Pairwise correlation inequalities and joint measurability

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A tribute to Professor Satyendra Nath Bose on his 125th birth anniversary

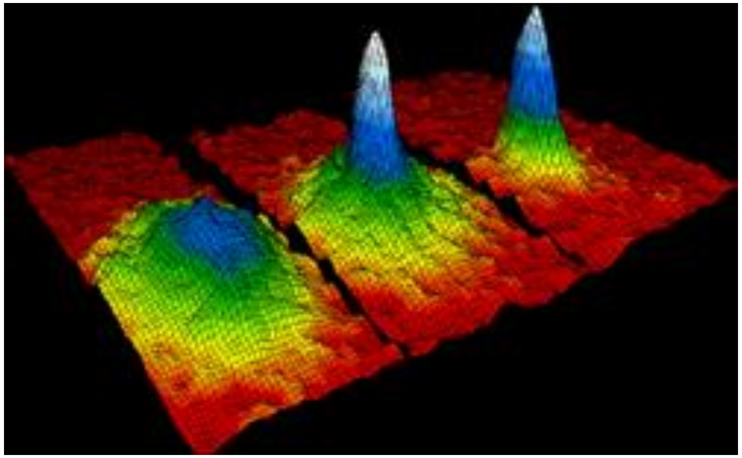

PHYSICS DEPARTMENT.
Dacca University.
Dated, the 4th June 1924.

Planck's law and hypothesis of light quanta,
S. N. Bose, 1924

Respected Sir, I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. You will see that I have tried to deduce the coefficient 8π ν^2/c^3 in Planck's Law independent of classical electrodynamics, only assuming that the ultimate elementary region in the phase-space has the content h^3. I do not know sufficient German to translate the paper. If you think the paper worth publication I shall be grateful if you arrange for its publication in *Zeitschrift für Physik*. Though a complete stranger to you, I do not feel any hesitation in making such a request. Because we are all your pupils though profiting only by your teachings through your writings. I do not know whether you still remember that somebody from Calcutta asked your permission to translate your papers on Relativity in English. You acceded to the request. The book has since been published. I was the one who translated your paper on Generalised Relativity.

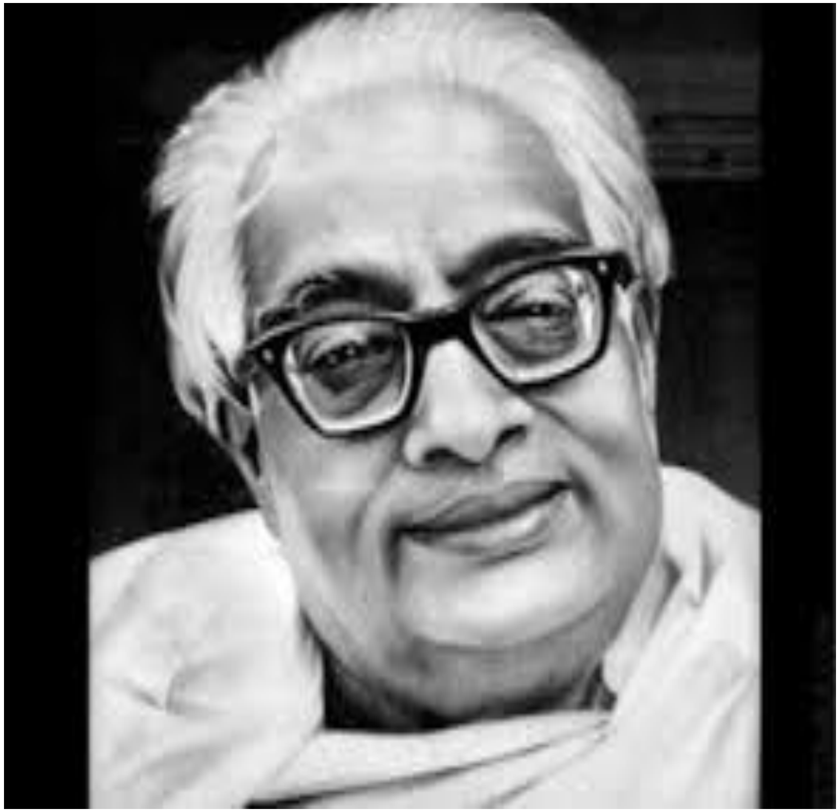
Bose's original approach struck Einstein

Bose's interpretation is now called Bose-Einstein statistics



Bose-Einstein Condensation

The class of particles that obey Bose–Einstein statistics, Bosons, was named after Bose, by Paul Dirac



Professor S. N. Bose came up with some difficulty, while teaching Planck's theory of radiation, and reached a conclusion that Maxwell-Boltzman statistics is inadequate to describe quantum statistics .

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A snapshot of foundational attitudes toward quantum mechanics

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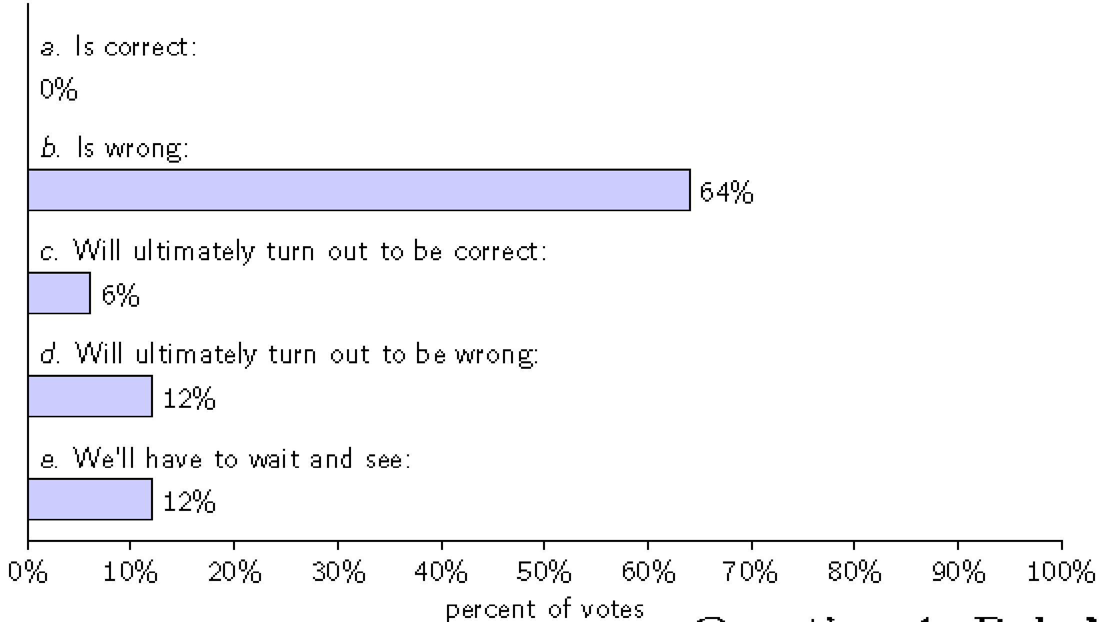
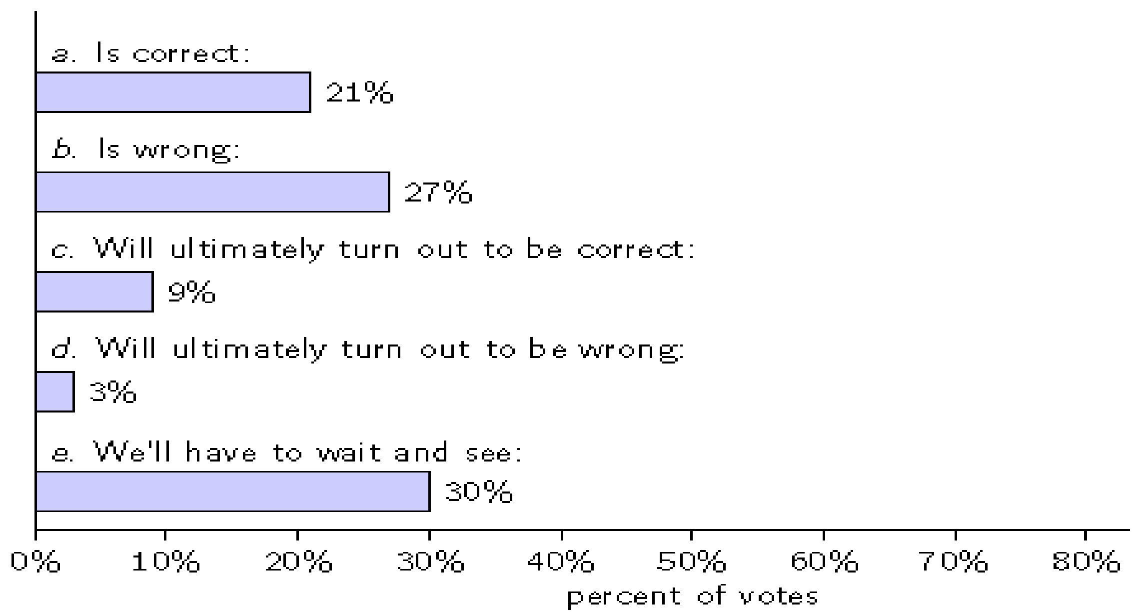
A survey probing respondents' views on various foundational issues in quantum mechanics was created by Schlosshauer, Kofler, and Zeilinger and then given to 33 participants at a quantum foundations conference. The participants completed a questionnaire containing 16 multiple-choice questions probing opinions on quantum- foundational issues. Participants included physicists, philosophers, and mathematicians.

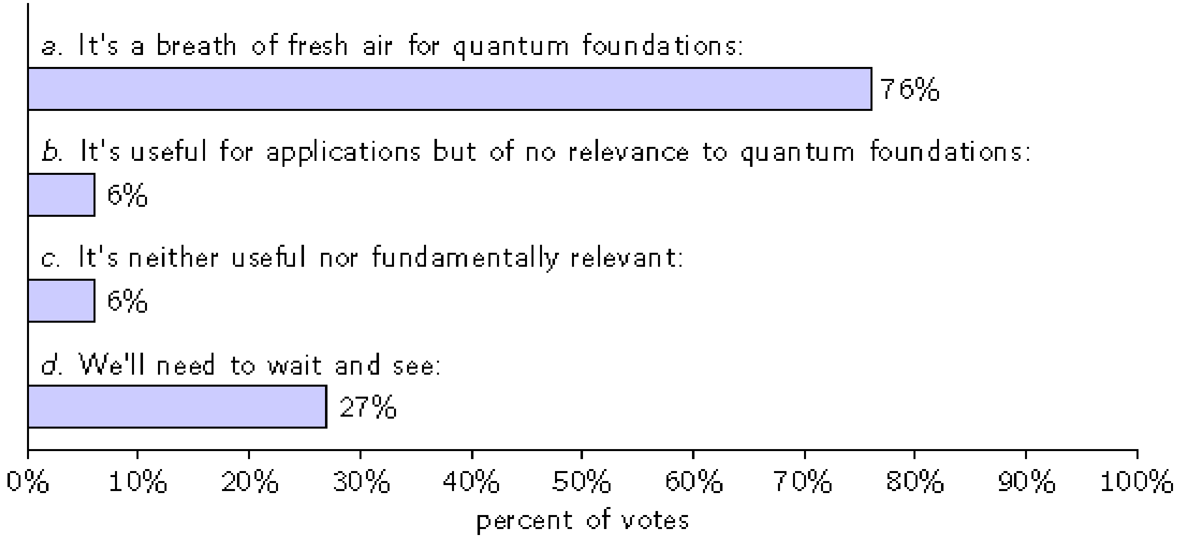
Question 3: Einstein's view of quantum mechanics



Question 4: Bohr's view of quantum mechanics

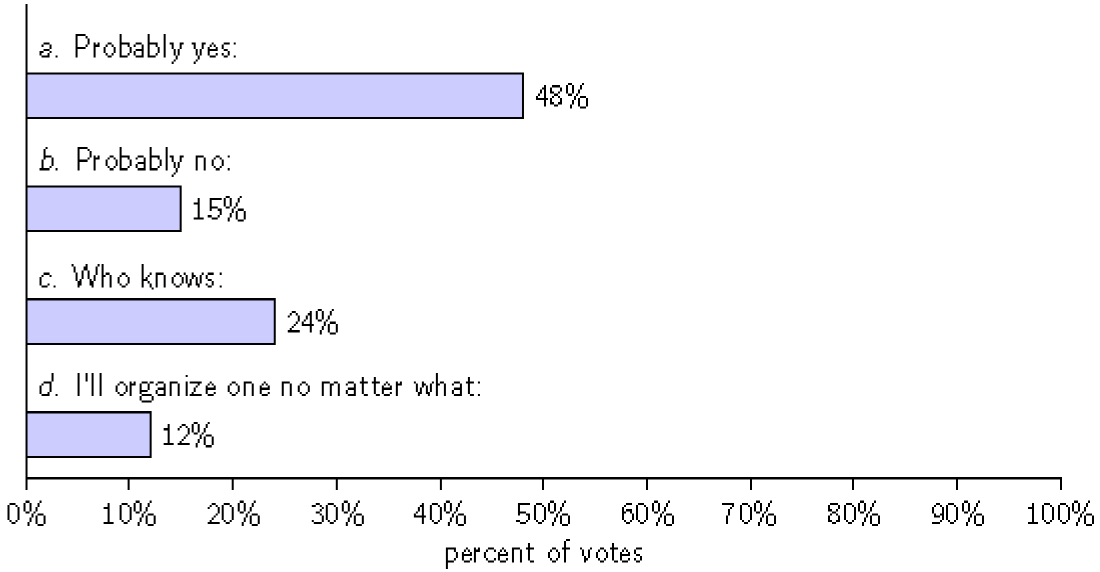


Question 7: What about quantum information?

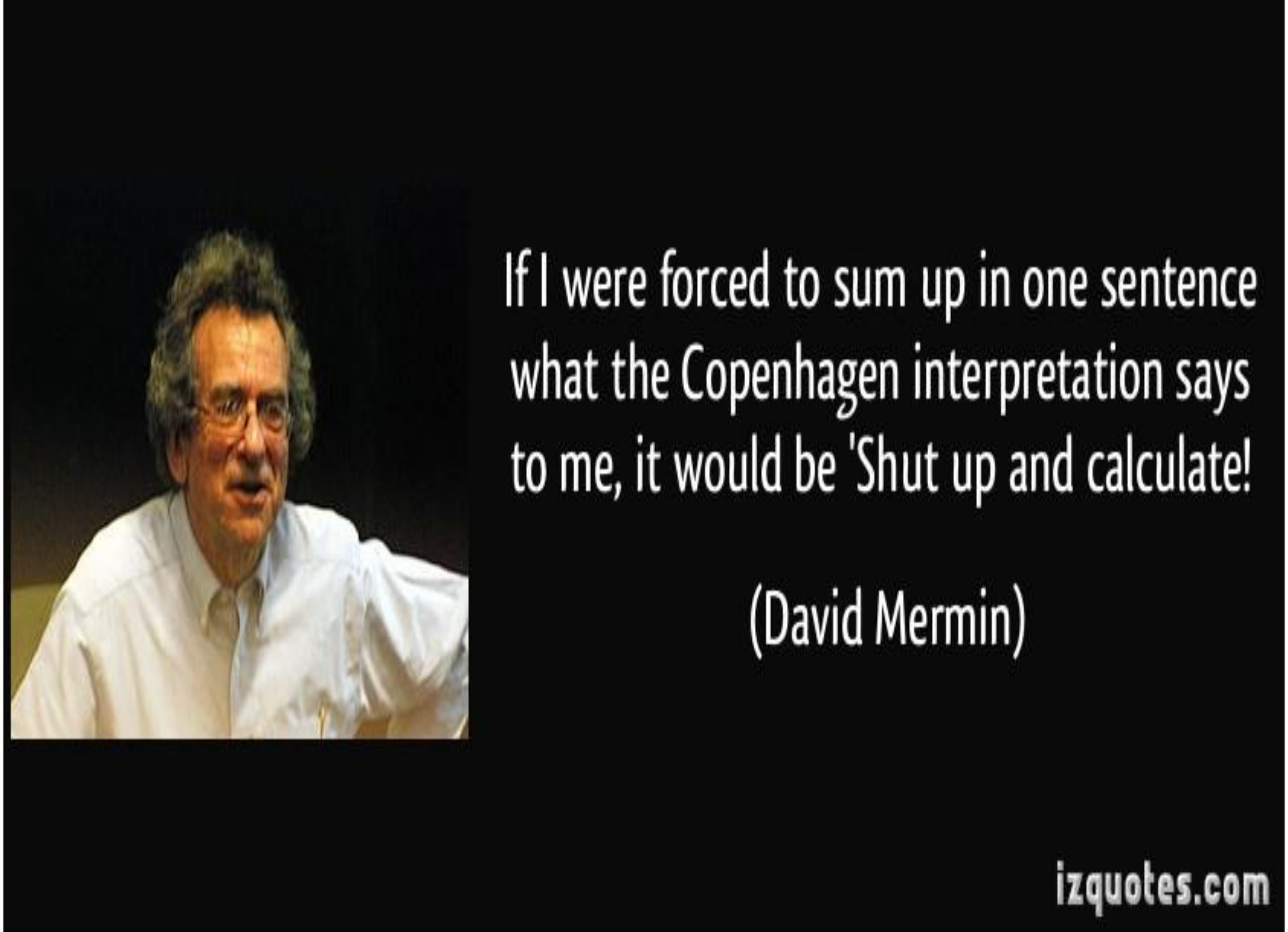


Evidently, there is broad enthusiasm—or at least open-mindedness—about quantum information, with three in four respondents regarding quantum information as "a breath of fresh air for quantum foundations." Indeed, it it hard to deny the impact quantum information theory has had on the field of quantum foundations over the past decade. It has inspired new ways of thinking about quantum theory and has produced information-theoretic derivations (reconstructions) of the structure of the theory. On the practical side, the quantum-information boom has helped fund numerous foundational research projects. Last but not least, quantum information has given foundational pursuits new legitimacy.

Question 16: In 50 years, will we still have conferences devoted to quantum foundations?



Should those who answered "probably yes" be proven right, then it would be fascinating to conduct another such poll 50 years from now. Notable write-ins included "I won't be here," and "I hope not."


If I were forced to sum up in one sentence
what the Copenhagen interpretation says
to me, it would be 'Shut up and calculate!
(David Mermin)
izquotes.com

"Different generations of physicists differed in the degree to which they thought that the interpretation of quantum mechanics remains a serious problem! I declared myself to be among those who feel uncomfortable when asked to articulate what we really think about the quantum theory, adding that, If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be *Shut up and calculate!*"

"..my professors – whom I viewed as agents of Copenhagen – when I was first learning quantum mechanics as a graduate student at Harvard, a mere 30 years after the birth of the subject said '*You'll never get a PhD if you allow yourself to be distracted by such frivolities,*' they kept advising me, '*so get back to serious business and produce some results.*' 'Shut up,' in other words, 'and calculate.' And so I did ...."

– David Mermin

- **Probabilities of measurement outcomes arising in the quantum framework turn out to be different from those arising in the traditional classical statistical scenario.**
- **This has invoked a wide range of debates on the quantum-classical worldviews of nature.**
- **Investigations by Bell, Kochen-Specker, Leggett-Garg tied the puzzling quantum features in terms of no-go theorems (Bell-CHSH inequality, Leggett-Garg Inequality..).**
- **Proofs of these no-go theorems essentially point towards the non-existence of a joint probability distribution for the outcomes of all possible measurements performed on a quantum system.**

- **Existence of joint probabilities in turn implies that the set of all Bell inequalities are satisfied (A. Fine, Phys. Rev. Lett. 48, 291 (1982)) when only compatible measurements are employed.**

Classical Moment Problem

Addresses the issue of finding a probability distribution given a set of moments.

Brings forth the fact

A given sequence of real numbers qualifies to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive.

Existence of joint probability distribution $\Longleftrightarrow$ **Moment matrix is positive**

J.A Sholat and J.D. Tamarkin, *The problem of moments,* AMS (1943)

N.J. Akhiezer, *The Classical Moment Problem*, Hofuer Publishing Co., (1965)

Classical Moment problem

Moment Inversion

↓

Are the probabilities moment determinate?

Moment Matrix Positivity

↓

Is the given sequence of “moments” admit *legitimate* probabilities?

H. S. Karthik, H. Katiyar, A. Shukla, T. S. Mahesh, A. R. Usha Devi and A. K. Rajagopal, Phys. Rev. A 87, 052118 (2013).

✓ Joint measurability of observables ensures joint probabilities

Non-joint measurability of physical observables gets traced back to Heisenberg's uncertainty principle and Bohr's notion of complementarity. These early studies imprinted that not all quantum measurements which can be carried out jointly. It is in this sense that they are incompatible/non-jointly measurable.

Sharp and unsharp measurements

- **In the classical world physical observables commute with each other and they can all be jointly measured. But in the quantum scenario, measurement of observables, which do not commute are usually declared to be *incompatible* in the quantum scenario.**

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- **Commuting observables can be measured jointly using projective valued (PV) measurements and their statistical outcomes can be discerned classically.**

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Are joint unsharp measurements possible?

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

Extended framework: Joint measurements of Positive Operator Valued Measures (POVMs)

Introduction of positive operator valued measures (POVMs) into physics:

- 1960s and 1970s – Ludwig, Davies, Helstrom, Holevo ...
- A notion of joint measurement of noncommuting observables could be formulated in the Hilbert-space formalism of quantum mechanics.
- A necessary requirement for joint measurability (though unsharp) is that there exists a joint probability distribution for the measurement outcomes of a set of compatible observables, such that it yields correct marginal probability distributions for the outcomes of all the subsets of observables.

See: P. Busch, M. Grabowski, and P. Lahti, *Operational Quantum Physics*, 2nd ed. (Springer, Berlin, 1997)

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

- **The orthodox notion of *sharp* projective valued (PV) measurements of self adjoint observables gets broadened to include *unsharp* measurements of POV observables.**

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

- **The orthodox notion of *sharp* projective valued (PV) measurements of self adjoint observables gets broadened to include *unsharp* measurements of POV observables.**
- **Do classical features emerge when one merely confines to measurements *compatible* unsharp observables? Is it possible to classify physical theories based on the *fuzziness* required for joint measurability?**

PHYSICAL REVIEW A **87**, 052125 (2013)

Degree of complementarity determines the nonlocality in quantum mechanics

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(Received 9 July 2012; published 20 May 2013)

Bohr's complementarity principle is one of the central concepts in quantum mechanics which restricts joint measurement for certain observables. Of course, later development shows that joint measurement could be possible for such observables with the introduction of a certain degree of unsharpness or fuzziness in the

PHYSICAL REVIEW A **89**, 022123 (2014)

Steering, incompatibility, and Bell-inequality violations in a class of probabilistic theories

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(Received 5 December 2013; published 24 February 2014)

We show that connections between a degree of incompatibility of pairs of observables and the strength of violations of Bell's inequality found in recent investigations can be extended to a general class of probabilistic physical models. It turns out that the property of universal uniform steering is sufficient for the saturation of a generalized Tsirelson bound, corresponding to maximal violations of Bell's inequality. It is also found that a limited form of steering is still available and sufficient for such saturation in some state spaces where universal uniform steering is not given. The techniques developed here are applied to the class of regular polygon state spaces, giving a strengthening of known results. However, we also find indications that the link between incompatibility and Bell violation may be more complex than originally envisaged.

$$|\langle \mathsf{A}_1\mathsf{B}_1\rangle_\eta + \langle \mathsf{A}_1\mathsf{B}_2\rangle_\eta + \langle \mathsf{A}_2\mathsf{B}_1\rangle_\eta - \langle \mathsf{A}_2\mathsf{B}_2\rangle_\eta| \leqslant 2.$$

$$\boxed{\lambda_{\mathrm{opt}} = 1} \longrightarrow \text{classical}$$

$$|\langle \mathsf{A}_1\mathsf{B}_1\rangle_\eta + \langle \mathsf{A}_1\mathsf{B}_2\rangle_\eta + \langle \mathsf{A}_2\mathsf{B}_1\rangle_\eta - \langle \mathsf{A}_2\mathsf{B}_2\rangle_\eta| \leqslant \frac{2}{\lambda_{\mathrm{opt}}}$$

$$\boxed{\lambda_{\mathrm{opt}} = \tfrac{1}{2}} \longrightarrow \text{GPT}$$

Tsirelson bound of $2\sqrt{2}$

$$\boxed{\lambda_{\mathrm{opt}} = \frac{1}{\sqrt{2}}} \longrightarrow \text{quantum}$$

Extended framework: Joint measurements of Positive Operator Valued (POV) observables

1. P. Busch, Phys. Rev. D **33**, 2253 (1986).
2. T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. **38**, 1133 (2008).
3. P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, No. v. 2 in Environmental Engineering, Springer, 1996.
4. M. M. Wolf, D. Perez-Garcia, and C. Fernandez, Phys. Rev. Lett. **103**, 230402 (2009).
5. N. Stevens and P. Busch, Phys. Rev. A **89**, 022123 (2014)
6. M. T. Quintino, T. Vertesi, and N. Brunner, Phys. Rev. Lett. **113**, 160402 (2014)
7. R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. **113**, 160403 (2014).

Positive Operator Valued (POV) observables and unsharp measurements

Def: POV observable $\mathbb{E}$ is a collection $\{E(x)\}$ of positive self-adjoint operators

$$0 \leq E(x) \leq 1$$

called *effects*.

The *effects* satisfy the condition

$$\sum_x E(x) = \mathbb{1}$$

($\mathbb{1}$ is the identity operator)

When a quantum system is prepared in the state ρ, measurement of the POV observable $\mathbb{E}$ gives an outcome x with probability

$$p(x) = \mathrm{Tr}[\rho\, E(x)]$$

Joint Measurability: Global Positive Operator Valued Measure (POVM)

Consider two POV observables

$$\mathbb{E}_i = \{E_i(x_i)\}, \quad i = 1, 2$$

These two POVMs are jointly measurable if there exists a *global* POVM

$$\mathbb{G} = \{G(x_1, x_2);\ 0 \leq G(\lambda) \leq \mathbb{1}, \sum_\lambda G(\lambda) = \mathbb{1}, \quad \lambda = \{x_1, x_2\}\}$$

such that the POV observables $\mathbb{E}_i$ can be realized as the marginals:

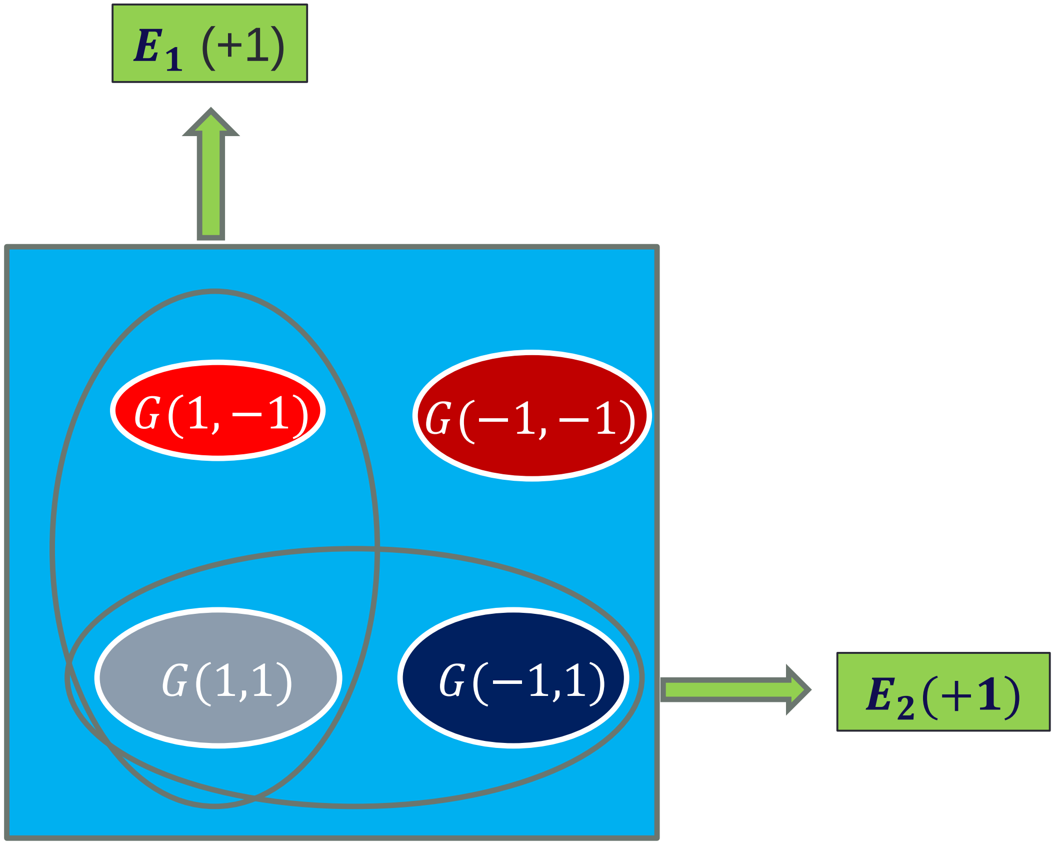
$$E_1(x_1) = \sum_{x_2} G(x_1, x_2), \quad E_2(x_2) = \sum_{x_1} G(x_1, x_2)$$

In general, if the effects $E_i(x_i)$ can be expressed as

$$E_i(x_i) = \sum_\lambda p(x_i|i, \lambda)\, G(\lambda) \quad \forall \quad i$$

where $\sum_{x_i} p(x_i|i, \lambda) = 1$, the fuzzy observables $\mathbb{E}_i$ are jointly measurable.

- Think of G as a common measurement device with four LEDs (corresponding to four outcomes); two of the LEDs correspond to the measurement outcome $+1$ for the binary POVMs E_1 and similarly for E_2.



Fuzzy measurements of noisy qubit observables

Positive operator valued *fuzzy* spin observables:

$$\text{Unsharp } x\text{-spin} \longrightarrow \{E_x(+1), E_x(-1)\}$$
$$\text{Unsharp } z\text{-spin} \longrightarrow \{E_z(+1), E_z(-1)\}$$

$$\begin{aligned} E_x(\pm 1) &= \frac{1}{2}[\mathbb{1} \pm \eta\,\sigma_x] \\ E_z(\pm 1) &= \frac{1}{2}[\mathbb{1} \pm \eta\,\sigma_z] \end{aligned}$$

$$0 \leq \eta \leq 1$$

$$\eta \longrightarrow \text{sharpness parameter}$$

- **PV measurements** $\longrightarrow \eta = 1 \Rightarrow$ **sharp measurement**

- **Joint measurability of** $\sigma_x,\ \sigma_z$ **requires** $\eta \leq \frac{1}{\sqrt{2}}$
- **Global POVM for pairwise joint measurabililty:**

$$G(x,z) = \frac{1}{4}\left[\mathbb{1} + \frac{x}{\sqrt{2}}\,\sigma_x + \frac{z}{\sqrt{2}}\,\sigma_z\right], \quad x, z = \pm 1.$$

- **Joint measurability of three orthogonal spin components** $\sigma_x,\ \sigma_y, \sigma_z$ **implies** $\eta \leq \frac{1}{\sqrt{3}}$
- **Three orthogonal qubit orientations are pairwise measurable but not tripplewise measurable iff** $\frac{1}{\sqrt{3}} \leq \eta \leq \frac{1}{\sqrt{2}}$.

Number of POVMs	Orientation of $\hat{n}_k$	η_{opt}
	Orthogonal axes	
$N = 3$	$\hat{n}_k \cdot \hat{n}_l = 0,\ k \neq l = 1, 2, 3$	$\frac{1}{\sqrt{3}}$
$N = 2$	$\hat{n}_1 \cdot \hat{n}_2 = 0$	$\frac{1}{\sqrt{2}}$
	Trine axes	
$N = 3$	$\hat{n}_k \cdot \hat{n}_l = -\frac{1}{2};\ k \neq l = 1, 2, 3$	$\frac{2}{3}$
$N = 2$	$\hat{n}_1 \cdot \hat{n}_2 = -\frac{1}{2}$	0.732

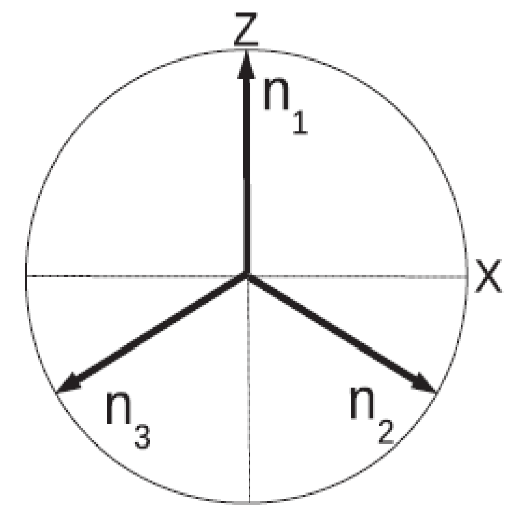
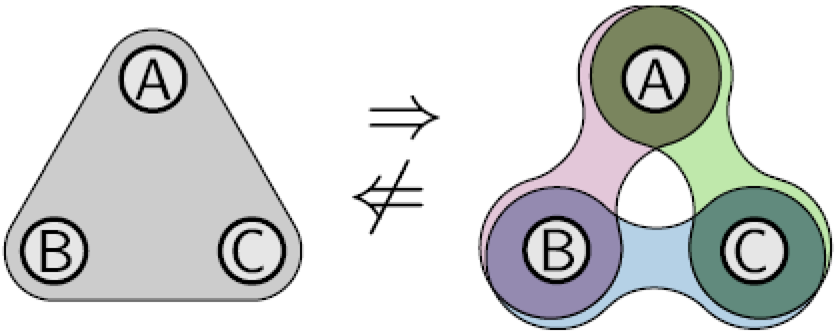


Table 1: Optimal value η_{opt} of the unsharpness parameter (evaluated using the necessary and sufficient conditions, below which the joint measurability of the qubit POVMs $\{E_{x_k}(a_k) = \frac{1}{2}(\mathbb{1} + \eta\, a_k\, \vec{\sigma} \cdot \hat{n}_k)\}$ for different orientations $\hat{n}_k$ are compatible.



Existence of a global observable for three observables A, B and C implies that their exist joint observables for each of the possible pairs $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ but the converse need not be true for unsharp observables

See: P. Busch, Phys. Rev. D 33, 2253 (1986),
T. Heinosaari, D. Reitzner, and P. Stano, Found. Phys. 38, 1133 (2008).

Necessary condition for joint measurability of N dichotomic POVMs with qubit orientations $\hat{n}_k,\ k = 1, 2, \ldots, N$

$$\eta \leq \frac{1}{N} \max_{x_1, x_2, \ldots, x_N} |\vec{m}_{x_1, x_2, \ldots, x_N}|$$

$$\vec{m}_{x_1, x_2, \ldots, x_N} = \sum_{k=1}^{N} x_k \, \hat{n}_k; \quad x_k = \pm 1$$

Sufficient condition:

$$\eta \leq \frac{2^N}{\sum_{\mathbf{a}} |\vec{m}_{\mathbf{a}}} |.$$

See: Ravi Kunjwal and Sibasish Ghosh, Phys. Rev. A 89, 042118 (2014)
Y. C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. 506, 1 (2011).

Joint measurability of equatorial qubit observables

Joint measurability of the POVMs

$$\{E_{\theta_k}(a_k = \pm 1) = \frac{1}{2}(\mathbb{1} + \eta\, a_k\, \sigma_{\theta_k})\}$$

with

$$\sigma_{\theta_k} = \sigma_x\, \cos(\theta_k) + \sigma_y\, \sin(\theta_k); \quad \theta_k = k\,\pi/N,\ k = 1, 2, \ldots, N$$

(correspond geometrically to the points on the circumference of the circle in the equatorial half-plane of the Bloch sphere, separated successively by an angle $\theta = \pi/N$)

$$\Rightarrow \eta_{\rm opt} = \frac{1}{N}\sqrt{N + 2\sum_{k=1}^{[\frac{N}{2}]}(N-2k)\, \cos\left(\frac{k\,\pi}{N}\right)}$$

Number of POVMs	η_{opt}
3	0.6666
4	0.6532
5	0.6472
6	0.6439
10	0.6392
20	0.6372
50	0.6367
100	0.6366

- In the large N limit, the degree of incompatibility (i.e., the cut-off value of the unsharpness parameter) approaches $\eta_{\mathrm{opt}} \to 0.6366$ and thus the POVMs associated with the set of all qubit observables σ_θ, $0 \leq \theta \leq \pi$ in the equatorial plane of the Bloch sphere are jointly measurable in the range $0 \leq \eta_{\mathrm{opt}}^{(\infty)} \leq \frac{2}{\pi} \approx 0.6366$.

Our work

- **We construct chained N term correlation inequality of Budroni et. al., Phys. Rev. Lett. 111, 020403 (2013) based on the positivity of a sequence of moment matrices.**
- **The correlation inequality is shown to be violated only when incompatible measurements of the observables are considered.**
- **The pairwise correlations necessarily obey the *classical bound* when jointly measurable set of POVMs is employed.**
- **However we identify that the compatibility is not sufficient to saturate the Tsirelson bound for $N > 3$.**
- **On the other hand, there exists a one-to-one equivalence between the *degree of incompatibility* (which quantifies the joint measurability) of the equatorial qubit observables and the optimal violation of a non-local steering inequality, proposed by Jones and Wiseman (Phys. Rev. A, 84, 012110 (2011)).**
- **We construct a *local* analogue of the steering inequality in a single qubit system and show that its violation is a mere reflection of measurement incompatibility.**

Chained correlation inequality

- **Consider N classical random variables X_k with outcomes $x_k = \pm 1$.**
- **Construct 4×4 moment matrices $M_k = \langle \xi_k \, \xi_k^T \rangle$ containing only pairwise moments of a set of three random variables each.**

$$\xi_k = \begin{pmatrix} 1 \\ x_1 \, x_k \\ x_k \, x_{k+1} \\ x_1 \, x_{k+1} \end{pmatrix}, \; k = 2, 3, \ldots N-1$$

and $\langle \cdot \rangle$ denotes expectation value.

Chained correlation inequality ...

The 4×4 moment matrix M_k has the form:

$$M_k = \begin{pmatrix} 1 & \langle X_1\, X_k \rangle & \langle X_k\, X_{k+1} \rangle & \langle X_1\, X_{k+1} \rangle \\ \langle X_1\, X_k \rangle & 1 & \langle X_1\, X_{k+1} \rangle & \langle X_k\, X_{k+1} \rangle \\ \langle X_k\, X_{k+1} \rangle & \langle X_1\, X_{k+1} \rangle & 1 & \langle X_1\, X_k \rangle \\ \langle X_1\, X_{k+1} \rangle & \langle X_k\, X_{k+1} \rangle & \langle X_1\, X_k \rangle & 1 \end{pmatrix}.$$

Chained correlation inequality ...

The moment matrix is real, symmetric and positive semidefinite by construction.

Chained correlation inequality . . .

- **The eigenvalues $\mu_i^{(k)}; i = 1, 2, 3, 4$ of the moment matrix:**

$$
\begin{aligned}
\mu_1^{(k)} &= 1 + \langle X_1\, X_k \rangle - \langle X_k\, X_{k+1} \rangle - \langle X_1\, X_{k+1} \rangle \\
\mu_2^{(k)} &= 1 - \langle X_1\, X_k \rangle + \langle X_k\, X_{k+1} \rangle - \langle X_1\, X_{k+1} \rangle \\
\mu_3^{(k)} &= 1 - \langle X_1\, X_k \rangle - \langle X_k\, X_{k+1} \rangle + \langle X_1\, X_{k+1} \rangle \\
\mu_4^{(k)} &= 1 + \langle X_1\, X_k \rangle + \langle X_k\, X_{k+1} \rangle + \langle X_1\, X_{k+1} \rangle.
\end{aligned}
$$

- **Positivity of the moment matrix implies that the eigenvalues $\mu_i^{(k)}$ are positive.**

Chained correlation inequality ...

For a set of $N-1$ **moment matrices** $M_2, M_3, \ldots, M_{N-1}$**, positivity condition** $\sum_{k=2,3,\ldots,N-1} \mu_i^{(k)} \geq 0$**, for the sum of eigenvalues** $\mu_i^{(k)}$**,** $i = 1,2,3,4$ **leads to four chained inequalities for pairwise moments:**

$$\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$2\sum_{k=1}^{N-2} \langle X_1\, X_k\rangle - \sum_{k=2}^{N-1}\langle X_k\, X_{k+1}\rangle + \langle X_1\, X_N\rangle - \langle X_1\, X_2\rangle \leq N-2$$

$$\sum_{i=1}^{N-1} \langle X_i\, X_{i+1}\rangle - \langle X_1\, X_N\rangle \leq N-2$$

$$-\sum_{k=2}^{N-1} \langle X_k\, X_{k+1}\rangle - 2\sum_{k=2}^{N-2}\langle X_1\, X_{k+1}\rangle - \langle X_1\, X_N\rangle + \langle X_1\, X_2\rangle \leq N-2.$$

Chained correlation inequality . . .

Of the four inequalities we find the generalized N-term Leggett-Garg/non-contextual/Bell inequality:
(S. Wehner, Phys. Rev. A 73, 022110 (2006); C. Budroni, T. Moroder, M. Kleinmann, and O. Gühne, Phys. Rev. Lett. 111, 020403 (2013))

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i\, X_{i+1}\rangle - \langle X_1\, X_N\rangle \leq N-2.$$

- **For $N = 3$, we have $\langle X_1\, X_2\rangle + \langle X_2\, X_3\rangle - \langle X_1\, X_3\rangle \leq 1$ (3 term Leggett-Garg Inequality).**
- **For $N = 5$, we have $\langle X_1\, X_2\rangle + \langle X_2\, X_3\rangle + \langle X_3\, X_4\rangle\langle X_4\, X_5\rangle - \langle X_1\, X_5\rangle \leq 3$ (5 term LGI)**

Chained correlation inequality ...

- **We replace the classical random variables by a set of N dichotomic qubit observables**

$$X_k = \vec{\sigma} \cdot \hat{n}_k, k = 1, 2, \ldots, N$$

and the classical probability distribution by an arbitrary single qubit density matrix.

- **The pairwise moments**

$$\langle X_k\, X_l \rangle \equiv \langle X_k\, X_l \rangle_{\text{seq}}$$

are obtained from sequential measurements of the observables – in the order in which they are written.

- **We obtain the chained inequality**

$$\mathcal{S}_N = \sum_{i=1}^{N-1} \langle X_i\, X_{i+1} \rangle_{\text{seq}} - \langle X_1\, X_N \rangle_{\text{seq}} \leq N - 2.$$

Chained correlation inequality . . .

- **Budroni et. al (Phys. Rev. Lett. 111, 020403 (2013)) have evaluated the Tsirelsen Bound for the linear combination of the pairwise correlations in the LHS of the chained N term inequality, when sequential sharp projective measurements are employed for suitably chosen orientations $\hat{n}_k$ for the qubit observables:**

$$\mathcal{S}_N^{(\text{quantum})} \leq N \cos\left(\frac{\pi}{N}\right),$$

- **Thus the classical bound $N-2$ on the chained N term inequality can get violated in the quantum framework.**

PRL **111,** 020403 (2013) PHYSICAL REVIEW LETTERS week ending 12 JULY 2013

Bounding Temporal Quantum Correlations

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Sequential measurements on a single particle play an important role in fundamental tests of quantum mechanics. We provide a general method to analyze temporal quantum correlations, which allows us to

Chained correlation inequality . . .

- **The Tsirelson bound,** $N\,\cos\left(\frac{\pi}{N}\right)$ **can be reached, when the system is prepared in a maximally mixed state** $\rho = I/2$**; and sequential projective measurements of qubit observables** $\vec{\sigma}\cdot\hat{n}_k$**, with unit vectors** $\hat{n}_k$ **equally separated by an angle** π/N **in a plane, one obtains pairwise correlations** $\langle X_k\, X_{k+1}\rangle = \hat{n}_k\cdot\hat{n}_{k+1} = \cos\left(\frac{\pi}{N}\right)$ **and** $\langle X_1\, X_{k+1}\rangle = \hat{n}_1\cdot\hat{n}_N = -\cos\left(\frac{\pi}{N}\right)$ **leading to the the Tsirelsen bound** $N\cos\left(\frac{\pi}{N}\right)$**.**
- **Do we get violation of the inequality if generalized compatible (but unsharp) qubit POVMs are employed?**

Chained correlation inequality ...

- **Using fuzzy qubit POVMs**

$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k), k = 1, 2, \ldots N\}$$

with successive unit vectors $\hat{n}_k$ separated by π/N in a plane, we obtain

$$\langle X_i\, X_{i+1}\rangle_{POVM} = \eta\, \langle X_i\, X_{i+1}\rangle_{\text{sharp}} = \eta\, \cos(\pi/N)$$

and

$$\langle X_1\, X_N\rangle_{POVM} = \eta\, \langle X_1\, X_N\rangle_{\text{sharp}} == -\eta\, \cos(\pi/N)$$

.

Chained correlation inequality . . .

- **The POVMs**

$$\{E_k(x_k) = \frac{1}{2}(I + \eta\, x_k\, \vec{\sigma}\cdot\hat{n}_k)$$

are all jointly measurable/compatible if the *unsharpness* parameter is less than the optimal value $0 \leq \eta \leq \eta_{\rm opt}$.

- **Maximum attainable value $\mathcal{S}_N^Q(\eta_{\rm opt}) = \eta_{\rm opt}\, N\, \cos\left(\frac{\pi}{N}\right)$, of the left hand side of the N term correlation inequality , when the qubit POVMs employed are jointly measurable :**

No. of POVMs employed	Classical bound N-2	Quantum bound $N\ \cos\left(\frac{\pi}{N}\right)$	Maximum achievable value $\mathcal{S}_N^Q(\eta_{\rm opt})$
3	1	1.5	1
4	2	2.83	1.85
5	3	4.05	2.62
6	4	5.20	3.35
10	8	9.51	6.08
20	18	19.75	12.59
50	48	49.90	31.77
100	98	99.95	63.62

- **Incompatible measurements are necessary, but are not sufficient to violate the chained N term temporal correlation inequality**

Some pertinent observations

- Recall that measurement of a single *grand* POVM leads to results of the measurements of all compatible POVMs. In other words, joint measurability entails a joint probability distribution for all compatible observables in the quantum framework. Also, there is Fine's result (A. Fine, Phys. Rev. Lett. 48, 291 (1982)): *Existence of joint probabilities in turn implies that the set of all Bell inequalities are satisfied.*

- Bell non-locality is not revealed when only compatible measurements are employed — even with an entangled state.

- Wolf et. al., (Phys. Rev. Lett. 103, 230402 (2009)) have shown that incompatible measurements of a pair of POVMs with dichotomic outcomes are necessary and sufficient for the violation of Clauser-Horne-Shimony-Holt (CHSH) Bell inequality.

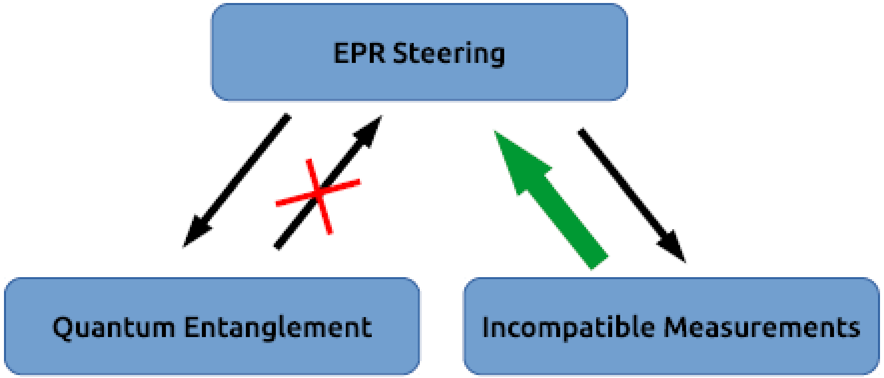
- Equivalence between steering and joint measurability:

  A set of POVMs are not compatible if and only if they can be employed for the task of non-local quantum steering.

  M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)

  R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014))

Steering and Joint Measurability are synonymous



Steering implies both entanglement and incompatible measurements at Bob's end

M. T. Quintino, T. Vértesi, and N. Brunner, Phys. Rev. Lett. 113, 160402 (2014)
R. Uola, T. Moroder, and O. Gühne, Phys. Rev. Lett. 113, 160403 (2014)

But Bell non-locality and joint measurability not synonymous (except in the N=4 CHSH case).

Chained correlation inequality . . .

- Find a steering protocol such that incompatibility of equatorial qubit measurements is both necessary and sufficient

Connection between joint measurability and time-like steering in single system is discussed in

- **H. S. Karthik, J. Prabhu Tej, A. R. Usha Devi, and A. K. Rajagopal, J. Opt. Soc. Am. B. 32, A34 (2015)**
- **M. Pusey, J. Opt. Soc. Am. B. 32, A56 (2015).**

Non-local steering

(E. Schrodinger, Proc. Camb. Phil. Soc. 31, 555563 (1935))

- Suppose Alice prepares a bipartite quantum state ρ_{AB} and sends a subsystem to Bob.
- If the state is entangled, and Alice chooses suitable local measurements, on her part of the state, she can affect Bob's quantum state remotely.
- How would Bob convince himself that his state is indeed steered by Alice's local measurements?

Contd...

- To verify that his (conditional) states are steered, Bob asks Alice to perform local measurements of the observables $\mathbf{X}_k = \sum_{a_k} a_k\, \Pi_{x_k}(a_k)$, on her part of the state and communicate the outcomes a_k in each experimental trial.

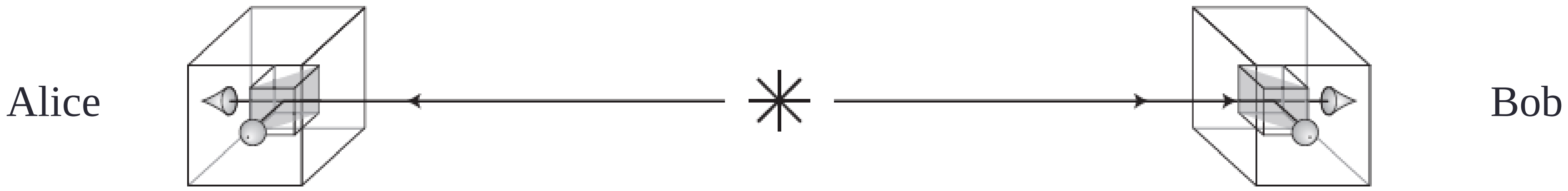
- If Bob's conditional reduced states (unnormalized) $\varrho^B_{a_k|x_k} = \mathrm{Tr}_A[\Pi_{x_k}(a_k) \otimes \mathbb{1}_B\, \rho_{AB}]$, admit a LHS decomposition viz.,

$$\varrho_{a_k|x_k} = \sum_\lambda p(\lambda)\, p(a_k|x_k, \lambda)\, \rho^B_\lambda$$

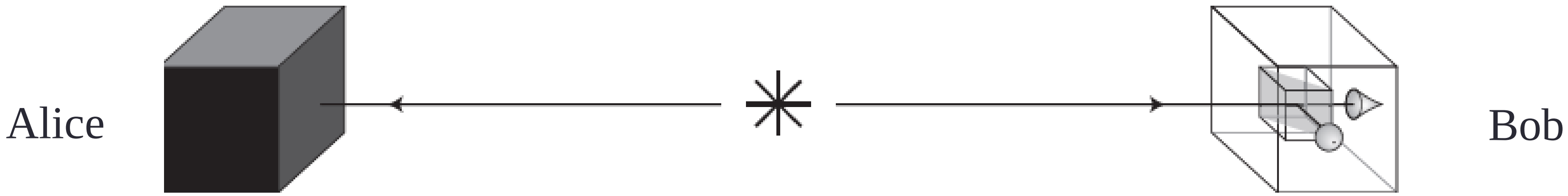
(where $0 \leq p(\lambda) \leq 1;\ \sum_\lambda p(\lambda) = 1$ and $0 \leq p(a_k|x_k, \lambda) \leq 1; \sum_{a_k} p(a_k|x_k, \lambda) = 1;\ (p_\lambda, \rho^B_\lambda)$ denote Bob's LHS ensemble), then Bob can declare that Alice is not able to steer his state through local measurements at her end.

- Incompatibility of Alice's local measurements too plays a crucial role in revealing steerability.

(N. Brunner, News and views, nature physics, 6, 842 (2010))



Trusted devices



Bob cannot trust Alice! Verifies if a 'steering inequality' is violated.

S. J. Jones and H. M. Wiseman, Phys. Rev. A. 84, 012110 (2011).

Linear steering inequality

- **Expectation value of a qubit observable**

$$\mathbf{S}_{\text{plane}} = \frac{1}{\pi}\int_0^{\pi} d\theta\, \alpha_\theta\, \sigma_\theta,$$

(where $\sigma_\theta = \sigma_x \cos(\theta) + \sigma_y \sin(\theta)$ $\rightarrow$ **equatorial qubit observable) is upper bounded by** $\frac{2}{\pi} \approx 0.6366$.

- **The Wiseman-Jones steering inequality:**

$$\frac{1}{\pi}\int_0^{\pi} d\theta\, \langle \sigma_\theta^A\, \sigma_\theta^B \rangle \leq \frac{2}{\pi}$$

$$\langle \sigma_\theta^B \rangle_{a_\theta} = \sum_{b_\theta = \pm 1} b_\theta\, p(b_\theta | a_\theta; \theta)$$

- **Violation of the inequality in any bipartite quantum state** ρ_{AB} **demonstrates non-local EPR steering phenomena.**

Linear steering inequality in the finite setting

With N evenly spaced equatorial measurements of σ_{θ_k} by Bob, conditioned by dichotomic outcomes a_k of Alice's measurements $\sigma^A_{\theta_k}$ $\Rightarrow$

$$
\begin{aligned}
\frac{1}{N}\sum_{k=1}^{N}\langle\sigma^A_{\theta_k}\,\sigma^B_{\theta_k}\rangle &\leq f(N) \\
f(N) &= \frac{1}{N}\left(\left|\sin\left(\frac{N\pi}{2}\right)\right| + 2\sum_{k=1}^{[N/2]}\sin\left[(2k-1)\frac{\pi}{2N}\right]\right).
\end{aligned}
$$

- **$f(N)$ is the maximum eigenvalue of the observable $\frac{1}{N}\sum_{k=1}^{N}\sigma_{\theta_k}$.**
- **Largest value of $f(N)$ is $f(2)\approx 0.7071$ for $N=2$.**
  Smallest value $f(\infty)=0.6366$ when $N\rightarrow\infty$.

S. J. Jones and H. M. Wiseman, Phys. Rev. A. 84, 012110 (2011)

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Experimental proof of nonlocal wavefunction collapse for a single particle using homodyne measurements

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A single quantum particle can be described by a wavefunction that spreads over arbitrarily large distances; however, it is never detected in two (or more) places. This strange phenomenon is explained in the quantum theory by what Einstein repudiated as 'spooky action at a distance': the instantaneous nonlocal collapse of the wavefunction to wherever the particle is detected. Here we demonstrate this single-particle spooky action, with no efficiency loophole, by splitting a single photon between two laboratories and experimentally testing whether the choice of measurement in one laboratory really causes a change in the local quantum state in the other laboratory. To this end, we use homodyne measurements with six different measurement settings and quantitatively verify Einstein's spooky action by violating an Einstein-Podolsky-Rosen-steering inequality by 0.042 ± 0.006. Our experiment also verifies the entanglement of the split single photon even when one side is untrusted.

PHYSICAL REVIEW A 95, 052105 (2017)

N-term pairwise-correlation inequalities, steering, and joint measurability

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Chained inequalities involving pairwise correlations of qubit observables in the equatorial plane are constructed based on the positivity of a sequence of moment matrices. When a jointly measurable set of positive-operator-valued measures (POVMs) is employed in the first measurement of every pair of sequential measurements, the chained pairwise correlations do not violate the classical bound imposed by the moment matrix positivity. We find that incompatibility of the set of POVMs employed in first measurements is only necessary, but not sufficient, in general, for the violation of the inequality. On the other hand, there exists a one-to-one equivalence between the degree of incompatibility (which quantifies the joint measurability) of the equatorial qubit POVMs and the optimal violation of a nonlocal steering inequality, proposed by Jones and Wiseman [S. J. Jones and H. M. Wiseman, Phys. Rev. A **84**, 012110 (2011)]. To this end, we construct a local analog of this steering inequality in a single-qubit system and show that its violation is a mere reflection of measurement incompatibility of equatorial qubit POVMs, employed in first measurements in the sequential unsharp-sharp scheme.

DOI: 10.1103/PhysRevA.95.052105

Implications of joint measurability on the the finite setting linear steering inequality:

$$\eta_{\rm opt} \leq f(N)$$

- A striking agreement between the degree of incompatibility $\eta_{\rm opt}$ and $f(N)$.
- This is a clear example of the intrinsic connection between steering and measurement incompatibility.

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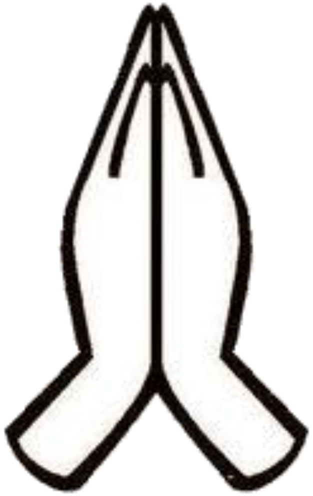

namaste