett +, PRL ,1995; PRA1996] defined by:

$$\mathcal{F}\left(\rho_{\psi,\Lambda}\right) \quad = \quad \max_{|\Psi\rangle} \langle \Psi \left| \rho_{\psi,\Lambda} \right| \Psi \rangle \,,$$

where the maximization is over all maximally entangled states $|\Psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$.

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  - `Teleportation fidelity:`

$$f(\rho_{\psi,\Lambda}) = \frac{\mathcal{F}(\rho_{\psi,\Lambda})\, d + 1}{d+1}$$

  - `Entanglement distillation: Yields typically depend on the singlet fraction.`

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However, maximizing $\mathcal{F}\left(\rho_{\psi,\Lambda}\right)$ over all $|\psi\rangle$ may not give the desired result. WHY? Because, singlet fraction can increase under LOCC [Badziag +, PRA, 2002, SB, PRA, 2002]. This brings us to the second step of the protocol.

- **Local post-processing:** Define,

$$\mathcal{F}^{*}\left(\rho_{\psi,\Lambda}\right) = \max_{L} \mathcal{F}\left(L\left(\rho_{\psi,\Lambda}\right)\right); L \in \text{TP-LOCC}$$

  - Why TP-LOCC? Because we do not want to throw away any particle
  - Unlike $\mathcal{F}$, $\mathcal{F}^{*}$ is a LOCC monotone [Verstraete & Verschelde, PRL 2003].

One-shot optimal singlet fraction

Definition:

$$\mathcal{F}(\Lambda) = \max_{|\psi\rangle} \mathcal{F}^*(\rho_{\psi,\Lambda}) = \mathcal{F}^*(\rho_{\psi_{\mathrm{opt}},\Lambda}) = \max_{L} \mathcal{F}(L(\rho_{\psi_{\mathrm{opt}},\Lambda}))$$

Question of interest: For a given quantum channel Λ find $\mathcal{F}(\Lambda)$ and figure out the protocol, that is, $|\psi_{\mathrm{opt}}\rangle$ and the relevant TP-LOCC.

Remark: If the channel is noiseless, the solution is obvious. What about noisy channels? Is $|\psi_{\mathrm{opt}}\rangle$ maximally entangled for noisy channels?

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For qubit depolarizing channel, $|\psi_{\mathrm{opt}}\rangle$ is ME [Horodeckis, 1999]; For amplitude damping channel (qubit) $|\psi_{\mathrm{opt}}\rangle$ turned out to be nonmaximally entangled; moreover, no local post-processing was necessary [SB & AG, PRA(RC), 2012].

Qubit channels (non-entanglement breaking)

R. Pal, SB & S. Ghosh, PRA, 2014

Theorem

For a qubit channel Λ*,*

$$\mathcal{F}(\Lambda) = \lambda_{\mathrm{max}}\left(\rho_{\Phi^+,\Lambda}\right), \tag{1}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $\rho_{\Phi^+,\Lambda} = (\mathcal{I} \otimes \Lambda)|\Phi^+\rangle\langle\Phi^+|$ *and* $\lambda_{\mathrm{max}}\left(\rho_{\Phi^+,\Lambda}\right)$ *is the maximum eigenvalue of the density matrix* $\rho_{\Phi^+,\Lambda}$*. Moreover, the following equalities hold:*

$$\mathcal{F}(\Lambda) = \mathcal{F}^*\left(\rho_{\psi_{\mathrm{opt}},\Lambda}\right) = \mathcal{F}\left(\rho_{\psi_{\mathrm{opt}},\Lambda}\right). \tag{2}$$

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$$\mathcal{F}(\Lambda) = \mathcal{F}^*\left(\rho_{\psi_{\rm opt},\Lambda}\right) = \mathcal{F}\left(\rho_{\psi_{\rm opt},\Lambda}\right). \quad (2)$$

The optimal case does not require any local post-processing.

What can we say about $|\psi_{\rm opt}\rangle$?

Qubit channels

Unital: $\Lambda(\mathcal{I}) = \mathcal{I}$; Non-unital: $\Lambda(\mathcal{I}) \neq \mathcal{I}$.

Theorem

The state $|\psi_{\rm opt}\rangle$ is maximally entangled if and only if Λ is unital, where $|\psi_{\rm opt}\rangle$ is the eigenvector corresponding to the largest eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}} = \left(\mathcal{I} \otimes \hat{\Lambda}\right) |\Phi^+\rangle \langle\Phi^+|$.

$\hat{\Lambda}$ is the map dual to Λ; that is, if $\Lambda = \{K_i\}$, then $\hat{\Lambda} = \left\{K_i^\dagger\right\}$.

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Theorem

For a nonunital channel Λ,

$$\mathcal{F}^* \left(\rho_{\Phi^+,\Lambda}\right) \quad < \quad \mathcal{F}(\Lambda) \tag{3}$$

Thus for a nonunital channel the optimal singlet fraction is achieved only by sending nonmaximally entangled states through the channel.

Optimal singlet fraction and Negativity

For any two-qubit density matrix ρ with negativity $\mathcal{N}(\rho)$ [Vidal and Wener, 1999] we have,

$$\mathcal{F}^*(\rho) \leq \frac{1}{2}\left[1 + \mathcal{N}(\rho)\right], \tag{4}$$

which immediately leads to,

$$\mathcal{F}(\Lambda) \quad \leq \quad \frac{1}{2}\left[1 + \mathcal{N}(\Lambda)\right]$$

where $\mathcal{N}(\Lambda) = \max_{\psi} \mathcal{N}(\rho_{\psi,\Lambda})$ is the maximum output negativity.

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Theorem

For any qubit channel Λ,

$$\mathcal{F}(\Lambda) = \frac{1}{2}\left[1 + N\left(\rho_{\Phi^+,\Lambda}\right)\right] \leq \frac{1}{2}\left[1 + \mathcal{N}(\Lambda)\right]$$

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However, for an amplitude damping channel (which is nonunital),

$$\mathcal{F}(\Lambda) = \frac{1}{2}\left[1 + N\left(\rho_{\Phi^+,\Lambda}\right)\right] < \frac{1}{2}\left[1 + \mathcal{N}(\Lambda)\right]$$

which means maximum output negativity is attained by a nonmaximally entangled state.

Higher dimensional channels

Characterization of quantum channels ($d \geq 3$) in terms of one-shot optimal singlet fraction remains open. Which properties carry over to higher dimensions?

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In a recent work [Pal and SB, available @ arXiv] we presented a family of quantum channels Ω in every finite dimension $d \geq 3$ with the following properties:

$$\mathcal{F}(\Omega) \geq \mathcal{F}^*(\rho_{\psi,\Omega}) > \mathcal{F}^*(\rho_{\Psi,\Omega})$$

where $\Psi \in \mathbb{C}^d \otimes \mathbb{C}^d$ is any maximally entangled state. And moreover,

$$\mathcal{N}(\Omega) \quad > \quad \mathcal{N}(\rho_{\Psi,\Omega}).$$

Some of the properties do indeed carry over – but a full characterization of channels, as we could do in the qubit case, looks like a hard problem (so far).