Computability theory and Bell non-locality

Antonio Acín[1], Ariel Bendersky[2], Gonzalo de La Torre[1],
Santiago Figueira[2] y *Gabriel Senno*[1]

[1]ICFO - The Institute of Photonic Sciences.
[2]Computer Science Department, FCEyN, University of Buenos Aires.

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Abstract

Two results relating computability theory and Bell non-locality:

- Inputs: *The computability loophole*.

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- Inputs: *The computability loophole*.
- Outputs: *Computability + non-locality* $\implies$ *signaling*.

The computability loophole

The CHSH game

$(x, y) \in_R \{0,1\}^2$

x y

$a \in \{0,1\}$

$b \in \{0,1\}$

$$a \oplus b = xy$$

The CHSH game

$(x, y) \in_{\mathrm{R}} \{0,1\}^2$

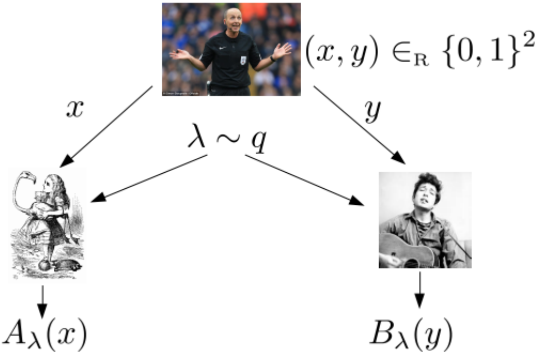
x $\quad$ y

$a \in \{0,1\}$ $\qquad$ $b \in \{0,1\}$

$$a \oplus b = xy$$

$$p_{win}^{\text{CHSH}} := \frac{1}{4} \sum_{a,b} \left[p(a \neq b | 1,1) + \sum_{(x,y) \neq (1,1)} p(a = b | x, y) \right]$$

Local strategies



$$p(a, b|x, y) = \sum_{\lambda} q(\lambda)\delta_{a=A_\lambda(x)}\delta_{b=B_\lambda(y)}.$$

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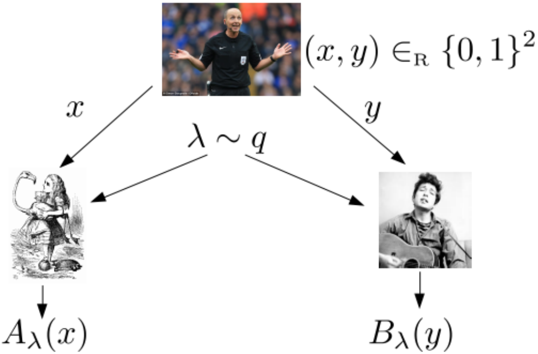

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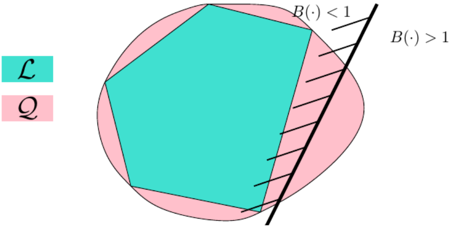
$$p_{win}^{\mathrm{CHSH}} \leq \frac{3}{4}, \text{ for every local strategy.}$$

Bell inequalities



- The CHSH inequality

$$p(=|P_1,P_1)+p(=|P_1,P_2)+p(=|P_2,P_1)+p(\neq|P_2,P_2)\leq 3$$

  is an example of a Bell inequality.
- In general,

$$\sum_{a,b,x,y} B_{a,b,x,y}\,p(a,b|x,y)\leq B_l.$$

Quantum strategies

- Quantum strategy:

$$p(a,b|x,y) = \langle\psi|\Pi_a^x\Pi_b^y|\psi\rangle$$

with $|\psi\rangle \in \mathcal{H}$, $\sum_a \Pi_a^x = \sum_b \Pi_b^y = \mathbb{I}_{\mathcal{H}}$ and $[\Pi_a^x, \Pi_b^y] = 0$.

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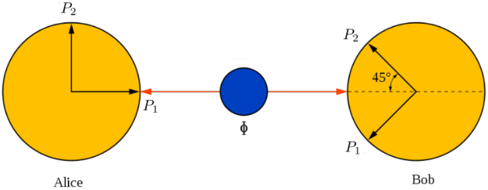
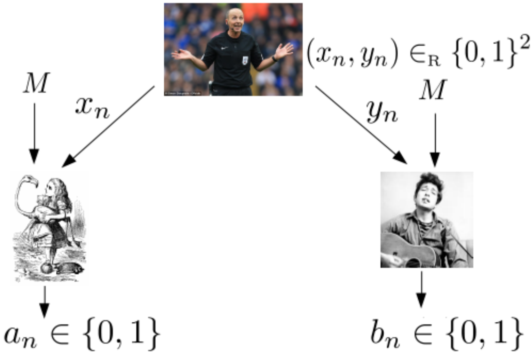
- For the CHSH game, preparing $|\psi^-\rangle =: \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and measuring in the following spin directions:



it is easy to see that, $p_{win}^{\text{CHSH}} = \cos^2(\pi/8) \approx 0,85$.

Memory scenario

$$M = [(x_1, y_1), \dots, (x_{n-1}, y_{n-1})]$$



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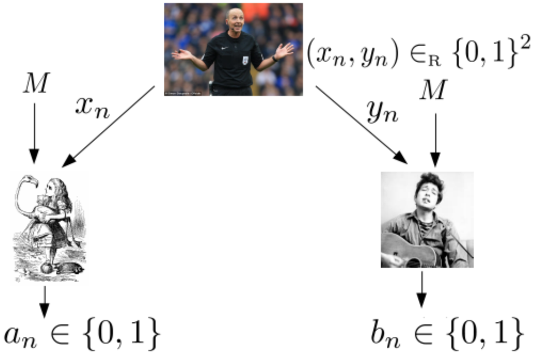

- Local strategies:
  $p(a_n, b_n | x_n, y_n) = \sum_\lambda q(\lambda) \delta_{a_n = A_\lambda(x_n, M)} \delta_{b_n = B_\lambda(y_n, M)}$.

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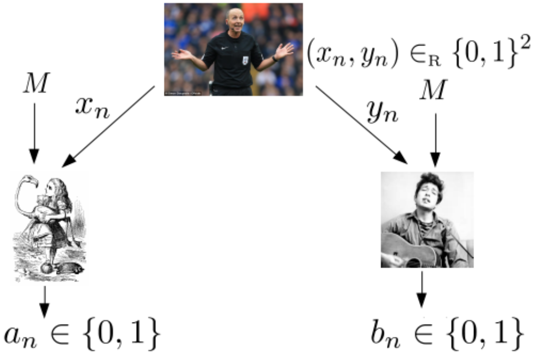

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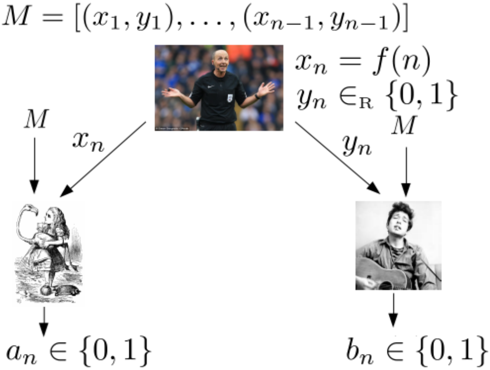
- $p_{win}^{\text{CHSH}} \leq 3/4$ [Barrett et al., Phys. Rev. A 66, 042111, 2002].

Memory scenario with computable inputs



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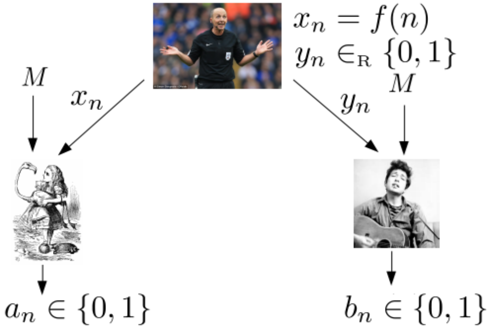

Theorem ([Bendersky, Senno, de la Torre, Figueira and Acín. **PRL** 116, 230402, 2016])

If the referee in the CHSH game with memory chooses (at least) one of the players' questions using a computable function $f : \mathbb{N} \to \{0, 1\}$*, there is a perfect local strategy (independent of* f*).*

Computable functions

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- For every computable $T : \mathbb{N} \to \mathbb{N}$, the class $\mathcal{C}_T$ of functions computable in time $O(T)$ is computably enumerable. This include the well-known complexity classes P, NP, EXP, BQP.

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- But, the class of all computable functions is *not* computably enumerable.

Predicting computable functions

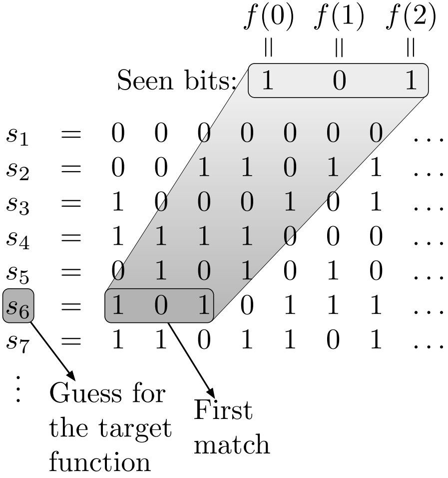
Functions in computably enumerable classes can be predicted in the following sense:

For every computably enumerable class $\mathcal{C}$ of computable functions there is a program $\mathcal{P}$ (called a *predictor for* $\mathcal{C}$) such that for every $f \in \mathcal{C}$,

$$(\exists n_0)(\forall n \geq n_0)\ f(n) = \mathcal{P}([f(0), \ldots, f(n-1)])$$

Predicting computable functions

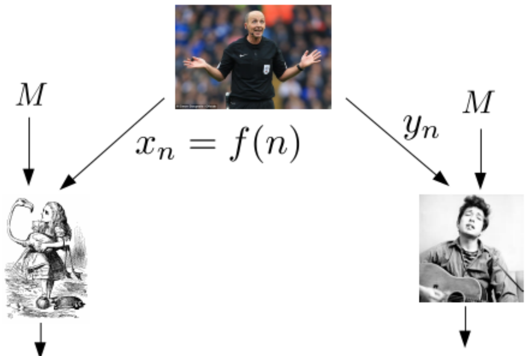
The predictor works as follows,



Perfect local strategy

Let $T : \mathbb{N} \to \mathbb{N}$ be some computable function, $f \in \mathcal{C}_T$ and $\mathcal{P}$ a predictor for $\mathcal{C}_T$.

$$M = [(x_1, y_1), \ldots, (x_{n-1}, y_{n-1})]$$



0

$$\begin{cases} 1 & \text{if } \mathcal{P}([x_1, \ldots, x_{n-1}]) = 1 \wedge y_n = 1 \\ 0 & \text{o.w.} \end{cases}$$

Technical details of the model

- Provided $f \in \mathcal{C}_T$, if Bob uses a predictor $\mathcal{P}$ for $\mathcal{C}_T$, after finitely many rounds he will start predicting x_i correctly.

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  - If $f(n)$ has a k-bits program, $M \leq O(\log(k))$.
- Also, the prediction algorithm is (almost) as efficient as f.
  - If f is computable in $O(T)$ time, then $\mathcal{P}$ runs in $O(T \cdot \log(T))$.

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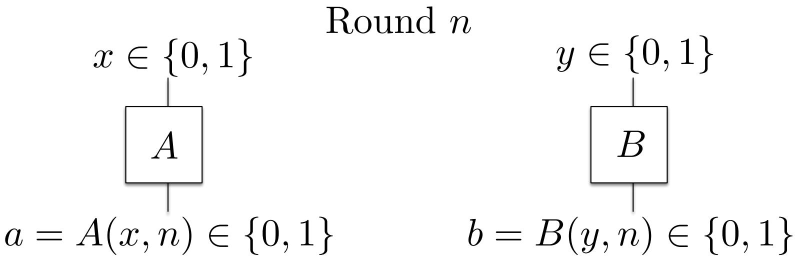
  - Rather circular to assume a non-local theory when testing non-locality.
  - Experimentally unverifiable.
- Nevertheless, the result I will talk about next implies that, under reasonable assumptions, the outputs from QRNGs are, in fact, uncomputable.

Computable non-locality allows for signaling

Deterministic boxes in a CHSH scenario



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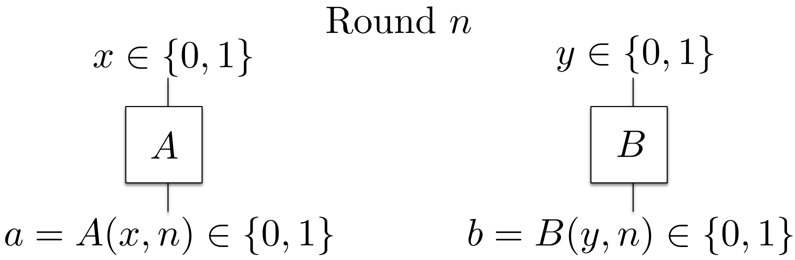

$$p(a,b|x,y) := \lim_{n\to\infty} \frac{|\{i \leq n \mid x_i = x, y_i = y, a_i = a, b_i = b\}|}{|\{i \leq n \mid x_i = x, y_i = y\}|}.$$

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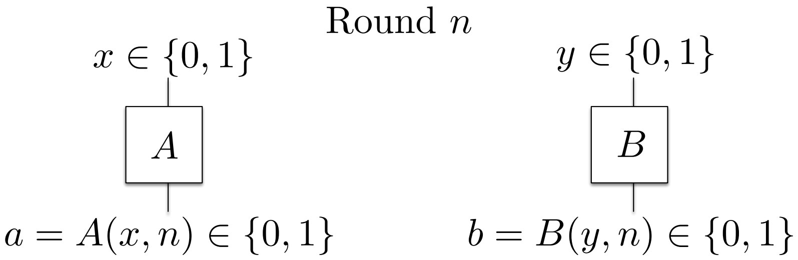

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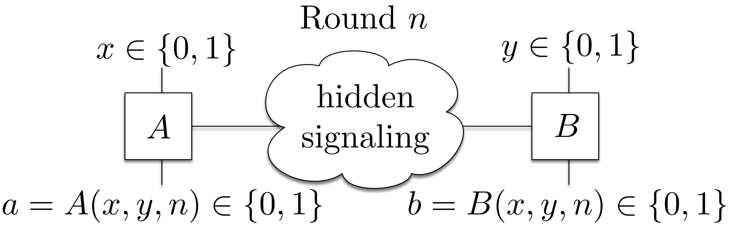
We will say that A and B are *non-local* if, when the inputs are chosen uniformly at random, p violates a Bell inequality with probability 1.

Determinism $\wedge$ non-locality $\implies$ violation of Parameter Independence



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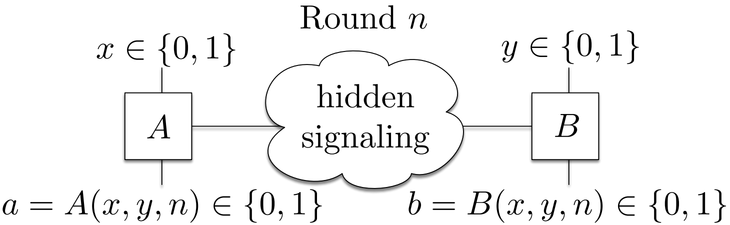

Lemma

If A and B are non-local,
$\exists^\infty n\ [\exists x\ A(x,0,n) \neq A(x,1,n) \vee \exists y\ B(0,y,n) \neq B(1,y,n)]$.

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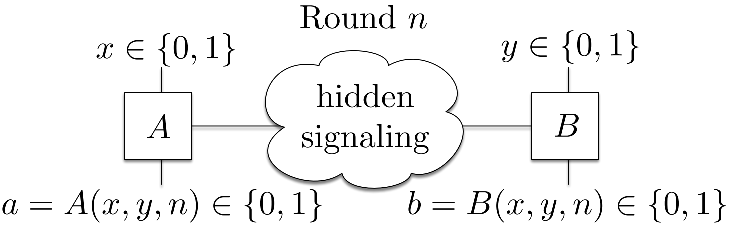

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Observation

Violation of Parameter Independence doesn't imply signaling (e.g.: Bohmian mechanics, Toner & Bacon, etc).

Using that hidden signaling for communicating

W.l.o.g., let's assume that

$$\exists^{\infty} n \exists y \; B(0, y, n) \neq B(1, y, n). \tag{1}$$

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If Alice and Bob had access to B, i.e. if they knew how to **compute** it, they could easily communicate: they just wait for the ns that verify (1) and, with the right choice of y_n, Bob can tell x_n. Thus, we assume B is hidden (it's Nature's secret). Can it be kept that way?

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Main Result

We give a protocol which, if B is a **computable function**, allows Alice to send a message to Bob with the sole knowledge of a bound on the computational complexity of B.

Learnability in the limit

A class computable functions $\mathcal{C}$ is *learnable in the limit* if there exists a program $\mathcal{P}$ (called a *learner for* $\mathcal{C}$) such that for every $f : \mathbb{N} \to \mathbb{N} \in \mathcal{C}$, there exists m such that for every $m \geq n$, on input (some coding of) $\{(0, f(0)), \ldots, (m, f(n))\}$ $\mathcal{P}$ outputs (the code of) a program that computes f.

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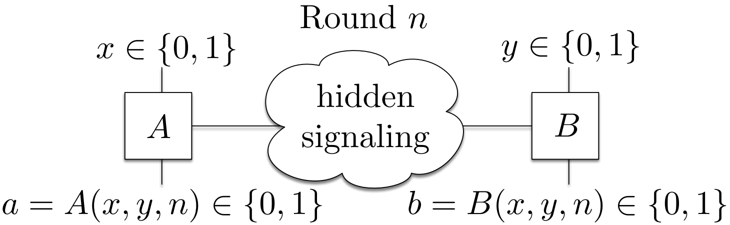
Learning in the limit allows communication



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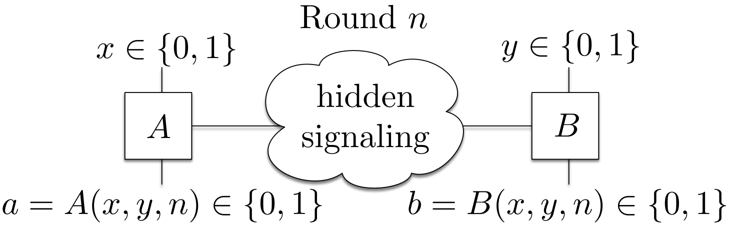

Restrictions

- In order for Bob to learn a program to compute the function B, he needs to know Alice's inputs x, at least until B has been learned.
- For every n, Bob will only see the value of B for just one pair of inputs (x_n, y_n).
- Bob will not be able to tell when he has effectively learned B.

The protocol $\mathcal{P}(t, m, S)$

Inputs:

1. a computable non-decreasing function t
2. the size of Alice's message m
3. a sequence $S \in \{(0,0),(0,1),(1,0),(1,1),1,\ldots,m\}^\infty$

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On each round n:

1. if $S(n) = (x, y)$, Alice inputs x and Bob inputs y. Then, Bob uses a learner for the class of functions computable in time $O(t)$ on input $((x_{i_1}, y_{i_1}, B(x_{i_1}, y_{i_1}, i_1)) \ldots (x, y, B(x, y, n)))$ (with i_k being the past learning rounds) to update his guess $\widetilde{B}$ of a program that computes B (**Learning round**).
2. if $S(n) = i \in \{1, \ldots, m\}$, Alice inputs the ith bit of her message, a_i and Bob y s.t. $\widetilde{B}(0, y, n) \neq \widetilde{B}(1, y, n)$ and makes the output of his box his new guess for a_i. If there is no such y, he inputs 0 (**Signaling round**).

Soundness of the protocol

For $\mathcal{P}(t, m, S)$ to be sound, it suffices that the following properties hold:

1. There exists a number of round n such that for all $m \geq n$, and $x, y \in \{0, 1\}$, we have $\widetilde{B}(x, y, m) = B(x, y, m)$, i.e. the learning process converges to B.
2. For every bit i of Alice's message and for infinitely many n, $S(n) = i \in \mathbb{N}$ and $\exists y \in \{0, 1\}.B(0, y, n) \neq B(1, y, n)$.

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Can we de-randominize?

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Theorem ([Bendersky, Senno, de la Torre, Figueira, Acín. Phys. Rev. Lett. 118, 130401, 2017])

If S is T-random, properties 1 and 2 hold.

Future directions

1. Randomness amplification

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2. Computability of the set of quantum correlations.
   - Very recent breaktrough results by Slofstra [arXiv:1606.03140, arXiv:1703.08618].