Quantum Entropy and Entanglement in Noncommutative Spaces

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Noncommutative Geometry: Physical and Mathematical Aspects of Quantum Space-Time and Matter

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Introduction

1. Irreducible Entropy from 3 Spins
2. Algebras, States, Entropy
3. Noncommutative Spaces
4. Spin from Bosons - Schwinger Construction
5. Entropy for Fuzzy Spaces
6. Summary

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Irreducible Entropy from 3 spin $\frac{1}{2}$'s

- An example: Three spin $\frac{1}{2}$'s (say neutrons) sitting at a point.
- The algebra of observables $\mathcal{A}$: spins S_i, their products, and linear combinations thereof.
- $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$.
- The full Hilbert space is 8-dimensional.

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Irreducible Entropy from 3 spin $\frac{1}{2}$'s

- Starting from the state

$$|\phi_{\frac{3}{2},\frac{3}{2}}\rangle = |\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle$$

  we can construct the spin $\frac{3}{2}$ representation easily.
- This is a unique 4-d subspace of the original 8-d Hilbert space.
- The projector to this subspace is uniquely defined.
- Simple matter to construct a density matrix:

$$\rho = \sum_m \lambda_m |\phi_{\frac{3}{2},m}\rangle\langle\phi_{\frac{3}{2},m}|, \quad \lambda_m \geq 0, \sum_m \lambda_m = 1. \tag{1}$$

- The von Neumann entropy $S(\rho)$ of ρ is simply $S = -\mathrm{Tr}\rho \log \rho$.
- We will have nothing more to say about this subspace, and ignore it henceforth.

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- The complement, also 4-dimensional, represents two copies of the spin-$\frac{1}{2}$ representation.
- There are *two* states with $j = m = \frac{1}{2}$:

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\begin{aligned}
|u^{(1)}_{\frac{1}{2},\frac{1}{2}}\rangle &= \sqrt{\frac{2}{3}}|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},-\frac{1}{2}}\rangle - \frac{1}{\sqrt{6}}|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},-\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle \\
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|u^{(2)}_{\frac{1}{2},\frac{1}{2}}\rangle &= \frac{1}{\sqrt{2}}|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},-\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle - \frac{1}{\sqrt{2}}|\psi_{\frac{1}{2},-\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle|\psi_{\frac{1}{2},\frac{1}{2}}\rangle
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- We can mix them by any $SU(2)$ matrix to get two other states with $j = m = \frac{1}{2}$:

$$|v^{(a)}_{\frac{1}{2},\frac{1}{2}}\rangle = |u^{(b)}_{\frac{1}{2},\frac{1}{2}}\rangle U_{ba}, \quad a, b = 1, 2 \quad \text{and} \quad U^{\dagger} U = \mathbf{1}.$$

- Thus there is an $SU(2)$ worth of ways for decomposing the 4-dimensional subspace into two spin-$\frac{1}{2}$ subspaces.
- there is no observable that distinguishes the $|u^{(a)}_{\frac{1}{2},\frac{1}{2}}\rangle$'s from the $|v^{(a)}_{\frac{1}{2},\frac{1}{2}}\rangle$'s.
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- How do we define density matrices in this subspace?
- Not obvious: there is no canonical projector to either of the spin $\frac{1}{2}$ subspaces!
- We could try using $P^{(a)} = \sum_m |u^{(a)}_{\frac{1}{2},m}\rangle\langle u^{(a)}_{\frac{1}{2},m}|$.
- Then write density matrices $\rho^{(a)}$ in each of the two spin-$\frac{1}{2}$ subspaces, with $\rho = \rho^{(1)} \oplus \rho^{(2)}$.
- However, because of the gauge redundancy, there is an $SU(2)$ worth of projectors $P^{(a)}(U)$.
- The corresponding $\rho^{(a)}(U)$ give the same expectation value for any observable A (independent of U).
- But the von Neumann entropy now depends on $u \in SU(2)$!
- This entropy is always non-zero: the quantum state is necessarily impure.

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Entropy Ambiguity

- Suppose we did define a density matrix as $\rho = \lambda_1 \rho_1 + \lambda_2 \rho_2$, $\lambda_1 + \lambda_2 = 1$.
- This corresponds to using the (non-canonical) projector $P^{(a)} = \sum_m |u^{(a)}_{\frac{1}{2},m}\rangle\langle u^{(a)}_{\frac{1}{2},m}|$,
- If we used the projector $P^{(a)}(U)$, we would find that the λ's have changed: $\lambda'_a = \sum_b \lambda_b |u_{ab}|^2$
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Algebraic Approach to Quantum Theory

- Algebra of observables $\mathcal{A}$. For our example, it is generated by S_i.
- States ω are positive linear functionals on $\mathcal{A}$.
- States ω form a convex set: the associated entropy is unique if the convex set is a simplex.
- The GNS construction gives us a canonical Hilbert space $\mathcal{H}_\omega$.
- $\mathcal{H}_\omega$ carries a representation π_ω of $\mathcal{A}$.
- In general π_ω is reducible, so $\mathcal{H}_\omega = \bigoplus_{r,j} \mathcal{H}_\omega^{(r,j)}$.
- When there is a degeneracy of representations ($r > 1$ for some j), we don't get a simplex!

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- Algebra of functions $\mathcal{F}$ on a space allows us to reconstruct the topological space (Gelfand-Naimark theorem) via GNS construction.
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Noncommutative Spaces: Examples

- Fuzzy S^2: $[X_i, X_j] = i\lambda\epsilon_{ijk}X_k$, $X_iX_i = R^2\mathbf{1}$.
- Moyal space: $[x_\mu, x_\nu] = i\theta_{\mu\nu}$.
- κ-Minkowski: $[x_\mu, x_\nu] = i\theta^\rho_{\mu\nu}x_\rho$.
- To reconstruct the space (time), we need to be given not just the algebra, but also the state.
- For many states, we will produce spaces that carry a non-trivial entropy!

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Fuzzy Sphere S_F^2

- Simple model for S_F^2 is by Schwinger construction.
- Start with a pair of oscillators $[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2.$
- Then $\hat{x}_i = \frac{1}{2}\hat{a}_\alpha^\dagger(\sigma_i)_{\alpha\beta}\hat{a}_\beta, \quad [\hat{x}_i, \hat{x}_j] = i\epsilon_{ijk}\hat{x}_k, \quad \hat{x}_i\hat{x}_i = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}+1\right).$

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Review of the Quantum Harmonic Oscillator

- Infinite-dimensional Hilbert space $\mathcal{H}$ spanned by a complete orthonormal basis $\{|n\rangle, n = 0, 1, \cdots, \infty\}$.
- The standard bosonic annihilation operator a acts as

$$a|n\rangle = n^{\frac{1}{2}}|n-1\rangle, \quad \forall n \geq 1 \quad \text{and} \quad a|0\rangle = 0$$

- The operator a is unbounded, and hence comes with a domain of definition:

$$\mathcal{D}_a = \{\sum_n c_n|n\rangle | \sum_n n|c_n|^2 < \infty\}$$

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$$[a, a^\dagger] = 1$$

(The oscillator algebra)

- The number operator N counts the number of quanta in a state, while the operators a and $a^\dagger$ destroy and create respectively a single quantum.
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Other representations of the oscillator algebra

- The Hilbert space $\mathcal{H}$ can split into two disjoint subspaces $\mathcal{H}_+ = \{\sum c_{2n}|2n\rangle \in \mathcal{H}\}$ and $\mathcal{H}_- = \{\sum c_{2n+1}|2n+1\rangle \in \mathcal{H}\}$: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.
- On the subspaces $\mathcal{H}_\pm$, the operators $b_\pm$ and its adjoint $b_\pm^\dagger$ can be defined as

$$b_+|2n\rangle = n^{\frac{1}{2}}|2n-2\rangle, \quad b_+^\dagger|2n\rangle = (n+1)^{\frac{1}{2}}|2n+2\rangle, \quad b_+|0\rangle = 0,$$

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Other representations of the oscillator algebra (Brandt-Greenberg, JMP 1969)

- On the domain $\mathcal{D}_N \cap \mathcal{H}_\pm$ we have $[b_\pm, b_\pm^\dagger] = 1$.
- So $(b_-, \mathcal{H}_-)$, $(b_+, \mathcal{H}_+)$ and $(a, \mathcal{H})$ are isomorphic to each other.
- In other words, there exist unitary operators $U_\pm$ such that $U_\pm b_\pm U_\pm^\dagger = a$.

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Reducible representations of the oscillator algebra

- Using the projection operators

$$\Lambda_+ = \sum_{n=0}^{\infty} |2n\rangle\langle 2n|, \quad \Lambda_- = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n+1|$$

  one can define an operator b

$$b = b_+\Lambda_+ + b_-\Lambda_-$$

- The b acts on the basis vectors $|n\rangle$ as

$$b|2n\rangle = n^{\frac{1}{2}}|2n-2\rangle, \quad b|2n+1\rangle = n^{\frac{1}{2}}|2n-1\rangle$$

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- Notice that both $|0\rangle$ and $|1\rangle$ are annihilated by b.
- The operator b satisfies $[N, b] = -2b$.
- A new number operator M can be defined as $M = b^\dagger b = \frac{1}{2}(N - \Lambda_-)$.
- It has the states $|n\rangle$ as eigenstates but each eigenvalue is two-fold degenerate.
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- This can generalized to construct an operator $b^{(k)}$ which lowers a state $|n\rangle$ by $k-$steps.
- Define projection operators Λ_i

$$\Lambda_i = \sum_{n=0}^{\infty} |kn+i\rangle\langle kn+i|, \quad i = 0, 1, \cdots k-1.$$

that project onto subspaces $\mathcal{H}_i = \{\sum_n c_{kn+i}|kn+i\rangle\}$.

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- This gives $K_1 K_2$ identical copies of the fuzzy sphere algebra.
- vN entropy $S = -\sum_{\alpha=1}^{K_1 K_2} \lambda_\alpha(u) \log \lambda_\alpha(u), \quad u \in U(K_1 K_2)$.
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Collaborators I

A. P. Balachandran and A. R. de Queiroz (Quantum Entropy and its Ambiguity) 1212.1239, 1302.4924
Nirmalendu Acharyya and Nitin Chandra (Entropy of Fuzzy Spaces) 1405.6471.