Nonassociative Differential Geometry: An Invitation.

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Particle Physics, Cosmology & Noncommutative Geometry

Nonassociative Geometry
- Algebras
- Dirac Operators
- Vector Fields & Differential Forms
- Boson Dynamics

Particle Physics, Cosmology & Noncommutative Geometry

Curious Features of our Current Models

$$\text{Diffeomorphisms} \quad \longleftrightarrow \quad \text{Coordinate Transformations}$$

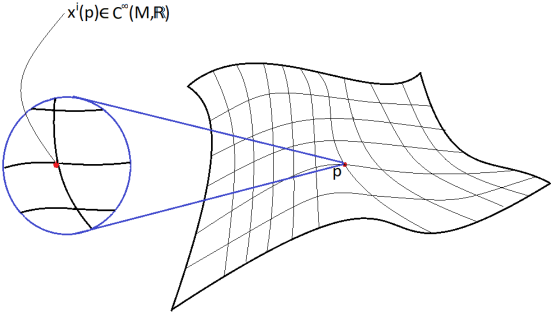
$$\text{Gauge Transformations} \quad \longleftrightarrow \quad ?$$

Curious Features of our Current Models

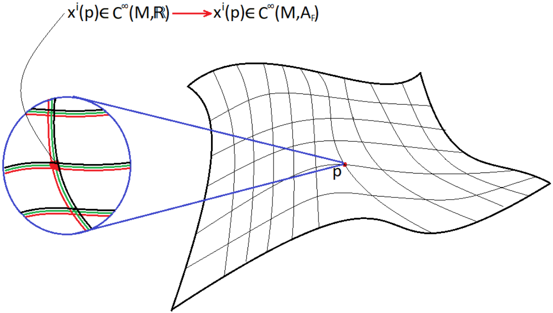
$$\Psi_{\mathrm{i}}^{\alpha}(\mathrm{x})$$

Noncommutative Geometry

Coordinatizing a Geometry



Coordinatizing a Geometry



When A_F is non-commutative & associative,

What happens to our usual geometric notions?

- manifold
- vector fields
- differential forms
- connections
- metric
- ...

Non-commutative geometry gives us the tools...

The standard model as an NCG

	SUSY	KK	GUT	NCG
Strong & electroweak Unification	×	×	✓	×
Gauge Gravity Unification	×	✓	×	✓
Gauge Higs Unification	×	✓	×	✓
Boson & Fermion Unification	✓	×	×	×
Constrains charges & representations	×	×	✓	✓
Avoids unobserved massive states	×	×	×	✓

Q: Why doesn't NCG help us make predictions?

A: It relies on too many assumptions:

Assumption	Reason
Unimodularity	Remove Unobserved U(1)
Symplectic Condition	Remove unwanted U(1)
Internal KO-dimension = 6	Remove Fermion Quadrupling
Three Particle Generations	Add particle trippling
Massless Photon Condition	Remove Unwanted Scalars
Chiral Theory	Match Observation
Associative Coordinate Algebra	???

Nonassociative Geometry

Example 1: Bison Algebra $\mathbb{B}_2$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	x_1	x_2	x_3	$-x_0$	x_5	$-x_6$	x_7	x_4
x_2	x_2	$-x_3$	$-x_0$	x_1	$-x_6$	x_7	x_4	$-x_5$
x_3	x_3	x_0	$-x_1$	x_2	x_7	$-x_4$	$-x_5$	$-x_6$
x_4	x_4	x_5	x_6	x_7	$-x_0$	$-x_1$	$-x_2$	$-x_3$
x_5	x_5	x_6	$-x_7$	x_4	$-x_1$	x_2	x_3	x_0
x_6	x_6	x_7	$-x_4$	$-x_5$	x_2	x_3	$-x_0$	$-x_1$
x_7	x_7	$-x_4$	x_5	x_6	$-x_3$	$-x_0$	$-x_1$	x_2

Example 1: Bison Algebra $\mathbb{B}_2$

Derivations on basis $\{x_0, x_2, x_6, x_4, x_3, x_1, x_5, x_7\}$

$$\delta_0(x) = \left(\begin{array}{c|cc|cc|cc} & & & & & & \\ \hline & & -2 & & & & \\ & 2 & & & & & \\ \hline & & & & 1 & & \\ & & & -1 & & & \\ \hline & & & & & & 1 \\ & & & & & -1 & \end{array}\right), \quad \delta_1(x) = \left(\begin{array}{c|cc|cc|cc} & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & 1 \\ & & & & & -1 & \\ \hline & & & & 1 & & \\ & & & -1 & & & \end{array}\right),$$

$$\delta_2(x) = \left(\begin{array}{c|cc|cc|cc} & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & & & 1 \\ & & & & & 1 & \\ \hline & & & & -1 & & \\ & & & -1 & & & \end{array}\right), \quad \delta_3(x) = \left(\begin{array}{c|cc|cc|cc} & & & & & & \\ \hline & & & & & & \\ & & & & & & \\ \hline & & & & 1 & & \\ & & & -1 & & & \\ \hline & & & & & & -1 \\ & & & & & 1 & \end{array}\right).$$

Bonus: No need for Unimodularity or Symplectic condition.
Unifies $U(1) \times SU(2)$ algebraically, automatically chiral.
Compatible with $J\Psi = \Psi = \Gamma\Psi$.

Example: Jordan Pati-Salam $M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_4(\mathbb{C})$

$$\boldsymbol{\pi}(a)\Psi = \left(\begin{array}{cc|c} q_L & & \\ & q_R & \\ \hline & & M \end{array}\right)\left(\begin{array}{cccc|cccc} & & & & \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ & & & & \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ & & & & \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ & & & & \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \\ \hline \overline{\Psi}_{11} & \overline{\Psi}_{21} & \overline{\Psi}_{31} & \overline{\Psi}_{41} & & & & \\ \overline{\Psi}_{12} & \overline{\Psi}_{22} & \overline{\Psi}_{32} & \overline{\Psi}_{42} & & & & \\ \overline{\Psi}_{13} & \overline{\Psi}_{23} & \overline{\Psi}_{33} & \overline{\Psi}_{43} & & & & \\ \overline{\Psi}_{14} & \overline{\Psi}_{24} & \overline{\Psi}_{34} & \overline{\Psi}_{44} & & & & \end{array}\right)$$

Example: Jordan Pati-Salam $M_2^+(\mathbb{C}) \oplus M_2^+(\mathbb{C}) \oplus M_4^+(\mathbb{C})$

$$\pi(a)\Psi = \frac{1}{2}\left\{ \left(\begin{array}{cc|c} q_L & & \\ & q_R & \\ \hline & & M \end{array} \right), \left(\begin{array}{cccc|cccc} & & & & \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ & & & & \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ & & & & \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ & & & & \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \\ \hline \overline{\Psi}_{11} & \overline{\Psi}_{21} & \overline{\Psi}_{31} & \overline{\Psi}_{41} & & & & \\ \overline{\Psi}_{12} & \overline{\Psi}_{22} & \overline{\Psi}_{32} & \overline{\Psi}_{42} & & & & \\ \overline{\Psi}_{13} & \overline{\Psi}_{23} & \overline{\Psi}_{33} & \overline{\Psi}_{43} & & & & \\ \overline{\Psi}_{14} & \overline{\Psi}_{24} & \overline{\Psi}_{34} & \overline{\Psi}_{44} & & & & \end{array} \right) \right\}$$

Bonus: No need for Unimodularity or Symplectic condition.
Compatible with $J\Psi = \Psi = \Gamma\Psi$.
Gauge group $\mathscr{G} = SU_L(2) \times SU_R(2) \times SU_c(4)$

Example 2: Exceptional Jordan Algebra $H_3(\mathbb{O})$

$$\begin{pmatrix} a & A & B \\ A^* & b & C \\ B^* & C^* & c \end{pmatrix}$$

Bonus: $SU(3) \times SU(2) \times U(1)$ as a subgroup, Three generations?
Compatible with $\mathbf{J\Psi} = \mathbf{\Psi} = \mathbf{\Gamma\Psi}$.

How do we build a Dirac operator on NCG?

Once we have an algebra A and representation H we:

1. Start with a big $n \times n$ matrix called D.
2. Set $D = D^\dagger$
3. Choose a KO-Dimension (along with a J and a Γ)
4. Set $\{D, \Gamma\} = 0$
5. Set $DJ = \varepsilon' JD$
6. Set $[[D, a], JbJ^*] = 0$ for all a, b in A
7. Set $\{[D, a], J[D, b]J^*\} = 0$ for all a, b in A
8. Fluctuate $D \to D_A = D + A + \varepsilon' JAJ^*$

Build an action: $S = \mathrm{Tr}[f(D_A/\Lambda)] + \overline{\Psi} D_A \Psi$

How do we usually Build a Dirac operator?

First off, the ingredients: Gamma matrices

$$\gamma^0 = \begin{pmatrix} & -\mathrm{i} & & \\ \mathrm{i} & & & \\ & & & -\mathrm{i} \\ & & \mathrm{i} & \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} & \sigma^1 & & \\ \sigma^1 & & & \\ & & & -\sigma^1 \\ & & -\sigma^1 & \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} & \sigma^2 & & \\ \sigma^2 & & & \\ & & & \sigma^2 \\ & & \sigma^2 & \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} & \sigma^3 & & \\ \sigma^3 & & & \\ & & & -\sigma^3 \\ & & -\sigma^3 & \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} & & \sigma^2 & \\ & & & -\sigma^2 \\ \sigma^2 & & & \\ & -\sigma^2 & & \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} & & \mathrm{i}\sigma^2 & \\ & & & -\mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & & & \\ & \mathrm{i}\sigma^2 & & \end{pmatrix},$$

These satisfy $\{\gamma^{\mathrm{I}}, \gamma^{\mathrm{J}}\} = 2\delta^{\mathrm{IJ}}$. We can then build:

$$\mathrm{J} = \mathrm{i}\gamma^0\gamma^2\gamma^4 \circ \mathrm{cc} = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix} \circ \mathrm{cc}, \quad \Gamma = \begin{pmatrix} \mathbb{I}_2 & & & \\ & -\mathbb{I}_2 & & \\ & & -\mathbb{I}_2 & \\ & & & \mathbb{I}_2 \end{pmatrix}$$

These satisfy: $\mathrm{J}^2 = 1$, $\mathrm{J}\mathrm{i}\gamma^{\mathrm{I}} = \mathrm{i}\gamma^{\mathrm{I}}\mathrm{J}$, $\mathrm{J}\Gamma = -\Gamma\mathrm{J} \longrightarrow$ KO 6.

How do we usually Build a Dirac operator?

Consider a connection on two different bases:

$$\nabla_\mu(V^\nu\partial_\nu) = \partial_\mu(V^\nu)\partial_\nu - V^\nu\Gamma^\rho_{\mu\nu}\partial_\rho = \underbrace{[\partial_\mu V^\nu - V^\rho\Gamma^\nu_{\mu\rho}]}_{D_\mu V^\nu}\partial_\nu$$

$$\nabla_\mu(V^A e_A) = \partial_\mu(V^A)e_A - V^A\omega^B_{\mu A}e_B = \underbrace{[\partial_\mu V^A - V^B\Gamma^A_{\mu B}]}_{D_\mu V^A}e_A$$

If these bases are related by vielbein $e_A = e^\mu_A\partial_\mu$, then:

$$-\omega^B_{\mu A}e_B = \nabla_\mu e_A = \nabla_\mu(e^\nu_A\partial_\nu) = \partial_\mu(e^\rho_A)\partial_\rho - e^\nu_A\Gamma^\rho_{\mu\nu}\partial_\rho$$

$$\rightarrow -\omega^B_{\mu A} = e^B_\rho\partial_\mu(e^\rho_A) - e^\nu_A\Gamma^\rho_{\mu\nu}e^B_\rho$$

How do we usually Build a Dirac operator?

Introduce the gamma matrices as a vielbein $\gamma^{a}_{b} = (\gamma^{A})^{a}_{b} e_{A}$:

$$\begin{aligned}\nabla_{\mu}(V^{b} W_{a} \gamma^{a}_{b}) &= \partial_{\mu}(V^{b} W_{a}) \gamma^{a}_{b} - V^{b} W_{a} \theta^{c}_{\mu b} \gamma^{a}_{c} + V^{b} W_{a} \theta^{a}_{\mu c} \gamma^{c}_{b} \\ &= \underbrace{[\partial_{\mu} V^{c} - \theta^{c}_{\mu b} V^{b}]}_{D_{\mu} V^{c}} \gamma^{a}_{c} W_{a} + V^{b} \gamma^{c}_{b} \underbrace{[\partial_{\mu} W_{c} + \theta^{a}_{\mu c} W_{a}]}_{D_{\mu} W_{c}}\end{aligned}$$

where $(\theta_{\mu})^{a}_{b} = \frac{1}{8}(\omega_{\mu AB}[\gamma^{A}, \gamma^{B}])^{a}_{b}$.

How do we usually Build a Dirac operator?

Now we have the tools to build the Dirac action:

$$\mathrm{S} = \mathrm{i}\int \boldsymbol{\varepsilon}_{\mathrm{A}_1\ldots\mathrm{A}_6}\mathrm{e}^{\mathrm{A}_1}\wedge\ldots\wedge\mathrm{e}^{\mathrm{A}_5}[\overline{\Psi}\boldsymbol{\gamma}^{\mathrm{A}_6}\mathrm{D}\Psi - \overline{\mathrm{D}\Psi}\boldsymbol{\gamma}^{\mathrm{A}_6}\Psi]$$

where $\mathrm{e}^{\mathrm{A}} = \mathrm{e}^{\mathrm{A}}_{\mu}\mathrm{dx}^{\mu}$ and $\mathrm{D} = \mathrm{dx}^{\mu}\mathrm{D}_{\mu}$.

How do we usually Build a Dirac operator?

$$\begin{aligned}S &= \mathrm{i}6!\int \mathrm{d}^6\mathrm{x}\mathrm{e}^{\mu}_{\mathrm{A}}[\overline{\Psi}\gamma^{\mathrm{A}}\mathrm{D}_{\mu}\Psi - \overline{\mathrm{D}_{\mu}\Psi}\gamma^{\mathrm{A}}\Psi] \\ &= \mathrm{i}6!\int \mathrm{d}^6\mathrm{x}[\overline{\Psi}\gamma^{\mathrm{A}}\mathrm{e}^{\mu}_{\mathrm{A}}\partial_{\mu}\Psi - \overline{\partial_{\mu}\Psi}\gamma^{\mathrm{A}}\mathrm{e}^{\mu}_{\mathrm{A}}\Psi - \tfrac{1}{8}\underbrace{\overline{\Psi}\mathrm{e}^{\mu}_{\mathrm{A}}\omega_{\mu\mathrm{BC}}\{\gamma^{\mathrm{A}},[\gamma^{\mathrm{B}},\gamma^{\mathrm{C}}]\}\Psi}_{\overline{\Psi}\Phi\Psi}]\end{aligned}$$

How do we usually Build a Dirac operator?

$$\mathrm{i}\mathrm{e}^{\mu}_{\mathrm{A}}\boldsymbol{\omega}_{\mu\mathrm{BC}}\{\gamma^{\mathrm{A}}[\gamma^{\mathrm{B}},\gamma^{\mathrm{C}}]\}\Psi = \begin{pmatrix} 0 & \Phi^{\dagger} & \mathrm{M}^{\dagger} & 0 \\ \Phi & 0 & 0 & \mathrm{N}^{\dagger} \\ \mathrm{M} & 0 & 0 & \Phi^{\mathrm{T}} \\ 0 & \mathrm{N} & \overline{\Phi} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{l}_{\mathrm{L}} \\ \mathrm{l}_{\mathrm{R}} \\ \bar{\mathrm{l}}_{\mathrm{L}} \\ \bar{\mathrm{l}}_{\mathrm{R}} \end{pmatrix}$$

The Derivation based Calculus

Tangent space for coordinate algebra $C^{\infty}(M,\mathbb{R})$

$$V = V^{\mu}\partial_{\mu}.$$

Tangent space for coordinate algebra $C^\infty(M, A_F)$

$$V = V^\mu \partial_\mu + V^I \delta_I = V^a \delta_a \equiv \delta_V .$$

Differential Forms

$$\delta_V(f+g) = \delta_V f + \delta_V g$$

Differential Forms

$$\delta_V(f+g) = \delta_V f + \delta_V g$$

$$\delta_{V+W} f = \delta_V f + \delta_W f.$$

$$\rightarrow df(\delta_V) \equiv \delta_V(f),$$

Differential Forms for coordinate algebra $C^\infty(M,\mathbb{R})$

Exact one Form: $df = \partial_\mu f \otimes dx^\mu$

General one Form: $\omega = \omega_\mu \otimes dx^\mu$

where: $dx^\mu(\partial_\nu) = \delta^\mu_\nu$

Differential Forms for coordinate algebra $C^\infty(M, A_F)$

Exact one Form: $df = \partial_\mu f \otimes dx^\mu + \delta_I f \otimes E^I \equiv \delta_a f \otimes E^a$

General one Form: $\omega = \omega_\mu \otimes dx^\mu + \omega_I \otimes E^I = \omega_a \otimes E^a$

where: $E^a(\delta_b) \equiv \delta^a_b$

Differential Forms (for Jordan coordinate algebras)

$$\omega_{\mathrm{a}}\mathrm{E}^{\mathrm{a}} \times \omega'_{\mathrm{b}}\mathrm{E}^{\mathrm{b}} \equiv (\omega_{\mathrm{a}}\omega'_{\mathrm{b}})\mathrm{E}^{\mathrm{a}} \wedge \mathrm{E}^{\mathrm{b}}, \quad \text{(product)},$$

$$\mathrm{d}^2 = 0, \quad \text{(nilpotency)},$$

$$\mathrm{d}(\omega_1 \times \omega_2) = \mathrm{d}(\omega_1) \times \omega_2 + (-1)^{|\omega_1|}\omega_1 \times \mathrm{d}(\omega_2), \quad \text{(Leibniz)}.$$

$$\omega = \omega_{1\ldots\mathrm{n}}\mathrm{E}^1 \wedge \ldots \wedge \mathrm{E}^{\mathrm{n}}.$$

Connection Forms

$$\text{Coordinate basis: } \nabla(\mathrm{V_a E^a}) = [\underbrace{\mathrm{dV_a}}_{!!!} - \Gamma_{\mathrm{a}}^{\mathrm{b}} \mathrm{V_b}]\mathrm{E^a}$$

$$\text{Frame basis: } \nabla(\mathrm{V_A E^A}) = [\underbrace{\mathrm{dV_A}}_{!!!} - \boldsymbol{\omega}_{\mathrm{A}}^{\mathrm{B}} \mathrm{V_B}]\mathrm{E^A}$$

$$\text{Spin basis: } \nabla(\mathrm{V}_\alpha \mathrm{W}^\beta \mathrm{E}_\beta^\alpha) = [\underbrace{\mathrm{dV}^\rho}_{!!!} - \boldsymbol{\theta}_\beta^\rho \mathrm{V}^\beta]\mathrm{E}_\rho^\alpha \mathrm{W}_\alpha + \mathrm{V}^\beta \mathrm{E}_\beta^\rho [\underbrace{\mathrm{dW}_\rho}_{!!!} + \boldsymbol{\theta}_\rho^\alpha \mathrm{W}_\alpha]$$

The Connection Γ^{c}_{ab}.

$$\Gamma_{\mu}{}^{\rho}{}_{\nu} E^{\mu} \wedge E^{\nu} = \left[e^{\rho}_{A} \partial_{\mu}(e^{A}_{\nu}) + e^{\rho}_{A} \omega_{\mu}{}^{A}{}_{B} e^{B}_{\nu} \right] E^{\mu} \wedge E^{\nu},$$

$$B_{\mu}{}^{K}{}_{I} E^{I} \wedge E^{\mu} = \left[e^{K}_{A} d_{\mu}(e^{A}_{I}) + (e^{K}_{A} \omega_{\mu}{}^{A}{}_{B} e^{B}_{I} - e^{K}_{A} \omega_{I}{}^{A}{}_{B} e^{B}_{\mu}) \right] E^{I} \wedge E^{\mu},$$

$$\Phi_{I}{}^{K}{}_{J} E^{I} \wedge E^{J} = e^{K}_{A} \omega_{I}{}^{A}{}_{B} e^{B}_{J} E^{I} \wedge E^{J},$$

The 'Dirac' Operator

$$\mathrm{i}\Phi_{\mathrm{IJK}}\{\gamma^{\mathrm{I}}[\gamma^{\mathrm{J}},\gamma^{\mathrm{K}}]\}\Psi = \begin{pmatrix} 0 & \phi^\dagger & \mathrm{M}^\dagger & 0 \\ \phi & 0 & 0 & \mathrm{N}^\dagger \\ \mathrm{M} & 0 & 0 & \phi^{\mathrm{T}} \\ 0 & \mathrm{N} & \overline{\phi} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{l}_{\mathrm{L}} \\ \mathrm{l}_{\mathrm{R}} \\ \bar{\mathrm{l}}_{\mathrm{L}} \\ \bar{\mathrm{l}}_{\mathrm{R}} \end{pmatrix}$$

Curvature of the Connection

$$\nabla\nabla(\mathrm{E}^{\mathrm{A}}) = -\nabla(\boldsymbol{\omega}^{\mathrm{A}}_{\mathrm{B}}\mathrm{E}^{\mathrm{B}}) = -(\mathrm{d}\boldsymbol{\omega}^{\mathrm{A}}_{\mathrm{B}} + \boldsymbol{\omega}^{\mathrm{A}}_{\mathrm{C}}\boldsymbol{\omega}^{\mathrm{C}}_{\mathrm{B}})\mathrm{E}^{\mathrm{B}} = -\mathrm{R}^{\mathrm{A}}_{\mathrm{B}}\mathrm{E}^{\mathrm{B}}.$$

Using the vielbein:

$$\mathrm{R}^{\mathrm{A}}{}_{\mathrm{B}} = \mathrm{e}^{\mathrm{A}}_{\mathrm{a}}\left[\mathrm{d}_{\mathrm{e}}(\boldsymbol{\Gamma}_{\mathrm{f}}{}^{\mathrm{a}}{}_{\mathrm{b}}) - \boldsymbol{\Gamma}_{\mathrm{c}}{}^{\mathrm{a}}{}_{\mathrm{b}}\boldsymbol{\Gamma}_{\mathrm{e}}{}^{\mathrm{c}}{}_{\mathrm{f}} + \boldsymbol{\Gamma}_{\mathrm{e}}{}^{\mathrm{a}}{}_{\mathrm{c}}\boldsymbol{\Gamma}_{\mathrm{f}}{}^{\mathrm{c}}{}_{\mathrm{b}}\right]\mathrm{e}^{\mathrm{b}}_{\mathrm{B}}\mathrm{E}^{\mathrm{e}}\wedge\mathrm{E}^{\mathrm{f}}.$$

Boson Dynamics

$$S_\Omega = \int \alpha_0 \star (\mathbb{I}) + \alpha_1 e^A e^B \star (R_{AB}) + \alpha_2 R^{AB} e^C e^D \star (R_{CD} e_A e_B) + \ldots$$

'Cosmological Constant' Term

$$
\begin{aligned}
S_{\Omega^0} &= \alpha_0 \int \star(\mathbb{I}) \\
&= \alpha_0 \int \varepsilon_{A_{(1)}\ldots A_{(n)}} e^{A_{(1)}}_{a_{(1)}} \ldots a^{A_{(n)}}_{a_{(n)}} e^{a_{(1)}} \wedge \ldots \wedge e^{a_{(n)}} \\
&= n!\alpha_0 \int |e| d^4x E^{(n-4)}
\end{aligned}
$$

'Einstein Hilbert' Mass Term

$$
\begin{aligned}
S_{\Omega^1} &= \alpha_1 \int e^C e^D \star (R_{CD}) \\
&= (n-2)!\alpha_1 \int |e| \left[R^{(4)} + 2\Phi_I{}^{(I|}{}_J \Phi_K{}^{J|K)} \right] d^4x E^{(n-4)}
\end{aligned}
$$

'Gauss–Bonnet' Term

$$\begin{aligned} S_{\Omega^2} &= \alpha_2 \int R^{AB} e^C e^D \star (R_{CD} e_A e_B) \\ &= 4(n-4)!\alpha_2 \int e \Bigg[\underbrace{R_{[ab]}{}^{[ab]} R_{[cd]}{}^{[cd]}}_{(1)} + \underbrace{R_{[ab][cd]} R^{[ab][cd]}}_{(2)} - \underbrace{4R^{[cd]}{}_{[ad]} R_{[cb]}{}^{[ab]}}_{(3)} \Bigg] d^4x E^{(n-4)} \end{aligned}$$

$$(1) = \mathrm{R}_{[ab]}{}^{[ab]}\mathrm{R}_{[cd]}{}^{[cd]}$$
$$= (\mathrm{R}^{(4)})^2 + 4\mathbf{\Phi}_{\mathrm{I}}{}^{(\mathrm{I}|}{}_{\mathrm{J}}\mathbf{\Phi}_{\mathrm{K}}{}^{\mathrm{J}|\mathrm{K})}(\mathrm{R}^{(4)})^2 + 4\mathbf{\Phi}_{\mathrm{I}}{}^{(\mathrm{I}|}{}_{\mathrm{J}}\mathbf{\Phi}_{\mathrm{K}}{}^{\mathrm{J}|\mathrm{K})}\mathbf{\Phi}_{\mathrm{M}}{}^{(\mathrm{M}|}{}_{\mathrm{N}}\mathbf{\Phi}_{\mathrm{P}}{}^{\mathrm{N}|\mathrm{P})}$$

$$
\begin{aligned}
(2) &= R^{[ab]}{}_{[ef]} R_{[ab]}{}^{[ef]} \\
&= R_{[\mu\nu][\rho\sigma]} R^{[\mu\nu][\rho\sigma]} - \Phi_{K}{}^{I}{}_{(M} \Phi_{N)}{}^{KJ} \Phi^{L}{}_{I}{}^{(M} \Phi^{N)}{}_{LJ} \\
&+ \left(\partial_{[\rho} B_{\sigma]}{}^{[IJ]} + B_{[\rho}{}^{[I}{}_{K} B_{\sigma]}{}^{K|I]} - B_{\mu}{}^{[IJ]} \Gamma_{[\rho}{}^{\mu}{}_{\sigma]} \right) \left(\partial^{[\rho} B^{\sigma]}{}_{IJ} + B^{[\rho|}{}_{IL} B^{\sigma]L}{}_{J} + B_{\nu IJ} \Gamma^{[\rho\sigma]\nu} \right) \\
&+ \frac{1}{2} \left(\partial_{\rho} \Phi_{M}{}^{[IJ]} + 2 B_{\rho}{}^{[I}{}_{K} \Phi_{M}{}^{K|J]} - B_{\rho}{}^{K}{}_{M} \Phi_{K}{}^{[IJ]} \right) \left(\partial^{\rho} \Phi^{M}{}_{[IJ]} + 2 B^{\rho}{}_{[I|L} \Phi^{ML}{}_{J]} - B^{\rho LM} \Phi_{L[IJ]} \right).
\end{aligned}
$$

$$
\begin{aligned}
(3) &= 4R^{[cd]}{}_{[ad]}R_{[cb]}{}^{[ab]} \\
&= 4\,{}^{(4)}R^{\mu\nu}\,{}^{(4)}R_{\mu\nu} + 4\left(\Phi_{[P|}{}^{I}{}_{K}\Phi_{J]}{}^{KJ} - \Phi_{K}{}^{IJ}\Phi_{[P}{}^{K}{}_{J]}\right)\left(\Phi^{[P|}{}_{I}{}^{N}\Phi^{M]}{}_{NM} - \Phi^{N}{}_{IM}\Phi^{[P}{}_{N}{}^{M]}\right) \\
&+ \left(\partial_\mu\Phi_{J}{}^{IJ} - 2B_\mu{}^{KJ}\Phi_{(K}{}^{I}{}_{J)} + B_\mu{}^{I}{}_{K}\Phi_{J}{}^{KJ}\right)\left(\partial^\mu\Phi^{M}{}_{IM} - 2B^\mu{}_{NM}\Phi^{(N}{}_{I}{}^{M)} + B^\mu{}_{I}{}^{N}\Phi^{M}{}_{NM}\right).
\end{aligned}
$$

Thankyou.