Quantum Control and Approximate Controllability of Infinite Dimensional Quantum Systems

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Outline

- Superconducting qubits
- Controllability of quantum systems
- Quantum control on the boundary

- Several technologies
  - Trapped Ions
  - NMR Quantum Computing
  - Superconducting Qubits
- Superconducting Qubits share many promising features
  - Large coherence times
  - Circuits can be printed
  - Good integration with current technology
  - Like “artificial atoms” where the different coupling constants can be crafted.

Josephson effect

- Predicted by Josephson $'62$. Two superconducting materials separated by a thin insulator. The effects are due to tunneling of Cooper Pairs.
  - DC Josephson effect
  - AC Josephson effect
- In the superconducting phase, electrons (Cooper Pairs) form a Bose-Einstein condensate. All the Pairs collapse in the groundstate.

$\psi_2 = \rho_2 e^{i\phi_2}$ $\psi_1 = \rho_1 e^{i\phi_1}$

  - ρ_i^2 Local density of conducting electrons
- There are two main equations (macroscopically) governing the Josephson effect.

$I = I_J \sin\phi$

$$V = \frac{\hbar}{2e}\dot{\phi}$$

ϕ: Phase difference $\phi_2 - \phi_1$

Josephson effect

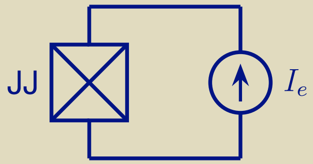
- DC Josephson effect
  - Suppose that there is no applied voltage to the junction:

$$\phi = const. \neq 0 \qquad I = I_J \sin\phi$$

- AC Josephson effect
  - Suppose that one applies a constant electric potential: $V = V_0 = const.$

$$\phi = \frac{2e}{\hbar} V t + const. \qquad I = I_J \sin\left(\frac{2e}{\hbar} V t + const.\right)$$

- There is another effect: Inverse AC effect
  - A DC current appears when an AC voltage is applied.



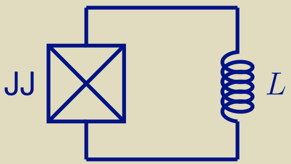
Current biased Josephson Junction

- We want to describe the dynamics of this quantum system, i.e. write a Hamiltonian.
  - Energy of a Capacitor: $e_C = \frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{\hbar}{2e}\right)^2\dot{\phi}^2$
  - Energy of the Junction:

$$e_J = \int P\,\mathrm{d}t = \int IV\,\mathrm{d}t = \frac{\hbar}{2e}\int I_J \sin\phi\,\dot{\phi}\,\mathrm{d}t = -\frac{\hbar I_J}{2e}\cos\phi$$

  - Energy of the Current Source: $e_I = \int -I_e V\,\mathrm{d}t = -\frac{\hbar I_e}{2e}\phi$
- The energy of the capacitance has the form of a quadratic kinetic term. Calling n the canonical conjugated momentum to ϕ and doing a Legendre transform:

$$H = \frac{1}{2}E_C n^2 - E_J\cos\phi - \frac{\hbar}{2e}I_e\phi$$



rf SQUID

- Another prototypical example
  - Energy of an Inductor: $e_L = \frac{1}{2}\frac{B^2}{\mu} \cdot Vol$
  - The Magnetic Field B is the total magnetic field traversing the inductance.
- Hamiltonian:

$$H = \frac{1}{2} E_C n^2 - E_J \cos\phi + E_L \frac{(\phi + \phi_e)^2}{2}$$

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Controllability of quantum systems

- Time dependent Schrödinger Equation

$$i\frac{\partial \Psi}{\partial t} = H(t)\Psi$$

- $H(t)$ is a family of self-adjoint operators
- The solution of the equation is given in terms of a unitary propagator
  - $U : \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathcal{H})$
  - $U(t,t) = \mathbb{I}_{\mathcal{H}}$
  - $U(t,s)U(s,r) = U(t,r)$
- $\Psi(t) = U(t,t_0)\Psi_0$ is a solution of Schrödinger's Equation with initial value Ψ_0

- Finite dimensional Quantum System $\mathcal{H} = \mathbb{C}^n$
- Simple situation. Linear controls:

$$i\frac{\partial\Psi}{\partial t} = (H_0 + c(t)H_1)\,\Psi$$

- H_0, H_1 self-adjoint operators (Hermitean matrices).
- $c \in \mathcal{C}$ Space of controls
- Use the controls to steer the state of the system from $\Psi_0 \to \Psi_f$.

- Ultimate Objective (not today): Find a curve $c(t) \subset \mathcal{C}$ that drives the system from $\Psi_0 \to \Psi_f$.
- Optimal control: The solution $\Psi(t) = U(t, t_0)\Psi_0$ must minimize some functional.
  - Minimal time
  - Minimal energy
- First of all: Decide wether or not the system is controllable.
  - If there exists $c(t) \subset \mathcal{C}$ such that for some T

$$\Psi_f = \Psi(T) = U(t, t_0)\Psi_0$$

- Study the dynamical Lie algebra:

$$\mathfrak{Lie}\{iH_0, iH_1\}$$

- The reachable set of Ψ_0 is the orbit through Ψ_0 of the exponential map of the dynamical Lie algebra.

- The finite dimensional quantum system is controllable if the dynamical Lie algebra is the Lie algebra of $U(N)$.

Harmonic Oscillator

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi + c(t)x\Psi = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

$$p\Psi = -i\frac{\mathrm{d}\Psi}{\mathrm{d}x}$$

$$q\Psi = x\Psi(x)$$

- Harmonic Oscillator algebra:

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$N = a^\dagger a$$

$$N|n\rangle = n|n\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Example: Truncation of the Harmonic Oscillator

H_0 Harmonic Oscillator

H_1

$$i\frac{\partial\Psi}{\partial t} = \boxed{-\frac{1}{2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{1}{2}x^2\Psi} + c(t)\boxed{x\Psi} = \left[\frac{1}{2}(p^2+q^2) + c(t)q\right]\Psi$$

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- Generators of the dynamic:

$$H_0 = N + \frac{1}{2}$$

$$H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Finite-dimensional approximation by the first n eigen-states

$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

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$$(H_0^n)_{ij} = \langle i|H_0|j\rangle \qquad (H_1^n)_{ij} = \langle i|H_0|j\rangle$$

- The finite dimensional approximation is controllable for all n

$$\dim \mathfrak{Lie}\{iH_0^n, iH_1^n\} = n^2$$

Controllability of the Harmonic Oscillator

- Generators of the dynamic:

$$H_0 = N + \frac{1}{2}$$

$$H_1 = \frac{1}{\sqrt{2}}(a^\dagger + a)$$

- Dynamical Lie Algebra of the Harmonic Oscillator

$$[a, a^\dagger] = \mathbb{I} \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger$$

$$[iH_0, iH_1] = -\frac{1}{\sqrt{2}}[N, a^\dagger + a] = -\frac{1}{\sqrt{2}}(a^\dagger - a) = ip = iH_2$$

Controllability of the Harmonic Oscillator

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$$[iH_0, iH_1] = iH_2 \quad [iH_0, iH_2] = iH_1 \quad [iH_1, iH_2] = i\mathbb{I} = iH_3$$

Controllability of the Harmonic Oscillator

- Generators of the dynamic:

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- Four dimensional Lie algebra!
- The infinite dimensional Harmonic Oscillator is not controllable.

Controllability of the Harmonic Oscillator

- Why is it controllable for finite dimensions?
- Consider the 3-level truncation ($n = 0, 1, 2$)

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \qquad a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

- What happens with the dynamical Lie algebra?

$$q = \frac{1}{\sqrt{2}}(a^\dagger + a) \qquad p = \frac{i}{\sqrt{2}}(a^\dagger - a)$$

$$[q, p] = i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Approximate Controllability: A linear control system is approximately controllable if for every Ψ_0, $\Psi_1 \in \mathcal{S}$ and every $\epsilon > 0$ there exist $T > 0$ and $c(t) \subset \mathcal{C}$ such that

$$\|\Psi_1 - U(T, t_0)\Psi_0\| < \epsilon$$

- Reasonable for infinite dimensions
- Hilbert Space is defined as equivalence classes of convergent sequences
- Is natural to expect this if one has exact controllability of every finite dimensional subsystem

Approximate controllability

- Consider the Linear Control System:

$$i\frac{\partial \Psi}{\partial t} = \left(H_0 + c(t)H_1\right)\Psi$$

  - H_0, H_1 are self-adjoint.
  - $\{\Phi_n\}_{n\in\mathbb{N}}$ O.N.B of eigenvectors of H_0
  - $\Phi_n \in \mathcal{D}(H_1)$ for every $n \in \mathbb{N}$

- The linear control system is approximately controllable with piecewise constant controls if [Chambrion, Mason, Sigalotti, Boscain 2009]:
  - $(\lambda_{n+1} - \lambda_n)_{n\in\mathbb{N}}$ are $\mathbb{Q}$-linearly independent.
  - $\langle H_1\Phi_n, \Phi_{n+1}\rangle \neq 0$ for any $n \in \mathbb{N}$

Existence of unitary propagators

- Study Hamiltonians of the form
  - $H(t) = \sum_{i=1}^{n} f_i(t) H_i$:
    - dom$H(t) = \mathcal{D}$
    - Self-adjoint for all t
    - H_i densely defined on $\mathcal{D}$ and symmetric
  - $f_i \in \mathcal{C}^\infty$, $i = 1, \ldots, n$
  - $\|H_i \Psi\| \leq K \left(\|H(t)\psi\| + \|\psi\| \right)$

Thm [Balmaseda, PP]: With the conditions above, there exists a strongly differentiable unitary propagator $U(t,s)$ that solves the time dependent Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t,s)\psi_s = -iH(t)U(t,s)\psi_s$$

Stability of the evolution

- Consider that the following Hamiltonians satisfy the previous conditions.

$$H_1(t) = \sum_{i=1}^{n} f_i(t) H_i, \quad H_2(t) = \sum_{i=1}^{n} g_i(t) H_i$$

Thm [Balmaseda, PP]: For every $T > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that $\|f_i - g_i\|_\infty < \delta$ implies

$$\|U_1(T,s)\psi_s - U_2(T,s)\psi_s\| < \epsilon$$

- This result is important for technical and experimental reasons
  - It allows to avoid other technical conditions like those appearing in the result of Chambrion et. al.
  - It guarantees that errors in the controls do not propagate dangerously to the solution.
- With these theorems one can prove that Hamiltonians of the type below are approximately controllable

$$H = -\left(\frac{\mathrm{d}}{\mathrm{d}\phi} - i\alpha\right)^2 + V_{\text{bounded}} + k\phi^2 + f_1(t)\phi + f_2(t)$$

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Quantum control on the boundary

- Time dependent Schrödinger Equation

$$i\frac{\partial\Psi}{\partial t} = H(t)\Psi$$

- $H(t)$ is a family of different self-adjoint extension of the same operator

$$\left(H, \mathcal{D}\left(c(t)\right)\right)$$

- Advantage: There is no need to apply an external field
- Problem: Even the existence of solutions of the dynamics is compromised.

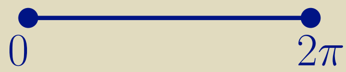
Quantum Control on the boundary

- Assumption:
  - The spectrum of $\left(H, \mathcal{D}\left(c\right)\right)$ only contains eigenvalues with finite degeneracy.
  - Then $\{\Phi_n^c\}_{n\in\mathbb{N}}$ forms a complete orthonormal base.
- Fix a reference extension $\left(H, \mathcal{D}\left(c_0\right)\right)$
  - Define the unitary operator

$$\begin{aligned} V_c\colon \quad & \mathcal{H} \to \mathcal{H} \\ & \Phi_n^c \to \Phi_n^0 \end{aligned}$$

  - One needs to require additionally that $V_c\colon \mathcal{D}(c) \to \mathcal{D}(c_0)$
- Using this unitary transformation one can transform the problem with time dependent domain into an equivalent one with time independent domain.

Example: Varying quasiperiodic boundary conditions



$$H_0 = -\frac{\mathsf{d}^2}{\mathsf{d}x^2}$$

$$\mathcal{D}_\alpha = \left\{ \phi \in \mathcal{L}^2 \middle| \|\frac{\mathsf{d}^2\phi}{\mathsf{d}x^2}\| < \infty, \quad \begin{array}{ll} \phi(0) & = e^{i2\pi\alpha}\phi(2\pi) \\ \phi'(0) & = e^{i2\pi\alpha}\phi'(2\pi) \end{array} \right\}$$

- This is a family of self-adjoint operators depending on α
  - Eigenvalues: $(n-\alpha)^2$
  - Eigenfunctions: $\phi_n(x) = e^{i\alpha x}e^{inx}$
- Assuming that the parameter α depends smoothly with time this is unitarily equivalent to:

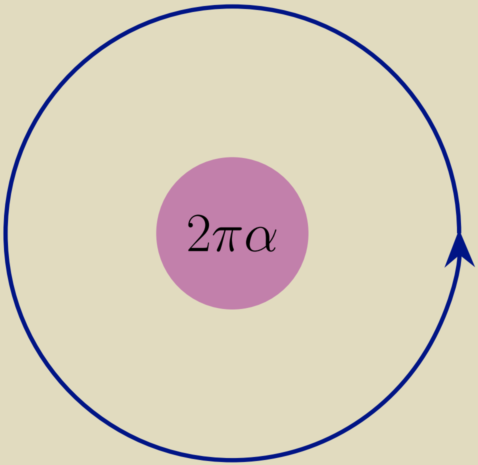
$$H(t) = \left[i\frac{\mathsf{d}}{\mathsf{d}x} - \alpha(t) \right]^2 + \dot{\alpha}(t)x$$

$\mathcal{D}_0 =$ "Periodic Boundary Conditions"

Example: Varying quasiperiodic boundary conditions

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Psi = [i\frac{\mathrm{d}}{\mathrm{d}\theta} - \alpha]^2\Psi + \theta\dot{\alpha}\Psi$$

Quantum Faraday Law



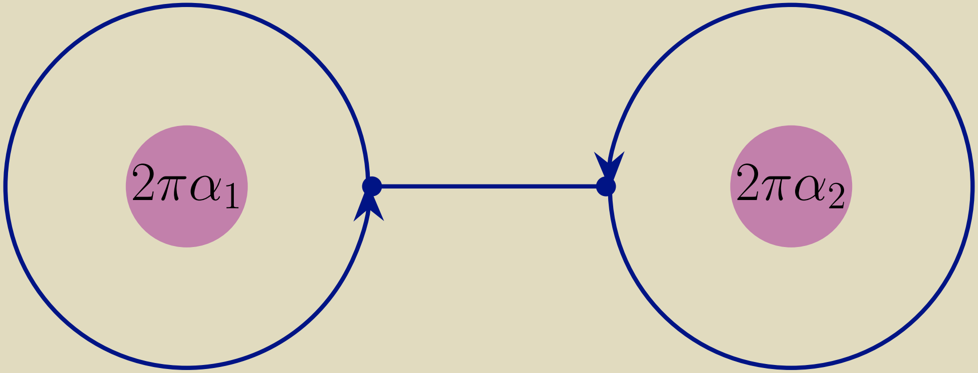
Particle Moving in a circular wire

Magnetic flux of intensity $2\pi\alpha$

- The magnetic vector potential α is related to the phase change of the wave function $\phi(0) = e^{i2\pi\alpha}\phi(2\pi)$.

Magnetic Laplacian on planar graphs

- One can generalise this to more general planar graphs.
- This points out that magnetic laplacians on planar graphs could be used to model superconducting qubits. This is ongoing research. Preliminary results show that the Josephson Junction can be modelled in these systems with δ-like interactions.



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