Dissipation and Noncommutativity

International Conference on "NONCOMMUTATIVE GEOMETRY : PHYSICAL AND MATHEMATICAL ASPECTS OF QUANTUM SPACE-TIME AND MATTER"

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My talk is based on the following work:

- **Connecting dissipation and non-commutativity :A Bateman system case study.** Sayan Kumar Pal, Partha Nandi, Biswajit Chakraborty.-(Phys. Rev. A 97, 062110, 2018)

Plan of the talk :

- Dissipative systems in classical and quantum mechanics
- Time-independent Lagrangian formulation of dissipative systems - The Bateman oscillator
- Hamiltonian formulation
- Path integral quantization of a generalized Bateman system embedded in the Moyal plane
- Canonical formulation
- Results and Discussions
- Summary

Introduction to Dissipative systems

The equation of motion for the one-dimensional damped harmonic oscillator (D.H.O) is

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \tag{1}$$

If $R = \frac{4\omega^2}{\gamma^2} > 1$, the motion is oscillatory with exponentially decaying amplitude. Otherwise, the motion is nonoscillatory i.e. overdamped.
Since the system (1) is dissipative, a straightforward Lagrangian description leading to a consistent canonical quantization is not available.

Time-independent Lagrangian formulation

We consider (1) along with its time reversed image

$$\ddot{y} - \gamma \dot{y} + \omega^2 y = 0 \qquad (2)$$

making the composite system (**Bateman oscillator**) conservative. The Lagrangian is:

$$L = \dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - \omega^2 xy \quad (\textit{indirect representation}) \qquad (3)$$

where x is the D.H.O coordinate and y corresponds to its time-reversed counterpart.
On introducing the rotated coordinates ($x_1 = \frac{x+y}{\sqrt{2}}$, $x_2 = \frac{x-y}{\sqrt{2}}$), the above Lagrangian can be written in a compact notation as

$$L = \frac{1}{2} g_{ij}\dot{x}_i\dot{x}_j - \frac{\gamma}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{\omega^2}{2} g_{ij}x_i x_j \qquad (4)$$

where g_{ij} is the pseudo - Euclidean metric: g_{11} = -$g_{22} = 1$ and $g_{12} = 0$.

Hamiltonian formulation of the problem

Now the Hamiltonian corresponding to (4) is :

$$H = \frac{1}{2}(p_1 - \frac{\gamma x_2}{2})^2 - \frac{1}{2}(p_2 + \frac{\gamma x_1}{2})^2 + \frac{1}{2}\omega^2(x_1^2 - x_2^2) \qquad (5)$$

Clearly we notice that the hamiltonian H is a difference of two positive hamiltonians H_1 and H_2 -

$$H = H_1 - H_2 \qquad (6)$$

so that it is not bounded from below.

We include additional interactions to (1) and (2)

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = -\epsilon y - \eta\ddot{y} \tag{7}$$

$$\ddot{y} - \gamma\dot{y} + \omega^2 y = -\epsilon x - \eta\ddot{x} \tag{8}$$

where, ϵ and η are constant parameters.

We call it the generalized coupled dho problem in 2D.

The corresponding Lagrangian and Hamiltonian written in terms of the (x_1, x_2) coordinates are given by-

$$L = \frac{(\eta+1)}{2}\dot{x}_1^2 + \frac{(\eta-1)}{2}\dot{x}_2^2 - \frac{\gamma}{2}(x_1\dot{x}_2 - x_2\dot{x}_1) - \frac{(\epsilon+\omega^2)}{2}x_1^2 - \frac{(\epsilon-\omega^2)}{2}x_2^2$$

$$H = \frac{p_1^2}{2(\eta+1)} + \frac{p_2^2}{2(\eta-1)} + \frac{\gamma}{2}\left(\frac{x_1 p_2}{\eta-1} - \frac{x_2 p_1}{\eta+1}\right) + \left(\frac{\gamma^2}{8(\eta-1)} + \frac{(\epsilon+\omega^2)}{2}\right)x_1^2 + \left(\frac{\gamma^2}{8(\eta+1)} + \frac{(\epsilon-\omega^2)}{2}\right)x_2^2$$

The positive definiteness of the Hamiltonian can now be ensured if we demand $\eta > 1$ and $\epsilon > \omega^2$.

Further we implement a canonical transformation given by:

$$x_1 \longrightarrow \left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{4}} x_1 \ , \quad p_1 \longrightarrow \left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{4}} p_1$$

$$x_2 \longrightarrow \left(\frac{\eta-1}{\eta+1}\right)^{\frac{1}{4}} x_2 \ , \quad p_2 \longrightarrow \left(\frac{\eta+1}{\eta-1}\right)^{\frac{1}{4}} p_2 \quad .$$

In terms of these transformed variables, the Hamiltonian can be rewritten as -

$$H = \frac{p_1^2}{2\mu} + \frac{p_2^2}{2\mu} + \frac{\gamma}{2\mu}(x_1 p_2 - x_2 p_1) + \frac{1}{2}\mu\omega_1^2 x_1^2 + \frac{1}{2}\mu\omega_2^2 x_2^2 \quad (9)$$

where $\mu = \sqrt{(\eta+1)(\eta-1)}$ and the frequencies are given by -

$$\omega_1^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon+\omega^2}{\eta+1} \ , \quad \omega_2^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon-\omega^2}{\eta-1} \quad .$$

Finally, we are in a stage to carry out the quantization effectively through the path integral scheme by promoting the phase space variables to the level of operators satisfying noncommutative Heisenberg algebra.

Noncommutative QM

- The NC Heisenberg algebra is:

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij} \; ; \quad [\hat{p}_i, \hat{p}_j] = 0 \; ; \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \qquad (10)$$

  Here θ denotes the spatial noncommutative parameter.
- Auxiliary Hilbert space $\mathcal{H}_c$ is defined as,

$$\mathcal{H}_c = Span\left\{ |n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle \; ; \; b = \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2\theta}} \right\}$$

- The quantum Hilbert space $\mathcal{H}_q$ is defined as the space of Hilbert-Schmidt operators acting on $\mathcal{H}_c$:

$$\mathcal{H}_q = \left\{ \psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}\left(\mathcal{H}_c\right), \; \mathrm{tr_c}(\psi^\dagger(\hat{x}_1, \hat{x}_2)\psi(\hat{x}_1, \hat{x}_2)) < \infty \right\}$$

- $\mathcal{H}_q$ furnishes a representation of the entire Heisenberg algebra through the action

$$\hat{X}_i|\psi) = |\hat{x}_i\psi) \; , \quad \hat{P}_i|\psi) = \frac{\hbar}{\theta}\left[\epsilon_{ij}\hat{x}_j, |\psi)\right]$$

- We can construct a state in quantum Hilbert space:

$$|z,\bar{z}) \equiv |x_1,x_2) = \frac{1}{\sqrt{2\pi\theta}}|z\rangle\langle z|. \quad (11)$$

where, $|z\rangle = e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle$ and $z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2)$ is a dimensionless complex number.

- The 'position' representation of a state $|\psi) = \psi(\hat{x}_1,\hat{x}_2)$ can be constructed as

$$(x_1,x_2|\psi) = \frac{1}{\sqrt{2\pi\theta}}tr_c(|z\rangle\langle z|\psi(\hat{x}_1,\hat{x}_2)) = \frac{1}{\sqrt{2\pi\theta}}\langle z|\psi(\hat{x}_1,\hat{x}_2)|z\rangle. \quad (12)$$

- We now introduce the normalized momentum eigenstates

$$|p) = \sqrt{\frac{\theta}{2\pi\hbar^2}}e^{i\sqrt{\frac{\theta}{2\hbar^2}}(\bar{p}b+pb^\dagger)} \;;\; \hat{P}_i|p) = p_i|p) \quad (13)$$

satisfying the completeness relation

$$\int d^2p\ |p)(p| = 1_Q\ . \quad (14)$$

Formalism - Path integral quantization

- The wave-function of a "free particle" on the noncommutative plane is given by

$$(z,\bar{z}|p) = \frac{1}{\sqrt{2\pi\hbar^2}} e^{-\frac{\theta}{4\hbar^2}\bar{p}p} e^{i\sqrt{\frac{\theta}{2\hbar^2}}(p\bar{z}+\bar{p}z)} \ . \qquad (15)$$

The completeness relation for the position eigenstates $|z,\bar{z})$ reads

$$\int 2\theta dz d\bar{z} \ |z,\bar{z}) \star (z,\bar{z}| = \int dx_1 dx_2 \ |x_1,x_2) \star (x_1,x_2| = 1_Q$$

- The propagation kernel on the noncommutative plane reads -

$$(z_f,t_f|z_0,t_0) = \lim_{n\to\infty} \int \prod_{j=1}^{n} (dz_j d\bar{z}_j) \ (z_f,t_f|z_n,t_n) \star_n (z_n,t_n|....|z_1,t_1) \star_1 (z_1,t_1|z_0,t_0) \ . \ ($$

-

$$
\begin{aligned}
(z_{j+1}, t_{j+1}|z_j, t_j) &= (z_{j+1}|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|z_j) \\
&= \int_{-\infty}^{+\infty} d^2 p_j \; e^{-\frac{\theta}{2\hbar^2}\bar{p}_j p_j} e^{i\sqrt{\frac{\theta}{2\hbar^2}}\left[p_j(\bar{z}_{j+1}-\bar{z}_j)+\bar{p}_j(z_{j+1}-z_j)\right]} \\
&\quad \times e^{-\frac{i}{\hbar}\epsilon[\frac{\bar{p}_j p_j}{2\mu}+\frac{\mu\theta}{4}(\omega_1^2-\omega_2^2)(\bar{z}_{j+1}^2+z_j^2)+\frac{\mu\theta}{4}(\omega_1^2+\omega_2^2)(2\bar{z}_{j+1}z_j+1)-\frac{i\gamma}{2\mu}\sqrt{\frac{\theta}{2}}(p_j\bar{z}_{j+1}-\bar{p}_j z_j)]}.
\end{aligned}
$$

- Substituting the above expression in eq.(16) and computing the star products, we obtain

$$
\begin{aligned}
(z_f, t_f|z_0, t_0) = \quad & \lim_{n\to\infty}\int \prod_{j=1}^{n}(dz_j d\bar{z}_j)\prod_{j=0}^{n} d^2 p_j \\
& \exp\left(\sum_{j=0}^{n}\left[\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}\left[p_j\left\{\left(1+\frac{i\epsilon\gamma}{2\mu}\right)\bar{z}_{j+1}-\bar{z}_j\right\}+\bar{p}_j\left\{z_{j+1}-\left(1+\frac{i\epsilon\gamma}{2\mu}\right)z_j\right\}\right]+\sigma p_j \bar{p}_j\right.\right. \\
& \qquad \left.+\frac{\theta}{2\hbar^2}\sum_{j=0}^{n-1}p_{j+1}\bar{p}_j-\frac{i}{\hbar}\epsilon V(\bar{z}_{j+1}, z_j)\right)
\end{aligned}
$$

where $\sigma = -\left(\frac{i\epsilon}{2\mu\hbar} + \frac{\theta}{2\hbar^2}\right)$ and
$V(\bar{z}_{j+1}, z_j) = \frac{\mu\theta}{4}(\omega_1^2 - \omega_2^2)(\bar{z}_{j+1}^2 + z_j^2) + \frac{\mu\theta}{4}(\omega_1^2 + \omega_2^2)(2\bar{z}_{j+1}z_j + 1)$.

- On executing the momentum integral, we obtain

$$(z_f, t_f|z_0, t_0) = \lim_{n\to\infty} A \int \prod_{j=1}^{n} (dz_j d\bar{z}_j) \exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right)$$

$$\times \exp\left(\frac{\theta}{2\hbar^2}\sum_{l=0}^{n}\sum_{r=0}^{n}\left\{\left(1+\frac{i\epsilon\gamma}{2\mu}\right)\bar{z}_{l+1}-\bar{z}_l\right\} M^{-1}_{lr}\left\{z_{r+1}-\left(1+\frac{i\epsilon\gamma}{2\mu}\right)z_r\right\}\right)$$

$$\times \exp\left(-\frac{i}{\hbar}\epsilon\sum_{j=0}^{n} V(\bar{z}_{j+1}, z_j)\right)$$

On taking the limit $\epsilon \to 0$ and performing the sum over k, we finally get -

$$(z_f, t_f|z_0, t_0) = A\exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right)\int_{z(t_0)=z_0}^{z(t_f)=z_f} \mathcal{D}z\mathcal{D}\bar{z}\exp\left(\frac{i}{\hbar}S\right)$$

where the action S is given as follows :

$$S = \int_{t_0}^{t_f} dt \quad \left[\frac{\theta}{2}\left\{\dot{\bar{z}}(t)+\frac{i\gamma}{2\mu}\bar{z}(t)\right\}\left(\frac{1}{2\mu}+\frac{i\theta}{2\hbar}\partial_t\right)^{-1}\left\{\dot{z}(t)-\frac{i\gamma}{2\mu}z(t)\right\}\right.$$

$$\left. -\frac{\mu\theta}{2}\left(\omega_1^2+\omega_2^2\right)\bar{z}(t)z(t)-\frac{\mu\theta}{4}(\omega_1^2-\omega_2^2)\left(z^2(t)+\bar{z}^2(t)\right)\right]$$

Results - General Solution

- The equations of motion following from above action are :

$$\ddot{x}_1 + \left\{\frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_2^2\right\}\dot{x}_2 + \left\{\omega_1^2 - \frac{\gamma^2}{4\mu^2}\right\}x_1 = 0 \quad . \qquad (17)$$

$$\ddot{x}_2 - \left\{\frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_1^2\right\}\dot{x}_1 + \left\{\omega_2^2 - \frac{\gamma^2}{4\mu^2}\right\}x_2 = 0 \quad . \qquad (18)$$

- In contrast, the classical equations of motion as obtained from the Hamiltonian (9) are -

$$\ddot{x}_1 + \frac{\gamma}{\mu}\dot{x}_2 + \left\{\omega_1^2 - \frac{\gamma^2}{4\mu^2}\right\}x_1 = 0 \quad . \qquad (19)$$

$$\ddot{x}_2 - \frac{\gamma}{\mu}\dot{x}_1 + \left\{\omega_2^2 - \frac{\gamma^2}{4\mu^2}\right\}x_2 = 0 \quad . \qquad (20)$$

- The system of equations (17) and (18) can be solved simultaneously to yield the following characteristic frequencies -

$$\nu_{\pm} = \sqrt{\frac{\Omega_1^2 + \Omega_2^2 + \gamma_1\gamma_2}{2} \pm \frac{1}{2}\sqrt{\gamma_1\gamma_2(2\Omega_1^2 + 2\Omega_2^2 + \gamma_1\gamma_2) + (\Omega_1^2 - \Omega_2^2)^2}} \quad (21)$$

where,

$$\gamma_1 = \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_2^2 \quad , \quad \gamma_2 = \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_1^2. \quad (22)$$

and,

$$\Omega_1^2 = \omega_1^2 - \frac{\gamma^2}{4\mu^2} \quad , \quad \Omega_2^2 = \omega_2^2 - \frac{\gamma^2}{4\mu^2}. \quad (23)$$

Canonical formalism

- We use the linear transformation-

$$\hat{X}_i = \hat{X}_i^{\,c} - \frac{\theta}{2\hbar}\epsilon_{ij}\hat{P}_j \tag{24}$$

connecting the phase space operators $(\hat{X}_i, \hat{P}_i) \longrightarrow (\hat{X}_i^{\,c}, \hat{P}_i)$.

- The Hamiltonian (9) now becomes :

$$\hat{H} = \frac{\hat{P}_1^{\,2}}{2\mu_1} + \frac{\hat{P}_2^{\,2}}{2\mu_2} + \frac{1}{2}\mu(\omega_1^2 \hat{X}_1^{c\,2} + \omega_2^2 \hat{X}_2^{c\,2}) + \frac{\gamma_2}{2}\hat{X}_1^c\hat{P}_2 - \frac{\gamma_1}{2}\hat{X}_2^c\hat{P}_1$$

where

$$\mu_1 = \frac{\mu}{(1 - \frac{\gamma\theta}{2\hbar} + \frac{\mu^2\theta^2\omega_2^2}{4\hbar^2})} \; ; \qquad \mu_2 = \frac{\mu}{(1 - \frac{\gamma\theta}{2\hbar} + \frac{\mu^2\theta^2\omega_1^2}{4\hbar^2})}$$

To diagonalise the Hamiltonian we introduce the following canonical transformation-

$$\begin{pmatrix} \hat{X}_1^c \\ \hat{X}_2^c \\ \hat{P}_1 \\ \hat{P}_2 \end{pmatrix} = \begin{pmatrix} acosu & 0 & 0 & \frac{1}{b}sinu \\ 0 & acosu & \frac{1}{b}sinu & 0 \\ 0 & -bsinu & \frac{1}{a}cosu & 0 \\ -bsinu & 0 & 0 & \frac{1}{a}cosu \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{\pi}_1 \\ \hat{\pi}_2 \end{pmatrix} \quad (25)$$

Therefore, we have

$$\hat{H} = \sigma_1^2 \hat{\pi}_1^2 + \sigma_2^2 \hat{\pi}_2^2 + k_1^2 \hat{q}_1^2 + k_2^2 \hat{q}_2^2 + \lambda_1 \hat{q}_1 \hat{\pi}_2 + \lambda_2 \hat{q}_2 \hat{\pi}_1 \ . \quad (26)$$

Setting $\lambda_1 = 0 = \lambda_2$, we get the diagonalised form of the Hamiltonian which basically is a 2D harmonic oscillator. And other coefficients depend on the ratio of a and b.

Spectrum

From canonical quantization we finally arrive at the diagonalised form of the Hamiltonian as :

$$\hat{H} = 2\hbar k_1\sigma_1(\hat{N}_1 + \frac{1}{2}) + 2\hbar k_2\sigma_2(\hat{N}_2 + \frac{1}{2}) \quad ; \ \hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1, \ \hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2$$

The energy eigen-frequencies are $\Omega_+ = 2k_1\sigma_1, \ \Omega_- = 2k_2\sigma_2$
Note: The real spectrum of the Hamiltonian is obtained by taking vanishing limits of η, ϵ which forces one to impose $n_1 = n_2$ for a physical subspace of the Hilbert space.

Restoring the Bateman form

- Finally, on re-writing the equations (17) and (18) in terms of the original coordinates x, y,

$$\ddot{x} + \eta\ddot{y} + (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar} - \frac{\epsilon\eta\theta}{\hbar})\dot{x} + (\frac{\epsilon\theta}{\hbar} - \frac{\eta\theta\omega^2}{\hbar})\dot{y} + \epsilon y + \omega^2 x = 0 \quad . \quad (27)$$

and,

$$\ddot{y} + \eta\ddot{x} - (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar} - \frac{\epsilon\eta\theta}{\hbar})\dot{y} - (\frac{\epsilon\theta}{\hbar} - \frac{\eta\theta\omega^2}{\hbar})\dot{x} + \epsilon x + \omega^2 y = 0 \quad , \quad (28)$$

Now on taking the limit $\eta = 0 = \epsilon$, we get-

$$\ddot{x} + (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar})\dot{x} + \omega^2 x = 0 \quad . \quad (29)$$

and,

$$\ddot{y} - (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar})\dot{y} + \omega^2 y = 0 \quad . \quad (30)$$

Discussions

. Even if $\gamma = 0$ initially, γ_R is non-zero: $\gamma_R = \frac{\theta\omega^2}{\hbar}$. This indicates that quantum effects along with NC can induce damping.

. On the other hand, if $\frac{\gamma^2}{4} > \omega^2$, we can fine-tune θ, taken as a free parameter, to the following value: $\theta_c = \frac{\gamma\hbar}{\frac{\gamma^2}{4} - \omega^2}$, so that $\gamma_R = 0$.

Spectrum of the Bateman system

The spectrum of the pure Bateman oscillator $(\eta = 0 = \epsilon)$ turns out to be

$$\lambda_{\pm}^{R} = i\frac{\gamma_R}{2} \pm \sqrt{\omega^2 - \frac{\gamma_R^2}{4}} \tag{31}$$

Summary

- Firstly we would like to mention that we have addressed two different problems at one stroke, namely the Landau problem with anisotropic oscillator potentials and the Bateman oscillators by considering our "master equations".
- We have successfully carried out the quantization of a dissipative system both in the path integral and canonical schemes.
- There is an additional NC contribution towards damping - a SHO in NC space can behave as a damped harmonic oscillator. Therefore, noncommutativity can lead to dissipation !
- An indication is shown that an original dissipative theory in commutative space can be mapped to a non-dissipative NC theory, hinting at a possible 'duality' between these two aspects!

THANK YOU

One can attempt to quantize this system following t'hooft.

$$H = \frac{1}{4\rho}(\rho + H)^2 - \frac{1}{4\rho}(\rho - H)^2 = H_1 - H_2 \qquad (32)$$

where we need to have $\{H_1, H_2\} = \{\rho, H\} = 0$ To get the lower bound for the Hamiltonian one thus imposes the constraint condition onto the Hilbert space: $H_2 |\psi\rangle = 0$, which projects out the states responsible for the negative part of the spectrum.

Therefore, $H|\psi\rangle = H_1|\psi\rangle = \left(\frac{1}{2}p_r^2 + \frac{\Omega^2}{2}r^2\right)|\psi\rangle$

H_1 thus reduces to the Hamiltonian for the linear harmonic oscillator $\ddot{r} + \Omega^2 r = 0$, where $\Omega = \sqrt{(\omega^2 - \frac{\gamma^2}{4})}$