Lessons from a 3-qubit experiment: from entanglement and thermalization to exact solutions

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Thermalization, Entanglement, Many-body systems, and all that jazz

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1. Entanglement within many-body quantum states drives subsystems to thermalization although the full state remain pure and of zero entropy.

2. Closed many-body systems maybe integrable or not, and it is believed that non-integrable systems with very few conserved laws rapidly thermalize.

3. Information diffusion within many-body systems could be exponentially fast and lead to "scrambling", "butterfly effect" and ...

to identify fundamental commonalities in the determinants of livelihoods across settings, suggesting that our approach could be used to fill in the large data gaps resulting from poor survey coverage in many African countries. In contrast to other recent approaches that rely on proprietary commercial data sets, our method uses only publicly available data and so is straightforward and nearly costless to scale across countries.

Although our model outperforms other sources of passively collected data (e.g., cellphone data, nightlights) in estimating economic well-being at the cluster level, we are currently unable to assess its ability to discern differences within clusters, as public-domain survey data assign identical coordinates to all households in a given cluster to preserve respondent privacy. In principle, our model can make predictions at any resolution for which daytime satellite imagery is available, though predictions on finer scales would likely be noisier. New sources of ground truth data, whether from more disaggregated surveys or novel crowdsourced channels, could enable evaluation of our model at the household level. Combining our extracted features with other passively collected data, in locations where such data are available, could also increase both household- and cluster-level predictive power.

Given the limited availability of high-resolution time series of daytime imagery, we also have not yet been able to evaluate the ability of our transfer learning approach to predict changes in economic well-being over time at particular locations. Such

STATISTICAL PHYSICS

Quantum thermalization through entanglement in an isolated many-body system

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Statistical mechanics relies on the maximization of entropy in a system at thermal equilibrium. However, an isolated quantum many-body system initialized in a pure state remains pure during Schrödinger evolution, and in this sense it has static, zero entropy. We experimentally studied the emergence of statistical mechanics in a quantum state and observed the fundamental role of quantum entanglement in facilitating this emergence. Microscopy of an evolving quantum system indicates that the full quantum state remains pure, whereas thermalization occurs on a local scale. We directly measured entanglement entropy, which assumes the role of the thermal entropy in thermalization. The entanglement creates local entropy that validates the use of statistical physics for local observables. Our measurements are consistent with the eigenstate thermalization hypothesis.

When an isolated quantum system is perturbed—for instance, owing to a sudden change in the Hamiltonian (a so-called quench)—the ensuing dynamics are determined by an eigenstate distribution that is induced by the quench (J). At any given time, the evolving quantum state will have amplitudes that depend on the eigenstates populated by the quench and the energy eigenvalues of the Hamiltonian. In many cases, however,

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Ergodic dynamics and thermalization in an isolated quantum system

C. Neill[1*†], P. Roushan[2†], M. Fang[††], Y. Chen[2†], M. Kolodrubetz[3], Z. Chen[1], A. Megrant[2], R. Barends[2], B. Campbell[1], B. Chiaro[1], A. Dunsworth[1], E. Jeffrey[2], J. Kelly[2], J. Mutus[2], P. J. J. O'Malley[1], C. Quintana[1], D. Sank[2], A. Vainsencher[1], J. Wenner[1], T. C. White[2], A. Polkovnikov[3] and J. M. Martinis[1,2]

Statistical mechanics is founded on the assumption that all accessible configurations of a system are equally likely. This requires dynamics that explore all states over time, known as ergodic dynamics. In isolated quantum systems, however, the occurrence of ergodic behaviour has remained an outstanding question[1–4]. Here, we demonstrate ergodic dynamics in a small quantum system consisting of only three superconducting qubits. The qubits undergo a sequence of rotations and interactions and we measure the evolution of the density matrix. Maps of the entanglement entropy show that the full system can act like a reservoir for individual qubits, increasing their entropy through entanglement. Surprisingly, these maps bear a strong resemblance to the phase space dynamics in the classical limit; classically, chaotic motion coincides with higher entanglement entropy. We further show that in regions of high entropy the full multi-qubit system undergoes ergodic dynamics. Our work illustrates how controllable quantum systems can investigate fundamental questions in non-equilibrium thermodynamics.

isolated quantum systems[1–4]. Do quantum systems exhibit ergodic behaviour in the sense of equation (1)? Do quantum systems act as their own bath to approach thermal equilibrium? Extensive experimental efforts have been made to address these fundamental questions[7–13].

Here, we investigate ergodic dynamics by considering a simple quantum model whose classical limit is chaotic[14–18]. This model describes a collection of spin-1/2 particles whose collective motion is equivalent to that of a single larger spin with total angular momentum j governed by the Hamiltonian

$$\mathcal{H}(t) = \frac{\pi}{2\tau} J_y + \frac{\kappa}{2j} J_z^2 \sum_{n=1}^{N} \delta(t - n\tau) \qquad (2)$$

where J_y and J_z are angular momentum operators. The sum over delta functions implies N applications of J_z^2, each at integer time steps. The angular momentum operators can be expressed in terms of the constituent spin-1/2 Pauli operators, for example, $J_z = (\hbar/2) \sum_i \sigma_z^{(i)}$. Setting $\tau = 1$, the first term in $\mathcal{H}$ causes each spin

Kicked tops to many-body spins

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$$H = \left(\frac{\hbar p}{\tau}\right) J_y + \left(\frac{\hbar \kappa}{2j}\right) J_z^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

(Kus, Scharf, Haake, 1987; Haake's book.)

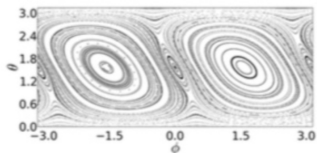$J^{x,y,z} = \sum_{l=1}^{2j} \sigma^{x,y,z}/2$, the unitary or Floquet operator:

$$U = \exp\left(-i\frac{\kappa}{8j} \sum_{l \neq l'=1}^{2j} \sigma_l^z \sigma_{l'}^z\right) \exp\left(-i\frac{\pi}{4} \sum_{l=1}^{2j} \sigma_l^y\right),$$

(Wang, Ghose, Sanders, and Hu, 2004)

Thermodynamic limit is also classical limt $j \to \infty$.

Classical Map: $X_n^2 + Y_n^2 + Z_n^2 = 1$
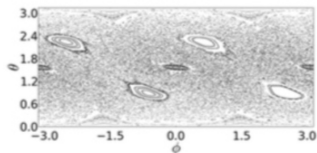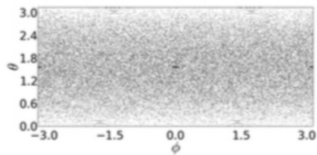
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$$X_{n+1} = Z_n \cos(\kappa X_n) + Y_n \sin(\kappa X_n)$$
$$Y_{n+1} = -Z_n \sin(\kappa X_n) + Y_n \cos(\kappa X_n)$$
$$Z_{n+1} = -X_n.$$

(a)

(b)

Results from the 3-Qubit experiment

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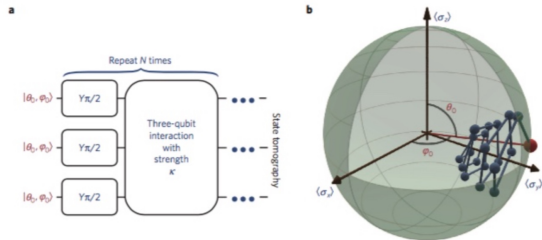
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Figure 1 | Pulse sequence and the resulting quantum dynamics. a, Pulse sequence showing first the initial state of the three qubits (equation (4)) followed by the unitary operations for a single time step (equation (3)). These operations are repeated N times before measurement. Single-qubit rotations are generated using shaped microwave pulses in 20 ns; the three-qubit interaction is generated using a tunable coupling circuit controlled using square pulses of length 5 ns for $\kappa = 0.5$ and 25 ns for $\kappa = 2.5$. **b,** The state of a single qubit is measured using state tomography and shown in a Bloch sphere. The initial state is shown in red with subsequent states shown in blue for N=1–20.

Results from the 3-Qubit experiment

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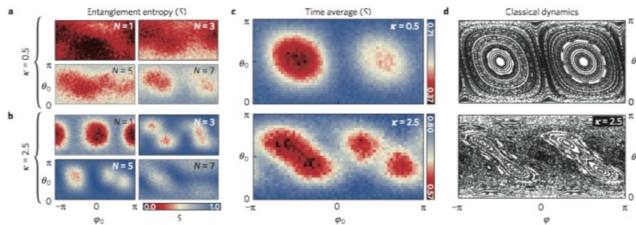**Figure 2 | Entanglement entropy and classical chaos. a,b,** The entanglement entropy (colour) of a single qubit (see equation (5)) averaged over qubits and mapped over a 31 × 61 grid of the initial state, for various time steps N and two values of interaction strength κ = 0.5 (**a**) and κ = 2.5 (**b**). The entanglement entropy of a single qubit can range from 0 to 1. **c,** The entanglement entropy averaged over 20 steps for κ = 0.5 and over 10 steps for κ = 2.5; for both experiments the maximum pulse sequence is ≈ 500 ns. The left/right asymmetry is the result of experimental imperfections and is not present in numerical simulations (see Supplementary Information). **d,** A stroboscopic map of the classical dynamics is computed numerically and shown for comparison. The map is generated by randomly choosing 5,000 initial states, propagating each state forwards using the classical equations of motion, and plotting the orientation of the state after each step as a point. We observe a clear connection between regions of chaotic behaviour (classical) and high entanglement entropy (quantum).

Results from the 3-Qubit experiment

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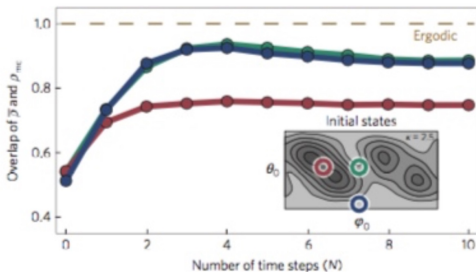Vaibhav Madhok
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Figure 4 | Ergodic dynamics. The overlap of the time-averaged three-qubit density matrix with a microcanonical ensemble (see equation (6)) versus number of time steps N, for $\kappa = 2.5$. We choose three different initial states, shown inset. A value of 1.0 indicates that the dynamics are fully ergodic.

Our analysis of experimental data: Correlations of various kinds

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$$|\psi_n\rangle = U^n|000\rangle$$

1. Entanglement of one qubit with other two.
2. Discord, concurrence between two qubits.
3. 3-Tangle between three qubits.
4. CHSH violation measure between 2 qubits ($M(\rho)$)

$$T_{ij} = \text{tr}(\sigma_i \otimes \sigma_j\rho), \;\; h_i : \text{eigenvalues of } TT^\dagger$$

$$M(\rho) = \max_{i<j} h_i + h_j$$

$M(\rho) > 1$: some setting violates CHSH inequality.

Small j: $j = 3/2$, (3 Qubits):

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(a) $\kappa = 0.5$ (b) $\kappa = 2.5$
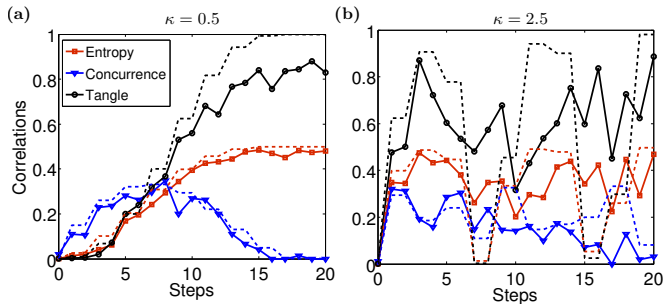
1. Rapid growth of correlations when system is chaotic.
2. Steps not seen in time averages indicate peculiar LU invariance between even-odd times.
3. Eventual domination of multipartite entanglement.
4. Dependence of decoherence on chaos.

Small j: $j = 3/2$, (3 Qubits):

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(a) $\kappa = 0.5$

(b) $\kappa = 2.5$
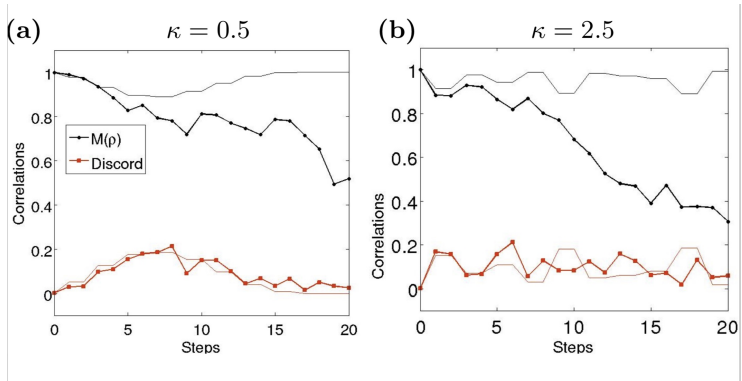
1. No violation of CHSH whether there is chaos or not: This is not true for typical 3-qubit pure states.

Small j: $j = 3/2$, (3 Qubits): A global view from simulations
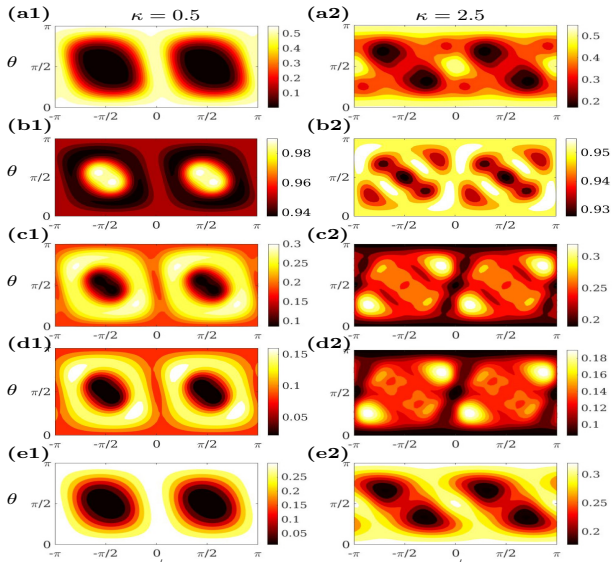
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Permutation symmetry and the Top

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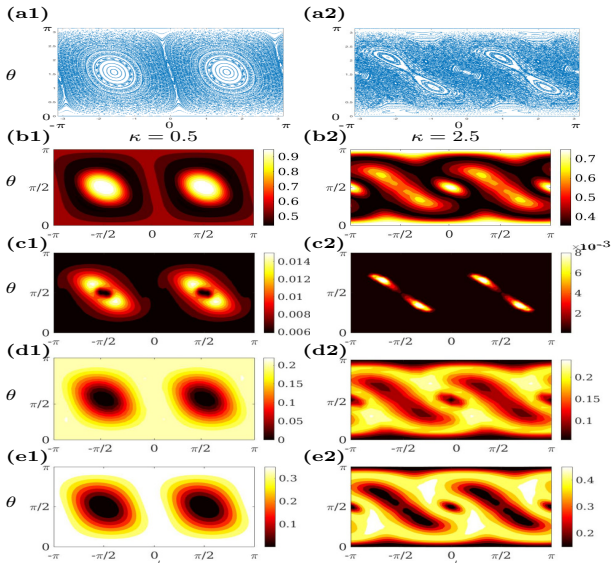
States of the Top are **permutation symmetric**

$$|\psi\rangle = a_0|000\rangle + a_1\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)+$$

$$a_2\frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle) + a_3|111\rangle$$

1. State of 2 qubits is a rank-3 $X-$ state.

2. State of 2 qubits do not violate any CHSH inequality.

3. Smaller entanglement in general: In an L qubit symmetric random state, the linear entropy of 1 qubit is on average $\langle S_L^{(1)} \rangle = (L-1)/(2L) = 1/3$ for $L = 3$.

Permutation symmetry and the Top

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From the 3-Qubit experiment

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It is interesting to know the generality of our results, as they could provide a generic framework for studying quantum dynamics. Numerical results suggest that ergodic behaviour breaks down only when the evolution is highly constrained by conservation laws; such systems are referred to as integrable and represent models that are fine tuned and consequently rare[3]. Our choice of Hamiltonian was motivated by the lack of conserved quantities, where only the total spin is conserved—not even energy is conserved. We believe that our simple and clear descriptions of thermalization merely lay the foundation upon which many fundamental questions in non-equilibrium thermodynamics can be experimentally investigated.

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Exact solution to the 3-qubit Top

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Additional parity or up-down symmetry: $[\otimes^{2j}\sigma_y,\ U] = 0$
4$-$dim permutation symm. subspace $= 2 \oplus 2$.

$$\{|000\rangle - i|111\rangle, |W\rangle + i|\overline{W}\rangle\}/\sqrt{2}$$
$$\{|000\rangle + i|111\rangle, |W\rangle - i|\overline{W}\rangle\}/\sqrt{2}$$

$$U^n|000\rangle = \frac{1}{2}(1 + i^n)\left(\alpha_n|000\rangle + i\beta_n|\overline{W}\rangle\right)$$
$$- \frac{i}{2}(1 - i^n)\left(\alpha_n|111\rangle + i\beta_n|W\rangle\right).$$

$$\alpha_n = T_n(\chi) - \frac{i}{2}U_{n-1}(\chi)\cos 2\kappa, \quad \beta_n = -\frac{\sqrt{3}}{2}U_{n-1}(\chi)\,e^{-2i\kappa},$$

where $\chi = \sin(2\kappa)/2$.

Exact solution to the 3-qubit Top

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Single qubit density matrix eigenvalues after time $2n$:
$\lambda_{2n}, 1 - \lambda_{2n}$,

$$\lambda_{2n} = \frac{1}{2} U_{2n-1}^2(\chi), \quad \chi = \frac{1}{2}\sin(2\kappa).$$

Tchebyshev Polynomials of 1st and 2nd kinds:

$$T_n(\cos\theta) = \cos(n\theta), \quad U_{n-1}(\cos\theta) = \sin(n\theta)/\sin\theta$$

Concurrence at time $2n$:

$$C_{2n} = |U_{2n-1}(\chi)| \left| \frac{1}{2}|U_{2n-1}(\chi)| - \sqrt{1 - \frac{3}{4}U_{2n-1}^2(\chi)} \right|$$

$\lambda_{2n-1} = \lambda_{2n}, \ C_{2n-1} = C_{2n}$: Steps and LU invariance

So is it chaotic?

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Exactly solvable chaotic models: Both rare and special.
$3 < \infty$, $3 < 4$: All-to-all interaction is same as nearest
neighbour for 3 particles.

$$U = \exp\left(i\frac{\kappa}{6}(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z\sigma_1^z)\right)$$
$$\times \exp\left(i\frac{\pi}{4}(\sigma_1^y + \sigma_2^y + \sigma_3^y)\right)$$

Can also be a small particle number of a kicked
transverse Ising model: an integrable model!
Entropy production can be ballistic even in integrable
models: (Cardy, Calaberese for TI model). (Rajarshi talk
as well).

Summary

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- Recent experiments explore thermalization and entanglement in small closed systems.
- One such experiment showed an approach to ergodic dynamics and made connections with the classical chaotic dynamics.
- Our analysis showed peculiar steps due to LU invariance and effects of permutation symmetry.
- Decoherence in the presence of chaos seems to be more drastic and may appear as thermalization in small systems.

Summary

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- Alhough the large number of qubits case is nonintegrable, the 3 qubit case is special and may even be construed as a small integrable model. Yet it is possible that thermalization may seem to occur.

- Small systems provide an interesting and surprisingly complex setting.

- Mindful of these effects as quantum computers are interacting many-body qubit systems.

Thats all folks!