Construction of Wigner functions from gauge equivalence classes of unitary irreducible representations of noncommutative quantum mechanics (NCQM)

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Noncommutative Geometry: Physical and Mathematical Aspects Of Quantum Space-Time and Matter

Work done in collaboration with Prof. Hishamuddin Zainuddin

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Summary of the main results

- Motivation and background discussion on Noncommutative Quantum Mechanics (NCQM).
- Introduction to the nilpotent Lie group G_{NC} and its unitary dual $\hat{G}_{\text{NC}}$.
- A class of unitarily equivalent representations (UIRs) of G_{NC} and their relation to 2-parameter classes of gauge potentials.
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• Noncommutative quantum mechanics, abbreviated as NCQM in the sequel, is the quantum mechanics in noncommutative configuration space.

• Focus on a nonrelativistic quantum mechanical system of 2-degrees of freedom. Here, we have 2 positions and 2 momenta coordinates denoted by q_1, q_2, p_1 and p_2. Denote an element of the 4-dimensional Abelian group of translations of $\mathbb{R}^4$ as (q_1, q_2, p_1, p_2). The Weyl-Heisenberg group is just a nontrivial central extension of this Abelian group, a generic element of which is denoted by $(\theta, q_1, q_2, p_1, p_2)$. The Weyl-Heisenberg Lie algebra, on the other hand, admits a realization of self adjoint differential operators on the smooth vectors of $L^2(\mathbb{R}^2)$, the commutation relations for which read as follows:

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$$[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}. \tag{1}$$

• Here, $\hat{Q}_i$'s and $\hat{P}_i$'s are the self-adjoint representations of the Lie algebra basis elements Q_i's and P_i's where $i = 1, 2$. Note that the noncentral basis elements Q_i's and P_i's correspond to the group parameters p_i's and q_i's, respectively, for $i = 1, 2$. Also, $\mathbb{I}$ stands for the identity operator on $L^2(\mathbb{R}^2)$ and the central basis element Θ of the algebra is mapped to scalar multiple of $\mathbb{I}$.

• In contrast to the well-known and much studied representation theory of the Weyl-Heisenberg group, if one considers 3 inequivalent local exponents (see [?]) of the Abelian group of translations in $\mathbb{R}^4$ and extend it centrally using them to obtain a 7-dimensional real Lie group denoted by G_{NC} in the sequel.

• The aim of introducing two other inequivalent local exponents besides the one used to arrive at the Weyl-Heisenberg group was to incorporate position-position and momentum-momentum noncommutativity as employed in the formulation of noncommutative quantum mechanics (NCQM).

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Representation of the corresponding Lie algebra g_{NC} reads:

$$\begin{aligned} [\hat{Q}_1, \hat{P}_1] &= [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \\ [\hat{Q}_1, \hat{Q}_2] &= i\vartheta\mathbb{I}, \text{ and } [\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}. \end{aligned} \tag{2}$$

Here, the central generators associated with the group parameters θ, ϕ and ψ are all mapped to scalar multiples of the identity operator $\mathbb{I}$ on $L^2(\mathbb{R}^2)$.

A quick recap of group extension

Given a connected and simply connected Lie group G, the local exponents ξ giving its central extensions are functions $\xi : G \times G \to \mathbb{R}$, obeying the following properties:

$$\xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g)$$
$$\xi(g, e) = 0 = \xi(e, g),\ \xi(g, g^{-1}) = \xi(g^{-1}, g).$$

We call the central extension trivial when the corresponding local exponent is simply a *coboundary* term, in other words, when there exists a continuous function $\zeta : G \to \mathbb{R}$ such that the following holds

$$\xi(g', g) = \xi_{cob}(g', g) := \zeta(g') + \zeta(g) - \zeta(g'g).$$

Two local exponents ξ and ξ' are *equivalent* if they differ by a coboundary term, i.e. $\xi'(g', g) = \xi(g', g) + \xi_{cob}(g', g)$. A local exponent which is itself a coboundary is said to be trivial and the corresponding extension of the group is called a trivial extension.

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Inequivalent local exponents to arrive at G_{NC}

We shall show that certain triple central extension of the abelian group of translations of $\mathbb{R}^4$ reproduces the noncommutative commutation relations (2). The relevant central extensions are executed using inequivalent local exponents that are enumerated in the following theorem:

Theorem

The three real valued functions ξ, ξ' and ξ'' on $G_T \times G_T$ given by

$$\begin{aligned}
\xi((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[q_1 p_1' + q_2 p_2' - p_1 q_1' - p_2 q_2'],\\
\xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[p_1 p_2' - p_2 p_1'],\\
\xi''((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[q_1 q_2' - q_2 q_1'],
\end{aligned}$$

are inequivalent local exponents for the group, G_T, of translations in $\mathbb{R}^4$.

Group composition rule for G_{NC}

The group G_{NC} is a 7-dimensional real nilpotent Lie group. Its group composition rule is given by (see [?])

$$
\begin{aligned}
&(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})(\theta', \phi', \psi', \mathbf{q}', \mathbf{p}') \\
&\quad = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{p}, \mathbf{q}' \rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p} \wedge \mathbf{p}'], \\
&\quad \psi + \psi' + \frac{\gamma}{2}[\mathbf{q} \wedge \mathbf{q}'], \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'),
\end{aligned} \tag{3}
$$

where α, β and γ some denote strictly positive dimensionful constants associated with the triple central extension. Here, $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. Also, in (3), $\langle ., . \rangle$ and $\wedge$ are defined as $\langle \mathbf{q}, \mathbf{p} \rangle := q_1 p_1 + q_2 p_2$ and $\mathbf{q} \wedge \mathbf{p} := q_1 p_2 - q_2 p_1$, respectively..

Coadjoint orbits of G_{NC} and the unitary dual $\hat{G}_{\text{NC}}$

There is a natural action of G_{NC} on its dual Lie algebra g^*_{NC} called the coadjoint action. This coadjoint action is given by

$$
\begin{aligned}
&Kg(p_1, p_2, q_1, q_2, \theta, \phi, \psi)(X_1, X_2, X_3, X_4, X_5, X_6, X_7) \\
&\quad = (X_1 - \frac{\alpha}{2} q_1 X_5 + \frac{\beta}{2} p_2 X_6, \;\; X_2 - \frac{\alpha}{2} q_2 X_5 - \frac{\beta}{2} p_1 X_6 \\
&\qquad , X_3 + \frac{\gamma}{2} q_2 X_7 + \frac{\alpha}{2} p_1 X_5, \; X_4 - \frac{\gamma}{2} q_1 X_7 + \frac{\alpha}{2} p_2 X_5, X_5, X_6, X_7)
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\tag{4}
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If one denotes the 3-polynomial invariants X_5, X_6 and X_7 by ρ, σ and τ, respectively, then the underlying coadjoint orbits can be classified based on the values of the triple (ρ, σ, τ) in the following ways:

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If one denotes the 3-polynomial invariants X_5, X_6 and X_7 by ρ, σ and τ, respectively, then the underlying coadjoint orbits can be classified based on the values of the triple (ρ, σ, τ) in the following ways:

- When $\rho \neq 0$, $\sigma \neq 0$ and $\tau \neq 0$ satisfying $\rho^2\alpha^2 - \gamma\beta\sigma\tau \neq 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,\sigma,\tau}$ are $\mathbb{R}^4$, considered as affine 4-spaces.
- When $\rho \neq 0$, $\sigma \neq 0$ and $\tau \neq 0$ satisfying $\rho^2\alpha^2 - \gamma\beta\sigma\tau = 0$, the coadjoint orbits are denoted by ${}^{\kappa,\delta}\mathcal{O}_2^{\rho,\zeta}$. For each ordered pair $(\kappa, \delta) \in \mathbb{R}^2$ along with $\rho \neq 0$ and $\zeta \in (-\infty, 0) \cup (0, \infty)$ satisfying $\rho = \sigma\zeta = \frac{\gamma\beta\tau}{\zeta\alpha^2}$, one obtains an $\mathbb{R}^2$-affine space to be the underlying coadjoint orbit ${}^{\kappa,\delta}\mathcal{O}_2^{\rho,\zeta}$.
- When $\rho \neq 0$, $\sigma \neq 0$, but $\tau = 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,\sigma,0}$ are $\mathbb{R}^4$-affine spaces.
- When $\rho \neq 0$, $\tau \neq 0$, but $\sigma = 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,0,\tau}$ are $\mathbb{R}^4$-affine spaces.
- When $\rho = 0$, $\tau \neq 0$ and $\sigma \neq 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{0,\sigma,\tau}$ are also $\mathbb{R}^4$-affine spaces.

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- When $\rho = \sigma = \tau = 0$, the coadjoint orbits are 0-dimensional points denoted by ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$. Every quadruple (c_1, c_2, c_3, c_4) gives rise to such an orbit.

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- When $\rho = \sigma = \tau = 0$, the coadjoint orbits are 0-dimensional points denoted by ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$. Every quadruple (c_1, c_2, c_3, c_4) gives rise to such an orbit.

- When $\rho \neq 0$ only but both σ and τ are taken to be identically zero, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,0,0}$ are $\mathbb{R}^4$-affine spaces.
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Unitary irreducible representations of G_{NC} and those of its Lie algebra g_{NC}

Since, G_{NC} is a connected, simply connected nilpotent Lie group, its unitary irreducible representations are in 1-1 correspondence with the underlying coadjoint orbits as corroborated by the method of orbit. There are nine distinct types of equivalence classes of unitary irreducible representations of G_{NC} and its Lie algebra g_{NC}:

Case: $\rho \neq 0, \sigma \neq 0, \tau \neq 0$ **with** $\rho^2\alpha^2 - \gamma\beta\sigma\tau \neq 0$

Unirreps of G_{NC}:

$$(U^{\rho}_{\sigma,\tau}(\theta,\phi,\psi,\mathbf{q},\mathbf{p})f)(\mathbf{r}) = e^{i\rho(\theta+\alpha p_1 r_1+\alpha p_2 r_2+\frac{\alpha}{2}q_1p_1+\frac{\alpha}{2}q_2p_2)}e^{i\sigma(\phi+\frac{\beta}{2}p_1p_2)} \times e^{i\tau(\psi+\gamma q_2 r_1+\frac{\gamma}{2}q_1q_2)}f\left(r_1+q_1, r_2+q_2+\frac{\sigma\beta}{\rho\alpha}p_1\right), \tag{5}$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$.

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where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$.

Reps of $\mathfrak{g}_{\text{NC}}$:

$$\hat{Q}_1 = r_1 + i\vartheta\frac{\partial}{\partial r_2}, \qquad \hat{Q}_2 = r_2,$$
$$\hat{P}_1 = -i\hbar\frac{\partial}{\partial r_1}, \qquad \hat{P}_2 = -\frac{\mathcal{B}}{\hbar}r_1 - i\hbar\frac{\partial}{\partial r_2}, \tag{6}$$

with the following identification:

$$\hbar = \frac{1}{\rho\alpha}, \;\; \vartheta = -\frac{\sigma\beta}{(\rho\alpha)^2} \text{ and } \mathcal{B} = -\frac{\tau\gamma}{(\rho\alpha)^2}. \tag{7}$$

$B := \frac{\mathcal{B}}{\hbar}$, here, can be interpreted as the constant magnetic field applied normally to the $\hat{Q}_1\hat{Q}_2$-plane.

Case: $\rho \neq 0, \sigma \neq 0, \tau \neq 0$ **with** $\rho^2\alpha^2 - \gamma\beta\sigma\tau = 0$

Unirreps of G_{NC}:

$$\begin{aligned}
&(U^{\kappa,\delta}_{\rho,\zeta}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r)\\
&\quad = e^{i\rho\left(\theta+\frac{1}{\zeta}\phi+\frac{\zeta\alpha^2}{\gamma\beta}\psi\right)+i\kappa q_1+i\delta q_2-i\rho\alpha r p_1-\frac{i\rho\alpha^2\zeta}{\beta}rq_2+\frac{i\rho\alpha}{2}(q_1p_1-q_2p_2)}\\
&\quad \times e^{i\rho\left(\frac{\alpha^2\zeta}{2\beta}q_1q_2-\frac{\beta}{2\zeta}p_1p_2\right)} f\left(r-q_1+\frac{\beta}{\alpha\zeta}p_2\right),
\end{aligned} \tag{8}$$

where $f \in L^2(\mathbb{R}, dr)$.

Reps of g_{NC}:

$$\begin{aligned}
&\hat{Q}_1 = -r, \quad \hat{Q}_2 = i\vartheta\frac{\partial}{\partial r},\\
&\hat{P}_1 = \hbar\kappa + i\hbar\frac{\partial}{\partial r}, \quad \hat{P}_2 = \hbar\delta + \frac{\hbar r}{\vartheta},
\end{aligned} \tag{9}$$

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Unirreps of G_{NC}:

$$
\begin{aligned}
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&\quad \times e^{i\rho\left(\frac{\alpha^2\zeta}{2\beta}q_1q_2-\frac{\beta}{2\zeta}p_1p_2\right)} f(r-q_1+\frac{\beta}{\alpha\zeta}p_2),
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where $f \in L^2(\mathbb{R}, dr)$.

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$$
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\end{aligned} \tag{9}
$$

Theorem

A 2-parameter (l,m) continuous family of unitarily equivalent irreducible representations, associated with the 4-dimensional coadjoint orbit $\mathcal{O}_4^{\rho,\sigma,\tau}$ of the connected and simply connected nilpotent Lie group G_{NC} due to a fixed nonzero triple (ρ,σ,τ) satisfying $\rho^2\alpha^2 - \tau\gamma\sigma\beta \neq 0$, is given by

$$
\begin{aligned}
&(U_{l,m}^{\rho,\sigma,\tau}(\theta,\phi,\psi,\mathbf{q},\mathbf{p})f)(r_1,r_2)\\
&= e^{i\rho\theta+i\sigma\phi+i\tau\psi}\, e^{i\rho\alpha p_1 r_1 + i\rho\alpha p_2 r_2 + \frac{i\rho^2\alpha^2\gamma(1-l)}{\tau\gamma\sigma\beta l-\rho^2\alpha^2} q_1 r_2 + i l\tau\gamma q_2 r_1 + i\left[\frac{\rho\alpha}{2}+\frac{\rho\alpha\tau\gamma\sigma\beta m(1-l)}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}\right]p_1 q_1}\\
&\times e^{i\left[\frac{\rho\alpha}{2}-\frac{l\tau\gamma\sigma\beta(1-m)}{\rho\alpha}\right]p_2 q_2 + i\left(m-\frac{1}{2}\right)\sigma\beta p_1 p_2 + i\left[\frac{\tau\gamma}{2}-\frac{\tau\gamma(1-l)(\tau\gamma\sigma\beta l-\tau\gamma\sigma\beta l m-\rho^2\alpha^2)}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}\right]q_1 q_2}\\
&\times f\left(r_1-\frac{(1-m)\sigma\beta}{\rho\alpha}p_2+\frac{\tau\gamma\sigma\beta(l+m-lm)-\rho^2\alpha^2}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}q_1,\, r_2+\frac{m\sigma\beta}{\rho\alpha}p_1-\frac{\tau\gamma\sigma\beta l(1-m)-\rho^2\alpha^2}{\rho^2\alpha^2}q_2\right), \qquad (10)
\end{aligned}
$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$. Here, $l \in \mathbb{R} \smallsetminus \left\{\frac{\rho^2\alpha^2}{\tau\gamma\sigma\beta}\right\}$ and $m \in \mathbb{R}$.

Self adjoint representation of $\mathfrak{g}_{\text{NC}}$ acting on the smooth vectors of $L^2(\mathbb{R}^2, dr_1 dr_2)$ is given by

$$\begin{aligned}
\hat{Q}_1^m &= r_1 - m\frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_2},\\
\hat{Q}_2^m &= r_2 + (1-m)\frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1^{l,m} &= \frac{\tau\gamma\rho\alpha(1-l)}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}r_2 - \frac{i}{\rho\alpha}\left[\frac{\tau\gamma\sigma\beta(l+m-lm)-\rho^2\alpha^2}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}\right]\frac{\partial}{\partial r_1},\\
\hat{P}_2^{l,m} &= \frac{l\tau\gamma}{\rho\alpha}r_1 + i\left[\frac{\tau\gamma\sigma\beta l(1-m)-\rho^2\alpha^2}{\rho^3\alpha^3}\right]\frac{\partial}{\partial r_2}.
\end{aligned} \tag{11}$$

Self adjoint representation of g_{NC} can be re-written as

$$\begin{aligned}
\hat{Q}_1^m &= r_1 - m\frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_2},\\
\hat{Q}_2^m &= r_2 + (1-m)\frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1^{l,m} &= \frac{\tau\gamma\rho\alpha(1-l)}{\tau\gamma\sigma\beta l - \rho^2\alpha^2}\hat{Q}_2^m - \frac{i}{\rho\alpha}\left[\frac{\tau\gamma\sigma\beta(1-l)}{\tau\gamma\sigma\beta l - \rho^2\alpha^2} + 1\right]\frac{\partial}{\partial r_1},\\
\hat{P}_2^{l,m} &= \frac{l\tau\gamma}{\rho\alpha}\hat{Q}_1^m - \frac{i}{\rho\alpha}\left(1 - \frac{l\tau\gamma\sigma\beta}{\rho^2\alpha^2}\right)\frac{\partial}{\partial r_2}.
\end{aligned} \tag{12}$$

Commutation relations:

$$\begin{aligned}
[\hat{Q}_1^m, \hat{P}_1^{l,m}] &= [\hat{Q}_2^{l,m}, \hat{P}_2^{l,m}] = \frac{i}{\rho\alpha}\mathbb{I},\\
[\hat{Q}_1^m, \hat{Q}_2^m] &= -\frac{i\sigma\beta}{\rho^2\alpha^2}\mathbb{I}, \quad [\hat{P}_1^{l,m}, \hat{P}_2^{l,m}] = -\frac{i\tau\gamma}{\rho^2\alpha^2}\mathbb{I}.
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\hat{P}_1^{l,m} &= \frac{\tau\gamma\rho\alpha(1-l)}{\tau\gamma\sigma\beta l - \rho^2\alpha^2}\hat{Q}_2^m - \frac{i}{\rho\alpha}\left[\frac{\tau\gamma\sigma\beta(1-l)}{\tau\gamma\sigma\beta l - \rho^2\alpha^2} + 1\right]\frac{\partial}{\partial r_1},\\
\hat{P}_2^{l,m} &= \frac{l\tau\gamma}{\rho\alpha}\hat{Q}_1^m - \frac{i}{\rho\alpha}\left(1 - \frac{l\tau\gamma\sigma\beta}{\rho^2\alpha^2}\right)\frac{\partial}{\partial r_2}.
\end{aligned} \tag{12}$$

<u>Commutation relations</u>:

$$\begin{aligned}
&[\hat{Q}_1^m, \hat{P}_1^{l,m}] = [\hat{Q}_2^{l,m}, \hat{P}_2^{l,m}] = \frac{i}{\rho\alpha}\mathbb{I},\\
&[\hat{Q}_1^m, \hat{Q}_2^m] = -\frac{i\sigma\beta}{\rho^2\alpha^2}\mathbb{I},\ \ [\hat{P}_1^{l,m}, \hat{P}_2^{l,m}] = -\frac{i\tau\gamma}{\rho^2\alpha^2}\mathbb{I}.
\end{aligned} \tag{13}$$

Inspired by the fact that the real parameters l and m do not contribute to the commutation relations of NCQM as has been verified in (13), we can thereby choose a continuous family of gauges using the noncommutative position operators $\hat{Q}_1$ and $\hat{Q}_2$ given in (11).

Definition

Associated with the UIRs (10) of G_{NC}, one can define the 2-parameter family of vector potentials $\mathbf{A}^{l,m} \equiv \left(-\frac{\tau\gamma\rho\alpha(1-l)}{\tau\gamma\sigma\beta l-\rho^2\alpha^2}\hat{Q}_2^m, -\frac{l\tau\gamma}{\rho\alpha}\hat{Q}_1^m\right)$ for a fixed nonzero triple (ρ,σ,τ) satisfying $\rho^2\alpha^2-\tau\gamma\sigma\beta\neq 0$, with $\hat{Q}_i^m$'s as given in (12), to be noncommutative vector potentials determining continuous family of NCQM gauges for $l\in\mathbb{R}\setminus\left\{\frac{\rho^2\alpha^2}{\tau\gamma\sigma\beta}\right\}$ and $m\in\mathbb{R}$. While writing the vector potential $\mathbf{A}^{l,m}$, its dependence on ρ, σ and τ is suppressed due to notational convenience.

• Landau gauge corresponds to $l = 1$, $m = 0$ so that the gauge potential is $\mathbf{A}^{1,0} \equiv \left(0, -\frac{\tau\gamma}{\rho\alpha}\hat{Q}_1^0\right) = \left(0, -\frac{\tau\gamma}{\rho\alpha} r_1\right)$ satisfying $\partial_1 A_2^{1,0} - \partial_2 A_1^{1,0} = -\frac{\tau\gamma}{\rho\alpha} :=$B.

Self adjoint representation of g_{NC} in the Landau gauge:

$$\begin{aligned}
\hat{Q}_1^0 &= r_1, \\
\hat{Q}_2^0 &= r_2 + \frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_1}, \\
\hat{P}_1^{1,0} &= -\frac{i}{\rho\alpha}\frac{\partial}{\partial r_1}, \\
\hat{P}_2^{1,0} &= \frac{\tau\gamma}{\rho\alpha} r_1 + \frac{i(\tau\gamma\sigma\beta - \rho^2\alpha^2)}{\rho^3\alpha^3}\frac{\partial}{\partial r_2}.
\end{aligned} \tag{14}$$

• Symmetric gauge corresponds to $l = \frac{\rho\alpha(\rho\alpha - \sqrt{\rho^2\alpha^2 - \gamma\beta\sigma\tau})}{\gamma\beta\sigma\tau} := l_s$ and $m = \frac{1}{2}$. Vector potential reads:

$$\mathbf{A}^{l_s,\frac{1}{2}} \equiv \left(\frac{(\rho\alpha - \sqrt{\rho^2\alpha^2 - \gamma\beta\sigma\tau})}{\sigma\beta}\hat{Q}_2^{\frac{1}{2}}, \frac{(\sqrt{\rho^2\alpha^2 - \gamma\beta\sigma\tau} - \rho\alpha)}{\sigma\beta}\hat{Q}_1^{\frac{1}{2}}\right) \tag{15}$$

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Self adjoint representation of $\mathfrak{g}_{\mathrm{NC}}$ in the Landau gauge:

$$\begin{aligned}
\hat{Q}_1^0 &= r_1,\\
\hat{Q}_2^0 &= r_2 + \frac{i\sigma\beta}{\rho^2\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1^{1,0} &= -\frac{i}{\rho\alpha}\frac{\partial}{\partial r_1},\\
\hat{P}_2^{1,0} &= \frac{\tau\gamma}{\rho\alpha}r_1 + \frac{i(\tau\gamma\sigma\beta - \rho^2\alpha^2)}{\rho^3\alpha^3}\frac{\partial}{\partial r_2}.
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$$\mathbf{A}^{l_s,\frac{1}{2}} \equiv \left(\frac{(\rho\alpha - \sqrt{\rho^2\alpha^2 - \gamma\beta\sigma\tau})}{\sigma\beta}\hat{Q}_2^{\frac{1}{2}}, \frac{(\sqrt{\rho^2\alpha^2 - \gamma\beta\sigma\tau} - \rho\alpha)}{\sigma\beta}\hat{Q}_1^{\frac{1}{2}}\right) \tag{15}$$

- It can be verified that the following holds:

$$\partial_1 A_2^{l_s,\frac{1}{2}} - \partial_2 A_1^{l_s,\frac{1}{2}} = \frac{2\hbar}{\vartheta}\left(\sqrt{1-\frac{B\vartheta}{\hbar}}-1\right) := \bar{B}, \tag{16}$$

where we chose $B = -\frac{\tau\gamma}{\rho\alpha}$, $\vartheta = -\frac{\sigma\beta}{\rho^2\alpha^2}$ and $\hbar = \frac{1}{\rho\alpha}$.

- Self adjoint representation of $\mathfrak{g}_{\text{NC}}$ in symmetric gauge:

$$\begin{aligned}
\hat{Q}_1^{\frac{1}{2}} &= r_1 - \frac{i\sigma\beta}{2\rho^2\alpha^2}\frac{\partial}{\partial r_2},\\
\hat{Q}_2^{\frac{1}{2}} &= r_2 + \frac{i\sigma\beta}{2\rho^2\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1^{l_s,\frac{1}{2}} &= \frac{(\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau}-\rho\alpha)}{\sigma\beta}r_2 - \frac{i}{2\rho^2\alpha^2}(\rho\alpha+\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})\frac{\partial}{\partial r_1},\\
\hat{P}_2^{l_s,\frac{1}{2}} &= \frac{(\rho\alpha-\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})}{\sigma\beta}r_1 - \frac{i}{2\rho^2\alpha^2}(\rho\alpha+\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})\frac{\partial}{\partial r_2}.
\end{aligned} \tag{17}$$

- It can be verified that the following holds:

$$\partial_1 A_2^{l_s,\frac{1}{2}} - \partial_2 A_1^{l_s,\frac{1}{2}} = \frac{2\hbar}{\vartheta}\left(\sqrt{1-\frac{B\vartheta}{\hbar}}-1\right) := \bar{B}, \qquad (16)$$

where we chose $B = -\frac{\tau\gamma}{\rho\alpha}$, $\vartheta = -\frac{\sigma\beta}{\rho^2\alpha^2}$ and $\hbar = \frac{1}{\rho\alpha}$.

- Self adjoint representation of g_{NC} in symmetric gauge:

$$\begin{aligned}
\hat{Q}_1^{\frac{1}{2}} &= r_1 - \frac{i\sigma\beta}{2\rho^2\alpha^2}\frac{\partial}{\partial r_2},\\
\hat{Q}_2^{\frac{1}{2}} &= r_2 + \frac{i\sigma\beta}{2\rho^2\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1^{l_s,\frac{1}{2}} &= \frac{(\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau}-\rho\alpha)}{\sigma\beta}r_2 - \frac{i}{2\rho^2\alpha^2}(\rho\alpha+\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})\frac{\partial}{\partial r_1},\\
\hat{P}_2^{l_s,\frac{1}{2}} &= \frac{(\rho\alpha-\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})}{\sigma\beta}r_1 - \frac{i}{2\rho^2\alpha^2}(\rho\alpha+\sqrt{\rho^2\alpha^2-\gamma\beta\sigma\tau})\frac{\partial}{\partial r_2}.
\end{aligned} \qquad (17)$$

On the construction of Wigner function

• Standard Wigner quasi-probability distribution for a system with 2-degrees of freedom (q_1, q_2):

$$W(|\chi\rangle\langle\lambda|; \mathbf{q}, \mathbf{p}) = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} e^{-\frac{i}{\hbar}p_1 r_1 - \frac{i}{\hbar}p_2 r_2}\, \overline{\lambda\left(\frac{1}{2}r_1 - q_1, \frac{1}{2}r_2 - q_2\right)} \times\chi\left(-\frac{1}{2}r_1 - q_1, -\frac{1}{2}r_2 - q_2\right) dr_1 dr_2, \tag{18}$$

where $\lambda, \chi \in L^2(\mathbb{R}^2, dr_1 dr_2)$.

• This is a quantum mechanical result. What is noncommutative quantum mechanical analog of (18). What if we incorporate the gauge parameters l and m into our study?

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where $\lambda, \chi \in L^2(\mathbb{R}^2, dr_1 dr_2)$.

• This is a quantum mechanical result. What is noncommutative quantum mechanical analog of (18). What if we incorporate the gauge parameters l and m into our study?

- The answer to the above question is given by:

$$W^{l,m}(|\chi_{\rho,\sigma,\tau}\rangle\langle\lambda_{\rho,\sigma,\tau}|;\mathbf{q}_{\text{nc}},\mathbf{p}_{\text{nc}};k_1,k_2,k_3)$$
$$= \frac{|\alpha|}{2\pi|k_1^2\alpha^2 - k_2k_3\beta\gamma|^{\frac{1}{2}}}\int_{\mathbb{R}^2} e^{-i\alpha p^l_{1,\text{nc}}r_1 - i\alpha p^l_{2,\text{nc}}r_2}\,\overline{\lambda_{k_1,k_2,k_3}\left(\frac{1}{2}r_1 - \frac{q^{l,m}_{1,\text{nc}}}{k_1}, \frac{1}{2}r_2 - \frac{q^{l,m}_{2,\text{nc}}}{k_1}\right)}$$
$$\times\,\chi_{k_1,k_2,k_3}\left(-\frac{1}{2}r_1 - \frac{q^{l,m}_{1,\text{nc}}}{k_1}, -\frac{1}{2}r_2 - \frac{q^{l,m}_{2,\text{nc}}}{k_1}\right)dr_1dr_2, \qquad (19)$$

- with the *"noncommuting coordinates"* $\mathbf{q}^{l,m}, \mathbf{p}^{l,m}$, in terms of the phase space coordinates k_1^*, k_2^*, k_3^* and k_4^* can be read off as

$$q^{l,m}_{1,\text{nc}} = \frac{[k_3\gamma k_2\beta l(1-m) - k_1^2\alpha^2]k_1^* + mk_1k_2\alpha\beta k_4^*}{(k_2\beta k_3\gamma l - k_1^2\alpha^2)},$$
$$q^{l,m}_{2,\text{nc}} = \frac{k_1\alpha[k_2\beta k_3\gamma(l+m-lm) - k_1^2\alpha^2]k_2^* + (1-m)k_2\beta(k_3\gamma k_2\beta l - k_1^2\alpha^2)k_3^*}{k_1\alpha(k_2\beta k_3\gamma - k_1^2\alpha^2)},$$
$$p^l_{1,\text{nc}} = \frac{k_1\alpha k_3\gamma(1-l)k_2^* + (k_1^2\alpha^2 - k_2\beta k_3\gamma l)k_3^*}{k_1^2\alpha^2 - k_2\beta k_3\gamma},$$
$$p^l_{2,\text{nc}} = \frac{k_1^2\alpha^2k_4^* - k_1\alpha k_3\gamma lk_1^*}{k_1^2\alpha^2 - k_2\beta k_3\gamma l}. \qquad (20)$$

- The answer to the above question is given by:

$$W^{l,m}(|\chi_{\rho,\sigma,\tau}\rangle\langle\lambda_{\rho,\sigma,\tau}|;\mathbf{q}_{\text{nc}},\mathbf{p}_{\text{nc}};k_1,k_2,k_3) = \frac{|\alpha|}{2\pi|k_1^2\alpha^2-k_2k_3\beta\gamma|^{\frac{1}{2}}}\int_{\mathbb{R}^2} e^{-i\alpha p^l_{1,\text{nc}}r_1-i\alpha p^l_{2,\text{nc}}r_2}\,\overline{\lambda_{k_1,k_2,k_3}\left(\frac{1}{2}r_1-\frac{q^{l,m}_{1,\text{nc}}}{k_1},\frac{1}{2}r_2-\frac{q^{l,m}_{2,\text{nc}}}{k_1}\right)} \times\chi_{k_1,k_2,k_3}\left(-\frac{1}{2}r_1-\frac{q^{l,m}_{1,\text{nc}}}{k_1},-\frac{1}{2}r_2-\frac{q^{l,m}_{2,\text{nc}}}{k_1}\right)dr_1dr_2, \tag{19}$$

- with the *"noncommuting coordinates"* $\mathbf{q}^{l,m}, \mathbf{p}^{l,m}$, in terms of the phase space coordinates k_1^*, k_2^*, k_3^* and k_4^* can be read off as

$$\begin{aligned}
q^{l,m}_{1,\text{nc}} &= \frac{[k_3\gamma k_2\beta l(1-m)-k_1^2\alpha^2]k_1^*+mk_1k_2\alpha\beta k_4^*}{(k_2\beta k_3\gamma l-k_1^2\alpha^2)},\\
q^{l,m}_{2,\text{nc}} &= \frac{k_1\alpha[k_2\beta k_3\gamma(l+m-lm)-k_1^2\alpha^2]k_2^*+(1-m)k_2\beta(k_3\gamma k_2\beta l-k_1^2\alpha^2)k_3^*}{k_1\alpha(k_2\beta k_3\gamma-k_1^2\alpha^2)},\\
p^l_{1,\text{nc}} &= \frac{k_1\alpha k_3\gamma(1-l)k_2^*+(k_1^2\alpha^2-k_2\beta k_3\gamma l)k_3^*}{k_1^2\alpha^2-k_2\beta k_3\gamma},\\
p^l_{2,\text{nc}} &= \frac{k_1^2\alpha^2k_4^*-k_1\alpha k_3\gamma lk_1^*}{k_1^2\alpha^2-k_2\beta k_3\gamma l}.
\end{aligned} \tag{20}$$

• Wigner function for NCQM in Landau gauge:

$$\begin{aligned} q^{1,0}_{1,\mathrm{nc}} &= k_1^*,\\ q^{1,0}_{2,\mathrm{nc}} &= \frac{k_1\alpha k_2^* + k_2\beta k_3^*}{k_1\alpha},\\ p^{1}_{1,\mathrm{nc}} &= k_3^*,\\ p^{1}_{2,\mathrm{nc}} &= \frac{k_1^2\alpha^2 k_4^* - k_1k_3\alpha\gamma k_1^*}{k_1^2\alpha^2 - k_2k_3\beta\gamma}. \end{aligned} \tag{21}$$

• Wigner function for NCQM in symmetric gauge:

$$\begin{aligned} q^{l_s,\frac{1}{2}}_{1,\mathrm{nc}} &= \frac{(k_1\alpha+\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_1^*-k_2\beta k_4^*}{2\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma}},\\ q^{l_s,\frac{1}{2}}_{2,\mathrm{nc}} &= \frac{(k_1^2\alpha^2-k_2\beta k_3\gamma+k_1\alpha\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_2^*+(k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_3^*}{2(k_1^2\alpha^2-k_2\beta k_3\gamma)},\\ p^{l_s}_{1,\mathrm{nc}} &= \frac{k_1\alpha(k_2\beta k_3\gamma-k_1^2\alpha^2+k_1\alpha\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_2^*+(k_1\alpha k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_3^*}{k_2\beta(k_1^2\alpha^2-k_2\beta k_3\gamma)},\\ p^{l_s}_{2,\mathrm{nc}} &= \frac{k_1\alpha k_2\beta k_4^*-k_1\alpha(k_1\alpha-\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_1^*}{k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma}}. \end{aligned} \tag{22}$$

• Wigner function for NCQM in Landau gauge:

$$\begin{aligned} q_{1,\mathrm{nc}}^{1,0} &= k_1^*, \\ q_{2,\mathrm{nc}}^{1,0} &= \frac{k_1\alpha k_2^* + k_2\beta k_3^*}{k_1\alpha}, \\ p_{1,\mathrm{nc}}^{1} &= k_3^*, \\ p_{2,\mathrm{nc}}^{1} &= \frac{k_1^2\alpha^2 k_4^* - k_1 k_3\alpha\gamma k_1^*}{k_1^2\alpha^2 - k_2k_3\beta\gamma}. \end{aligned} \tag{21}$$

• Wigner function for NCQM in symmetric gauge:

$$\begin{aligned} q_{1,\mathrm{nc}}^{l_s,\frac{1}{2}} &= \frac{(k_1\alpha+\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_1^*-k_2\beta k_4^*}{2\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma}}, \\ q_{2,\mathrm{nc}}^{l_s,\frac{1}{2}} &= \frac{(k_1^2\alpha^2-k_2\beta k_3\gamma+k_1\alpha\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_2^*+(k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_3^*}{2(k_1^2\alpha^2-k_2\beta k_3\gamma)}, \\ p_{1,\mathrm{nc}}^{l_s} &= \frac{k_1\alpha(k_2\beta k_3\gamma-k_1^2\alpha^2+k_1\alpha\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_2^*+(k_1\alpha k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_3^*}{k_2\beta(k_1^2\alpha^2-k_2\beta k_3\gamma)}, \\ p_{2,\mathrm{nc}}^{l_s} &= \frac{k_1\alpha k_2\beta k_4^*-k_1\alpha(k_1\alpha-\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma})k_1^*}{k_2\beta\sqrt{k_1^2\alpha^2-k_2\beta k_3\gamma}}. \end{aligned} \tag{22}$$

Some proposals

Inspired by Fedosov's deformation quantization, one can consider the following problem:

- Fedosov started with an arbitrary finite dimensional symplectic manifold and then considers the Weyl algebra bundle whose base manifold is the underlying symplectic manifold and the fiber at one point is the associative algebra of formal power series in $\hbar$ whose coefficients are the smooth functions on the respective tangent spaces which are of course symplectic vector spaces. Then he studied various geometric aspects of this Weyl algebra bundle by constructing appropriate connections on this bundle using the symplectic connection of the underlying symplectic manifold.
- We like to start with a nilpotent Lie group, find its unitary dual. Then we would look at the foliation of the dual algebra into coadjoint orbits. There will be an appropriate base manifold over which the fibres are the coadjoint orbits and the fibre bundle is precisely the dual algebra associated with the nilpotent group that we started with.

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Now inspired by Fedosov's construction, we want to replace these fibres (coadjoint orbits) with formal power series in parameters that label them (in fact from Kirillov's theory these coadjoint orbits are in 1-1 correspondence with the unitary dual of the nilpotent Lie group that we started with). So, if one of the coadjoint orbits represent an unirrep of the nilpotent group being labeled by 3 parameters, then the corresponding associative algebra will be a formal power series in those 3 parameters. We can term such a bundle as the corresponding nilpotent lie algebra bundle and see how Fedosov's construction turns out in this context.

- In the proposed construction, the base manifold of the "nilpotent Lie algebra bundle" will precisely be the deformation parameter space. And from Kirilov's theory, these parameters describe the smooth foliations of the coadjoint orbits in the dual Lie algebra. I am not sure, if under such construction where the deformation parameters describe such smooth foliation, may address the convergence issues of a formal power series.

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• There are at present 2 worked out (partially) examples that may help one to delve further into such construction. One is the Heisenberg group and the other is the kinematical symmetry group of noncommutative quantum mechanics G_{NC}.

Thank you for your patience!