Noncommutative geometry and Quantum physics on manifolds with boundary

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Motivations: Manifolds with boundary

- Quantum physics on Riemannian manifolds with boundary exhibit novel phenomena even in commutative geometric framework. There is a physical need to understand them, motivated by quantum blackholes and edge states in quantum hall effect, topological insulators, ...
- One expects interesting interplay between noncommutativity and boundary effects. We will explore with specific example..
- In addition such a study is motivated by the fact that in several situations like Quantum Hall effect or near horizon geometry of blackhole need to be understood in the quantum space time context.
- It can also be used to understand white dwarfs which exceed the Chandrasekar limit substantially....
- In the case bose gas and BE condensate the new ideas can inhibit formation of BE condensate...

Riemann....

- It has beeen pointed out by many about the role of noncommutative geometries when we incorporate quantum gravity-Sergio Doplicher... Noncommutative geometry is expected to play a role in near horizon geometry (TRG) of a blackhole as well as near big bang singularity and cosmological constant(STEINACKER).
- The need to go beyond the conventional notions of geometry was anticipated by non other than Riemann himself, given lack of knowledge of physics at infintesimal length scales. As pointed out by Riemann "the metric relations of space in the infinitely small do not conform to hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena...." - Clifford.
- First we will briefly explain issues quantum physics in manifolds with boundary described by commutative geometry with Riemannian metric.

Riemannian manifolds with boundary

- We want to ask whether Laplace Beltrami operator is self adjoint in the space of square integrable functions in this manifold.
- The theory of space of deficiency indices due to Von Neumann is the conventional framework for this question. But that is not helpful due to the divergence of indices.
- We are interested in the Laplace-Beltrami operator on these manifolds to obtain appropriate energy functionals.
- We have to make sure that Laplacian a self adjoint operator on suitable domain of $\mathcal{L}^2$ functions. That will be translated to appropriate boundary conditions.
- The operator is symmetric if $\int \phi^* \{-\Delta^2\} \chi = \int \{-\Delta^2 \phi\}^* \chi$.
- This leads to the boundary term which should vanish with suitable boundary conditions to ensure domains of the operator and its hermitian conjugate are same: $\mathcal{D}_{\Delta^2} = \mathcal{D}_{\Delta^{2\dagger}}$.

Laplace-Beltrami

- It was shown by Asorey, Ibort, Marmo to use the structure of the boundary condions:

$$\phi - i\partial_n\phi = U(\phi + i\partial_n\phi)$$

  where ∂_n stands for the normal derivative to the boundary.
- It was pointed out by them that allowed self-adjoint extensions can be identified with the unitary matrix U.
- $U = \pm 1$ correspond to Neumann and Dirichlet conditions.
- They are extreme points and generic boundary conditions are interesting and bring novel features.
- One of the interesting feature is the possible existence of edge states localised close to the boundary with negative eigenvalue.

Conditions for edge states

- To appreciate further one should divide the Riemannian manifold into a collar neighbourhood and outside called bulk.
- The contribution of the bulk is always positive, whereas that of the collar is semibounded below if a gap condition is obeyed.
- Without getting into details we will establish these in a simple example of $\mathrm{R}^2 \; - \;$ (Disc). We will use a simplified condition known as 'Robin' boundary condition since it preserves rotational invariance.
- This condition is characterised by

$$\psi(\mathrm{r},\theta) \; + \; \kappa \, \partial_{\mathrm{r}}\psi(\mathrm{r},\theta) \; = \; 0.$$

2D Laplacian on R^2 – Disc

- We are interested in finding the bound state solutions of the time-independent Schroedinger equation

$$-\triangle\psi(\mathrm{r},\Phi) = \Lambda\psi(\mathrm{r},\Phi)\,.$$

The solution to the above equation is to be obtained subject to the Robin boundary condition

$$\kappa\psi + \partial_{\vec{\mathrm{n}}}\psi\ |_{\mathrm{R_b}} = 0\,,$$

- $\mathrm{R_b}$ is radius of the disc. We note that κ has dimensions of $(\mathrm{length})^{-1}$ and is a constant for rotational invariance. This parameter would not be present if we use the usual Dirichlet or Neumann boundary conditions (which are the limits of Robin boundary condition corresponding, respectively, to κ^{-1} or $\kappa \to 0$)
- We will see the interplay of R and κ, decides the number of 'boundstates'.

2+1 dimensional spacetime

- In terms of polar coordinates, our eigenvalue equation is:

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial\psi}{\partial \mathrm{r}}\right)+\frac{1}{\mathrm{r}^2}\frac{\partial^2\psi}{\partial\theta^2}=\lambda\psi\,,$$

where we have replaced $\Lambda \rightarrow -\lambda$. For bound states $\lambda > 0$.

- We solve using the ansatz $\psi(\mathrm{r},\theta)=\mathrm{e}^{\mathrm{i}\mathrm{n}\theta}\mathrm{R}(\mathrm{r})$ (with n an integer) resulting in the equation

$$\frac{\mathrm{d}^2\mathrm{R}}{\mathrm{d}\mathrm{r}^2}+\frac{1}{\mathrm{r}}\frac{\mathrm{d}\mathrm{R}}{\mathrm{d}\mathrm{r}}-\frac{\mathrm{n}^2\mathrm{R}}{\mathrm{r}^2}=\lambda\mathrm{R}\,.$$

- We know that the Laplacian (and this can directly be checked), $\left(\hat{\mathrm{O}}=\frac{\mathrm{d}^2}{\mathrm{d}\mathrm{r}^2}+\frac{1}{\mathrm{r}}\frac{\mathrm{d}}{\mathrm{d}\mathrm{r}}-\frac{\mathrm{n}^2}{\mathrm{r}^2}\right)$ is self-adjoint on the domain

$$\mathrm{D}(\hat{\mathrm{O}})=\{\psi|\psi\,\text{continuous and}\,\psi\in\mathcal{L}^2(\mathrm{r_b},\infty),\kappa\psi(\mathrm{r_b})-\psi'(\mathrm{r_b})=0\}$$

2D Laplacian on $\mathrm{R}^2 - \mathrm{D}$: Eigenvalues and solutions

- Explicite solutions are given by modified Bessel's function $\mathrm{I_n}(\sqrt{\lambda}\mathrm{r})$ and $\mathrm{K_n}(\sqrt{\lambda}\mathrm{r})$. Square-integrability on $(\mathrm{R_b}, \infty)$ with measure rdr ($\mathrm{R_b}$ being the boundary), rules out the exponentially growing $\mathrm{I_n}(\sqrt{\lambda}\mathrm{r})$ solution:

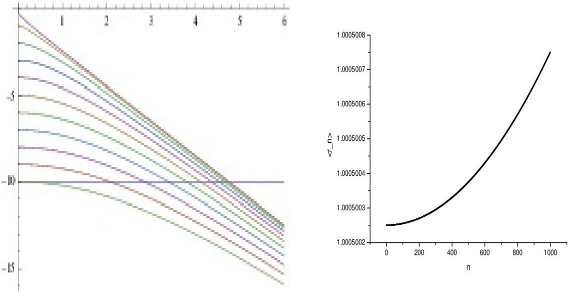
$$\mathrm{R(r)} = \mathrm{cK_n}(\sqrt{\lambda}\mathrm{r}) \,,$$

  where c is a normalization constant.
- Impose the boundary condition at $\mathrm{r} = \mathrm{R_b}$:

$$\kappa = \frac{\sqrt{\lambda}\mathrm{K'_n}(\sqrt{\lambda}\mathrm{R_b})}{\mathrm{K_n}(\sqrt{\lambda}\mathrm{R_b})} \,.$$

- Since $\mathrm{K'_n}(\mathrm{r}) < 0$, we find that there will be bound solutions only for $\kappa < 0$. Graphically the above equation (for different values of n) as a function of $\sqrt{\lambda}$ shows only a finite number of bound solutions for a given value of κ and $\mathrm{R_b}$, given by the intersection of the above mentioned curves with the horizontal line representing the constant value of κ,

2D Laplacian on $R^2 - D$: Eigenvalues and solutions



solutions and expectation values $< r_n >, \kappa = 100, R_b = 1$.

We have various energy levels as well as the spread of wavefunction near the boundary. It is clear all the bound state solutions are close to R_b. Using approximations: $K_\alpha(x) \approx \frac{\Gamma(\alpha)}{2}\left(\frac{2}{x}\right)^\alpha$ we can also establish the number of boundstates as: $N \simeq \kappa R_b$.

Fuzzy Disc - Brief review

- We define $\mathcal{A}_\theta(\mathrm{R}^2)$ as functions with quantised coordinates $\hat{x}$ and $\hat{y}$, such that: $[\hat{x}, \hat{y}] = \frac{i\theta}{2}$.
- $a \; \{a^\dagger\} = \hat{x} \pm i\hat{y}$ are the annihilation and creation operators serving the role of complex coordinates z.
- We define the map between functions on R^2 and operators on the Moyal plane, $f(z, \bar{z}) = \sum f^T_{mn} \bar{z}^m z^n$ can be associated with: $\hat{f} = \sum f_{mn} {a^\dagger}^m a^n$. (We have used normal ordering).
- Inverse maps from operators on Moyal plane to functions can be given using coherent states. $a\,|z\rangle = z\,|z\rangle$. Inverse map from operators $\hat{f}$ to functions:

$$f(z, \bar{z}) = \langle z|\hat{f}|z\rangle$$

- One uses number operator basis $N\,|n\rangle = a^\dagger a|n\rangle = n|n\rangle$.

Review - contd..

- In this this the operator can be written as $\hat{f} \;=\; \sum f_{mn}|m\rangle\,\langle n|$.
- We need the inner product between the bases:

$$\langle z|n\rangle \;=\; e^{-\frac{|z|^2}{2\theta}}\frac{\bar{z}^n}{\sqrt{n!\theta^n}}.$$

- This can be used to relate the operators in the two series.

$$f_{mn} \;=\; \sum_{0}^{\min(m,n)} f^T_{m-q,n-q}\frac{\sqrt{m!n!\theta^{m+n}}}{\theta^q q!}$$

- We can obtain a new expression for the function $f(z,\bar{z})$

$$f(z,\bar{z}) \;=\; \langle z|\hat{f}|z\rangle \;=\; e^{\frac{|z|^2}{\theta}}\sum f_{mn}\frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}}$$

- We have set up the machinary required for Moyal plane from where the fuzzy disc should be carved out.

Fuzzy Disc

- Fuzzy Disc sub algebra (as explained by Lizzi) can be obtained by the projection operator $\hat{P}_\theta^N = \sum^N |n\rangle\langle n|$.
- This function for this operator can be computed as:

$$P_\theta^N(z, \bar{z}) = \langle z|\hat{P}_\theta^N|z\rangle = e^{\frac{-r^2}{\theta}} \sum^N \frac{r^{2n}}{n!\theta^n}$$

- The finite sum can be summed without difficulty as:

$$P_\theta^N = \frac{\Gamma(N+1, \frac{r^2}{\theta})}{\Gamma(N+1)}$$

- $\Gamma(N, r)$ is incomplete Gamma function. In the limit $N \to \infty$, $\theta \to 0$ with the requirement $N\theta = R^2$ gives functions (subclass) on disc.
- The projection gives truncated function:

$$f_\theta^N = e^{\frac{-r^2}{\theta}} \sum^N f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}}$$

Derivatives

- The derivative in Moyal plane $\partial_z f = \frac{1}{\theta}\langle z|[\hat{f}, a^\dagger]|z\rangle$ should be modified to:

$$\partial_z f_\theta^N = \frac{1}{\theta}\langle z|P_\theta^N[P_\theta^N \hat{f} P_\theta^N, a^\dagger]P_\theta^N|z\rangle$$

- The eigenvalue equation for the Laplacian can be written as:

$$-\Delta_N^2 f_\theta^N = -\frac{4}{\theta^2}P_\theta^N[a, [P_\theta^N \hat{f} P_\theta^N, a\dagger]]P_\theta^N = \lambda^2 f_\theta^N$$

- If $k_{n,m}$ satisfy $J_n(kR) = 0$ then we can write down the eigensolutions of the 'Fuzzy' Laplacian. They are:

$$f_{n,m}^N = e^{in\phi}\left(\frac{k_{n,m}^2}{4}\right)^{n/2} e^{-Nr^2}\sum_n^N \sum_0^{j-n}\left(\frac{-k_{n,m}^2}{4N}\right)^s \frac{N^{j-n}}{s!(n+s)!(j-s-n)!} r^{2j-n}$$

- In this framework Dirichlet boundary conditions are automatically imposed.

Robin boundary conditions

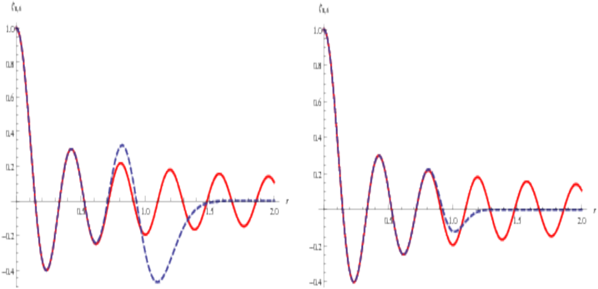
- Robin boundary conditions are: $(f + \kappa\partial_r f)|_R = 0$. This translates $(J_n(kr) + \kappa\partial_r J_n(kr))\,|_R = 0$.
- If $k_{n,m}$ staisfies the above equation then the eigenvalue is $(\frac{k_{n,m}}{R})^2$. These are still positive and go over to smoothly to Dirichlet condition.
- The novel solutions are with $k^2 < 0$. They satisfy

$$(I_n(ikr) + \kappa\partial_r I_n(ikr))\,|_R = 0.$$

  These are finite in number upto $n < n_{max}$.
- The eigenvectors are:

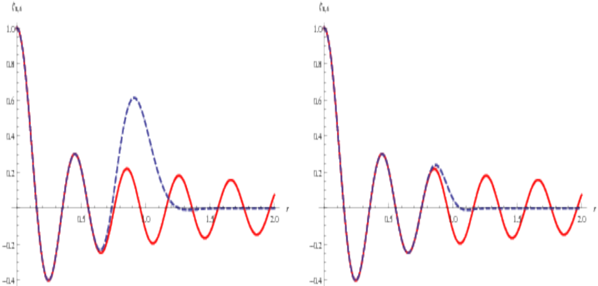
$$I_n(ikr)e^{in(\phi+\pi/2)} = e^{in(\phi+\pi/2)} \left(\frac{ik_{|n|}r)}{2}\right)^{|n|} \sum \frac{(ik_{|n|})^s\ r^{2s}}{2^{2s}s!(|n|+s)!}$$

Figures



Comparison of commutative (red) and fuzzy (blue) with robin boundary conditions N=10, κ = 5, N=30, κ= 5 and positive energy

Figures



Comparison of commutative (red) and fuzzy (blue) with robinboundary conditions N=10, κ = .1, N=30, κ= .1 and positive energy

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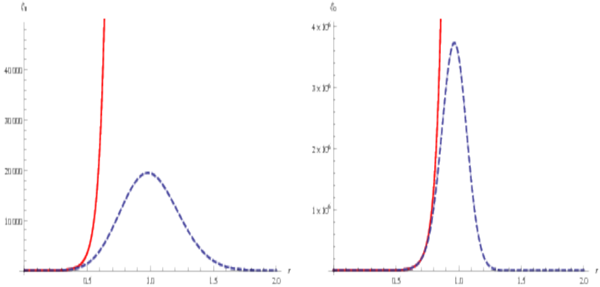


Comparison of commutative (red) and fuzzy (blue) with robinboundary conditions N=10, κ = .1, N=30, κ= .1 and negative energy=edge states

Edge states

- We can truncate the series using the projectors and obtain 'eigen spectrum'. The edge bound states are the novel whose width is fized by the scale κ from the boundary.
- These are really the edge states in the fuzzy case unlike those obeying 'Dirichlet' which have nonzero value close to the boundary. This is because they are unstable and will move inwards to the 'bulk' under perturbation.
- To obtain the correct boundary in the fuzzy case one should apply a suitable infinite potential in the Moyal plane.
- Following Scholtz etal we will explain this.

Potential well in Moyal plane

- One goes back to Moyal plane. We need to carve out infinite potential well. We consider piecewise potential.
- Define

$$\mathrm{P_{N,M}} = \sum_{\mathrm{N}}^{\mathrm{N+M}} |\mathrm{n}\rangle\langle\mathrm{n}|$$

Then we can create 'well' by considering the potential

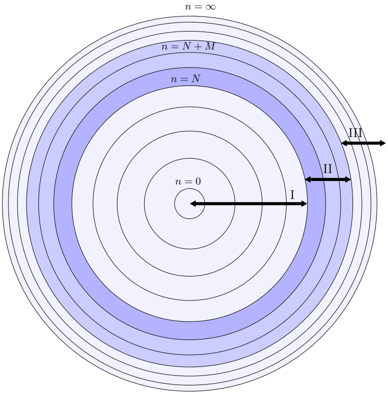
$$\mathrm{V} \;=\; \mathrm{V_N P_{0,N}} \;+\; \mathrm{V_{N,M} P_{N,M}} \;+\; \mathrm{V_{M,\infty} P_{M,\infty}}$$

- We get in the limit $\mathrm{N} \to \infty,\ \theta \to 0$ such that $\mathrm{N}\theta \;=\; \mathrm{R}^2$ a disc if $\mathrm{V_{N,M}} \;= \infty, \mathrm{V_N} \;=\; 0.$
- We solve the equations in different regions and match the solutions at the 'boundaries'

Potential well

- Non-commutative potential well $V_q(\hat{x}_1, \hat{x}_2)$ (potential $\neq 0$ for radius $r_{N+1} \leqslant r \leqslant r_{N+M}$)

$$V_q(\hat{x}_1, \hat{x}_2) = V_2(M) Q_2^q$$



Heuristic picture of Moyal plane

Solutions

- Since the Schrodinger equation with constant potential is easy we will get:

$$\langle n| \quad \hat{\psi} \quad |n+m\rangle = c_1\left(m, \frac{k^2r^2}{4n}\right)\sqrt{m}n^{\frac{-m}{2}}L_n^m\left(\frac{-k^2r^2}{4n}\right)$$
$$+c_2\left(m, \frac{k^2r^2}{4n}\right)\sqrt{\frac{2\pi}{m!}}e^{-n}n^{\frac{2n+m+1}{2}}U\left(n+1, 1-m, \frac{-k^2r^2}{4n}\right)$$

- Here L_n^m and U are the associate Laguerre Polynomials and the Kummer functions. They go to in the limit to Bessel functions of the appropriate kind.
- One needs to match the solutions at the boundaries for the problem.
- In the above potential the role of M indicates how the solutions from one region decays into other region. This measures the freedom of Robin boundary condition parameter κ in the commutative limit.

NC Disc

- To see commutative limit: Define $\theta = \mathrm{r}^2/(2(\mathrm{N}+1))$ taking $\mathrm{n} = \mathrm{N}+1$ as the boundary, we get

$$(\mathrm{N}+\mathrm{m})\frac{\mathrm{L}_{\mathrm{N}+1}^{\mathrm{m}-1}(-\frac{\mathrm{r}^2\lambda}{2(\mathrm{N}+1)})}{\mathrm{L}_{\mathrm{N}+1}^{\mathrm{m}}(-\frac{\mathrm{r}^2\lambda}{2(\mathrm{N}+1)})} + \frac{\mathrm{r}^2\lambda}{2(\mathrm{N}+1)}\frac{\mathrm{U}(\mathrm{N}+2, 2-\mathrm{m}, \frac{\mathrm{r}^2\lambda}{2(\mathrm{N}+1)})}{\mathrm{U}(\mathrm{N}+2, 1-\mathrm{m}, \frac{\mathrm{r}^2\lambda}{2(\mathrm{N}+1)})} = -\mathrm{g}. \tag{4.1}$$

The commutative limit leads:

$$\mathrm{r}\sqrt{\frac{\lambda}{2}}\left\{\frac{\mathrm{I}_{\mathrm{m}-1}(\mathrm{r}\sqrt{2\lambda})}{\mathrm{I}_{\mathrm{m}}(\mathrm{r}\sqrt{2\lambda})} + \frac{\mathrm{K}_{\mathrm{m}-1}(\mathrm{r}\sqrt{2\lambda})}{\mathrm{K}_{\mathrm{m}}(\mathrm{r}\sqrt{2\lambda})}\right\} = -\mathrm{g}.$$

where $\mathrm{g} = \theta \mathrm{V}$ and V is the depth of the potential well.

- Exactly identical equation as given in TRG,Rakesh.
- Provides the correct number of ‘edge states’ or ‘localised bound states’ near the boundary.

Squashed fuzzy sphere

- Blackhole descriptions require some input from quantum mechanics at some level to explain the Statistical mechanics behaviour. The horizon as boundary has played a role at that point.
- In principle stars at white dwarf stage also may require such a modelling. Reason being the the limit on the mass of of dwarf comes with a balance between statistics pressure of electrons and gravity. But if the limit is violated it can be speculated to ‘less than fermion’ behaviour of electrons and increased mass.
- Such a behaviour is natural in NC geometry was known. But can a Moyal disc which is analogous to QHE has constant noncommutativity everywhere. On the other hand squashed fuzzy spheres seem to be a natural candidate (Stefan Andronache and Harold Steinacker).
- Fuzzy sphere is described by the algebra $[x_i, x_j] = i\lambda\epsilon_{ijk}\, x_k$. We can consider x_i to be proportional to angular momentum generators.

Squashed fuzzy sphere

- Since $\sum x_i^2 = r^2$ we look for projection on to x_1, x_2 plane treating x_3 as dependent. $x_3 = \pm\sqrt{r^2 - x_1^2 - x_2^2}$.
- Then Poisson structure (upper hemisphere) is:

$$\omega = \frac{2\ r}{\lambda\sqrt{r^2 - x_1^2 - x_2^2}}\, dx_1 \wedge dx_2$$

- This leads to the NC algebra:

$$[x_i, x_j] = \theta_{ij} = \frac{\sqrt{r^2 - x_1^2 - x_2^2}}{r}\lambda\ \epsilon_{ij}$$

- The interesting outcome is the 'effective magnetic field ' varies and at the edge we reach almost commutative space.
- Can be mapped to Landau level problem..
- Spectrum can be worked out for this situation and would be interesting to check the degeneracy and pressure can be modelled to see the effect....Ongoing work..(Discussions with Banibrata)

Solitons in NC space

- Noncommutative solitons in $2+1$ dimensional field theories were considered by Gopakumar, Minwalla, Strominger. They use the Hamiltonian (in rescaled variable)

$$\mathrm{H} = \int \mathrm{d}^2\mathrm{z}(\phi(-\Delta^2)\phi + \theta \mathrm{V}_*(\phi))$$

- In the infinite θ limit the potential term dominates and minimising that leads to soliton solutions.
- If $\mathrm{V}_*(\phi) = -\mathrm{m}^2\phi * \phi + \lambda\phi * \phi * \phi$, then $\mathrm{V}' = -2\mathrm{m}\phi + 3\lambda\phi * \phi$ which can be solved using projectors.
- In the finite θ case, derivative term along with the potential term gives for large θ soliton solutions.
- We also know there are no solitons in $\theta = 0$ commutative case.
- There exists a critical θ_c below which the NC solitons are unstable and we end up with commutative limits.

Solitons in Fuzzy Disc

- The situation continues in fuzzy disc with Dirichlet boundary conditions. There are bulk solitons for large θ limit and large radius R.
- There are also angular solitons in the fuzzy disc with Dirichlet boundary conditions.
- The situation changes with Robin boundary conditions.
- With this boundary condition the contribution of the Laplacian is no longer positive definite. This is because of the negative energy eigen modes localised close to the boundary.
- They are bounded below and finite in number. Now there are two length parameters θ and κ. The NC parameter contributes to the potential where as κ contributes to the Laplacian.
- We have in addition to the bulk solitons edge solitons localised close to the boundary.

Solitons in Fuzzy Disc

- These solitons survive even for low θ unlike the bulk ones since they depend mainly on the boundary properties.
- There is no criical θ_c for NC space which support solitons.
- They are low mass solitons since the θ play a small role.
- Riemannian manifold with boundary bring new features even in commutative theory due to the novel boundary conditions.
- Noncommutative deformations of BTZ blackhole leads to Noncommutative cylinder $R_+ \otimes S^1$ with the horizon at the boundary.

BTZ deformations

- It is defined by the relation: $[\mathrm{z}, \mathrm{e}^{\mathrm{i}\phi}] \; = \; \mathrm{e}^{\mathrm{i}\phi}$. Here z corresponds to the 'time'.
- The quantum theory on NC cylinder can be related to that of Disc through conformal transformations in the commutative limit.
- The discretisation of the cylinder and field theory simulations show novel features.
- Novel stripe phases where translational invariance is broken can be seen.
- There are also stripes on the boundary which can be due to novel boundary conditions.
- Based on works in collaboration with Rakesh Tibrewala, V. Parameswaran Nair, Sanatan Digal, Kumar Gupta, Xavier Martin Rahul Srivastava and Chaoba Yendrembam.

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