TWISTING REALITY AND FODO

Andrzej Sitarz

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Plan

1. Real spectral triples and NCG.
2. The Geometries with softened reality.
3. Geometries with no (apparent) reality
4. Twisted reality
5. Fluctuations
6. Conformally transformed Dirac operators
7. Twisting and twisting
8. Conclusions & Future

GEOMETRY IS MORE THAN TOPOLOGY

Classical differential geometry:

- an orientable manifold M, smooth functions, $C^\infty(M)$,
- differential algebra $\Omega(M)$, metric $g^{\mu\nu}$, Laplace operator Δ,
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Problems are to calculate:

- the eigenvalues of the Dirac operator
- the invariants of the manifolds/structures

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THE SPECTRAL TRIPLE

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5. ...+ conditions of „analysis” type

THEOREM [CONNES]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

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A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36, 6194, (1995)

Commutative geometries

which satisfy Connes' axioms are in 1:1 correspondence with Riemannian spin manifolds with a given spin structure and metric.

A. Connes, *On the spectral characterization of manifolds*, J. Noncom. Geom. 7, 1–82 (2013)

Remark

Classical (real) spectral triples are *slightly* richer than spin geometries – as they describe (for example) geometries with torsion.

Genuine noncommutative real spectral triples

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How to construct them?

There is so far no general method. Only examples.

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Theorem (Paschke & Sitarz, LMP, 77, 3,(2006))

There are four inequivalent equivariant spin structures on the 2-dimensional noncommutative torus, with a unique choice of equivariant Dirac operator for each spin structure:

$$d^{+}_{\mu,\nu} = \tau_\mu \mu + \tau_\nu \nu,$$

which satisfies the Hochschild cycle condition, provided that $\tau_\mu \tau_\nu^* \neq \tau_\mu^* \tau_\nu$*. The spectrum of the equivariant Dirac operator depends on the spin structure.*

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More results followed (J-J. Venselaar).

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Problem: Why only flat or round geometries ?

Are there any interesting NC Geometries ?

A softer version of *geometry*?

The facts:

1. for the examples of q-deformed algebras (Podleś spheres, $SU_q(2)$) - there are no spectral geometries in the exact sense – but – there are geometries in which some of the commutation relations are satisfied up to compact operators:

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Remark: Leads to nontrivial classical "triples".

Recent examples of new NC geometries

Geometries from NC circle bundles

Take M a compact Riemannian spin manifold, on which S^1 acts freely and isometrically. Assume that the lenght of fibre is constant. **Aim:** express the Dirac operator on the total space using the Dirac on the base space and the $U(1)$ connection ω. Amman & Bär (1998); LD+AS (Comm.Math.Phys, 318, 1, 111-130 (2013))

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Conformal deformations of NC tori and toric manifold

A family of conformally rescaled Dirac operators on the noncommutative 2-torus for which the Gauss-Bonnet formula holds:

$$D_h = hDh, \qquad h^2 D^2 h^2,$$

where $h \in JC^\infty(\mathbb{T}^2_\Theta)J$, so it is in the commutant, $h > 0$, was introduced by Connes and Tretkoff, by M.Khalkhali et al, LD,AS. All good properties (Hochschild cocycle etc) hold.

Recent examples of new NC geometries

Partial conformal deformations

If you take a torus with the metric $dx^2 + k^{-2}(x,y)dy^2$ (that is, for instance the usual „round" torus embedded in $\mathbb{R}^3$) the Dirac operator is:

$$D = -i\sigma^1\partial_x - i\sigma^2\left(k\,\partial_y + \tfrac{1}{2}\partial_y(k)\right),$$

Same is possible with NC torus and the Gauss-Bonnet holds (LD+AS, Asymmetric noncommutative torus, SIGMA 11 (2015) 075-086).

These are examples of new spectral geometries that do not satisfy (or at least not in the obvious sense) the axioms of first-order condition.

NEW: REALITY TWISTED BY AN AUTOMORPHISM

Let A be a complex $*$-algebra and let (H, π) be a (left) representation of A on a complex vector space H. A linear automorphism ν of H defines an algebra automorphism

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is an algebra map too, and hence it defines a new representation (H, π^{ν}) of A. The map ν is an isomorphism that intertwines (H, π) with (H, π^{ν}).
We could also require that $\pi^{\nu}(a) \in \pi(A)$ so for faithful π the map $\bar{\nu}$ defines an (algebra) automorphism of A

Twisted reality

Definition (Twisted Real Spectral Triple)

Let A be a $*$-algebra, (H, π) a representation of A, D a linear operator on H, and let ν be a linear automorphism of H. We say that the triple (A, H, D) admits a ν-*twisted real structure* if there exists an anti-linear map $J : H \to H$ such that $J^2 = \epsilon\, \mathrm{id}$, and, for all $a, b \in A$,

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TWISTED REALITY

If (A, H, D) admits a grading operator $\gamma : H \to H$:

$$\gamma^2 = \mathrm{id}, \quad [\gamma, \pi(a)] = 0, \quad \gamma D = -D\gamma, \quad \nu^2\gamma = \gamma\nu^2,$$

then the twisted real structure J is also required to satisfy

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In case of H being a Hilbert space the automorphism ν is also assumed to be densely defined and selfadjoint, with the requirement that $\bar{\nu}$ maps $\pi(A)$ into bounded operators.

The signs $\epsilon, \epsilon', \epsilon''$ determine the KO-dimension modulo 8 in the usual way and the operator J is antiunitary.

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Remark

This is an extension not a replacement. In the case of $\nu = \mathrm{id}$ we get the usual, well known, spectral triples.

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Let Ω_D^1 be a bimodule of one forms:

$$\Omega_D^1 := \{\sum_i \pi(a_i)[D, \pi(b_i)] \mid a_i, b_i \in A\}.$$

The standard fluctuaction (= gauge transform) of a spectral triple (A, H, D) consist of

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For our case of ν-twisted real spectral triple we set the fluctuated Dirac operator D_α to be:

$$D_\alpha := D + \alpha + \epsilon' \nu J\alpha J^{-1}\nu,$$

with the requirement that $\alpha + \epsilon' \nu J\alpha J^{-1}\nu$ is selfadjoint.

Fluctuations

Proposition

If (A, H, D) with $J \in \mathrm{End}(H)$ is a ν-twisted real spectral triple, then (A, H, D_α) with (the same) J is also a ν-twisted real spectral triple.

If (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.

The composition of twisted fluctuations is a twisted fluctuation.

Fluctuations

Proposition

If (A, H, D) with $J \in \mathrm{End}(H)$ is a ν-twisted real spectral triple, then (A, H, D_α) with (the same) J is also a ν-twisted real spectral triple.
If (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.
The composition of twisted fluctuations is a twisted fluctuation.

Proof

As a perturbation of D by a bounded selfadjoint operator, the fluctuated Dirac operator D_α is selfadjoint, has bounded commutators with $\pi(a) \in A$ and has compact resolvent.
We show that a fluctuation of the fluctuated Dirac operator is also a fluctuation. In other words, that

$$\Omega^1_{D_\alpha} = \Omega^1_D, \qquad \alpha \in \Omega^1_D.$$

Proof (continued)

We compute:

$$
\begin{aligned}
[\nu J\alpha J^{-1}\nu, \pi(a)] &= \nu J\alpha J^{-1}\nu\pi(a) - \pi(a)\nu J\alpha J^{-1}\nu \\
&= \nu J\alpha J^{-1}\nu\pi(a) - \nu\pi(\hat{\nu}^{-1}(a))J\alpha J^{-1}\nu \\
&= \nu J\alpha J^{-1}\nu\pi(a) - \nu J\alpha J^{-1}\pi(\hat{\nu}(a))\nu \\
&= \nu J\alpha J^{-1}\nu\pi(a) - \nu J\alpha J^{-1}\nu\pi((a))\nu = 0.
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Therefore for any $\alpha \in \Omega^1_D$ and $a \in A$ we have:

$$
[D_\alpha, \pi(a)] = [D, \pi(a)] + [\alpha, \pi(a)] \in \Omega^1_D.
$$

Proof (continued)

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$$\begin{aligned}
(\alpha + \epsilon' \nu J \alpha J^{-1} \nu) J\nu &= (\alpha J \nu + \epsilon' \nu J \alpha J^{-1} \nu J \nu \\
&= \alpha J \nu + \epsilon' \nu J \alpha \\
&= \epsilon' \nu J \left(\alpha + \epsilon' J^{-1} \nu^{-1} \alpha J \nu \right) \\
&= \epsilon' \nu J \left(\alpha + \epsilon' \nu J \alpha J^{-1} \nu \right). \qquad \square
\end{aligned}$$

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Since D itself satisfies it, just check that

$$\begin{aligned}(\alpha + \epsilon'\nu J\alpha J^{-1}\nu)J\nu &= (\alpha J\nu + \epsilon'\nu J\alpha J^{-1}\nu J\nu \\ &= \alpha J\nu + \epsilon'\nu J\alpha \\ &= \epsilon'\nu J\left(\alpha + \epsilon' J^{-1}\nu^{-1}\alpha J\nu\right) \\ &= \epsilon'\nu J\left(\alpha + \epsilon'\nu J\alpha J^{-1}\nu\right). \qquad \square\end{aligned}$$

Thus like in the usual case of the real spectral triples the twisted fluctuations form a semigroup.

EXAMPLE: CONFORMAL PERTURBATIONS

Let us assume that we have a real spectral triple (A, H, D, J) with reality operator J and fixed signs ϵ, ϵ'. Let $k \in \pi(A)$ be a positive and invertible bounded operator such that k^{-1} is also bounded, and let us denote by $k^J := Ad_J(k) = JkJ^{-1}$.

Example: conformal perturbations

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Proposition

If (A, H, D) with J is a real spectral triple, which satisfies order one condition, then for:

$$D_k = k^J D k^J, \qquad \nu(h) = k^{-1} k^J h,$$

the triple (A, H, D_k) with J is a ν-twisted real spectral triple. If furthermore (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.

Example: conformal perturbations

Proof

Since k and k^J are bounded operators it is clear that $\bar{\nu}$ sends bounded operators to bounded operators, and $\forall a \in A$:

$$\boxed{\bar{\nu}(\pi(a)) = k^{-1}\pi(a)k.}$$

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We show now that D_k satisfies the twisted order-one condition :

$$J\pi(b)J^{-1}[D_k, \pi(a)] = J\pi(b)J^{-1}JkJ^{-1}[D, \pi(a)]JkJ^{-1}$$
$$= k^J[D, \pi(a)]k^J J(k^{-2}\pi(b)k^2)J^{-1} = [D_k, \pi(a)]\, J\bar{\nu}^2(\pi(b))J^{-1}.$$

EXAMPLE: CONFORMAL PERTURBATIONS

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Next we check compatibility between J and ν:

$$\nu J \nu = k^{-1}JkJ^{-1}Jk^{-1}JkJ^{-1} = J.$$

Example 1: conformal perturbations

Proof (ctd)

Finally, if $JD = \epsilon' DJ$ then for D_k we have:

$$JD_k = Jk^J J^{-1} JDk^J = \epsilon' kDJk^J = \epsilon' k(k^J)^{-1} D_k (k^J)^{-1} kJ,$$

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REMARK

In the 'classical' case of a manifold M and (commutative) $A = C^\infty(M)$ with Ad_J being the complex conjugation, the conformal twists are always trivial as $JkJ^{-1} = k$ for a positive k and hence $\nu = \mathrm{id}$.

The (ν, ρ) twisting

Definition ((ν, ρ)-twisted ST)

We say that (A, H, D, J) is a (ν, ρ)-type twisted real spectral triple if:

(1) for all $a \in A$, the commutators $[D, a]_\rho$ are bounded,

(2) νJ preserves the domain of D,

(3) $DJ\nu = \epsilon' \nu JD$ and $\nu J \nu = J$ and $\nu^2 \gamma = \gamma \nu^2$,

(4) the (ν, ρ)-twisted first-order condition holds:

$$[[D, a]_\rho, b]_{\rho \circ \nu^{-2}} = 0$$

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Examples

(1) The (ν, id)-type spectral triple (untwisted) with twisted reality of [Brzezinski, Ciccola, Dabrowski,Sitarz]

(2) The $(1, \rho)$-type twisted real spectral triple of [Landi, Martinetti].

Twisting and untwisting

Theorem

Let A be a $*$-algebra and $\tilde{\pi} : A \to B(H)$ be a $*$-representation of A on a Hilbert space H. Let $J : H \to H$ be a $\mathbb{C}$-antilinear isometry such that $J^2 = \epsilon$ and that the zero order condition is satisfied. Let ρ be an algebra automorphism, and let ν be a bounded operator on H with the bounded inverse such that

(a) ν implements an algebra automorphism $\hat{\nu}$ of A in representation $\tilde{\pi}$ and $\rho = \hat{\nu}^{-2}$, or

(b) ν is a unitary operator such that ν^{-2} implements ρ in representation $\tilde{\pi}$

Let

$$\pi_\nu : A \to B(H), \qquad a \mapsto \nu^{-1}\tilde{\pi}(a)\nu, \tag{1}$$

be the induced representation of A...

Twisting and untwisting

... and set

$$\pi = \begin{cases} \tilde{\pi}, & \textit{in case (a)}, \\ \pi_\nu, & \textit{in case (b)}, \end{cases}$$

so that π is always a $*$-representation. Assume further that

$$\nu J \nu = J.$$

For an operator $\tilde{D}$ on H, set

$$D = \nu \tilde{D} \nu,$$

Then:

(1) (π, D, J, ν^2) satisfy conditions of a spectral triple with a ν^2-twisted real structure if and only if $(\tilde{\pi}, \tilde{D}, J, \rho)$ satisfy conditions of real ρ-twisted spectral triple.

Twisted and untwisted

We can summarise here three different kinds of twisted reality conditions obtained by the conformal twisting of a real spectral triple (A, H, π, D, J) in the following table:

$(A, H, \pi, k'Dk', J)$	$(A, H, \pi, kk'Dkk', J)$	(A, H, π, kDk, J)
spectral triple with the ν-twisted real structure and first-order condition	real ρ-twisted spectral triple	twisted spectral triple with real structure and un-twisted first-order condition
$\nu = k^{-1}k'$	$\rho = \mathrm{Ad}_{u^2}$	$\nu = kk'^{-1}$

Here $k = \pi(u) \in \pi(A)$, where $u \in A$ is invertible and such that k is positive with bounded inverse, $k' = JkJ^{-1}$ and we have $\nu JD = \epsilon' JD\nu$, and $\nu J\nu = J$ in the first and the third cases.

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THANK YOU !

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