Two Variable Trace Formula

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Outline

Notations

- $\mathcal{H} \equiv$ Separable Hilbert space
- $\mathcal{B}(\mathcal{H}) \equiv$ Set of bounded operators
- $\mathcal{B}_1(\mathcal{H}) \equiv$ Set of trace class operators
- $\mathcal{B}_2(\mathcal{H}) \equiv$ Set of Hilbert-Schmidt class operators
- $\mathcal{B}_p(\mathcal{H}) \equiv$ Schatten-p class operators
- $\|\cdot\|_p \equiv$ Schatten-p norm
- $\mathcal{P}([a,b]) \equiv$ Set of polynomials with complex coefficients on $[a,b]$
- $C([a,b]) \equiv$ Set of continuous functions on $[a,b]$
- $\sigma(H) \equiv$ Spectrum of the operator H

Introduction

- Let H and H_0 be two possibly unbounded self-adjoint operators in a separable Hilbert space $\mathcal{H}$ such that $V = H - H_0 \in \mathcal{B}_1(\mathcal{H})$. Then Krein proved that there exists a unique real-valued $L^1(\mathbb{R})$- function ξ with support in the interval $[a, b]$ such that

$$\operatorname{Tr}\left[\phi\left(H_0 + V\right) - \phi\left(H_0\right)\right] = \int_a^b \phi'(\lambda)\, \xi(\lambda)\, d\lambda, \tag{1}$$

  for a large class of functions ϕ (where $a = min\{\inf \sigma(H), \inf \sigma(H_0)\}$ and $b = max\{\sup \sigma(H), \sup \sigma(H_0)\}$).

- The function ξ is known as Krein's spectral shift function and the relation (1) is called Krein's trace formula.

- The original proof of Krein [4] uses analytic function theory.

- Later, Voiculescu approached the trace formula (1) from a different direction.

Introduction

- If H and H_0 are bounded, then Voiculescu [6] proved that

$$\mathrm{Tr}\left[p\left(H\right) - p\left(H_0\right)\right] = \lim_{n \longrightarrow \infty} \mathrm{Tr}\left[p\left(H_n\right) - p\left(H_{0,n}\right)\right], \tag{2}$$

  where p is a polynomial and H_n, $H_{0,n}$ are finite-dimensional approximation of H and H_0 respectively (constructed by adapting Weyl-von Neumann theorem).

- Then one constructs the spectral shift function in the finite dimensional case and finally the formula is extended to the infinite dimensional case.

- Later Sinha and Mohapatra [5] used a similar method to get the same result for the unbounded self-adjoint case.

- More recently, Potapov, Skripka and Sukochev [2] has proven the trace-formula for all orders, obtaining a kind of Taylor's theorem under trace.

Introduction

- It is natural to ask similar questions for a pair of commuting self-adjoint n-tuples, particularly an appropriate adaptation of Krein's formula (1) to two and higher dimensions.
- Here our aim is to formulate a relevant question for a pair of commuting bounded self-adjoint tuples and use finite-dimensional approximation to get a trace formula.
- Before going to the main result in this talk first we start with an approximation result which we have adapted from the proof of Weyl-von Neumann-Berg Theorem.

Approximation Results

- A result due to Weyl and von Neumann [3] proves that for a self-adjoint operator A that given $\epsilon > 0$, $\exists K \in \mathcal{B}_2(\mathcal{H})$ such that $\|K\|_2 < \epsilon$ and $A + K$ has pure point spectrum.

- Later Berg extended this to an n-tuples of bounded commuting self-adjoint operators $(A_1, A_2, \ldots, A_n)$, which says that given $\epsilon > 0$, $\exists \, \{K_j\}_{j=1}^n$ of compact operators such that $\|K_j\| < \epsilon \; \forall j$ and $\{A_j - K_j\}_{j=1}^n$ is a commuting family of bounded self-adjoint operators with pure point spectra.

- Here we extend in the next theorem the ideas of the proof of Berg's result as given in [1].

- It is worth mentioning that Voiculescu [7] had earlier obtained related (though not the same) results.

Approximation Results

Theorem 1

Let $\{A_i\}_{1\leq i\leq n}$ be a commuting family of bounded self-adjoint operators in an infinite-dimensional separable Hilbert space $\mathcal{H}$. Then there exists a sequence $\{P_N\}$ of finite-rank projections such that $\{P_N\}\uparrow I$ as $N\longrightarrow\infty$ and such that there exists a commuting family of bounded self-adjoint operators $\{B_i^{(N)}\}_{1\leq i\leq n}$ with the properties that for $p\geq n$ and for each i $(1\leq i\leq n)$, as $N\longrightarrow\infty$,

(i) $P_N B_i^{(N)} P_N = B_i^{(N)} P_N,$ (ii) $\left\| A_i - B_i^{(N)} \right\|_p \longrightarrow 0,$

(iii) $\|[A_i, P_N]\|_p \longrightarrow 0,$

(iv) $\left\| P_N A_i P_N - B_i^{(N)} P_N \right\|_p \longrightarrow 0$ and (v) $\{B_i^{(N)} P_N\}\uparrow A_i.$

Sketch of the Proof of Theorem 1:

- Without loss of generality we assume that $0 \leq A_i \leq I$ for all $1 \leq i \leq n$, and we start with the representation for each i,

$$A_i = \sum_{k=1}^{\infty} 2^{-k} E_k^{(i)},$$

where $E_k^{(i)} = E_{A_i}\left(\bigcup_{j=1}^{2^{k-1}} (2^{-k}(2j-1), 2^{-k}(2j)]\right)$ with E_{A_i} the spectral measure associated to the bounded self-adjoint operator A_i.

- Next set for $N \in \mathbb{N}$ (the set of natural numbers),

$$\mathcal{L}_N \equiv \text{span}\{\left[\prod_{k=1}^{N} \prod_{i=1}^{n} \left(E_k^{(i)}\right)^{\epsilon}\right] f_j \mid \ 1 \leq j \leq N; \ \epsilon = \pm 1\},$$

where $\{f_1, f_2, \ldots.., f_N, \ldots...\}$ be a countable orthonormal basis of $\mathcal{H}$ and $\left(E_k^{(i)}\right)^1 = E_k^{(i)}$ and $\left(E_k^{(i)}\right)^{-1} = I - E_k^{(i)}$.

- Then $\mathcal{L}_N$ is a finite dimensional subspace of $\mathcal{H}$ and it has the following properties:

  (a) $\mathcal{L}_N \subseteq \mathcal{L}_{N+1}$, (b) $\overline{\left(\bigcup_{N=1}^{\infty} \mathcal{L}_N \right)} = \mathcal{H}$,

  (c) $\dim\left(\mathcal{L}_N\right) \leq N\left(2^n - 1\right)^N + N$.

- Set P_N to be the finite rank projection associated with the finite dimensional subspace $\mathcal{L}_N$ and observe that $\{P_N\}$ increases to I.

- Next define

$$B_i^{(N)} = \sum_{k=1}^{N} 2^{-k} E_k^{(i)} + \sum_{k=N+1}^{\infty} 2^{-k} E_k^{(i)} (I - P_k).$$

- Then $\{B_i^{(N)}\}_{1 \leq i \leq n}$ is a commuting family of bounded self-adjoint operators.

- Furthermore, $A_i - B_i^{(N)} = \sum_{k=N+1}^{\infty} 2^{-k} E_k^{(i)} P_k$ and

$$\left\| A_i - B_i^{(N)} \right\|_n \leq \sum_{k=N+1}^{\infty} 2^{-k} \|P_k\|_n \leq \sum_{k=N+1}^{\infty} 2^{-k} \left[k\{1 + (2^n - 1)^k\} \right]^{\frac{1}{n}}$$
$$= \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} \left[2^{-nk} + \left(1 - 2^{-n}\right)^k \right]^{\frac{1}{n}}$$
$$\leq \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} 2^{-k} + \sum_{k=N+1}^{\infty} k^{\frac{1}{n}} \left[\left(1 - 2^{-n}\right)^{\frac{1}{n}} \right]^k,$$

where we have used that for $a, b > 0$, $(a + b)^{\frac{1}{n}} \leq (a^{\frac{1}{n}} + b^{\frac{1}{n}})$.

- Since for fixed n, $(1 - 2^{-n})^{\frac{1}{n}} < 1$, and since $\sum_{k=1}^{\infty} k^{\frac{1}{n}} \alpha^k < \infty$ for $\alpha < 1$, it follows from the above that for each i $(1 \leq i \leq n)$ and any $p \geq n$,

$$\left\| A_i - B_i^{(N)} \right\|_p \leq 2^{\left(1 - \frac{n}{p}\right)} \left\| A_i - B_i^{(N)} \right\|_n^{\frac{n}{p}} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty. \square$$

Remark

The choice that $0 \leq A_i \leq I$ does not materially affect the calculations of the above theorem. For if $C_i \in \mathcal{B}(\mathcal{H})$ $(1 \leq i \leq n)$, then we can set

$$A_i = (2\|C_i\|)^{-1} C_i + \frac{1}{2} I$$

so that $0 \leq A_i \leq I$ and thus $C_i = 2\|C_i\|(\sum 2^{-k} E_k^{(i)} - \frac{1}{2} I)$. Thus choosing

$$B_i^{(N)} = 2\|C_i\|\{\sum_{k=1}^{N} 2^{-k} E_k^{(i)} + \sum_{k=N+1}^{\infty} (I - P_k) E_k^{(i)} - \frac{1}{2} I\}$$

one has $\|[C_i, B_i^{(N)}]\|_p = 2\|C_i\| \|[A_i, B_i^{(N)}]\|_p \to 0$ as $N \to \infty$ for $p \geq n$.

Trace Formula in Finite Dimension

Theorem 2

Let P and Q be two finite-dimensional projections in a (infinite dimensional separable) Hilbert space $\mathcal{H}$. Assume furthermore the two commuting pairs of bounded self-adjoint operator tuples $\left(H_1^0, H_2^0\right)$ and (H_1, H_2) are acting in the common reducing subspaces $P\mathcal{H}$ and $Q\mathcal{H}$ respectively. Also let $\sigma\left(H_1\right), \sigma\left(H_2\right), \sigma\left(H_1^0\right), \sigma\left(H_2^0\right)$ be in $[a, b]$ and let ϕ, ψ be in $C^1([a, b])$. Then

$$\begin{aligned} &\mathrm{Tr}\Big\{ Q\Big(\phi(H_1) - \phi(H_1^0)\Big) P\Big(\psi(H_2) - \psi(H_2^0)\Big) Q\Big\} \\ &= \int_{[a,b]^2} \phi'(x)\ \psi'(y)\ \xi(x, y)\ dxdy, \end{aligned} \tag{3}$$

where $\xi(x, y) = \mathrm{Tr}\Big\{ Q\Big[E_{H_1}(x) - E_{H_1^0}(x)\Big] P\Big[E_{H_2}(y) - E_{H_2^0}(y)\Big] Q\Big\}$ and $E_{H_1}(\cdot)$, $E_{H_2}(\cdot)$, $E_{H_1^0}(\cdot)$, $E_{H_2^0}(\cdot)$ are the spectral measures of the operators H_1, H_2, H_1^0, H_2^0 respectively.

Sketch of the Proof of Theorem 2:

By the spectral theorem of self-adjoint operators, Fubini's theorem and performing integration by-parts appropriately we can prove the above theorem. $\square$

- In the next theorem we gave an equivalent description of the expression on the left hand side of the equation (3), in terms of divided differences and a $\mathcal{B}_2(\mathcal{H})$-valued spectral measure.

Trace Formula in Finite Dimension

Theorem 3

Under the hypotheses of Theorem 2,

$$\mathrm{Tr}\Big\{Q\Big(\phi(H_1)-\phi(H_1^0)\Big)P\Big(\psi(H_2)-\psi(H_2^0)\Big)Q\Big\}$$

$$=\int_{[a,b]^2}\int_{[a,b]^2}\left\{\frac{\phi(x_1)-\phi(x_2)}{x_1-x_2}\right\}\left\{\frac{\psi(y_1)-\psi(y_2)}{y_1-y_2}\right\}\bullet$$

$$\Big\langle\Big(H_1-H_1^0\Big),\ PE_{\underline{H}^0}(dx_2\times dy_1)\Big(H_2-H_2^0\Big)E_{\underline{H}}(dx_1\times dy_2)Q\Big\rangle_2,$$

where we have written $\underline{H}^0=(H_1^0,H_2^0)$, $\underline{H}=(H_1,H_2)$; $E_{\underline{H}^0}(\cdot)$ and $E_{\underline{H}}(\cdot)$ are the associated spectral measures of the operators tuples $\underline{H}^0$ and $\underline{H}$ respectively on the Borel sets of $[a,b]^2$, and where $\langle\cdot,\cdot\rangle_2$ denotes the inner product of the Hilbert space $\mathcal{B}_2(\mathcal{H})$.

- Note that the above theorem (Theorem 3) can also be extended in an infinite dimensional setting.

Sketch of the Proof of Theorem 3:

In $\mathcal{H}$, using the ideas of double spectral integrals, trace properties, Fubini's theorem and the fact that (H_1^0, H_2^0), (H_1, H_2) are two commuting pairs of self-adjoint operators, we can prove the above theorem. $\square$

Main Theorem

Theorem 4

Let (H_1^0, H_2^0) and (H_1, H_2) be two commuting pairs of bounded self-adjoint operators in a separable Hilbert space $\mathcal{H}$ such that $H_j - H_j^0 \equiv V_j \in \mathcal{B}_2(\mathcal{H})$ and such that $\sigma(H_j)$, $\sigma(H_j^0) \subseteq [a, b]$ for $j = 1, 2$. Then there exists a unique complex Borel measure μ on $[a, b]^2$ such that

$$\mathrm{Tr}\Big\{\Big(\phi(H_1) - \phi(H_1^0)\Big)\Big(\psi(H_2) - \psi(H_2^0)\Big)\Big\} = \int_{[a,b]^2} \phi'(x)\ \psi'(y)\ \mu(dx \times dy),$$

where $\phi, \psi \in \mathcal{P}([a, b])$.

Sketch of the Proof of Theorem 4:

- **(Finite Dimensional Reduction):**

  Applying approximation results (Theorem 1) to the pairs (H_1^0, H_2^0) and (H_1, H_2) we get two commuting pairs of finite dimensional self-adjoint operators $\left(H_1^{0(N)}, H_2^{0(N)}\right)$ and $\left(H_1^{(N)}, H_2^{(N)}\right)$ in $P_N^0\mathcal{H}$ and $P_N\mathcal{H}$ respectively, such that

$$\left\|[H_j^0, P_N^0]\right\|_p, \ \left\|P_N^0 H_j^0 P_N^0 - H_j^{0(N)} P_N^0\right\|_p \longrightarrow 0 \text{ as } N \longrightarrow \infty \text{ for } p \geq 2,\ j = 1, 2,$$

  and

$$\left\|[H_j, P_N]\right\|_p, \ \left\|P_N H_j P_N - H_j^{(N)} P_N\right\|_p \longrightarrow 0 \text{ as } N \longrightarrow \infty \text{ for } p \geq 2,\ j = 1, 2,$$

  where P_N^0, P_N are projections increasing to I (i.e. $P_N^0, P_N \uparrow I$).

- Applying the above results we show that

$$\mathrm{Tr}\Big\{\Big(\phi(H_1)-\phi(H_1^0)\Big)\Big(\psi(H_2)-\psi(H_2^0)\Big)\Big\}$$
$$=\lim_{N\longrightarrow\infty}\mathrm{Tr}\Big\{P_N\Big(\phi(H_1^{(N)})-\phi(H_1^{0(N)})\Big)P_N^0\Big(\psi(H_2^{(N)})-\psi(H_2^{0(N)})\Big)\Big\},\tag{4}$$

for $\phi,\psi\in\mathcal{P}([a,b])$.

- **(Finite Dimensional Trace formula):** Using Theorem 2 we get

$$\begin{aligned}
&\mathrm{Tr}\Big\{P_N\Big(\phi(H_1^{(N)}) - \phi(H_1^{0(N)})\Big)P_N^0\Big(\psi(H_2^{(N)}) - \psi(H_2^{0(N)})\Big)\Big\} \\
&= \int_a^b\int_a^b \phi'(x)\ \psi'(y)\ \xi_N(x,y)\ dxdy \\
&= \int\limits_{[a,b]^2} \phi'(x)\ \psi'(y)\ \mu_N(dx\times dy),
\end{aligned} \tag{5}$$

where

$$\xi_N(x,y) = \mathrm{Tr}\Big\{P_N\left[E_{H_1^{(N)}}(x) - E_{H_1^{0(N)}}(x)\right]P_N^0\left[E_{H_2^{(N)}}(y) - E_{H_2^{0(N)}}(y)\right]P_N\Big\}$$

and μ_N is a complex Borel measure on $[a,b]^2$ such that

$$\mu_N(\Delta) = \int\limits_{\Delta}\xi_N(x,y)dxdy, \qquad \text{for a Borel subset} \quad \Delta\subseteq[a,b]^2.$$

- **(Finite Dimensional Trace formula):** Moreover, using Theorem 3 we have that

$$\mathrm{Tr}\Big\{P_N\Big(\phi(H_1^{(N)}) - \phi(H_1^{0(N)})\Big)P_N^0\Big(\psi(H_2^{(N)}) - \psi(H_2^{0(N)})\Big)\Big\}$$

$$= \int_{[a,b]^2}\int_{[a,b]^2}\left\{\frac{\phi(x_1)-\phi(x_2)}{x_1-x_2}\right\}\left\{\frac{\psi(y_1)-\psi(y_2)}{y_1-y_2}\right\} \bullet$$

$$\Big\langle P_N^0 V_1^{(N)} P_N,\ P_N^0 E_{\underline{H}^{0(N)}}(dx_2\times dy_1) V_2^{(N)} E_{\underline{H}^{(N)}}(dx_1\times dy_2) P_N\Big\rangle_2, \tag{6}$$

where $V_j^{(N)} = H_j^{(N)} - H_j^{0(N)}$ for $j=1,2$ and $E_{\underline{H}^{0(N)}}(dx_2\times dy_1) = E_{H_1^{0(N)}}(dx_2)E_{H_2^{0(N)}}(dy_1)$ and $E_{\underline{H}^{(N)}}(dx_1\times dy_2) = E_{H_1^{(N)}}(dx_1)E_{H_2^{(N)}}(dy_2)$ are $\mathcal{B}_2(\mathcal{H})$-valued spectral measures of the operators tuples $\underline{H}^{0(N)}$ and $\underline{H}^{(N)}$ respectively on the Borel sets of $[a,b]^2$.

- Combining (5) and (6) we get

$$\int_{[a,b]^2} \Psi(x,y)\mu_N(dx \times dy)$$

$$= \int_{[a,b]^2} \int_{[a,b]^2} \left\{ \frac{\phi(x_1) - \phi(x_2)}{x_1 - x_2} \right\} \left\{ \frac{\psi(y_1) - \psi(y_2)}{y_1 - y_2} \right\} \bullet$$

$$\left\langle P_N^0 V_1^{(N)} P_N,\ P_N^0 E_{\underline{H}^{0(N)}}(dx_2 \times dy_1) V_2^{(N)} E_{\underline{H}^{(N)}}(dx_1 \times dy_2) P_N \right\rangle_2$$

$$= \int_{[a,b]^2} \int_{[a,b]^2} \frac{\int_{x_2}^{x_1} \int_{y_2}^{y_1} \Psi(x,y)\,dx\,dy}{(x_1 - x_2)(y_1 - y_2)} \bullet$$

$$\left\langle P_N^0 V_1^{(N)} P_N,\ P_N^0 E_{\underline{H}^{0(N)}}(dx_2 \times dy_1) V_2^{(N)} E_{\underline{H}^{(N)}}(dx_1 \times dy_2) P_N \right\rangle_2,$$

where $\Psi(x,y) = \phi'(x)\psi'(y)$ and $\phi, \psi \in \mathcal{P}([a,b])$.

- For fixed N, we can extend the above equality to an arbitrary polynomial Ψ in two variables by taking suitable linear combination of products of polynomials in one-variables, x and y and then extend the above equality from $\mathcal{P}([a,b]^2)$ to $C([a,b]^2)$ by Stone-Weierstrass theorem.

- Next we show that for all $\Psi \in C([a,b]^2)$,

$$\left| \int_{[a,b]^2} \Psi(x,y)\, \mu_N(dx \times dy) \right| < C\, \|\Psi\|_\infty,$$

for some constant C $(< \infty)$ and hence by applying Helley's theorem we conclude that there exists a subsequence μ_{N_k} of μ_N such that μ_{N_k} converges weakly to a unique complex Borel measure μ on $[a,b]^2$, that is,

$$\lim_{k \longrightarrow \infty} \int_{[a,b]^2} \Psi(x,y)\, \mu_{N_k}(dx \times dy) = \int_{[a,b]^2} \Psi(x,y)\, \mu(dx \times dy)\, \forall\, \Psi \in C([a,b]^2).$$

Regarding Absolute Continuity of the Measure μ:

- Unlike in one variable Krein's trace formula we note that the measure μ which appears in Theorem 5 is not necessarily absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. In fact the measure μ may be singular with respect to the product Lebesgue measure on $\mathbb{R}^2$ as shown in Theorem 6.
- Indeed, if we assume that the measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$, then there exists a function $\xi \in L^1([a,b]^2)$ such that $\mu(dx \times dy) = \xi(x,y)dxdy$.
- Next from the main theorem (Theorem 4) in particular situation where $H_1 = H_2 = H$ and $H_1^0 = H_2^0 = H^0$ we get

$$\begin{aligned} &\mathrm{Tr}\Big\{\Big(\phi(H) - \phi(H^0)\Big)\Big(\psi(H) - \psi(H^0)\Big)\Big\} \\ &= \int_a^b \int_a^b \phi'(x)\ \psi'(y)\ \xi(x,y)\ dxdy, \end{aligned} \tag{7}$$

where $\phi, \psi \in \mathcal{P}([a,b])$.

- On the other hand we will show separately in the next theorem bellow (a counterexample theorem) that

$$\mathrm{Tr}\Big\{\Big(\phi(H)-\phi(H^0)\Big)\Big(\psi(H)-\psi(H^0)\Big)\Big\} = \int_a^b \phi'(t)\,\psi'(t)\,\eta(t)\,dt, \quad (8)$$

for suitable $\phi, \psi \in \mathcal{P}([a,b])$.

- Combining (7) and (8) we conclude that

$$\xi(x,y) = \delta(x-y)\;\eta(y),$$

which contradicts the fact that $\xi \in L^1([a,b]^2)$ and hence the measure μ is not absolutely continuous.

- Before going to discuss counterexample theorem in an infinite dimensional setup let us start with the similar type of theorem in finite dimension.

Theorem 5

Let H and H^0 be two self-adjoint operators in a finite dimensional Hilbert space $\mathcal{H}$ such that $\sigma(H) \cup \sigma(H^0) \subseteq [a, b]$. Then the trace formula

$$\mathrm{Tr}\Big\{\Big(\psi(H) - \psi(H^0)\Big)\Big(\phi(H) - \phi(H^0)\Big)\Big\} = \int_a^b \psi'(t)\ \phi'(t)\ \eta(t)\ dt$$

holds for a suitable class of functions $\phi, \psi : [a, b] \mapsto \mathbb{R}$ in $C^1([a, b])$ and

$$\eta(t) = \mathrm{Tr}\Big\{\Big(H - H^0\Big)\Big(E_{H^0}(t) - E_H(t)\Big)\Big\}.$$

Sketch of the Proof of Theorem 5:

- By spectral theorem and performing integration by-parts we get

$$\mathrm{Tr}\Big\{\Big(H - H^0\Big)\Big(\phi(H) - \phi(H^0)\Big)\Big\} = \int_a^b \phi'(t)\,\eta(t)\,dt,$$

  where $\eta(t) = \mathrm{Tr}\Big\{\Big(H - H^0\Big)\Big(E_{H^0}(t) - E_H(t)\Big)\Big\}$ and ϕ is a real-valued continuously differentiable function on $[a, b]$.

- Next consider the real-valued continuously differentiable function $\psi : [a, b] \to [\psi(a), \psi(b)]$ such that $\psi' \neq 0$. Moreover, ψ is invertible and ψ^{-1} is also continuously differentiable.

- Now let $G^0 = \psi(H^0)$ and $G = \psi(H)$. Then both G and G^0 are bounded self-adjoint operators and therefore by applying the above argument to G, G^0 and interchanging the spectral variable $\lambda = \psi(t)$ of G, G^0 to the spectral variable t of H, H^0 we conclude

$$
\begin{aligned}
&\mathrm{Tr}\Big\{\Big(\psi(H)-\psi(H^0)\Big)\Big(\phi(H)-\phi(H^0)\Big)\Big\}\\
&=\mathrm{Tr}\Big\{\Big(G-G^0\Big)\Big(\phi\circ\psi^{-1}(G)-\phi\circ\psi^{-1}(G^0)\Big)\Big\}\\
&=\int_a^b \phi'(t)\,\tilde{\eta}\big(\psi;t\big)\,dt,
\end{aligned}
\tag{9}
$$

where

$$
\tilde{\eta}\big(\psi;t\big)=\mathrm{Tr}\Big\{\Big(\psi(H)-\psi(H^0)\Big)\Big(E_{H^0}(t)-E_H(t)\Big)\Big\},
$$

and once again by performing integration by-parts and using the spectral theorem we conclude

$$
\int_a^b \tilde{\eta}\big(\psi;t\big)\,dt=\int_a^b \psi'(t)\,\eta(t)\,dt.
$$

- Finally by considering the function

$$\tilde{\psi}(t) = \begin{cases} \psi(t) & \text{for} \quad a \leq t \leq \alpha \\ \psi(\alpha) & \text{for} \quad \alpha \leq t \leq b, \end{cases}$$

where $a \leq \alpha \leq b$ and noting the fact that $\tilde{\eta}\big(\tilde{\psi}; t\big) = \tilde{\eta}\big(\psi; t\big)$ for $a \leq t \leq \alpha$ and $\tilde{\eta}\big(\tilde{\psi}; t\big) = 0$ for $\alpha \leq t \leq b$ we conclude that

$$\int_a^\alpha \psi'(t)\, \eta(t)\, dt = \int_a^\alpha \tilde{\eta}\big(\psi; t\big)\, dt,$$

and thus $\tilde{\eta}\big(\psi; t\big) = \psi'(t)\, \eta(t)$ almost everywhere and therefore the result follows by combining with equation (9).

Counterexample Theorem:

Theorem 6

Let H and H^0 be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H^0 = V \in \mathcal{B}_2(\mathcal{H})$ and

$\sigma(H) \cup \sigma(H^0) \subseteq [a, b]$. Then there exists a function $\eta \in L^1([a, b])$ such that

$$\mathrm{Tr}\Big\{\Big(\phi(H) - \phi(H^0)\Big)\Big(\psi(H) - \psi(H^0)\Big)\Big\} = \int_a^b \phi'(t)\,\psi'(t)\,\eta(t)\,dt, \quad (10)$$

for a suitable class of functions $\phi, \psi : [a, b] \mapsto \mathbb{R}$ in $C^1([a, b])$.

Sketch of the Proof of Theorem 6:

- **(Finite Dimensional Reduction):**
  - Using Weyl-Von Neumann theorem we conclude that there exists a sequence $\{P_N\}$ of finite rank projections such that

$$\left\|(I-P_N)H^0P_N\right\|_2, \|(I-P_N)V\|_2, \|(I-P_N)HP_N\|_2 \longrightarrow 0 \qquad (11)$$

    as $N \longrightarrow \infty$.

  - Using the above results we show that

$$\begin{aligned}&\mathrm{Tr}\Big\{\Big(\phi(H)-\phi(H^0)\Big)\Big(\psi(H)-\psi(H^0)\Big)\Big\}\\ &= \lim_{N\longrightarrow\infty} \mathrm{Tr}\Big\{P_N\Big(\phi(P_NHP_N)-\phi(P_NH^0P_N)\Big)\\ &\qquad\qquad P_N\Big(\psi(P_NHP_N)-\psi(P_NH^0P_N)\Big)P_N\Big\},\end{aligned}$$

    for $\phi,\psi \in \mathcal{P}([a,b])$.

- **(Finite Dimensional Formula):**

  Using Theorem 5 we conclude that for a suitable class of functions $\phi, \psi : [a,b] \mapsto \mathbb{R}$,

$$\mathrm{Tr}\Big\{ P_N \Big(\phi(P_N H P_N) - \phi(P_N H^0 P_N) \Big) P_N \Big(\psi(P_N H P_N) - \psi(P_N H^0 P_N) \Big) P_N \Big\}$$

$$= \int_a^b \phi'(t)\ \psi'(t)\ \eta_N(t)\ dt,$$

  where

$$\eta_N(t) = \mathrm{Tr}\Big\{ P_N \Big(P_N H P_N - P_N H^0 P_N \Big) P_N \Big(E_{P_N H^0 P_N}(t) - E_{P_N H P_N}(t) \Big) \Big\}.$$

- Finally to guarantee the existence of the function $\eta \in L^1([a,b])$ we show that the sequence $\{\eta_N\}$ is a Cauchy sequence in $L^1([a,b])$. Indeed, we have shown that for $f \in L^\infty([a,b])$ and

  $g(t) = \int_a^t f(\lambda)\, d\lambda,$

$$\begin{aligned}
&\left\|\eta_N - \eta_M\right\|_{L^1([a,b])} \\
&= \sup_{0\neq f\in L^\infty([a,b])} \frac{\left|\int_a^b f(t)\, \left[\eta_N(t) - \eta_M(t)\right]\, dt\right|}{\|f\|_\infty} \\
&\leq \|V\|_2 \left\{ \|P_N H(I-P_N)\|_2 + \left\|P_N H^0(I-P_N)\right\|_2 + \|P_M H(I-P_M)\|_2 \right. \\
&\qquad \left. + \left\|P_M H^0(I-P_M)\right\|_2 + \|P_N V P_N - P_M V P_M\|_2 \right\},
\end{aligned}$$

  which converges to zero as $N, M \longrightarrow \infty$ (by (11)) and therefore $\{\eta_N\}$ is a Cauchy sequence in $L^1([a,b])$.

Remark

In the counterexample theorem (Theorem 6) we can extend the theorem for any $\phi, \psi \in C^1([a,b])$. *The only restriction of the function* ψ *in Theorem 6 is the following:* $\psi' \neq 0$ *and* $\psi^{-1} \in C^1([a,b])$. *Now if we assume that* $\psi' = 0$ *for some subset of* $[a,b]$, *then since* ψ *is continuously differentiable* $\psi' = 0$ *on some interval* $\Delta \subseteq [a,b]$ *and hence* ψ *is constant on* Δ. *Let* $\Delta^c = \bigcup_{i=1}^{\infty} \delta_i$ *where* δ_i *is an interval of* $[a,b]$ *for* $i \geq 1$ *and consider the function* $\tilde{\psi}|_{\Delta^c} = \sum_{i=1}^{\infty} \psi|_{\delta_i}$. *Therefore by applying Theorem 6 corresponding to the function* $\tilde{\psi}|_{\Delta^c}$ *we will have the final conclusion because both left hand side and right hand side of* (10) *are equals to zero whenever* ψ *is constant.*

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THANK YOU