Multipartite pure states can almost never be manipulated by LOCC

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joint work with

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Atoms, Light, and Molecules
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MPQ

Introduction

Multipartite LOCC transformations

Conclusion & Outlook

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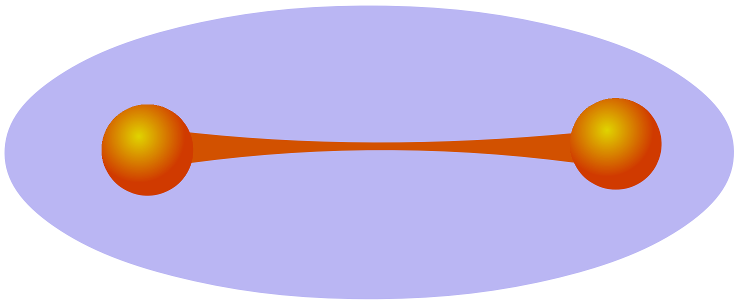
Bipartite entanglement

→ very well understood

→ have a maximally entangled state

$$|\phi^+\rangle \propto \sum_{i=0}^{d-1} |i\rangle|i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$$

$$S(|\phi^+\rangle\langle\phi^+|) = 0 \qquad S(\rho_A) = \log(d)$$



Many applications:

→ teleportation Bennett et al., PRL 1993

→ measurement based Quantum Key Distribution (QKD) Ekert, PRL 1991

→ super dense coding Bennett, Wiesner, PRL 1992

→ ...

⇒ $|\phi^+\rangle$ **is most useful state.**

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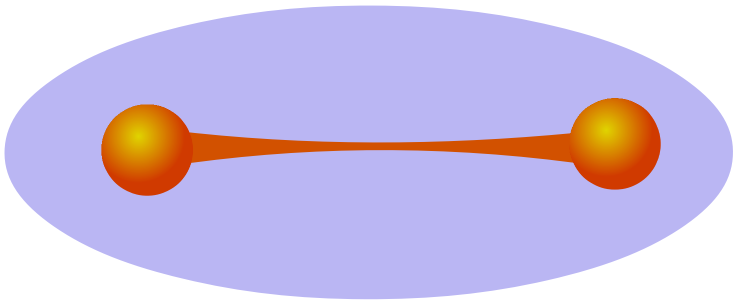

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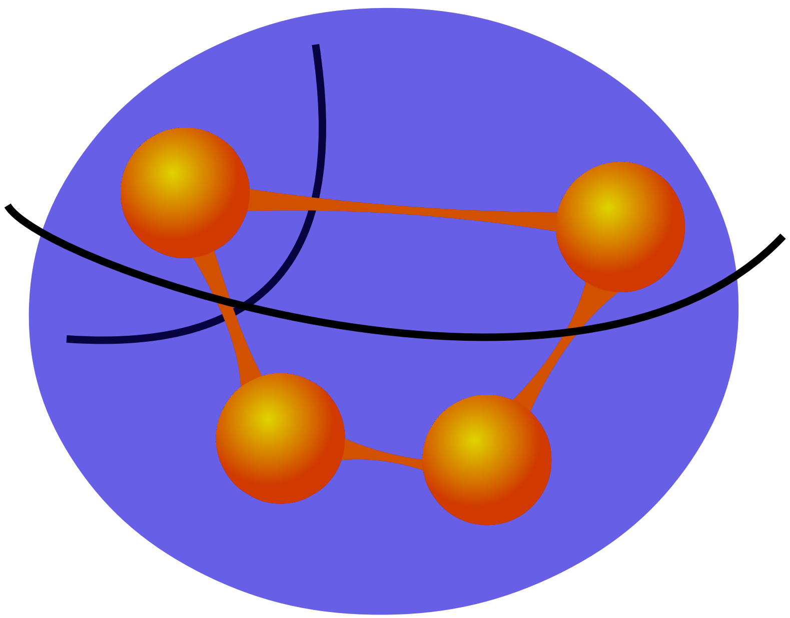
Multipartite Entanglement

A pure state is multipartite entangled if it is entangled in all bipartitions.



In contrast to bipartite entanglement, **multipartite entanglement is much more complicated,** because

→ number of parameters grows exponentially with number of subsystems

→ many different kinds of multipartite entanglement (e.g. no ~~THE maximally entangled state~~)

Multipartite states and applications

Several applications of multipartite entangled states are known:

quantum computation[1], quantum error correction[2], quantum metrology[3], quantum secret sharing[4], applications in condensed matter theory[4]

Others?

Need better understanding of multipartite entanglement
→ systematic approach!

1) e.g. Raussendorf, Briegel, PRL 2001; 2) see e.g. Nielsen, Chuang, Cambridge Univ. Press 2010; 3) e.g. Giovannetti et al., PRL 2006; 4) e.g. Hillery et al., PRA 1999; 5) e.g. Cirac, Verstraete, J. Phys. A: Math. Theor. 2009

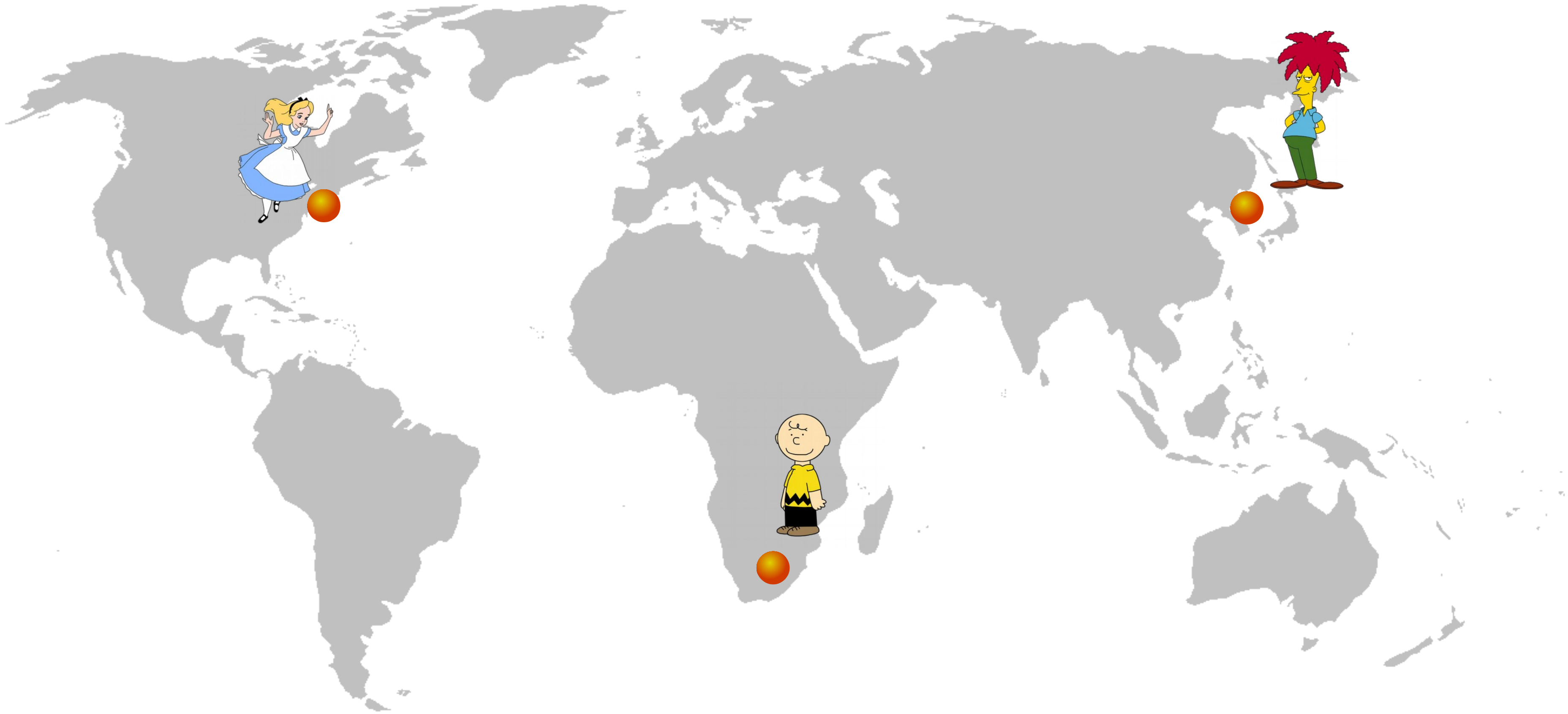
Introduction

Multipartite LOCC transformations

Conclusion & Outlook

LOCC

Local Operations (LO) and Classical Communication (CC)



Pictures: disney.wikia.com, clownopedia.wikia.com, shauntmax30.com, commons.wikimedia.org

LOCC

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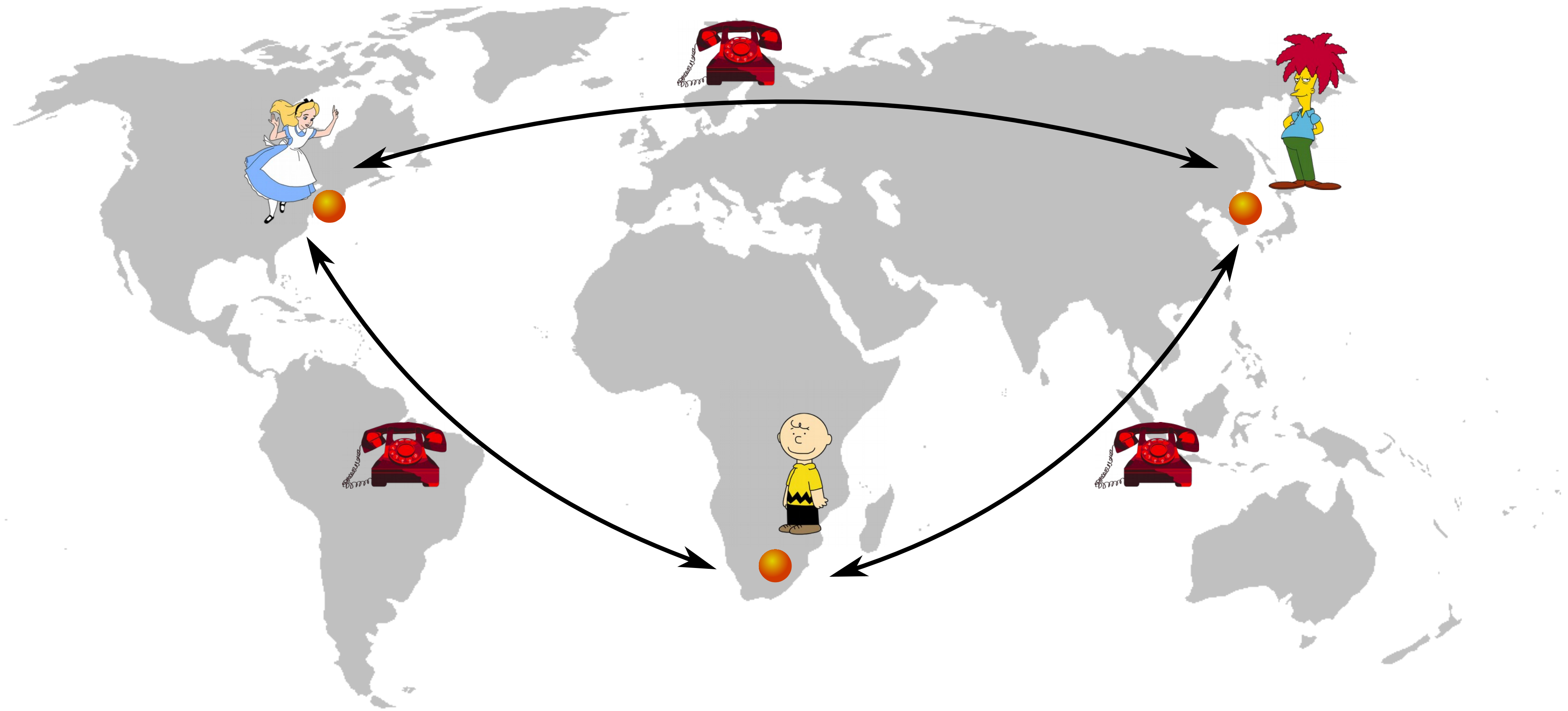

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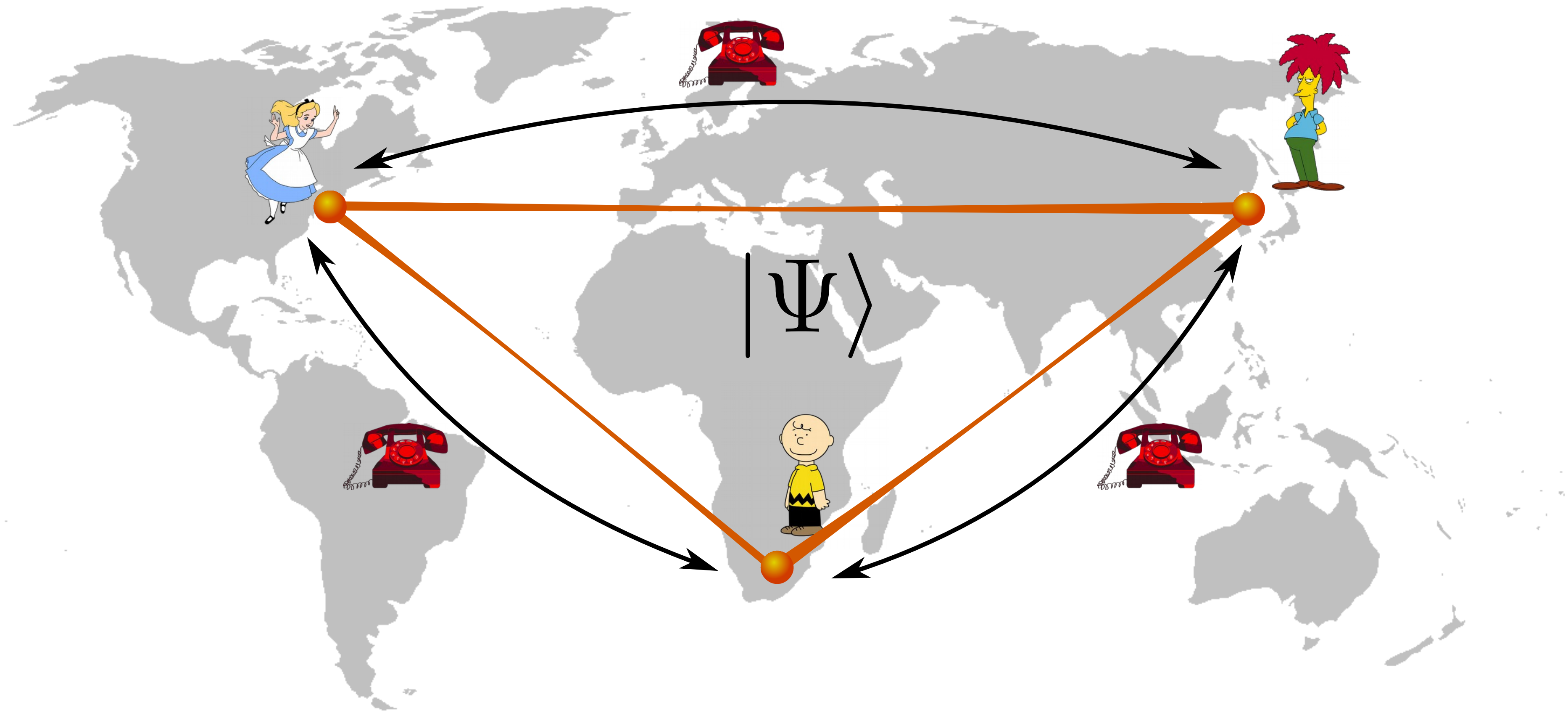
LOCC

Local Operations (LO) and Classical Communication (CC)

Spatially separated parties use LOCC to manipulate entangled quantum states



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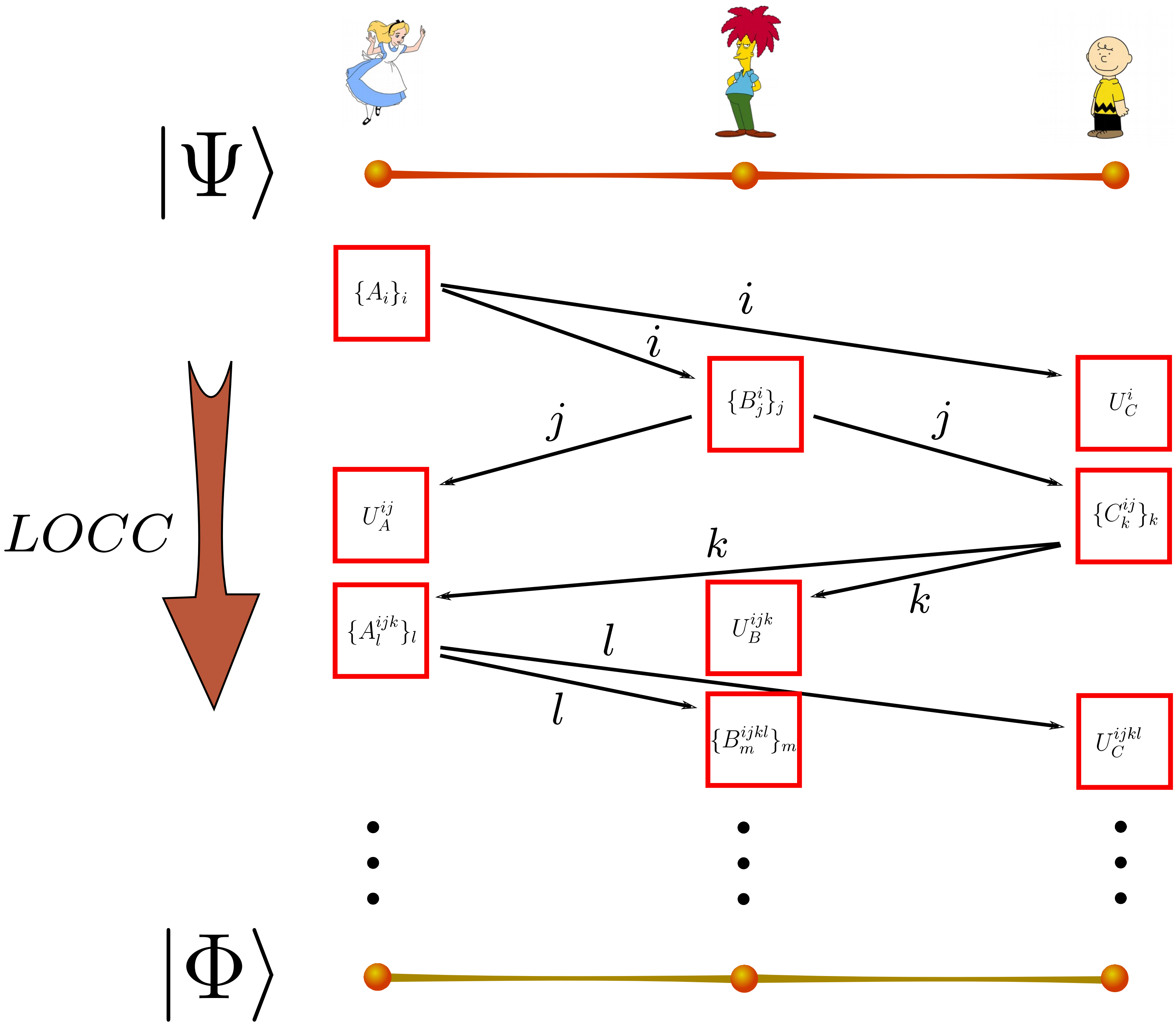

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Entanglement is a resource under LOCC!

→ entanglement cannot increase under LOCC operations

→ Alice, Bob, ... can use the resource entanglement to overcome LOCC restriction

$$|\Psi\rangle \xrightarrow{LOCC} |\Phi\rangle \Rightarrow |\Psi\rangle \text{ at least as entangled as } |\Phi\rangle$$

$$\text{(at least as useful as)}$$

Understanding LOCC is important to understand entanglement!

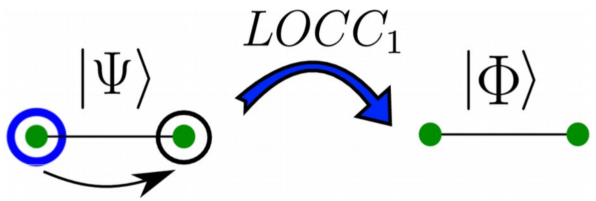
Single out most relevant states!

Bipartite pure state LOCC transformations

Have Schmidt decomposition: $$|\Psi\rangle = U_A \otimes U_B \sum_{i=0}^{d-1} \sqrt{\lambda_i^\Psi} |i\rangle|i\rangle \ \in \mathbb{C}^d \otimes \mathbb{C}^d$$

$$\vec{\lambda}(\Psi) = (\lambda_0^\Psi, \ldots, \lambda_{d-1}^\Psi)$$

$\rightarrow$ all LOCC protocols are equivalent to



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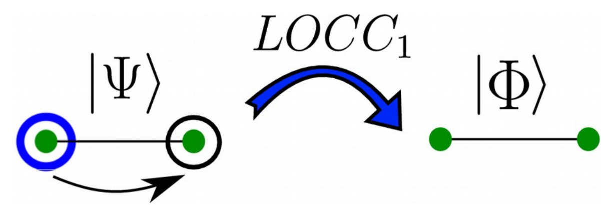
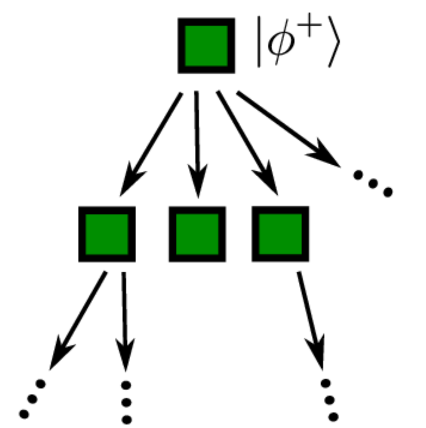


$$|\Psi\rangle \xrightarrow{LOCC} |\Phi\rangle \Leftrightarrow \vec{\lambda}(\Psi) \prec \vec{\lambda}(\Phi)$$

Nielsen, PRL 1999

$\vec{\lambda}(\Psi) \prec \vec{\lambda}(\Phi)$ if $\sum_{i=0}^{k} \lambda_i^{\Psi\downarrow} \leq \sum_{i=0}^{k} \lambda_i^{\Phi\downarrow}$ $, \forall k \in \{0, \ldots, d-1\}$

$|\phi^+\rangle$ is indeed the maximally entangled state



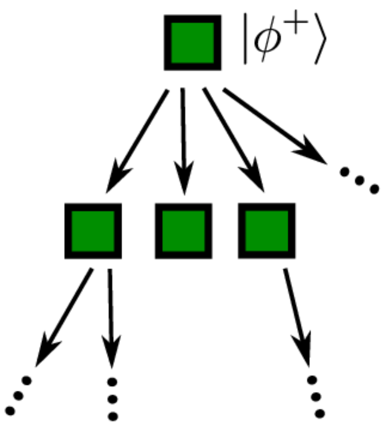
Multipartite pure state LOCC transformations

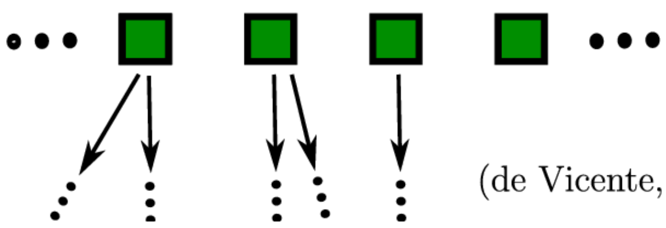
No Schmidt decomposition for multipartite states!

→ a lot more difficult to find multipartite LOCC hierarchy

Example:

Maximally entangled state → Maximally Entangled Set (MES)





(de Vicente, Spee, Kraus, PRL, 2013)

Multipartite pure state LOCC transformations

Different approaches:

→ characterize all LOCC transformations for small systems
e.g. three qubits,... (e.g. Turgut et al, PRA 2010,...)

→ look at special states (e.g. graph states, GHZ type states,...) (e.g. Cui, Chitambar, Lo, PRA 2010,...)

→ look at more general transformations

→ stochastic LOCC (SLOCC) transformations (e.g. Dür, Vidal, Cirac, PRA 2000)

→ deterministic separable (SEP) transformations (e.g. Gour, Wallach, NJP 2011)

→ look at finite-round LOCC ($LOCC_{\mathbb{N}}$)

(Spee, de Vicente, DS, Kraus, PRL 2017 & de Vicente, Spee, DS, Kraus, PRA 2017)

→ ...

LOCC transformations of fully entangled states

We are interested in LOCC transformations of fully entangled states.

$|\Psi\rangle$ fully entangled if $\rho_i = \mathrm{Tr}_{\neq i}(|\Psi\rangle\langle\Psi|)$ has full rank for all i.

For states that are not fully entangled we can map the problem to a system with less dimensions.

The local symmetries of a fully-entangled $|\Psi\rangle$ are elements of the group,

$$\widetilde{G}_\Psi = \{g = g_1 \otimes \ldots \otimes g_n \mid g|\Psi\rangle = |\Psi\rangle\} \subset GL(\mathcal{H}_n)$$

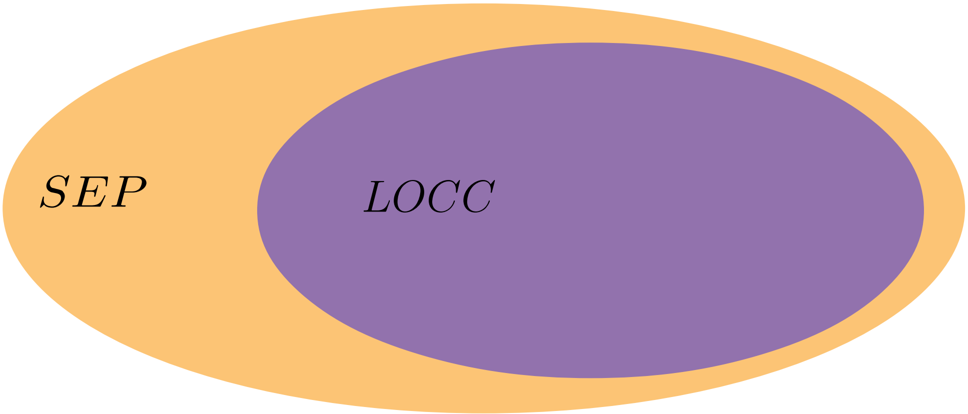
$\widetilde{G}_\Psi$ **is fundamental for LOCC transformations of** $|\Psi\rangle$**!**

Gour, Wallach, NJP 2011

Separable Transformations

$$\Lambda_{SEP}(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$$

$$K_i = K_i^{(1)} \otimes \ldots \otimes K_i^{(n)}$$



LOCC operations are a subset of separable (SEP) operations.

→ SEP is more powerful than LOCC, even for pure states. Chitambar et al., Commn. Math. Phys. 2014; Hebenstreit, Spee, Kraus, PRA 2016

→ SEP does not have a clear physical meaning

→ easier to deal with SEP than with LOCC

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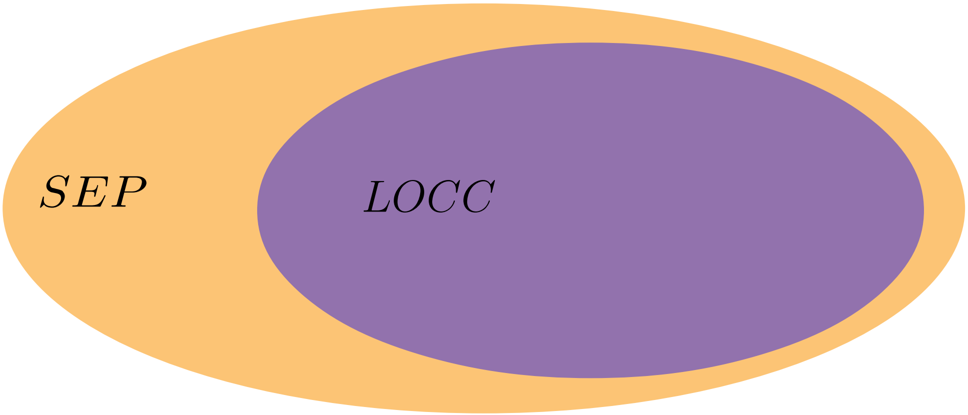

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Hebenstreit, Spee, Kraus, PRA 2016

$\rightarrow$ SEP does not have a clear physical meaning

$\rightarrow$ easier to deal with SEP than with LOCC

$|\Psi\rangle, |\Phi\rangle$ fully entangled :

$|\Psi\rangle \xrightarrow{LOCC/SEP} |\Phi\rangle$ only if $|\Phi\rangle = h_1 \otimes \ldots \otimes h_n |\Psi\rangle$ for some invertible h_i

Dür, Vidal, Cirac, PRA 2000

Local transformations are determined by local symmetries

Local symmetries determine SEP and LOCC transformations:

Gour, Wallach, NJP 2011

$$|\Psi\rangle \xrightarrow{SEP} h|\Psi\rangle \Leftrightarrow \text{ there exists a } m \in \mathbb{N}, \text{ probabilities } \{p_i\}_{i=1}^m \text{and}$$

$$\{S_i\}_{i=1}^m \subset \tilde{G}_\Psi : \sum_{i=1} p_i S_i^\dagger H S_i = r\mathbb{I}$$

$$H = h^\dagger h, r = \| h|\Psi\rangle \|^2 / \| |\Psi\rangle \|^2$$

→ bipartite states, GHZ states, W states, cluster states,... have many symmetries

→ for many multi-qudit systems almost all states have only finitely many symmetries

Gour, Wallach, NJP 2011

→ for states with $\widetilde{G}_\Psi = \{\mathbb{I}\}$ only trivial transformations are possible.

Gour, Kraus, Wallach, JMP 2017

Characterization of LOCC transformations of almost all multi-qudit states

DS, Wallach, Gour, Kraus, in preparation 2017

We characterize LOCC transformations of almost all states (i.e. a full-measure subset) of n>3 qudits with local dimension d>2 and certain tripartite systems.

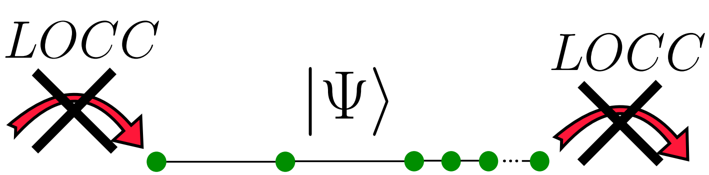
Characterization of LOCC transformations of almost all multi-qudit states

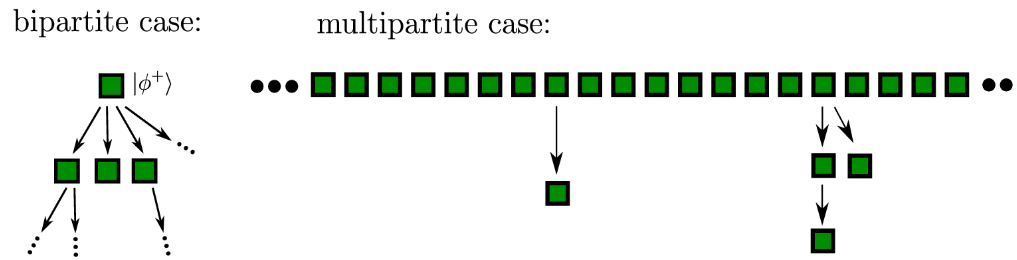
DS, Wallach, Gour, Kraus, arXiv:1711.11056

We characterize LOCC transformations of almost all states (i.e. a full-measure subset) of n>3 qudits with local dimension d>2 and certain tripartite systems.

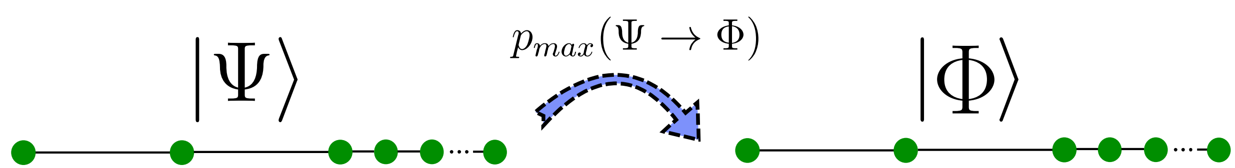
In particular, for almost all multi-qudit states $|\Psi\rangle$:

→ nontrivial deterministic LOCC transformations to/from $|\Psi\rangle$ from/to other (fully entangled) states not possible (the state is isolated).





→ have explicit expression for optimal probability to transform $|\Psi\rangle$ via LOCC to an other fully entangled state $|\Phi\rangle$.



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DS, Wallach, Gour, Kraus, arXiv:1711.11056

We characterize LOCC transformations of almost all states (i.e. a full-measure subset) of n>3 qudits with local dimension d>2 and certain tripartite systems.

This follows from a more general result .

We use **algebraic geometry, the theory of Lie groups** and **geometric invariant theory** to show that

$$\widetilde{G}_{\Psi} = \{\mathbb{I}\}$$

for almost all multi-qudit states $|\Psi\rangle$.

Results of Gour, Kraus, Wallach, JMP 2017
on local transformations of states
with trivial symmetries apply!

Almost all (n>4)-qubit states have only trivial local symmetries

Gour, Kraus, Wallach, JMP 2017

Theorem:

For n > 4 qubits there exists an open and full-measure subset of the Hilbert space whose elements have only trivial symmetries .

→ proof: use **algebraic geometry** and **geometric invariant theory** and so-called **homogeneous SL-invariant polynomials** (SLIPS)

→ SLIPS are not explicitly known for higher dimensions.

→ **Was not clear if this result can be generalized to higher dimensions.**

If yes: What is the relation between d and n?

Almost all multi-qudit states have only trivial local symmetries

DS, Wallach, Gour, Kraus, arXiv:1711.11056

Theorem:

For any number n > 3 and local dimension d > 2 there exists an open and full-measure subset of the Hilbert space whose elements have only trivial symmetries. Such a set also exists for n = 3 and d = 4 (and d = 5, 6 as we show numerically).

Proof this without explicitly using SLIPs and

→ methods from Gour, Kraus, Wallach, JMP 2017

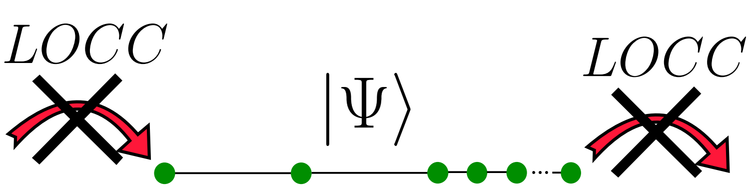
→ new methods from the **theory of Lie groups** and **geometric invariant theory**

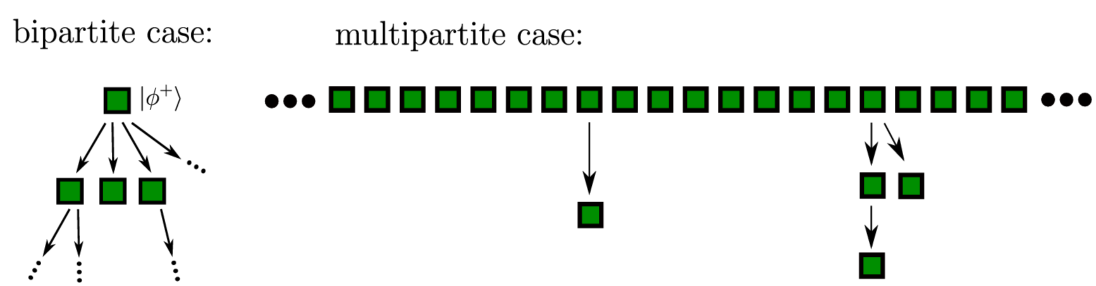
Implications for entanglement theory

This theorem from Gour, Kraus, Wallach, JMP 2017 applies to almost all states of n>3 qudits and n>4 qubits and tripartite systems with local dimension d=4,5,6.

Theorem:

A fully entangled state $|\Psi\rangle$ with $\widetilde{G}_\Psi = \{\mathbb{I}\}$ can be deterministically obtained from or transformed to a fully entangled state $|\Phi\rangle$ iff $|\Phi\rangle = u_1 \otimes \ldots \otimes u_n |\Psi\rangle$ for some local unitaries u_i .





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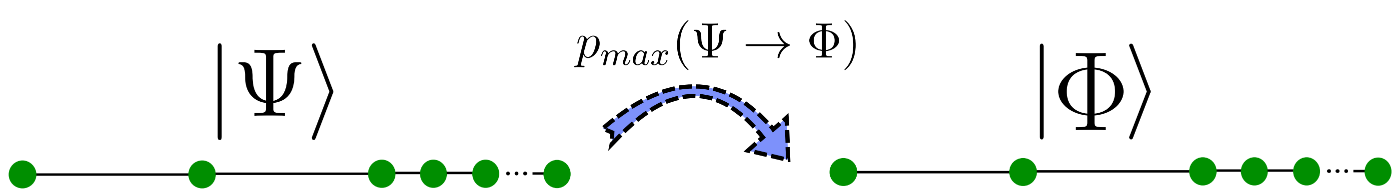
This theorem from Gour, Kraus, Wallach, JMP 2017 applies to almost all states of n>3 qudits and n>4 qubits and tripartite systems with local dimension d=4,5,6.

Theorem:

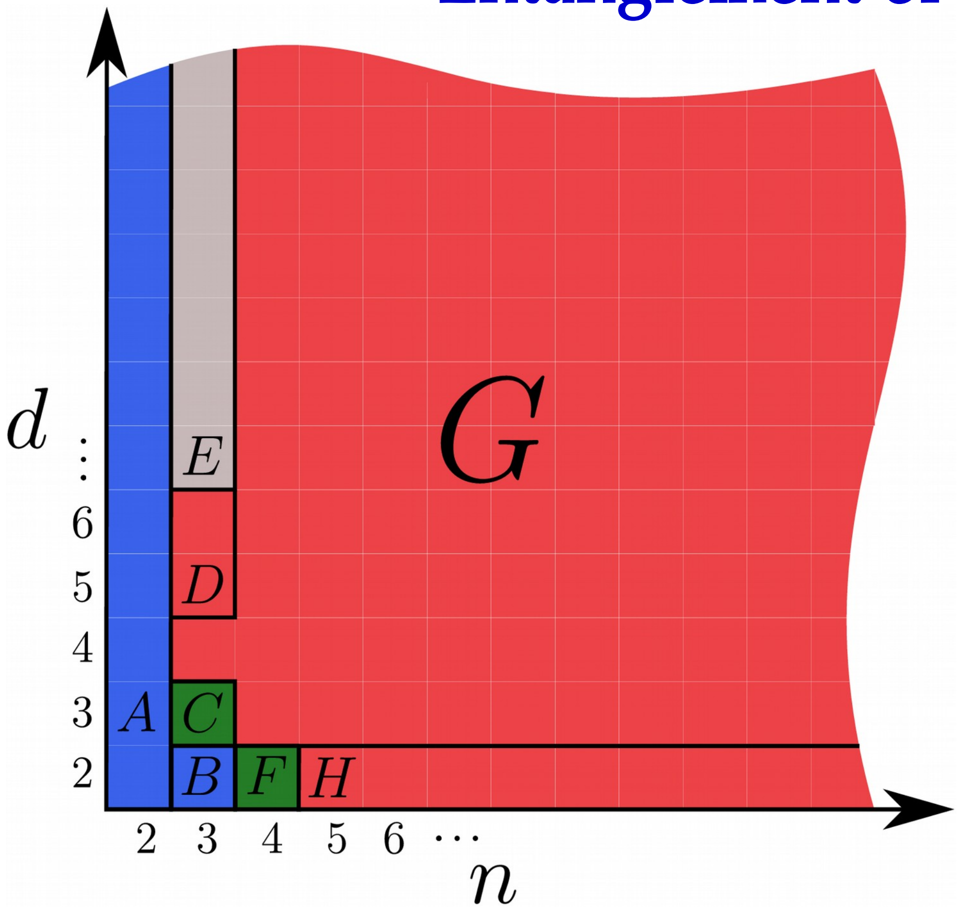
Let $|\Psi\rangle$ be a fully entangled normalized state with $\widetilde{G}_\Psi = \{\mathbb{I}\}$.
Then the optimal probability to transform $|\Psi\rangle$ via LOCC or SEP to a normalized state $|\Phi\rangle = h|\Psi\rangle$ is

$$p_{max}(\Psi \to \Phi) = \frac{1}{\lambda_{max}(h^\dagger h)},$$

where $\lambda_{max}(X)$ denotes the maximum eigenvalue of X.



Entanglement of multi-qudit pure states



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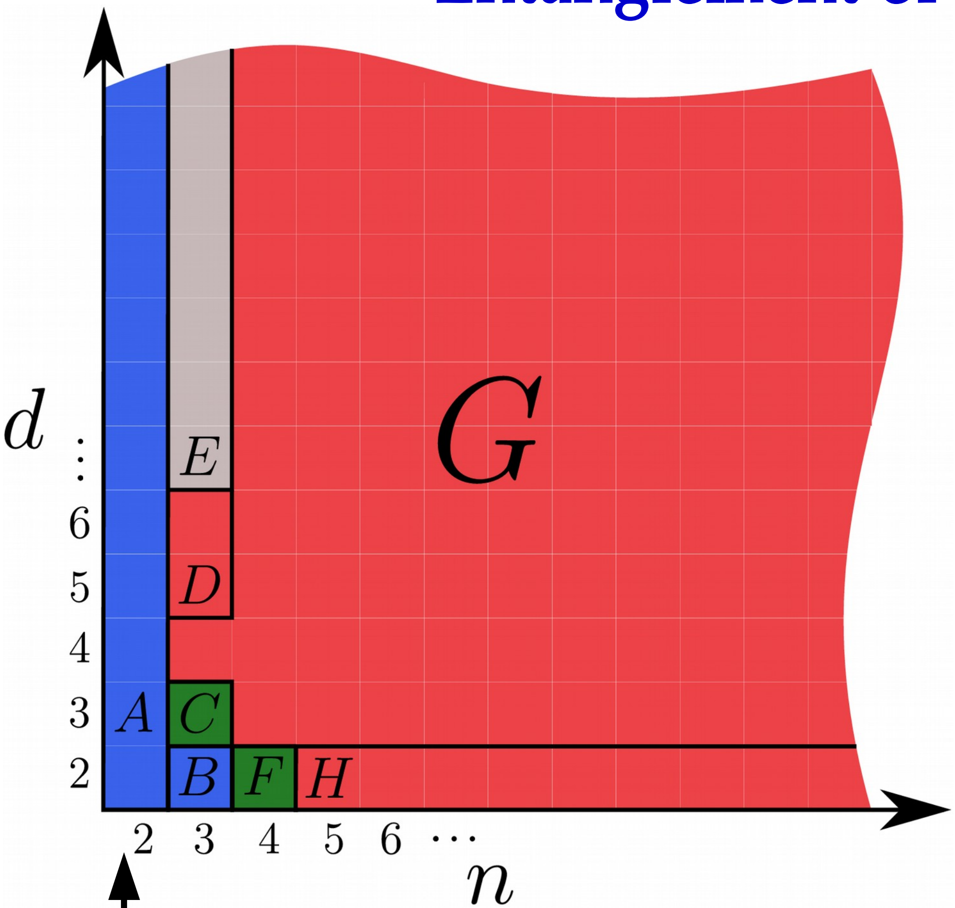

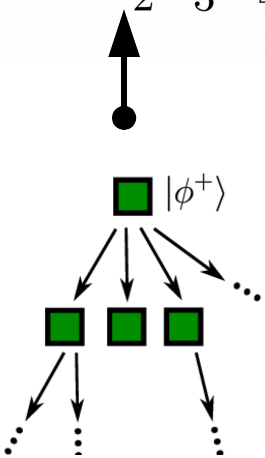

A: **LOCC characterized**, all states convertible,

1 max. ent. state (Nielsen, PRL 1999)

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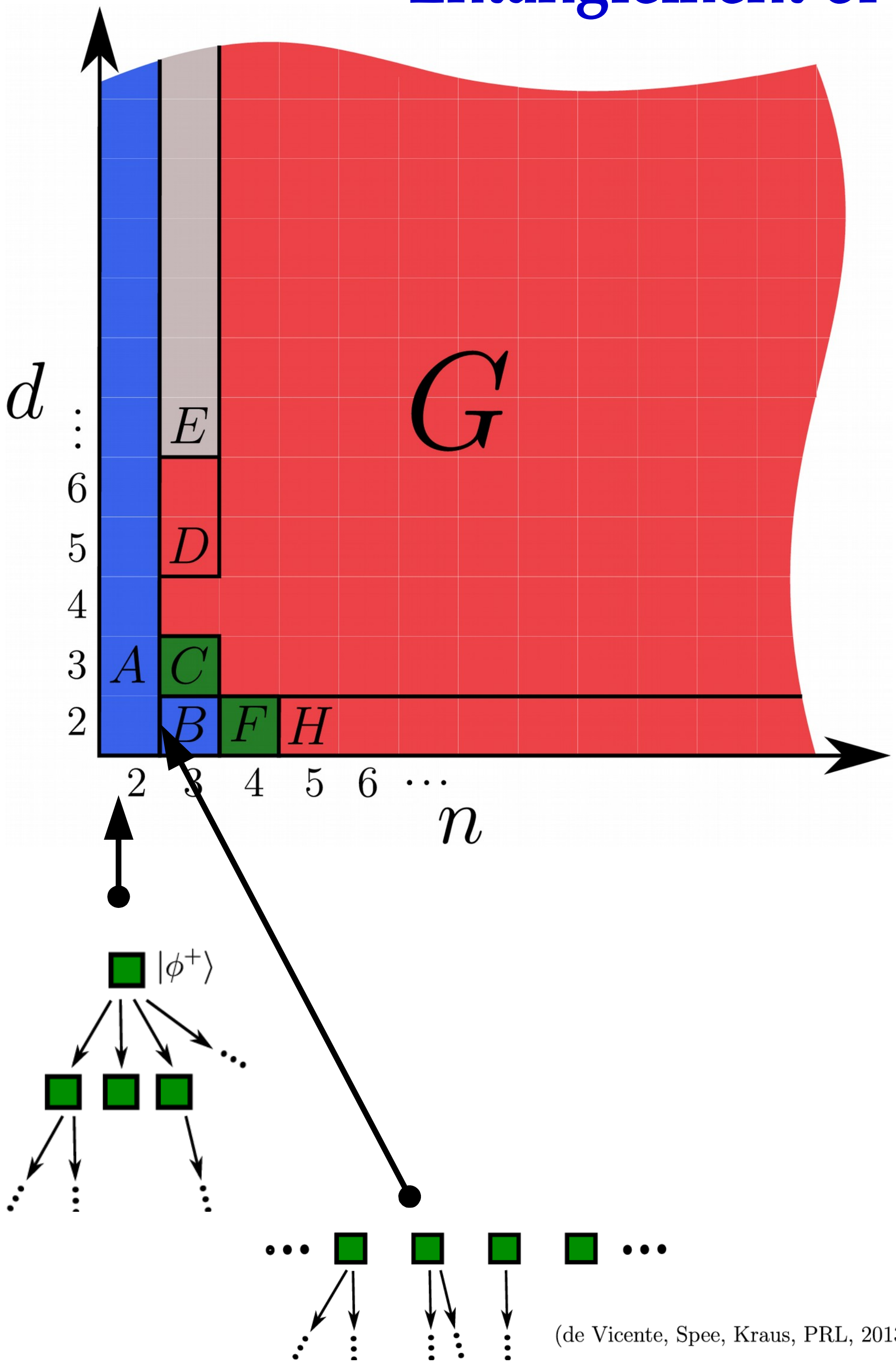

(de Vicente, Spee, Kraus, PRL, 2013)

A: **LOCC characterized**, all states convertible,

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B: **LOCC characterized**, all states convertible,

infinite MES (0-measure)

(Turgut et al, PRA 2010; Kintas, Turgut, JMP 2010; de Vicente et al , PRL 2013)

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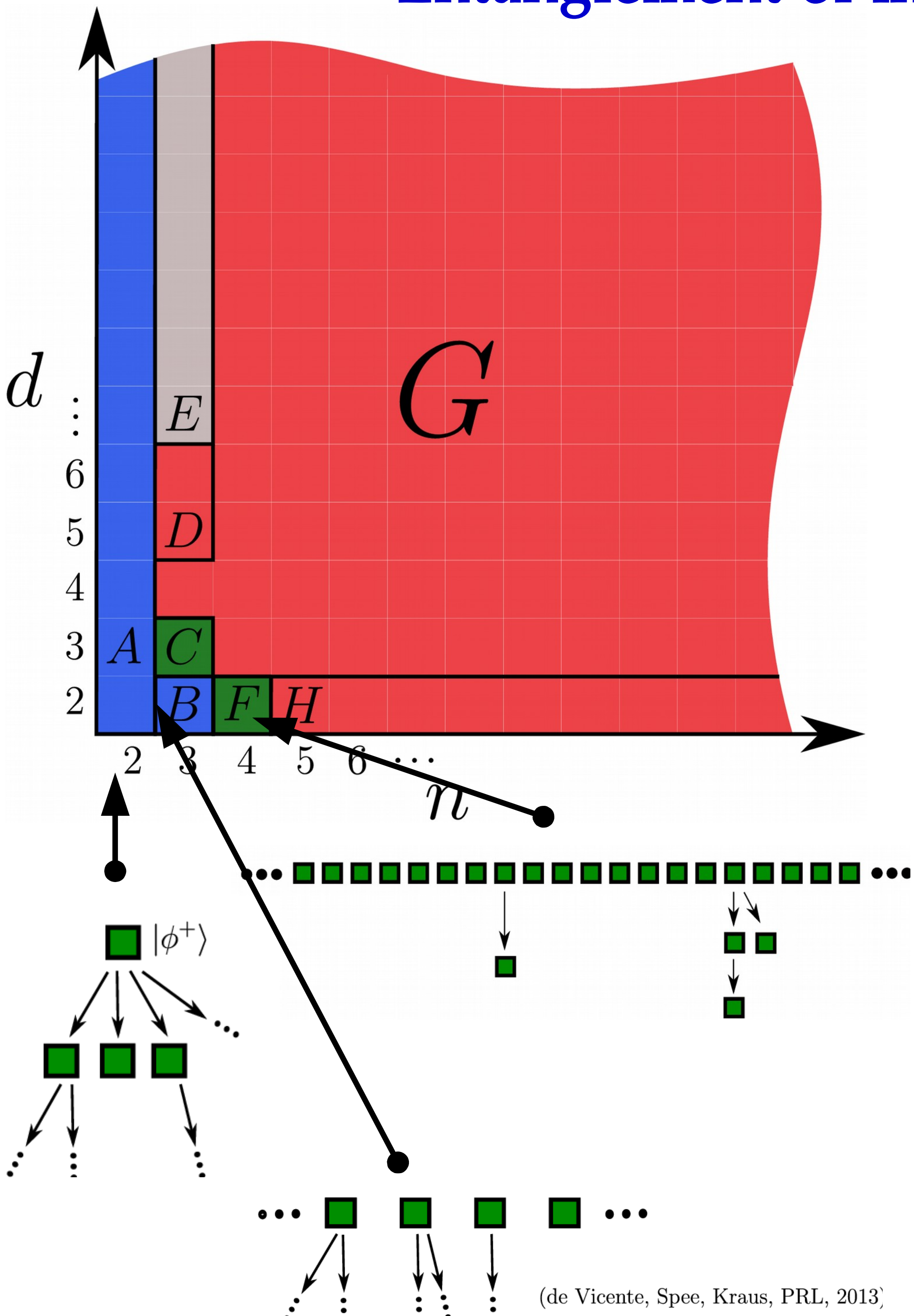

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F: LOCC of generic states characterized, **full measure MES,**

almost all states isolated (de Vicente et al , PRL 2013; DS et al, PRA 2015)

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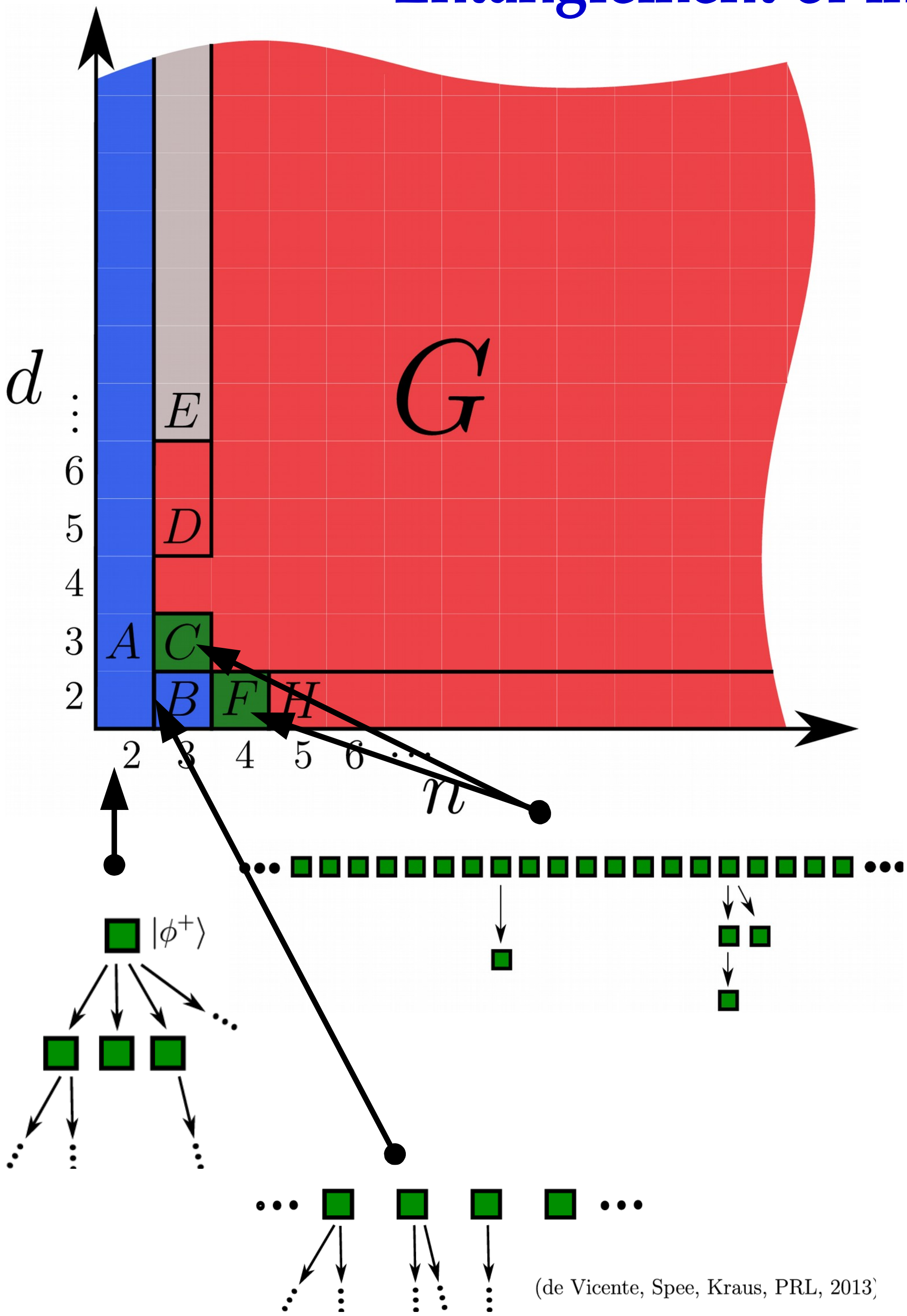

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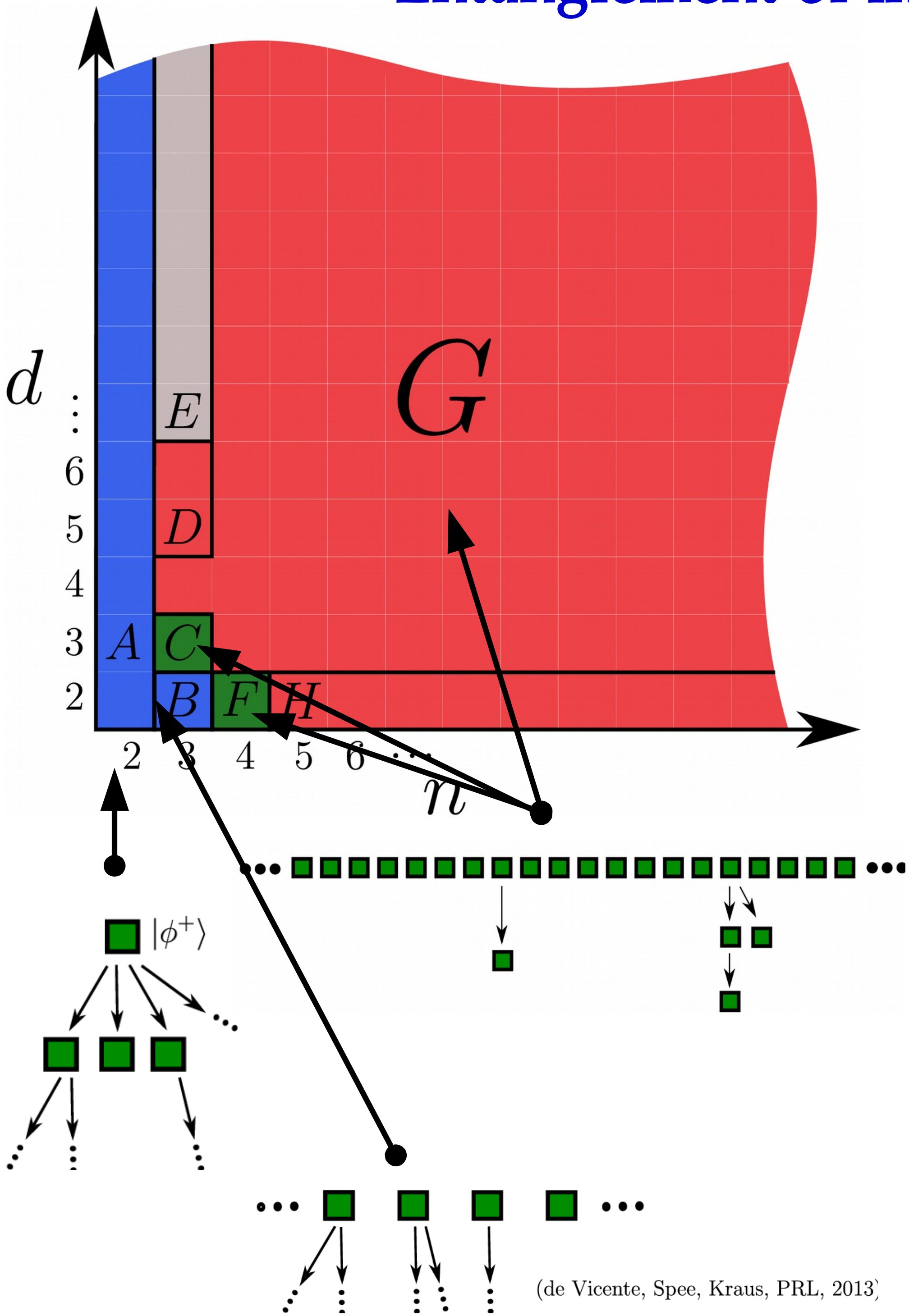

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G,H,D :**full-measure MES, almost all states isolated,**

generically optimal conversion prob. known,

generically LOCC = SEP

(Gour, Kraus, Wallach, JMP 2017; DS, Wallach, Gour, Kraus, in prep. 2017)

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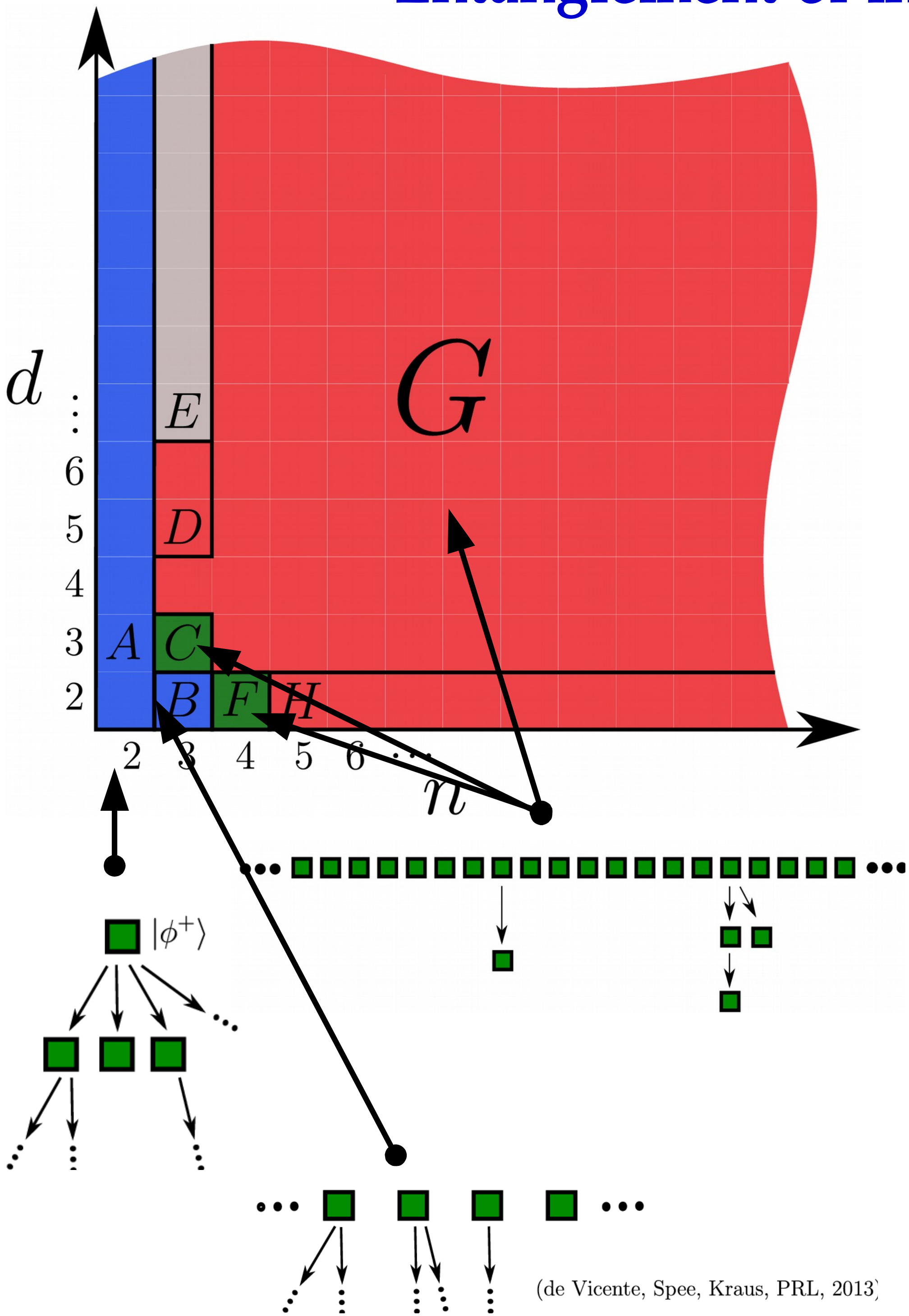

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(Gour, Kraus, Wallach, JMP 2017; DS, Wallach, Gour, Kraus, in prep. 2017)

E: generically finite stabilizer; unknown if trivial

(Gour, Wallach, NJP 2011)

Introduction

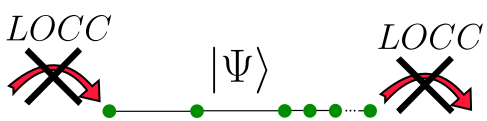
Multipartite LOCC transformations

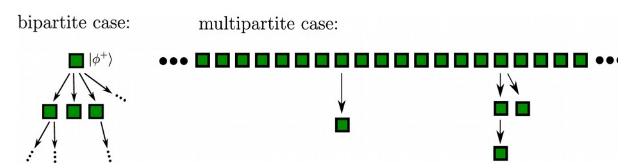
Conclusion & Outlook

Conclusion

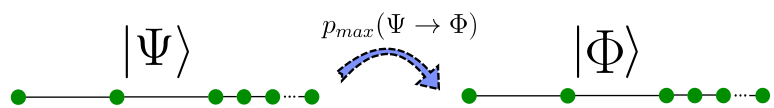
→ **characterization of LOCC and SEP transformations of almost all multi-qudit states**

→ non-trivial deterministic transformations are almost never possible

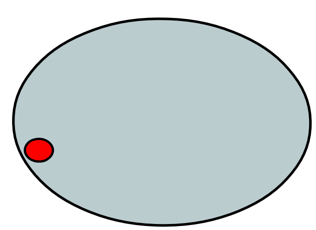




→ have optimal protocol for probabilistic transformations

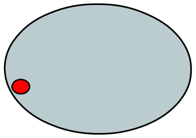


→ in terms of entanglement transformations **only a 0-measure subset of states is relevant** (compare to e.g. condensed matter physics)



Outlook

→ local transformations of 0-measure subset of states that can be transformed (e.g. graph states, matrix product state, ...)



→ LOCC transformations of states of heterogenous systems

→ optimal probabilistic local transformations of quantum states

→ apply new mathematical methods to other physical problems

Thank You!

arXiv:1711.11056

Almost all multi-qudit states have only trivial local symmetries

We show that for a generic multi-qudit state $|\Psi\rangle$ the equation

$$g_1 \otimes \ldots \otimes g_n |\Psi\rangle = |\Psi\rangle$$

only has the trivial solution

$$g_1 \otimes \ldots \otimes g_n = \mathbb{I}$$

Surprising?

→ **d^n equations** for **$d^2 n$ variables**

→ However, the proof is highly nontrivial.

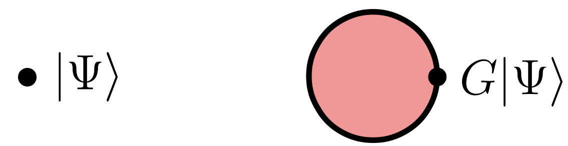
Outline of the proof

Notation & Preliminaries:

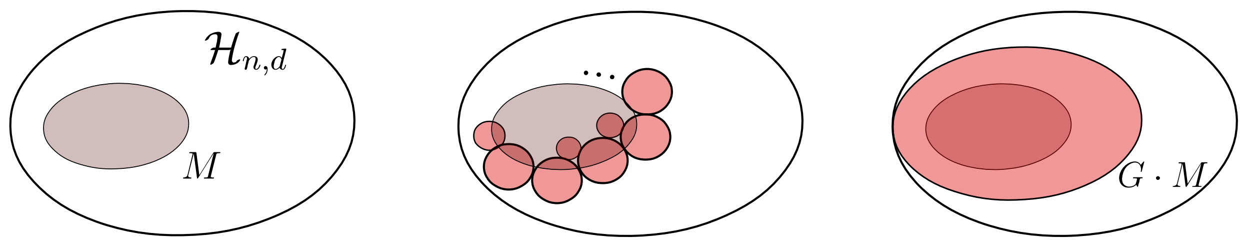
$\mathcal{H}_{n,d} \equiv (\mathbb{C}^d)^{\otimes n}$

$G \equiv SL(d)^{\otimes n} \subset GL(d)^{\otimes n}$

$G|\Psi\rangle \equiv \{g|\Psi\rangle \mid g \in G\}$



$M \subset \mathcal{H}_{n,d} : G \cdot M \equiv \bigcup_{|\Psi\rangle \in M} G|\Psi\rangle$

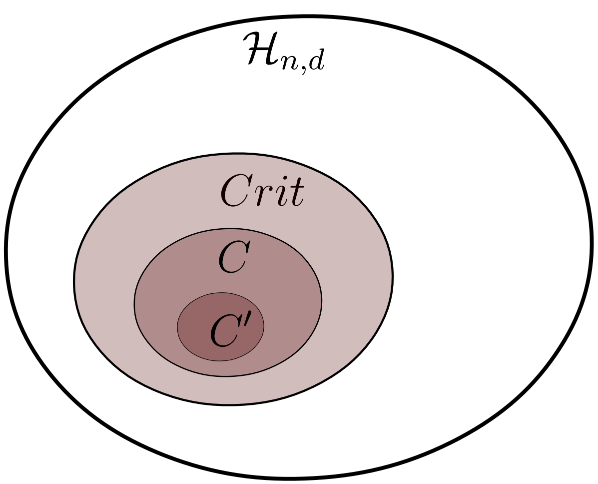


$Crit \equiv \{|\Psi\rangle \in \mathcal{H}_{n,d} \mid \rho_i = \mathrm{Tr}_{\neq i}(|\Psi\rangle\langle\Psi|) \;\propto\; \mathbb{I},\ \forall i\} \subset \mathcal{H}_{n,d}$

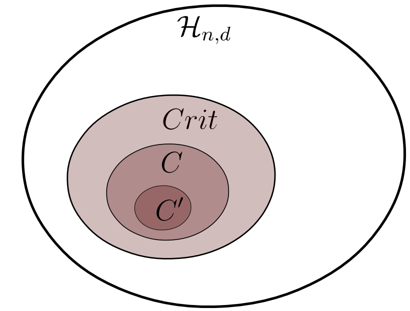
e.g. Bell states, GHZ states, graph states, . . .

$C = \{|\Psi\rangle \in Crit \mid \widetilde{G}_\Psi \text{ is finite}\} \subset Crit$

$C' = \{|\Psi\rangle \in C \mid \widetilde{G}_\Psi = \{\mathbb{I}\}\} \subset C$



Outline of the proof



Main steps:

1. $\widetilde{K}_\Psi \equiv \{u \in U(d)^{\otimes n} \mid u|\Psi\rangle = |\Psi\rangle\}$
   If $|\Psi\rangle \in Crit$ and $\widetilde{K}_\Psi$ is finite then $\widetilde{K}_\Psi \equiv \widetilde{G}_\Psi$.
   $\Rightarrow \; C' = \{|\Psi\rangle \in C \mid \widetilde{G}_\Psi = \{\mathbb{I}\}\} = \{|\Psi\rangle \in C \mid \widetilde{K}_\Psi = \{\mathbb{I}\}\}$

2.
(a) C is a connected smooth manifold of $\mathcal{H}_{n,d}$
and $U(d)^{\otimes n}$ acts differentiably on C.
(b) $G \cdot C$ is open and full measure in $\mathcal{H}_{n,d}$.
(see Gour, Kraus, Wallach, JMP 2017)

3. $2.a$ + Principal Orbit Type Theorem:
   $\Rightarrow$ If $C' \neq \emptyset$ then C' is open and full measure in C

4. construct for n, d described before a $|\Psi\rangle$ with $\widetilde{K}_\Psi = \{\mathbb{I}\}$.
   $\Rightarrow C' \neq \emptyset \quad \Rightarrow C'$ is open and full measure in C

5. $G \cdot C'$ is open and full measure in $\mathcal{H}_{n,d}$
   and $\widetilde{G}_\Phi = \{\mathbb{I}\}$ for all $|\Phi\rangle \in G \cdot C'$.

□