Lorentzian spectral triples, causality and distance

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Introduction

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Lorentzian (Spin) Geometry: Commutative

The motivating example: complete globally hyperbolic spacetimes M.
By an improved version of a classical result of Geroch, there exist smooth functions $T : M \to \mathbb{R}$ (called causal functions) such that:

- It is increasing along future directed timelike curves;
- Its level sets are global (smooth) Cauchy surfaces;
- They can be used to define global coordinates such that the metric takes the form $g = -N^2 dT^2 + g_S$ with bounded $N > 0$.
- As a consequence they satisfy $g(\nabla T, \nabla T) < 0$.
- We have $p_1 \leqslant p_2$ iff $T(p_2) - T(p_1) \leqslant 0$ for all causal T.

Later strengthened result:

- automatic existence of steep functions T' satisfying $g(\nabla T', \nabla T') \leqslant -c_{T'}^2$ for some fixed $c_{T'} > 0$. They are necessarily unbounded.

The splitting of the metric naturally gives a reflection $r : M \to M$ sending g to its "Wick-rotated" riemannian counterpart g^r. Our assumption means that M is complete with respect to it.

To the metric g there correspond:

- A spin bundle with space of sections $\Gamma(M,S)$, a Clifford action c such that $c(u)c(v)+c(v)c(u)=2g(u,v)1_S$ $(u,v\in T^*M)$.
- For any local basis $x=(x^0,\cdots,x^{n-1})$ one defines curved gamma matrices $\gamma^\mu=c(dx^\mu)$: γ^0 is anti-hermitian, the γ^i's are hermitian, $[\gamma^\mu,\gamma^\nu]_+=2g^{\mu\nu}$.

The natural inner product on $\Gamma(M,S)$ comes from an indefinite non-degenerate bilinear form $<\cdot,\cdot>_S$ on S:

$$<f_1,f_2>=\int_M <f_1(p),f_2(p)>_S \sqrt{|g|}d^nx.$$

To the reflection r there corresponds the fundamental symmetry $J=iN^{-1}c(dT)$ and the scalar product

$$(f_1,f_2)=<f_1,Jf_2>=\int_M <f_1(p),j_Sf_2(p)>_S \sqrt{|g|}d^nx,$$

and one sets $\mathcal{H}=L^2(M,S)$.

- $D=-ic\circ\nabla^S$ is such that iD is essentially Krein-selfadjoint on the smooth functions on $C_0^\infty(M)=\mathcal{A}$ (and on an appropriate unital $\widetilde{\mathcal{A}}\subset C_b^\infty(M)$).
- Since $ic(dT)=[D,T]$, we see that $J=\gamma_0^{flat}=-N^{-1}[D,T]$.
- This construction actually works for any signature of the metric g.

Franco, Eckstein:

- If (M', g') is a complete pseudo-riemannian manifold and J' is such that $N' = -J'[D', T']$ for some smooth functions N', T', then J' has lorentzian signature and the metric admits a global splitting so that M' is globally hyperbolic;
- Given J, a smooth function $T : M \to \mathbb{R}$ is causal if and only if there is $N > 0$ such that $N = -J[D, T]$.
- Given J, a smooth function $T : M \to \mathbb{R}$ is steep if and only if there is $N > 0$ such that $N = -J([D, T] + i\gamma)$ with the parity operator $\gamma = -i^{1+n/2}\gamma_0 \cdots \gamma_n$.

Letting $\mathcal{L} = \{\text{Cas. future orient. paths } \mathcal{P} : p_1 \to p_2\}$, a lorentzian "distance" can be defined on M by the formula:

$$d(p_1, p_2) = \begin{cases} \sup_{\mathcal{L}} \int_{\mathcal{P}} ds, & \text{for } p_1 \preccurlyeq p_2, \\ 0, & \text{for } p_1 \not\preccurlyeq p_2. \end{cases}$$

For $p_1 \preccurlyeq p_2 \preccurlyeq p_2$ and (v, w) timelike vectors, we have

- If $d(p_1, p_2), d(p_2, p_1) \geqslant 0$ then $d(p_1, p_2) = 0$, (antisymmetry);
- $d(p_1, p_3) \geqslant d(p_1, p_2) + d(p_2, p_3)$, (reverse triangle inequality);
- $|g(v, w)| \geqslant \sqrt{-g(v, v)}\sqrt{-g(w, w)}$, (reverse Cauchy-Schwartz).

The "Gelfand-dualised" version of the distance reads,

$$d(p, q) = \inf_{T \in \mathcal{F}} \left\{ [T(p) - T(q)]^+ \right\},$$

where $[c]^+ = \max\{0, c\}$ and $\mathcal{F}$ is the set of all smooth steep functions. Since these are unbounded, this makes sense only for pure states. However (forget about antisymmetry and the reverse triangle inequality), we may define

$$d(\omega_1, \omega_2) = \begin{cases} \inf_{\mathcal{F} \cap \mathcal{D}(\omega_1, \omega_2)} \{[\omega_2(f) - \omega_1(f)]^+\}, & \text{for } \mathcal{D}(\omega_1, \omega_2) \neq \emptyset, \\ +\infty, & \text{for } \mathcal{D}(\omega_1, \omega_2) = \emptyset. \end{cases}$$

with $\mathcal{D}(\omega_1, \omega_2) = \mathcal{F} \cap L^1(M, d\mu_{\omega_1}) \cap L^1(M, d\mu_{\omega_2})$.

Lorentzian Spectral Triples

A Lorentzian spectral triple is given by $(\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \mathcal{H}, D, J)$ with:

- A Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \cdot)$.
- A non unital pre-C*-algebra $\mathcal{A}$ with a faithful *-representation π on $\mathcal{B}(\mathcal{H})$.
- A preferred unitization $\widetilde{\mathcal{A}}$ of $\mathcal{A}$, which is also a pre-C*-algebra, with a compatible faithful *-representation π on $\mathcal{B}(\mathcal{H})$ and such that $\mathcal{A}$ is an ideal of $\widetilde{\mathcal{A}}$.
- An unbounded operator D, densely defined on $\mathcal{H}$, such that:
  - $\forall a \in \widetilde{\mathcal{A}}$ the operator $[D, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$,
  - with $\langle D \rangle^2 := \frac{1}{2}(DD^* + D^*D)$, $\forall a \in \mathcal{A}$ the operator $\pi(a)(1 + \langle D \rangle^2)^{-\frac{1}{2}}$ is compact, .
- A bounded operator J on $\mathcal{H}$ with $J^2 = 1$, $J^* = J$, $[J, \pi(a)] = 0$, $\forall a \in \widetilde{\mathcal{A}}$ and such that:
  - $D^* = -JDJ$ on $\text{Dom}(D) = J\text{Dom}(D^*) \subset \mathcal{H}$;

Then: $(J\gamma)^* = -J\gamma \Rightarrow iJ\gamma$ is selfadjoint. Moreover, $(iJ\gamma)^2 = -J\gamma J\gamma = 1$.
The operator J is a fundamental symmetry which turns the Hilbert space $\mathcal{H}$ into a Krein space with (indefinite) inner product $< \cdot, \cdot >= (\cdot, J\cdot)$.
The condition $D^* = -JDJ \doteq -D^\dagger$ means that iD is Krein-selfadjoint.

- There is a distinguished selfadjoint operator T such that: $\mathrm{Dom}(T) \cap \mathrm{Dom}(D)$ is dense in $\mathcal{H}$, it holds $\left(1 + T^2\right)^{-\frac{1}{2}} \in \widetilde{\mathcal{A}}$ and there exist an operator $N \in \widetilde{\mathcal{A}}$ such that $N \geqslant 0$ and $N = -J[D, T]$ holds.

A Lorentzian spectral triple is even if there exists a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $[\gamma, \pi(a)] = 0\ \forall a \in \widetilde{\mathcal{A}}$, $\gamma J = -J\gamma$ and $\gamma D = -D\gamma$.

Casual Structure for LST's

Let $\mathcal{C}$ be the convex cone of all selfadjoint elements $T \in \widetilde{\mathcal{A}}$ such that

$$\forall \ \phi \in \mathcal{H}, \qquad (\phi, J[D, \pi(a)]\phi) \leqslant 0,$$

If $\overline{\mathrm{Span}_{\mathbb{C}}(\mathcal{C})} = \overline{\widetilde{\mathcal{A}}}$ then $\mathcal{C}$ is called a causal cone.
It induces a partial order relation on $\mathfrak{S}(\mathcal{A})$ by

$$\forall \omega_1, \omega_2 \in \mathfrak{S}(\mathcal{A}), \qquad \omega \preccurlyeq \omega_2 \qquad \text{iff} \qquad \forall \, T \in \mathcal{C}, \quad \omega_1(T) \leqslant \omega_2(T).$$

- In the commutative case the unitisation $\widetilde{\mathcal{A}}$ has to be carefully chosen so that the set of causal functions is really a causal cone.

Unbounded Multipliers

An unbounded multiplier of a C^*-albebra $\mathcal{B}$ (or an unbounded element affiliated to $\mathcal{B}$) is a closed $\mathcal{B}$-linear map $R : \mathcal{J} \to \mathcal{B}$, where $\mathcal{J}$ is a dense left ideal in $\mathcal{B}$, with a densely defined R^* and such that $(1 + R^*R)$ has dense range. We write $R\eta\mathcal{B}$ and $UM(\mathcal{B})$.

- $R\eta\mathcal{B}$ iff there exists $z \in M(\mathcal{B})$ (the multiplier of $\mathcal{B}$) with $||z|| \leqslant 1$ and

$$(x \in D(R),\ y = Rx) \Leftrightarrow (\text{there is}\, b \in \mathcal{B}\ :\ x = (1 - z^*z)^{1/2}b \text{ and } y = zb),$$

- If such a z exists, it is unique and called the z-transform of R (we write z_R).
- An element $z \in M(\mathcal{B})$ is the z-transform of some $R\eta\mathcal{B}$ if and only if $||z|| \leqslant 1$ and $\overline{(1 - z^*z)\mathcal{B}} = \mathcal{B}$.
- $z_R^* = z_{R^*}$, $(1 + R^*R)^{-1} = (1 - z_R z_R^*)^{-1}$ and $R = z_R(1 - z_R z_R^*)^{-1/2}$ on $(1 - z_R z_R^*)^{-1/2}\mathcal{B}$, which is a core for R.
- $M(\mathcal{B}) \subset UM(\mathcal{B})$ but the two sets coincide if $\mathcal{B}$ is unital.
- Any representation π of $\mathcal{B}$ on $\mathcal{K}$ extends to a map $\hat{\pi}$ from $UM(\mathcal{B})$ to the closed (unbounded) operators on $\mathcal{K}$.

R is $\widetilde{\mathcal{A}}$-affiliated to $\mathcal{A}$ if $R\eta\mathcal{A}$ and $z_R \in \widetilde{\mathcal{A}}$.

Regular LST's

- A LST $(\mathcal{A}, \widetilde{\mathcal{A}}, \pi, \mathcal{H}, D, J)$ is said to be regular (RLST) whenever there exists a preferred $\widetilde{\mathcal{A}}$-affiliated selfadjoint operator T and a positive $N \in \widetilde{\mathcal{A}}$ such that $N = -J[D, T]$.

Suppose that $0 < N \in \widetilde{\mathcal{A}}$ commutes with the preferred T. This includes Franco's Temporal LST, where $N \in \mathcal{C}(\widetilde{\mathcal{A}})$. If N is invertible, $JN^{-1/2}DN^{-1/2}$ is selfadjoint and $[JN^{-1/2}DN^{-1/2}, T] = -i1$. But $T\eta\mathcal{A}$ is bounded if $\mathcal{A}$ is unital, and boundedness of T is incompatible with exponentiability to the corresponding Weyl relations. This would rule out compact noncomm. lorentzian manifolds. We are thus led to

- A selfadjoint T $\widetilde{\mathcal{A}}$-affiliated to $\mathcal{A}$ is temporal if $\mathrm{Dom}(T) \cap \mathrm{Dom}(D)$ is dense in $\mathcal{H}$, and there exists $0 \leqslant N \in \widetilde{\mathcal{A}}$ (so $[N, J] = 0$) such that $N = -J[D, T]$ on $\mathrm{Dom}([D, T])$. We indicate the set of all such operators by $\mathcal{T}_D^J$.

This can be seen as the analog of the causal cone $\mathcal{C}$. Notice that $\mathcal{T}_D^J \neq \emptyset$.

When are we are entitled to call $\mathcal{T}_D^J$ a casual cone? Ideally, we should ask that $\overline{\widetilde{\mathcal{A}}} \subset C^*(\mathcal{T}_D^J)$. However, due to the presence of unbounded elements it is highly problematic to give a precise meaning to such a requirement. Woronowicz gave a notion of C^*-algebras generated by unbounded affiliated elements which appears to perfectly suit this context and in particular the lorentzian Moyal.
The second difficulty concerns the need to evaluate states of the C^*-algebra on unbounded elements. Our solution rests on the following definition:

- Let ω be a state on the C^*-algebra $\mathcal{A}$, $(\pi_\omega, \phi_\omega)$ the corresponding GNS representation and vector and T $\widetilde{\mathcal{A}}$-affiliated to $\mathcal{A}$. We say $\omega \in \mathrm{Dom}(T)$ if $\phi_\omega \in \mathrm{Dom}(|\widehat{\pi_\omega}(T)|)$, where $\widehat{\pi}_\omega$ is the canonical extension of π_ω to $UM(\mathcal{A})$. In this case we set $\omega(\widehat{\pi}_\omega) = (\phi_\omega, \widehat{\pi_\omega}(T)\phi_\omega)$.

If a state ω is a vector state in the representation π defining the spectral triple with corresponding vector ψ and $T \in \mathcal{T}_D^J$ is in the domain of ω, we have that $\psi \in \mathrm{Dom}(|\widehat{\pi}(T)|)$ and $(\psi, \widehat{\pi}(T)\psi) = (\phi_\omega, \widehat{\pi}_\omega(T)\phi_\omega) = \omega(T)$.

Causal Order Revisited

For $\omega_1, \omega_2 \in \mathfrak{S}(\mathcal{A})$, we are now ready to introduce

- $D(\omega_1, \omega_2) = \{T \in \mathcal{T}_J^D \ : \ \omega_1(|T|), \omega_2(|T|) < \infty\}$,
- ω_1, ω_2 are Causally Related whenever for all $T \in D(\omega_1, \omega_2)$ there holds $\omega_1(T) \leqslant \omega_2(T)$.

Since for a (R)LST $\mathcal{C} \subset D(\omega_1, \omega_2)$ for all ω_1, ω_2, it is clear that if two states are casually related then they are comparable according to the partial order previously introduced. Still, the two notions need not be equivalent.

A General Distance Formula

- A selfadjoint T $\widetilde{\mathcal{A}}$-affiliated to $\mathcal{A}$ is steep whenever $\mathrm{Dom}(T) \cap \mathrm{Dom}(D)$ dense in $\mathcal{H}$ and there exist an operator $N > 0$ such that $[J, N] = 0$ and $N = -J([D, T] + i\gamma)$. We denote by $\widetilde{\mathcal{T}}_J^D$ the set of all such operators.
- $\widetilde{D}(\omega_1, \omega_2) = D(\omega_1, \omega_2) \cap \widetilde{\mathcal{T}}_J^D$.

The set $\widetilde{\mathcal{T}}_J^D$ is not empty in typical cases, precisely because we include unbounded operators. It is not difficult to prove that $\widetilde{\mathcal{T}}_J^D \subset \mathcal{T}_J^D$. Next, we set:

- Given two states ω_1, ω_2 on $\mathcal{A}$, their distance is given by:

$$d(\omega_1, \omega_2) = \begin{cases} \inf_{\widetilde{D}(\omega_1,\omega_2)} \{[\omega_2(f) - \omega_1(f)]^+\}, & \text{for } \widetilde{D}(\omega_1, \omega_2) \neq \emptyset, \\ +\infty, & \text{for } \widetilde{D}(\omega_1, \omega_2) = \emptyset. \end{cases}$$

This formula reduces to the ordinary one for (sufficiently regular) commutative lorentzian manifolds.

The Moyal Lorentzian (R)LST

- $\mathcal{H}_0 := L^2(\mathbb{R}^{1,1}) \otimes \mathbb{C}^2$ with the usual positive definite inner product $\langle \psi, \phi \rangle = \int d^2x \, (\psi_1^* \phi_1 + \psi_2^* \phi_2)$ with $\psi = (\psi_1, \psi_2)$, $\phi = (\phi_1, \phi_2)$.
- $\mathcal{A}$ is the space of Schwartz functions $S(\mathbb{R}^{1,1})$ with the Moyal $\star$ product

$$(f \star g)(x) := \frac{1}{\pi^2} \int_{\mathbb{R}^4} d^2s \; d^2t \; f(x+s) \, g(x+t) \, e^{-2i\sigma(s,t)}, \quad f, g \in S(\mathbb{R}^2).$$

where $\sigma(\cdot, \cdot)$ denotes the standard symplectic form. $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ is defined by the left multiplication:

$$\pi(f) = L(f) \otimes 1, \qquad \pi(f)\psi = (L(f)\psi_1, L(f)\psi_2) = (f \star \psi_1, f \star \psi_2),$$

is faithful and $\overline{\mathcal{A}} = \mathbb{K}(\mathcal{H}_0)$. We will identify states on $\mathcal{A}$ and $\pi(\mathcal{A})$. Any pure state $\omega \in S(\mathcal{A})$ is a vector state: there is a vector $\psi \in \mathcal{H}_0$ such that $\omega(f) =< \psi, \pi(f)\psi >$ for all $f \in \mathcal{A}$.

- $D := -i\partial_\mu \otimes \gamma^\mu$ (with $\mu = 0, 1$) is the flat Dirac operator on $\mathbb{R}^{1,1}$ where:

$$\gamma^0 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

- $J := i\gamma^0$ and $\gamma = -\gamma_0\gamma_1 = \mathrm{diag}(1, -1)$.

Consider light-cone coordinates and derivatives

$$x_+ := \frac{x_0 + x_1}{\sqrt{2}}, \quad x_- := \frac{x_0 - x_1}{\sqrt{2}}, \qquad \partial_+ := \frac{\partial_0 + \partial_1}{\sqrt{2}}, \quad \partial_- := \frac{\partial_0 - \partial_1}{\sqrt{2}}.$$

The lorentzian inner product then looks $x \cdot y = -x_+ y_- - x_- y_+$ and

$$D = \sqrt{2}\begin{pmatrix} 0 & \partial_+ \\ \partial_- & 0 \end{pmatrix}, \quad [D, \pi(f)]\,\psi = \sqrt{2}\begin{pmatrix} \partial_+ f \star \psi_2 \\ \partial_- f \star \psi_1 \end{pmatrix}.$$

The operator $J[D, \pi(a)]$ of the causal constraint is

$$J[D, \pi(f)] = -\sqrt{2}\begin{pmatrix} L(\partial_- f) & 0 \\ 0 & L(\partial_+ f) \end{pmatrix},$$

and $\mathcal{T}_D^J$ is the set of all $f \in C^1(M)$ such that

$$\forall \psi_1 \in L^2(\mathbb{R}^{1,1}), \int d^2x \; \psi_1^*((\partial_- f) \star \psi_1) = \int d^2x \; \psi_1^* \star (\partial_- f) \star \psi_1 \geqslant 0,$$

and

$$\forall \psi_2 \in L^2(\mathbb{R}^{1,1}), \int d^2x \; \psi_2^*((\partial_+ f) \star \psi_2) = \int d^2x \; \psi_2^* \star (\partial_+ f) \star \psi_2 \geqslant 0.$$

Coordinate Operators and Translations

Set $z = \frac{x_0 + ix_1}{\sqrt{2}}, \bar{z} = \frac{x_0 - ix_1}{\sqrt{2}}$ and consider Wigner's transition eigenfunctions

$$h_{mn} := \frac{1}{\sqrt{m!\,n!}} \bar{z}^{\star m} \star h_{00} \star z^{\star n}, \quad m, n \in \mathbb{N},\ h_{00} = \sqrt{\frac{2}{\pi}} e^{-(x_0^2 + x_1^2)}.$$

They form an orthonormal basis of $L^2(\mathbb{R}^{1,1})$ and their linear span $\mathcal{L}$ of the h_{mn}'s constitutes an invariant dense domain of analytic vectors for the symmetric operators $L(x_+), L(x_-)$ (or $L(x_0), L(x_1)$) which are then essentially self-adjoint on $S(\mathbb{R}^{1,1})$. Since $UM(\mathbb{K}) = B(\mathcal{H})$, their closure is trivially affiliated to $\mathcal{A}$. One obtains a representation of the Heisenberg algebra:

$$[L(x_0), L(x_1)] = iI, \qquad [L(x_-), L(x_+)] = iI.$$

and the useful relations

$$x_+ \star f = x_+ f - \frac{i\partial_-}{2} f, \qquad x_- \star f = x_- f - \frac{i\partial_+}{2} f,$$
$$f \star x_+ = f x_+ + \frac{i\partial_-}{2} f, \qquad f \star x_- = f x_- + \frac{i\partial_+}{2} f,$$

Translations $(\alpha_\kappa f)(x) := f(x+\kappa)$ with $f \in S(\mathbb{R}^{1,1})$ and $\kappa \in \mathbb{R}^{1,1}$ define a $*$-automorphism of the algebra $\mathcal{A}$ implemented by

$$L(\alpha_\kappa f) = \mathrm{Ad}\, U_\kappa\, L(f), \qquad U_\kappa(x) := L(e^{i(-\kappa_1 x_0 + \kappa_0 x_1)}) = L(e^{i(\kappa_- x_+ - \kappa_+ x_-)}),$$

Moreover, one has

$$\frac{d}{dt} L\left(\alpha_{t\kappa}(f)\right)_{|_0} = L\left(\frac{d}{dt} f(x+t\kappa)_{|_0}\right) = L(\kappa_- \partial_- f + \kappa_+ \partial_+ f)$$

$$L\left(\alpha_\kappa(x_\pm)\right) = L\left(x_\pm + \kappa_\pm\right), \qquad \frac{d}{dt} L(\alpha_{t\kappa}(x_\pm))_{|_t} = L(\frac{d}{dt}\alpha_{t\kappa}(x_\pm)_{|_t}) = \kappa_\pm I.$$

as operators on $S(\mathbb{R}^{1,1})$. From this one gets

$$\pm L(\partial_\pm f) = i[L(x_\pm), L(f)], \qquad \forall f \in \mathcal{A}.$$

- For $\kappa \in \mathbb{R}^{1,1}$, the κ-translated of a state ω is $\omega_\kappa \ := \ \omega \circ \alpha_\kappa$;
- We say a state ω is smooth whenever $|\omega(x_+^m x_-^n)|, |\omega(x_-^m x_+^n)| < +\infty$ for any $m, n \in \mathbb{N}$.

Any smooth state can be decomposed into a convex combination of pure states which will again be smooth. Moreover, pure smooth states are given by $\psi = (\psi_1, \psi_2) \in \mathcal{H}_0$ such that $\psi_1, \psi_2 \in S(\mathbb{R}^{1,1})$.

Causal Relations between Translated States

Proposition Suppose ω is a any smooth state and let ω_κ be its translated by $\kappa \in \mathbb{R}^{1,1}$. Then these states are casually related with $\omega \preccurlyeq \omega_k$ if and only if $\kappa \in V_+ = \{\kappa_+, \kappa_- \geqslant 0\}$, the closed forward light-cone.

Sketch of Proof (for pure states, easily generalised) We start by showing that under the stated assumptions for each $f \in D(\omega_\kappa, \omega)$ we have $\omega_\kappa(f) - \omega(f) \geqslant 0$. Suppose first that ω is pure. From the Fundamental Theorem of Calculus we get

$$\omega_\kappa(f) - \omega(f) = \int_0^1 dt\, (k_+ \omega_{t\kappa}(\partial_+ f) + k_- \omega_{t\kappa}(\partial_- f)),$$

and the result follows immediately from the characterisation of the convex cone $\mathcal{T}_D^J$ and the fact that all pure states are vector states.
Conversely, for $\kappa \notin V_+$ at least one of κ_+, κ_- is stricly negative, say $\kappa_+ < 0$. Observe that $f = x_+ \in D(\omega_\kappa, \omega)$ and

$$\omega_\kappa(f) - \omega(f) = \frac{1}{2} \int_0^1 dt\, k_+\, \omega_{t\kappa}(\partial_+ f_+) = \kappa_+.$$

Distance between Translated States

Proposition Suppose ω and ω_κ are as above. Then $d(\omega, \omega_k) = \sqrt{2\kappa_+\kappa_-}$.

Sketch of Proof (for pure states, easily generalised).

$$J([D, \pi(f)] + i\gamma) = -\begin{pmatrix} \sqrt{2}\, L(\partial_- f) & i \\ -i & \sqrt{2}\, L(\partial_+ f) \end{pmatrix},$$

and from the condition that the bilinear form it defines is negative definite we infer that the hermitian bilinear form on $\mathbb{C}^2$ defined by

$$\begin{pmatrix} \sqrt{2}\,(\psi_1, (\partial_- f) \star \psi_1) & -i(\psi_1, \psi_2) \\ i(\psi_2, \psi_1) & \sqrt{2}\,(\psi_2, (\partial_+ f) \star \psi_2) \end{pmatrix},$$

is positive definite. This is equivalent to

$$(\psi_1, (\partial_- f) \star \psi_1) \geqslant 0, \quad 2(\psi_1, (\partial_- f) \star \psi_1)(\psi_2, (\partial_+ f) \star \psi_2) - |(\psi_1, \psi_2)|^2 \geqslant 0,$$

from which we easily deduce that

$$(\psi, (\partial_+ f) \star \psi) \geqslant 0, \quad (\psi, (\partial_+ f) \star \psi_i)(\psi, (\partial_- f) \star \psi) \geqslant \frac{1}{2}.$$

are also valid for any $\psi \in L^2(\mathbb{R}^{1,1})$ with unit norm.

Moreover, with $\psi_{t\kappa} = U_{t\kappa} \star \psi \in L^2(\mathbb{R}^{1,1})$ and $\omega_{t\kappa} = \omega \circ \alpha_{t\kappa} = (\psi_{t\kappa}, \cdot\, \psi_{t\kappa})$ with $k \in V_+, t \in [0,1]$ and $||\psi_{t\kappa}|| = 1$, we have

$$[\omega_\kappa(f) - \omega(f)]^+ = \omega_\kappa(f) - \omega(f) = \int_0^1 \, dt \, (\psi^*_{t\kappa}, (\kappa_+\partial_+ + \kappa_-\partial_-)f \star \psi_{t\kappa}) \, .$$

However, it follows from the inequalities above that for each $t \in [0,1]$ the vector $(\omega_{t\kappa}(\partial_+ f), \omega_{t\kappa}(\partial_- f))$ is timelike. By assumption so is κ, thus we can use the reverse Schwartz inequality to obtain

$$\omega_\kappa(f) - \omega(f) \geqslant \sqrt{2\kappa_+\kappa_-} \int_0^1 dt \sqrt{2\, \omega_{t\kappa}(\partial_+ f)\, \omega_{t\kappa}(\partial_- f)} \geqslant \sqrt{2\kappa_+\kappa_-},$$

and thus deduce the fundamental inequality

$$d(\omega, \omega_\kappa) \geqslant \sqrt{2\kappa_+\kappa_-}.$$

Finally, we see that the inf is attained if we choose

$$f = \frac{\kappa_-}{\sqrt{2\kappa_+\kappa_-}}\, x_+ + \frac{\kappa_+}{\sqrt{2\kappa_+\kappa_-}}\, x_- \in \widetilde{D}(\omega, \omega_\kappa),$$

which obviously satisfies the required condition.

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