SOME FINITE QUANTUM RIEMANNIAN GEOMETRIES

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Quantum spacetime hypothesis:

Quantum Gravity??? → Quantum Riemannian geometry (NCRG) → Classical Riem. Geom.

$\lambda_{Planck} \to 0$

- Visible in 3D quantum gravity
- Born reciprocity as a key idea for quantum gravity

Cla. Qua. Grav. 1988

Swap posn/mom

Quantum Phase space should contain quantum spacetime

	Position	Momentum
Gravity	Curved	Noncommutative
Cogravity	Noncommutative	Curved
Quantum Gravity	Both	Both

…=> Bicrossproduct quantum groups acting on quantum spacetime eg

(w/ Ruegg) PLB 1994

30 years later we have a fairly good idea what should be quantum Riemannian geometry on any algebra

1. Quantum differentials on an algebra A

LTCC lectures 2011
(and 2019 book w/ Beggs)

Classically, $C^\infty(M)=\Omega^0(M)\subset\Omega(M)=\oplus_i\Omega^i(M)$

Ω^1 space of 1-forms, e.g. `differentials' $\mathrm{d}f=\sum_i\frac{\partial f}{\partial x^i}\mathrm{d}x^i$

$f\mathrm{d}g=(\mathrm{d}g)f\in\Omega^1$

$\wedge:\Omega\otimes_A\Omega\to\Omega,\quad \mathrm{d}(\omega\wedge\eta)=(\mathrm{d}\omega)\wedge\eta+(-1)^{|\omega|}\omega\wedge\mathrm{d}\eta$

`graded Leibniz rule'

$\omega\wedge\eta=(-1)^{|\omega||\eta|}\eta\wedge\omega,\quad \mathrm{d}^2=0$

- algebra *A* over k, we drop the (graded) commutativity but keep:

Ω^1	*a*((d*b*)*c*)=(*a*(d*b*))*c*	`bimodule'
$\mathrm{d}:A\to\Omega^1$	d(*ab*)=(d*a*)*b*+*a*(d*b*)	`Leibniz rule'
$\{\sum a\mathrm{d}b\}=\Omega^1$		`surjectivity'
$\ker\mathrm{d}=k.1$		(`connected')

- require this to extend to a DGA $\Omega=T_A\Omega^1/\mathcal{I}=\oplus_n\Omega^n,\quad \mathrm{d}^2=0$
- inner if exists $\theta\in\Omega^1,\quad \mathrm{d}=[\theta,\ \}$

Nice problem: take your favourite algebra and classify all differential structures (perhaps with some symmetry)

Thm (w/ Tao) Pac. J. Math 2016

pre-Lie algebra $\circ : \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$

transl. inv., connected, classical dim $\Omega^1(U(\mathfrak{g}))$ <=> $(x\circ y)\circ z-(y\circ x)\circ z=x\circ(y\circ z)-y\circ(x\circ z)$

$$[x,y]=x\circ y-y\circ x$$

$$[x,\mathrm{d}y]=\mathrm{d}(x\circ y)$$

$\Rightarrow\ \Omega(U(\mathfrak{g}))$ by skew-symmetrisation of the basic one-forms

- Example $\mathfrak{g}=\mathrm{Vect}(M)$ and torsion free flat connection

$$x\circ y=\nabla_x y,\quad \nabla_{[x,y]}z=\nabla_x\nabla_y z-\nabla_y\nabla_x z$$

$$A=U(\mathrm{diff}(M))\qquad [x,y]=\nabla_x y-\nabla_y x$$

- Example $\mathfrak{g}=V,\quad [\ ,\ ]=0,\quad (V,\circ)$ comm associative algebra

e.g. $V=\mathbb{C}.x,\quad x\circ x=\lambda x,\quad A=\mathbb{C}[x],\quad [x,\mathrm{d}x]=\lambda\mathrm{d}x$

=> $\mathrm{d}f(x)=\dfrac{f(x)-f(x-\lambda x)}{\lambda}\mathrm{d}x$

Propn X discrete set $\Omega^1(C(X))$ <=> directed graphs on X

$$\Omega^1 = \mathrm{span}_k\{\omega_{x\to y}\}$$

$$f.\omega_{x\to y} = f(x)\omega_{x\to y}, \omega_{x\to y}.f = f(y)\omega_{x\to y} \qquad \mathrm{d}f = \sum_{x\to y}(f(y)-f(x))\omega_{x\to y}$$

If a graph is bidirected, define

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y}\otimes_{C(X)}\omega_{y\to x} \qquad g_{x\to y}\in k$$ `metric arrow lengths'

$$g_{x\to y} = g_{y\to x}$$ (optional `edge-symmetric' case of `metric edge lengths')

- Example: Cayley graph on ad-stable set generators $\mathcal{C}$ of a group X

edges: $x\to xa, a\in\mathcal{C}$ => left-invariant 1-forms: $e_a = \sum_{x\in X}\omega_{x\to xa}$

$$e_a f = R_a(f)e_a, \quad \mathrm{d}f = \sum_{a\in\mathcal{C}}\partial^a(f)e_a$$

$$\partial^a = R_a - \mathrm{id}$$

$$g = \sum_{a,b} g_{a,b}e_a\otimes e_b; \quad g_{a,b}\in C(X)$$

Quantum metrics

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes_A \mathrm{d}x^\nu$$

$$g \in \Omega^1 \underset{A}{\otimes} \Omega^1$$

$$\wedge(g) = 0$$ (optional `quantum symmetric')

invertible in the sense exists inverse: $(\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

$$((\ ,\ )\otimes \mathrm{id})(\omega \otimes g) = \omega = (\mathrm{id} \otimes (\ ,\ ))(g \otimes \omega), \quad \forall \omega \in \Omega^1$$

$$a(\omega, \eta) = (a\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta a)$$ `bimodule map (tensorial)'

need this to be able to contract/ `raise/lower' via metric, eg to have well defined contraction:

$$(\ ,\ )\otimes \mathrm{id} : \quad \Omega^1 \underset{A}{\otimes} \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1$$

" $T_{\mu\nu\rho} \mapsto g^{\mu\nu} T_{\mu\nu\rho}$ "

but

$$(\omega, g^1)g^2 = \omega \qquad g = g^1 \underset{A}{\otimes} g^2$$

$$\Longrightarrow (\omega, g^1)g^2 a = \omega a = (\omega a, g^1)g^2 = (\omega, a g^1) g^2$$

$$\Longrightarrow \qquad ag = ga, \quad \forall a \in A$$ need metric to be central

Connections and curvature

Classically, a connection assigns a covariant derivative

$$\nabla \mathrm{d}x^\mu = -\Gamma^\mu{}_{\nu\rho}\mathrm{d}x^\nu \otimes_A \mathrm{d}x^\rho$$

(Christoffel symbols)

Similarly for any differential algebra $(A, \Omega^1, \mathrm{d})$

bimodule connection: $\nabla : \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$ $\quad \sigma : \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$

$$\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega \qquad \nabla(\omega f) = \sigma(\omega \otimes \mathrm{d}f) + (\nabla\omega)f$$

(Quillen, Karoubi,…) (Michor, Dubois-Violette, …)

such connections extend to tensor products

$$\omega \otimes \eta \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta)$$

more generally $\nabla_E : E \to \Omega^1 \otimes_A E, \quad \sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$

$${}_A\mathcal{E}_A = \{(E, \nabla_E, \sigma_E)\}$$

is a monoidal category by $\otimes_A$

`metric compatible' now makes sense $\nabla g = 0$ but is quadratic

- torsion free also makes sense $T_\nabla : \Omega^1 \to \Omega^2 \qquad T_\nabla = \wedge\nabla - \mathrm{d}$
- quantum Levi-Civita connection (QLC) $T_\nabla = \nabla g = 0$
- Curvature

$$R_\nabla : \Omega^1 \to \Omega^2 \underset{A}{\otimes} \Omega^1 \qquad R_\nabla = (\mathrm{d} \underset{A}{\otimes} \mathrm{id} - (\wedge \underset{A}{\otimes} \mathrm{id})(\mathrm{id} \underset{A}{\otimes} \nabla))\nabla$$

- Laplacian $\Delta : A \to A, \quad \Delta = (\ ,\ )\nabla \mathrm{d}$
- *-compatibility in *-algebra case

$$[\mathrm{d}, *] = 0, \quad g^\dagger = g, \quad \sigma \dagger \nabla * = \nabla; \quad \dagger = \mathrm{flip}(* \otimes *)$$

- Naive Ricci tensor depends on a lift map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$ then take a trace => $\mathrm{Ric} \in \Omega^1 \otimes_A \Omega^1 \qquad S = (\ ,\ )\mathrm{Ric} \in A$

$\boxed{\text{Don't know conserved Einstein, Stress-Energy tensor ...!}}$

2. Scalar fields on one edge

$$X = \bullet_0 \longleftrightarrow \bullet_1 = \mathbb{Z}_2$$

$$\Omega^1 = \mathrm{sp}\{\omega_{0\to 1}, \omega_{1\to 0}\} = C(\mathbb{Z}_2).e_1; \quad \mathcal{C} = \{1\}, \quad e_1 = \omega_{0\to 1} + \omega_{1\to 0}$$

$$e_1 f = R(f)e_1, \quad R(f)(x) = f(x+1), \quad \mathrm{d}f = \partial f e_1, \quad \partial = R - \mathrm{id} \qquad \Omega^2 = 0$$

$$g = a e_1 \otimes e_1 = a(0)\omega_{0\to 1} \otimes \omega_{1\to 0} + a(1)\omega_{1\to 0} \otimes \omega_{0\to 1}$$

$$a(0) = g_{0\to 1}, a(1) = g_{1\to 0} \in \mathbb{R} \setminus \{0\}$$

Exists QLC iff $a(1) = \pm a(0)$ Focus on edge-symmetric case a constant

=> 1-parameter connection $\nabla e_1 = b e_1 \otimes e_1, \quad b = (1-q, 1-q^{-1}); \quad |q| = 1$

=> Laplacian $\Delta f = (\ ,\ )\nabla \mathrm{d} f = -(\partial f)(q + q^{-1})/a$

=> Action $S_f = \sum_{\mathbb{Z}_2} \mu f^*(\Delta + m^2) f = (q + q^{-1})|f(1) - f(0)|^2 + am^2\left(|f(0)|^2 + |f(1)|^2\right)$

$\mu = a > 0$

Fourier modes $f = f_0 + f_1 \phi; \quad \phi(i) = (-1)^i \qquad f(0) = f_0 + f_1 \qquad f(1) = f_0 - f_1$

QFT $Z = 2\int \mathrm{d}f_0 \mathrm{d}f_1 e^{\imath S_f} \qquad S_f = 4(q + q^{-1}) f_1^2 + 2am^2(f_0^2 + f_1^2).$

=>

$$\langle f(0)f(1)\rangle = \langle f(1)f(0)\rangle = \langle f_0^2 - f_1^2\rangle = \frac{\imath}{4}\left(\frac{1}{am^2} - \frac{1}{am^2 + 2(q + q^{-1})}\right)$$

$$\langle f(0)f(0)\rangle = \langle f(1)f(1)\rangle = \langle f_0^2 + f_1^2\rangle = \frac{\imath}{4}\left(\frac{1}{am^2} + \frac{1}{am^2 + 2(q + q^{-1})}\right)$$

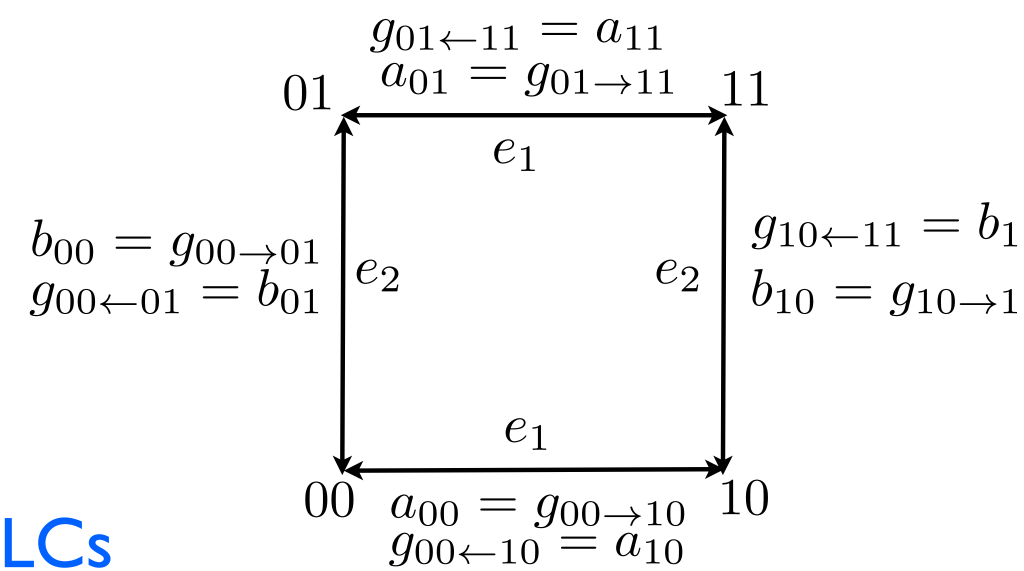
3. Quantum gravity on a square

arXiv:1810.10831

e.g. $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\qquad e_a^2 = 0,\ e_a e_b + e_b e_a = 0$ $\qquad \mathrm{d}e_a = 0$

=> metric $g = a e_1 \otimes e_1 + b e_2 \otimes e_2$

for generic functions *a,b*



We suppose g is edge-symmetric: $\partial^1 a = \partial^2 b = 0$

=> 1-parameter moduli space of QLCs

$$\nabla\omega = \theta\otimes\omega - \sigma(\omega\otimes\theta)$$

$$\theta = e_1 + e_2$$

$$Q = (q, q^{-1}, q^{-1}, q) \qquad |q| = 1$$

$$\sigma = \begin{pmatrix} -Q^{-1} & 0 & 0 & \frac{a(R_1\alpha-1)}{b} \\ 0 & \alpha-1 & \beta & 0 \\ 0 & \alpha & \beta-1 & 0 \\ \frac{b(R_2\beta-1)}{a} & 0 & 0 & Q \end{pmatrix}$$

cf `8-vertex R-matrix'

$$\alpha = \left(\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}}\right) \qquad \beta = \left(1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1\right)$$

with curvature e.g.

$$R_\nabla e_1 = \Big(Q^{-1}R_1\alpha - Q\alpha + (1-\alpha)(R_1\beta - 1) + \frac{R_2 a}{a}(R_2\beta - 1)(R_2R_1\alpha - 1)\Big)\mathrm{Vol}\otimes e_1$$

$$+\Big(Q^{-1}(1-\alpha) + \alpha(R_2\alpha - 1) + Q^{-1}\frac{R_1 b}{a}(\beta^{-1} - 1)) + \frac{b}{a}(R_2\beta - 1)R_2\beta\Big)\mathrm{Vol}\otimes e_2$$

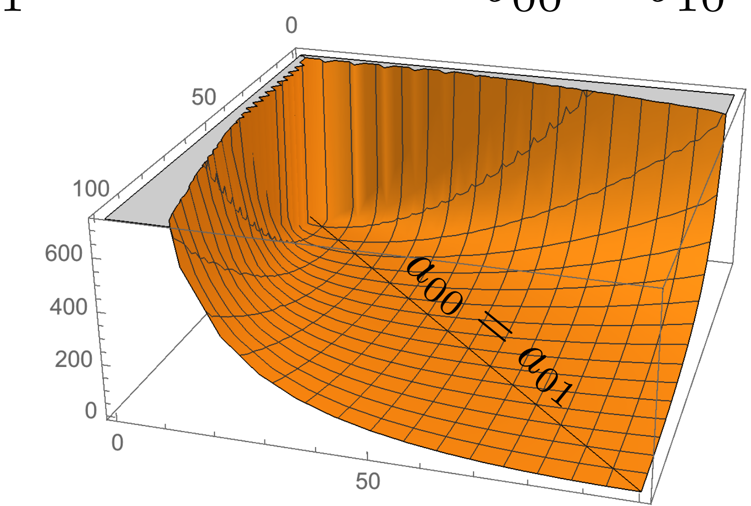
and quantum Ricci scaler curvature for the antisymm lift,

$$S = -\frac{1}{4ab}\Big((3+q+(1-q)\chi)\frac{\partial_2 a}{\alpha} + (1-q^{-1} - (3+q^{-1})\chi)\frac{\partial_1 b}{\beta}\Big) \qquad \chi = (1,-1,-1,1)$$

Choice of measure $\mu = |\det g| = ab$ (in Eucl. case *a*, *b*>0) =>

$$\int S = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2}\mu S = (a_{00} - a_{01})^2\Big(\frac{1}{a_{00}} + \frac{1}{a_{01}}\Big) + (b_{00} - b_{10})^2\Big(\frac{1}{b_{00}} + \frac{1}{b_{10}}\Big)$$

measures the `energy' in the gravitational field, minimised at *a*, *b* constant (i.e. on `rectangules')

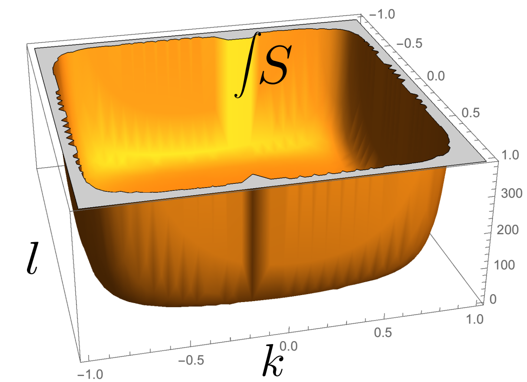


Momentum mode expansion for the gravitational field:

$$\phi(i,j) = (-1)^i = (1,1,-1,-1), \quad \psi(i,j) = (-1)^j = (1,-1,1,-1)$$

$$a = k_0 + k_1\psi, \quad b = l_0 + l_1\phi$$

$$\int S = 8\left(\frac{k_0k^2}{1-k^2} + \frac{l_0l^2}{1-l^2}\right) \qquad \begin{array}{ll} k = k_1/k_0 & |k| < 1 \\ l = l_1/l_0 & |l| < 1 \end{array}$$



Full quantisation of metric (Minkowski case $-a = k_0 + k_1\psi$)

$$S_g = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu S = k_0\alpha(k) - l_0\alpha(l); \quad \alpha(k) = \frac{8k^2}{1-k^2}$$

=> partition function $|Z|^2$

$$Z = 2\int_{-1}^{1} \mathrm{d}k \int_0^L \mathrm{d}k_0 k_0 e^{\frac{\imath}{G}k_0\alpha(k)} = 4G^2\int_0^1 \mathrm{d}k \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=\alpha(k)} \frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha} = 4G^2\int_0^\infty \mathrm{d}\alpha \frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)$$

$k = \sqrt{\frac{\alpha}{8+\alpha}}$ control IR divergence by $0 \le k_0, l_0 \le L$ (still dvges at $\alpha = 0$)

=>

$$\langle k_0^m\rangle := \frac{\int_{-1}^1 \mathrm{d}k \int_0^L \mathrm{d}k_0 k_0^{m+1} e^{\frac{\imath}{G}k_0\alpha(k)}}{\int_{-1}^1 \mathrm{d}k \int_0^L \mathrm{d}k_0 k_0 e^{\frac{\imath}{G}k_0\alpha(k)}} = \frac{2}{m+2}L^m \qquad \langle k_0^m k^n\rangle = 0$$

=>

$$\langle a_{00}\rangle = \langle a_{01}\rangle = \langle k_0(1\pm k)\rangle = \frac{2}{3}L \qquad \langle a_{00}a_{01}\rangle = \langle a_{00}a_{00}\rangle = \langle a_{01}a_{01}\rangle = \langle k_0^2(1\pm k^2)\rangle = \frac{L^2}{2}$$

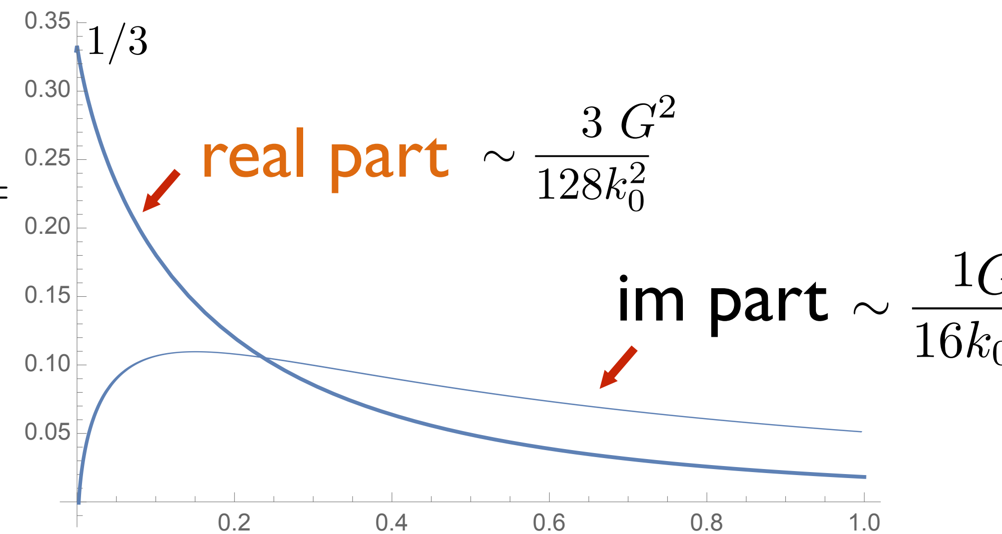
=>

$$\boxed{\frac{\Delta a_{00}}{\langle a_{00}\rangle} = \frac{\sqrt{\langle a_{00}^2\rangle - \langle a_{00}\rangle^2}}{\langle a_{00}\rangle} = \frac{1}{\sqrt{8}}}$$

Quantize relative fluctuations k, l only (regard average lengths $k_0, l_0 > 0$ as background coupling constants)

$$Z(k_0, l_0) = 4k_0 l_0 \int_{-1}^{1} \mathrm{d}k \int_{-1}^{1} \mathrm{d}l\, e^{\frac{i}{G}k_0\alpha(k) - \frac{i}{G}l_0\alpha(l)}$$

$$\langle k^2\rangle = \frac{\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}l\, e^{\frac{i}{G}k_0\alpha(k)-\frac{i}{G}l_0\alpha(l)}\, k^2}{\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}l\, e^{\frac{i}{G}k_0\alpha(k)-\frac{i}{G}l_0\alpha(l)}} =$$



=>

$$\langle a_{00}\rangle = \langle a_{01}\rangle = \langle k_0(1\pm k)\rangle = k_0$$

$$\langle a_{00}^2\rangle = \langle a_{01}^2\rangle = k_0^2(1+\langle k^2\rangle), \quad \langle a_{00}a_{01}\rangle = k_0^2(1-\langle k^2\rangle)$$

=> $\Delta a_{00}/\langle a_{00}\rangle = \sqrt{\langle k^2\rangle} = 1/\sqrt{3}$ in \`deep qg' limit $k_0 \to 0$ or $G \to \infty$

4. Curved space scalar QFT on a square

QLC => 1-parameter geometric quantum Laplacian

$$\Delta f = (\ ,\ )\nabla(\partial_i f e_i) = -\frac{2}{a}\partial_1 f - \frac{2}{b}\partial_2 f + \partial_i f(\ ,\ )\nabla e_i = \left(\frac{Q^{-1} - R_2\beta}{a}\right)\partial_1 f - \left(\frac{Q + R_1\alpha}{b}\right)\partial_2 f$$

$$S_f = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu f(\Delta + m^2) f$$

Has symmetry (Mink case) $f = f_0 + f_1\phi + f_2\psi + f_3\chi$

$$q \leftrightarrow -q^{-1}; \quad k_0 \leftrightarrow -l_0; \quad k \leftrightarrow l; \quad f_1 \leftrightarrow f_2$$

$q = \pm 1$ for the action to have real coefficients E.g. $q = 1$ =>

$$S_f = 4\big(4k_0(f_2^2 + f_3^2) + 2kk_0(f_0f_2 + f_1f_3) + 2ll_0(f_0f_1 + f_2f_3) + 4ll_0f_1f_3 + 4kk_0f_2f_3 + m^2k_0l_0\left(f_0^2 + f_1^2 + f_2^2 + f_3^2 + 2l(f_0f_1 + f_2f_3) + 2k(f_0f_2 + f_1f_3) + 2kl(f_1f_2 + f_0f_3)\right)\big).$$

Quadratic in the f_i => QFT similar to the flat space case where $k = l = 0$

Remains to compute Hawking effect

5. Geometry of the integer line

$\mathbb{Z}$ arXiv:1811.06264

$$\cdots \bullet_{i-1} \longleftrightarrow \bullet_i \longleftrightarrow \bullet_{i+1} \cdots \qquad \mathcal{C} = \{\pm 1\} \qquad e_+ = \sum_i \omega_{i\to i+1}, \qquad e_- = \sum_i \omega_{i\to i-1}$$

$$R_\pm(f)(i) = f(i\pm 1) \qquad e_\pm f = R_\pm(f)e_\pm,\ \mathrm{d}f = (\partial_\pm f)e_\pm \qquad \partial_\pm = R_\pm - \mathrm{id}$$

$$\Omega^1 = C(\mathbb{Z}).\{e_\pm\} \qquad \Omega^2 = C(\mathbb{Z}).\mathrm{Vol}; \qquad \mathrm{Vol} = e_+\wedge e_-$$

We suppose g is edge-symmetric: $\bullet_i \overset{a_i}{-} \bullet_{i+1}$

$$g = ae_+\otimes e_- + R_-ae_-\otimes e_+$$

$$a(i) = g_{i\to i+1} = (R_-a)(i+1) = g_{i+1\to i} = a_i > 0$$

=> Generically unique QLCs (could be others for specific a)

$$\nabla e_+ = (1-\rho)e_+\otimes e_+, \qquad \nabla e_- = (1-R_-^2\rho^{-1})e_-\otimes e_-; \qquad \rho_i = a_{i+1}/a_i$$

$$R_\nabla e_+ = \partial^-\rho\mathrm{Vol}\otimes e_+, \qquad R_\nabla e_- = -\partial^+\left(\frac{1}{R_-^2\rho}\right)\mathrm{Vol}\otimes e_-$$ + antisymmetric lift =>

$$\mathrm{Ricci} = \frac{1}{2}\left(\partial^+\left(\frac{1}{R_-\rho}\right)e_+\otimes e_- + \partial^- R_-\rho e_-\otimes e_+\right) \qquad S = \frac{1}{2a}\left(-\partial^-\left(\frac{1}{\rho}\right) + R_-(\rho\partial^-\rho)\right)$$

=>

$$S_g = -2\sum_{\mathbb{Z}}\mu S = \mathrm{Const.} - \frac{1}{2}\sum_i \rho(i)(\rho(i+1)+\rho(i-1)-2\rho(i)) = \mathrm{Const.} - \frac{1}{2}\sum_{\mathbb{Z}}\rho\Delta_{\mathbb{Z}}\rho$$

$$\Delta\phi = (\ , \ )\nabla \mathrm{d}\phi = -\frac{1+R_{-}\rho}{a}\Delta_{\mathbb{Z}}\phi$$

$$S_\phi = \frac{1}{2}\sum \mu\bar{\phi}(\frac{\Delta}{2} - m^2)\phi = \sum_i \frac{1+\rho_{i-1}}{4}\bar{\phi}_i(2\phi_i - \phi_{i+1} - \phi_{i-1}) - \frac{1}{2}\sum_i a_i m^2 |\bar{\phi}_i|^2$$

Real scalar field theory on flat background; restrict to n modes

$$\phi = (\cdots, 0, 0, \phi_0, \cdots, \phi_{n-1}, 0, 0, \cdots)$$

$$B_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & \vdots & & & \cdots & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$S_\phi = \frac{1}{2}\left(\bar{\phi}B_n\phi - am^2\bar{\phi}\phi\right)$$

$$Z = \int_{-\infty}^{\infty} \mathrm{d}\phi_0 \cdots \mathrm{d}\phi_{n-1} e^{\frac{\imath}{\beta}S_\phi} = \frac{(2\pi\imath\beta)^{\frac{n}{2}}}{\sqrt{D_n}}$$

$$D_n = \det(B_n - am^2) = \sum_{k=0}^{n}\left(-am^2\right)^k \binom{k+n+1}{2k+1}$$

=> $$\langle\phi_i\phi_j\rangle = \langle\phi_j\phi_i\rangle = \imath\frac{D_i D_{n-1-j}}{D_n}, \quad i \le j$$

(i) Continuum phase: $am^2 < 4$. In this case we write

$$2 - am^2 + ma^2\sqrt{am^2 - 4} = 2e^{\imath x}; \quad 2\sin(x) = \sqrt{am^2\left(4 - am^2\right)}, \quad 2\cos(x) = 2 - am^2$$

=> $$D_i = \frac{\sin(x(i+1))}{\sin(x)}$$ => 2-point function is Green fn

$$(\Delta_{\mathbb{Z}} + am^2)_{ik}\langle\phi_k\phi_j\rangle = -\imath\delta_{ij}$$

$$\langle\phi_0\phi_i\rangle = \imath\frac{D_{n-1-i}}{D_n} \to \imath, \quad \langle\phi_0\phi_{\frac{n-1}{d}}\rangle = \imath\frac{D_{(n-1)(1-\frac{1}{d})}}{D_n} \to \imath\left(1-\frac{1}{d}\right)$$

if we first set $x \to 0$ and then set $n \to \infty$ (correlations in the boundary layer and between that and the `bulk')

(ii) Discrete phase $am^2 > 4$. In this case we let $y > 0$ be defined by either of

$$2-am^2 \mp \sqrt{am^2\,(am^2-4)} = -2e^{\pm y}; \quad 2\sinh(y) = \sqrt{am^2(am^2-4)}, \quad 2\cosh(y) = am^2-2$$

=> $$D_i = (-1)^i\frac{\sinh(y(i+1))}{\sinh(y)} \qquad \langle\phi_i\phi_j\rangle \sim -\imath(-1)^{i-j}\frac{e^{y|(i-j)|}}{2\sinh(y)}$$

<u>Plane wave solutions to</u> $\Delta_{\mathbb{Z}}\phi = -am^2\phi$ let $\frac{m\sqrt{a}}{2} = \sin(\frac{m_0\sqrt{a}}{2}) = \sin(\frac{x}{2})$

$$\phi(j) = \alpha e^{-\imath m_0 j\sqrt{a}} + \bar{\alpha}e^{\imath m_0 j\sqrt{a}} = \alpha e^{-\imath x j} + \bar{\alpha}e^{\imath x j}$$

Alternative Hamiltonian quantisation

$$\Phi(j) = Ae^{-\imath m_0 j\sqrt{a}} + A^\dagger e^{\imath m_0 j\sqrt{a}} = e^{\imath Hj\sqrt{a}}\Phi(0)e^{-\imath Hj\sqrt{a}}; \quad \Phi(0) = A + A^\dagger$$

$[A, A^\dagger] = 1$ and $H = m_0(A^\dagger A + \frac{1}{2})$ => $\langle 0|T\phi(i)\phi(j)|0\rangle = e^{\imath x|i-j|}$

path integral theory ~ imag part

Hawking effect on the integer line

$\frac{\Delta}{2}\phi = m^2\phi$ curved wave eqn is $\phi_i = 2(1 - c_{i-1}m^2)\phi_{i-1} - \phi_{i-2}; \quad c_i = \frac{a_{i-1}a_i}{a_{i-1} + a_i}$

- Suppose $a_i = a$ constant for $i \leq 0$ and $a_i = a\rho_0 \cdots \rho_{n-1} = b$, constant $i \geq n$

- Suppose normalised incoming plane wave with $\sin(\frac{x}{2}) = \frac{m\sqrt{a}}{2}$

$$\phi = a^{\text{in}}\bar{\phi}_{\text{in}} + \bar{a}^{\text{in}}\phi_{\text{in}} \qquad \phi_{\text{in}}(i) = \frac{e^{\imath x i}}{\sqrt{\sin(x)}}; \quad i \leq 1$$

- Solve the wave equation through $i = 0, \cdots, n$

- Match to normalised outgoing plane wave with $\sin(\frac{y}{2}) = \frac{m\sqrt{b}}{2}$

$$\phi = a^{\text{out}}\bar{\phi}_{\text{out}} + \bar{a}^{\text{out}}\phi_{\text{out}} \qquad \phi_{\text{out}}(i) = \frac{e^{\imath y(i-n-1)}}{\sqrt{\sin(y)}}; \quad i \geq n$$

=> full solution has two different parametrizations related by Bogoliubov transformation $a^{\text{out}} = a^{\text{in}} f + \bar{a}^{\text{in}} g$ or $\phi_{\text{out}} = \bar{f}\phi_{\text{in}} - g\bar{\phi}_{\text{in}}$ for some $f, g \in \mathbb{C}; \quad |f|^2 - |g|^2 = 1$

- Assume the corresponding quantum fields with $[A, A^\dagger] = 1, \ |0 \text{ in}\rangle$

$$\Phi = A\bar{\phi}_{\text{in}} + A^\dagger \phi_{\text{in}} = B\bar{\phi}_{\text{out}} + B^\dagger \phi_{\text{out}}$$

and $[B, B^\dagger] = 1$, $H_{\text{out}} = \frac{y}{\sqrt{b}}(B^\dagger B + \frac{1}{2})$, vacuum $|0 \text{ out}\rangle$ and $B = Af + A^\dagger g$

=> $|0 \text{ in}\rangle$ from the point of view of later time has occupation number

$$\langle N \rangle = \langle 0 \text{ in}|B^\dagger B|0 \text{ in}\rangle = \langle 0 \text{ in}|A\bar{g}A^\dagger g|0 \text{ in}\rangle = \langle 0 \text{ in}||g|^2 AA^\dagger|0 \text{ in}\rangle = |g|^2$$

Example Step function metric $a_i = \begin{cases} a & i \leq 0 \\ b & i \geq 1 \end{cases}$ $\quad \gamma = 2\left(1 - \frac{abm^2}{a+b}\right)$

$$a^{\text{out}} = \frac{1}{\sqrt{-(q - q^{-1})(p - p^{-1})}} \left(a^{\text{in}}(q^{-1}(\gamma - p) - 1) + \bar{a}^{\text{in}}(q(\gamma - p) - 1) \right) \qquad q = e^{\imath x}, \ p = e^{\imath y}$$

$$\langle N \rangle = \frac{1}{\sqrt{\rho(4 - am^2)(4 - am^2\rho)}} \left(1 + \rho - \frac{am^2\rho(3 - 2\rho + 3\rho^2)}{2(\rho+1)^2} - \frac{1}{2}\sqrt{\rho\,(4 - am^2)\,(4 - am^2\rho)} \right)$$

$$= (\sqrt{\rho} - \sqrt{\rho}^{-1})^2/4$$ in continuum limit $a \to 0$ (not thermal)

6. Conclusions

- For graphs, edge-symmetric metrics are good and seem to allow a QLC, typically with a circle parameter - remains to find a general theorem

- For square graph we did first quantum gravity computations, in a path integral approach & finding a uniform relative uncertainty $\Delta a/\langle a\rangle$
  - is this connected to cosmological constant?
  - remains to study the joint matter-gravity system
  - remains to do the Hamiltonian quantisation

- For the line graph we found its natural calculus is 2D and a generic metric has curvature.
  - we found a natural Einstein-Hilbert action, remains to quantise
  - some degree of match between path integral of scalar field and Hamiltonian quantisation
  - found a frequency-independent Hawking effect; the continuum limit sees only the ratio ρ of out/in metrics

$\boxed{\text{Lots more models!}}$

Appendix: NCRG works over any field *k* e.g. $\mathbb{F}_2$

'digital geometry' (w/ A. Pachol) *arXiv:1807.08492 (math.dg)*

Algebra	Relations	dim A	dim Ω^1	# metrics	# QLCs	# $R_\nabla=0$	# Ricci $=0$
$\mathbb{F}_2\mathbb{Z}_2$	$x^2=0$	2	1	2	1	1	1
$\mathbb{F}_2(\mathbb{Z}_2)$	$x^2=x$	2	1	1	1	1	1
$\mathbb{F}_4$	$x^2=1+x$	2	1	3	1	1	1
	$x^2=y^2=xy=0$	3	2	0	-	-	-
$\mathbb{F}_2(\mathbb{Z}_3)$	$x^2=x,\ y^2=y,\ xy=0$	3	2	1	4	1	3
	$x^2=x,\ y^2=xy=0$	3	2	0	-	-	-
$\mathbb{F}_2\mathbb{Z}_3$	$x^3=x+x^2$	3	2	3	12	1	3
	$x^3=0$	3	2	0	-	-	-
$\mathbb{F}_8$	$x^3=1+x^2$	3	2	7	40	13	18
	$x^2=x, y^2=yx=0, xy=y$	3	2	0	-	-	-

All digital algebras dim <4 that admit a parallelisable diff calculus with top form degree 2. We see 9 which are Ricci flat but not flat

Common phenomena e.g. in dim 3:

(i) $\Delta=0$ *if and only if* $\underline{\dim}=0$.

(ii) If $\Delta\neq 0$ *and* $\mathrm{Tr}(\Delta)=1$ *then* Δ *has one mode with eigenvalue 1 and two with eigenvalue 0*