Perturbative algebraic quantum field theory on DFR quantum spacetime

Gerardo Morsella

Tor Vergata University, Roma

joint work with S. Doplicher, N. Pinamonti

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Outline

Introduction

- [Doplicher, Fredenhagen, Roberts '95]: QM + GR $\Rightarrow$ uncertainties Δq^μ satisfy Spacetime Uncertainty Relations (STUR)
- Minkowski spacetime replaced by a Quantum (noncommutative) Spacetime $\mathcal{E}$ (C*-algebra generated by q^μ)
- QFT on QST has remarkable properties [Bahns, Doplicher, Fredenhagen, Piacitelli '01,'03,'04,...]
- It can also serve as a (partial) substitute of inflation [Doplicher, M., Pinamonti '13]
- Perturbative algebraic quantum field theory (pAQFT): renormalization on curved spacetime, construction of algebras of interacting observables, quantum gravity... [Hollands, Wald, Brunetti, Fredenhagen, Dütsch, Rejzner... '01 on]

Aim of this talk:

Adapt pAQFT to QST to obtain a more manageable vperturbation expansion and study some consequences

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QM+GR $\Rightarrow$ energy E localized in region of radius $R \simeq E^{-1}$ not hidden by TS only if $R \gtrsim R_E \simeq E$, i.e. $R \gtrsim \lambda$ (Planck length). But if only one coordinate is well localized, TS will not form

[DFR] analysis:

- quantum state localized in region supp f of sizes Δq^μ, $\mu = 0, \dots, 3$

$$\omega_f(A) = \langle e^{i\varphi(f)}\Omega, Ae^{i\varphi(f)}\Omega \rangle$$

  energy $E \simeq 1/\min_\mu\{\Delta q^\mu\} \implies$ energy density ρ
- solution of linearized Einstein equations with source ρ given by retarded potential: $g_{\alpha\beta}(\Delta q^\mu)$
- if signals from supp f have to be observable TS should not form: $g_{00} > 0$

Spacetime Uncertainty Relations (STURs)

$$\Delta q^0 \sum_{j=1}^{3} \Delta q^j \geq \lambda^2, \qquad \sum_{i<j=1}^{3} \Delta q^i \Delta q^j \geq \lambda^2$$

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Quantum Spacetime

STURs can be realized by assuming that Δq^μ's are standard deviations of quantum operators q^μ satisfying suitable commutation relations, as for Heisenberg uncertainty relations

Quantum Conditions

$$[q^\mu, q^\nu] = i\lambda^2 Q^{\mu\nu}, \quad [q^\rho, Q^{\mu\nu}] = 0,$$

$$Q_{\mu\nu}Q^{\mu\nu} = 0, \quad \left(\frac{1}{4}Q^{\mu\nu}(*Q)_{\mu\nu}\right)^2 = 1$$

- Noncommutative C*-algebra $\mathcal{E}$ of Quantum Spacetime (QST) generated by q^μ's replaces algebra of functions on Minkowski
- It is equipped with action of the Poincaré group $q^\mu \to \Lambda^\mu_\nu q^\nu + a^\nu$
- $\mathcal{E}$ has nontrivial center $Z(\mathcal{E})$ = functions on a manifold $\Sigma \simeq T\mathbb{S}^2 \times \mathbb{Z}_2$ and $\mathcal{E} \simeq C_0(\Sigma, \mathcal{K})$, $\mathcal{K}$ = compact operators

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Optimal localization on QST

In an irreducible representation q^μ is a Lorentz transform of Schroedinger's (x_1, x_2, p_1, p_2)

$$\Downarrow$$

There exists states of optimal localization ω on $\mathcal{E}$, minimizing

$$\sum_\mu (\Delta q^\mu)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta p_1)^2 + (\Delta p_2)^2$$

given by translates of the harmonic oscillator ground states
They are the best approximation of points on QST

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Free quantum fields on QST

φ free (scalar) field on Minkowski can be defined on QST through Weyl-von Neumann-Moyal quantizazion

$$\varphi(q) = \int d^4k\, \check{\varphi}(k) \otimes e^{ikq}$$

(formal) element of $\mathfrak{F} \otimes \mathcal{E}$, $\mathfrak{F}$ field algebra

- it satisfies Klein-Gordon equation (derivatives on $\mathcal{E}$ defined by $\partial_\mu \varphi(q) := \frac{\partial}{\partial x^\mu} \varphi(q + x\mathbb{1})$)
- ω_x, ω_y optimally localized states around $x, y \implies$ $[\mathrm{id} \otimes \omega_x(\varphi(q)), \mathrm{id} \otimes \omega_y(\varphi(q))]$ falls off as a Gaussian of width λ for large spacelike $x - y$

Locality is lost at distances small w.r.t. λ, but recovered as $\lambda \to 0$

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(Perturbative) interacting fields on QST

Several (inequivalent) possibilities of defining perturbative interacting fields

- Hamiltonian approach (interaction picture) with interaction Lagrangian defined by $:\varphi(q)^n:$ [DFR]
- Yang-Feldman equation and quasi-planar Wick products [Bahns, Doplicher, Fredenhagen, Piacitelli '02 & '04]
- Hamiltonian approach with interaction defined by quantum Wick product $:\varphi^n(q):_Q$, which yields UV-finite (IR-cutoff) theory to all orders [Bahns, Doplicher, Fredenhagen, Piacitelli '03]

$:\varphi^n(q):_Q$ defined by generalizing point-splitting to QST:
e.g., for $n = 2$

$$:\varphi^2:(x) := \lim_{y \to x} \varphi(x)\varphi(y) - \langle \Omega, \varphi(x)\varphi(y)\Omega \rangle$$

limit $y \to x$ has to be performed in a way compatible with the STURs

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Quantum Wick product

- Introduce quantum coordinates of independent events

$$q_j^\mu := \mathbb{1} \otimes \cdots \otimes q^\mu \otimes \cdots \otimes \mathbb{1}, \quad j = 1, \ldots, n$$

tensor product of Z-moduli $\implies [q_j^\mu, q_k^\nu] = i\lambda^2 Q^{\mu\nu}\delta_{jk}$

- introduce center of mass and relative coordinates

$$\bar{q}^\mu := \frac{1}{n}\sum_j q_j^\mu, \qquad \xi_{jk}^\mu := q_j^\mu - q_k^\mu$$

identification of commutators $\implies [\bar{q}^\mu, \xi_{jk}^\nu] = 0$, $[\xi_{jk}^\mu, \xi_{jk}^\nu] = 2i\lambda^2 Q^{\mu\nu}$

- evaluating optimally localized state on ξ_{jk}^μ yields a map $E^{(n)} : \mathcal{E}^{\otimes_Z n} \to \mathcal{E} \simeq C^*(\bar{q}^\mu)$
- define quantum Wick product as

$$:\varphi^n(q):_Q := E^{(n)}(:\varphi(q_1)\ldots\varphi(q_n):)$$
$$= \int d^4k_1 \ldots d^4k_n :\check{\varphi}(k_1)\ldots\check{\varphi}(k_n): e^{-\frac{\lambda^2}{4}\sum_j |k_j - \frac{1}{n}\sum_l k_l|^2} e^{i\sum_j k_j q}$$

Feynmann rules for quantum Wick product

S-matrix is equivalent to the one of a non-local QFT on commutative Minkowski with interaction Hamiltonian

$$H_I(t) = \int_{x^0=t} d^3\boldsymbol{x} \int dx_1 \dots dx_n \, e^{-\frac{1}{2\lambda^2}\sum_j |x_j - x|^2} \delta(\bar{x} - x) : \varphi(x_1) \dots \varphi(x_n) :$$

$(\bar{x} := \frac{1}{n}\sum_j x_j,\ x = (t, \boldsymbol{x}))$

Feynmann rules for this theory are modified [Piacitelli 2004]:

- time ordering is done w.r.t. to $\bar{x}^0$, not x_j^0
- vertices of Feynmann diagrams become fat ($x \to x_1, \dots, x_n$)
- propagator between x_j and y_k depends also on $\bar{x}^0 - \bar{y}^0$

Resulting perturbation theory is manifestly unitary but not easy to handle (e.g., pass to momentum space...)

We look for a more manageable formulation

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Observables as functionals

Proposed by [Brunetti, Dütsch, Fredenhagen 2009] to construct perturbatively algebras of observables (or fields) for the interacting theory defined by Lagrangian

$$L = L_0 + L_I = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + L_I$$

With $M(=\mathbb{R}^4)$ spacetime, define:

- $\mathcal{C} := C^\infty(M,\mathbb{R}) \cap \mathcal{S}'(M)$ field configurations
- $\mathcal{F} := \{F : \mathcal{C} \to \mathbb{C} \ : \ F^{(n)} := \frac{\delta^n F}{\delta\phi^n} \in \mathcal{E}'(M^n), WF(F^{(n)}) \cap (M^n \times (\bar{V}^n_+ \cup \bar{V}^n_-)) = \emptyset\}$ observables
- $\mathcal{F}_{\text{loc}} := \{F \in \mathcal{F} \ : \ F^{(1)} \in \mathcal{D}(M), \text{supp}\, F^{(n)} \subset \{x_1 = x_2 = \cdots = x_n\}\}$ local observables
- $\mathcal{F}_{\text{reg}} := \{F \in \mathcal{F} \ : \ F^{(n)} \in \mathcal{D}(M^n)\}$ regular observables

E.g.: $F(\phi) = \int_{M^n} K(x_1,\dots,x_n)\phi(x_1)\dots\phi(x_n)$ with $K \in \mathcal{D}(M^n)$ regular, $L_I(\phi) = \int_M g(x)\phi(x)^4$ local but not regular

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Free fields algebra 1/2

Algebra of free fields defined by deforming the pointwise product $M : \mathcal{F}_{\text{reg}} \otimes \mathcal{F}_{\text{reg}} \to \mathcal{F}_{\text{reg}}$ as

$$F *_{\frac{i}{2}\Delta} G = M \circ e^{\int_{M^2} \frac{i}{2}\Delta(x-y)\frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

(formal power series), with $\Delta := \Delta_R - \Delta_A$ free field commutator function

Then:

$$[\phi(x), \phi(y)]_{*_{\frac{i}{2}\Delta}} = \frac{i}{2}\Delta(x-y)$$

$\Rightarrow$ $(\mathcal{F}_{\text{reg}}, *_{\frac{i}{2}\Delta})$ is isomorphic to the *-algebra generated by the free scalar quantum field φ on Fock space

Formal series can sometimes be replaced by convergent ones by requiring bounds for $F^{(n)}$ [Bahns, Rejzner 2017]

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Problem: as $WF(\Delta) \subset M^2 \times ((\bar{V}_+ \times \bar{V}_-) \cup (\bar{V}_- \times \bar{V}_+))$, $\Delta(x_1 - y_1)\Delta(x_2 - y_2)F^{(2)}(x_1, x_2)G^{(2)}(y_1, y_2)$ makes no sense for $F, G \in \mathcal{F} \Rightarrow *_{\frac{i}{2}\Delta}$ cannot be extended to $\mathcal{F}$, but $L_I \notin \mathcal{F}_{\mathrm{reg}}$

Solution: since for the 2-point function Δ_+

$$WF(\Delta_+) = \{(x, y, p, q) \,:\, y = x + tp,\, p^2 = 0, p_0 > 0, q = -p\}$$

it is possible to define the product on $\mathcal{F}$

$$F *_{\Delta_+} G = M \circ e^{\int_{M^2} \Delta_+(x-y) \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

and $F \mapsto e^{\frac{1}{2}\int_{M^2} H(x-y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)}} F =: \alpha_H(F)$, with $H = \Delta_+ - \frac{i}{2}\Delta = \frac{1}{2}(\Delta_+ + \Delta_-)$ defines an isomorphism $\alpha_H : (\mathcal{F}_{\mathrm{reg}}, *_{\frac{i}{2}\Delta}) \to (\mathcal{F}_{\mathrm{reg}}, *_{\Delta_+})$

E.g.

$$\alpha_H^{-1}(\phi(x)\phi(y)) = \phi(x) *_{\frac{i}{2}\Delta} \phi(y) - \Delta_+(x-y) =: \phi(x)\phi(y) :$$

$\Rightarrow (\mathcal{F}, *_{\Delta_+})$ algebra generated by Wick monomials

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$$F *_{\Delta_+} G = M \circ e^{\int_{M^2} \Delta_+(x-y) \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

and $F \mapsto e^{\frac{1}{2}\int_{M^2} H(x-y) \frac{\delta^2}{\delta\phi(x)\delta\phi(y)}} F =: \alpha_H(F)$, with $H = \Delta_+ - \frac{i}{2}\Delta = \frac{1}{2}(\Delta_+ + \Delta_-)$ defines an isomorphism $\alpha_H : (\mathcal{F}_{\text{reg}}, *_{\frac{i}{2}\Delta}) \to (\mathcal{F}_{\text{reg}}, *_{\Delta_+})$

E.g.

$$\alpha_H^{-1}(\phi(x)\phi(y)) = \phi(x) *_{\frac{i}{2}\Delta} \phi(y) - \Delta_+(x - y) =: \phi(x)\phi(y) :$$

$\Rightarrow (\mathcal{F}, *_{\Delta_+})$ algebra generated by Wick monomials

Interacting fields 1/2

The time-orderd product of two local functionals with disjoint supports is defined as

$$F \cdot_T G := M \circ e^{\int_{M^2} \Delta_F(x-y) \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

with the Feynmann propagator Δ_F, and satisfies the casual factorization property

$$F \cdot_T G = F *_{\Delta_+} G \ \text{ if supp } F \text{ is earlier than supp } G$$

The map $(F_1, \dots, F_n) \mapsto F_1 \cdot_T \dots \cdot_T F_n$ is then extended to all $F_j \in \mathcal{F}_{\text{loc}}$ by induction on n, mantaining causal factorization (plus other requirements) [Epstein, Glaser 1973].

The non-uniqueness of the extension gives rise to the renormalization group

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Interacting fields 2/2

Given an interaction s.t. $\alpha_H L_I \in \mathcal{F}_{\text{loc}}$ the (IR-cutoff) S-matrix is defined by

$$S(L_I) = \alpha_H^{-1} T(e^{iT^{-1}\alpha_H L_I}) = \sum_{n=0}^{+\infty} \frac{i^n}{n!} \alpha_H^{-1}(\alpha_H L_I \cdot_T \cdots \cdot_T \alpha_H L_I)$$

and it is unitary in $(\alpha_H^{-1}(\mathcal{F}), *_{\frac{i}{2}\Delta})$: $S(L_I) *_{\frac{i}{2}\Delta} \overline{S(L_I)} = 1$ (while it is not in general if $L_I \in \mathcal{F}_{\text{reg}}$)

Interacting fields are defined by the Bogoliubov map

$$\begin{aligned} R_{L_I}(F) &:= S(L_I)^{-1} *_{\frac{i}{2}\Delta} \alpha_H^{-1}(\alpha_H F \cdot_T \alpha_H S(L_I)) \\ &= S(L_I)^{-1} *_{\frac{i}{2}\Delta} \left[\sum_{n=0}^{+\infty} \frac{i^n}{n!} \alpha_H^{-1}(\alpha_H F \cdot_T \alpha_H L_I \cdot_T \cdots \cdot_T \alpha_H L_I) \right] \end{aligned}$$

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Outline

Non-unitarity

For a $:\varphi^n(q):_Q$ interaction on QST the effective interaction on commutative spacetime is

$$L'_{I,\mathrm{eff}}(\varphi) = \frac{1}{(\sqrt{2\pi}\lambda)^{4(n-1)} n^2} \int_M dx\, g(x) \times \int_{M^n} dx_1 \ldots dx_n\, e^{-\frac{1}{2\lambda^2}\sum_j |x_j - x|^2} \delta(\bar{x} - x) : \varphi(x_1) \ldots \varphi(x_n) :$$

Problem: turning on the pAQFT machinery, T-products order w.r.t. $x_1^0, \ldots, x_n^0 \Rightarrow$ Feynmann diagrams computed using Filk rules $\Rightarrow$ resulting $S(L'_I)$ is non unitary

Key observation: the limit $g \to 1$ of L'_I is equivalent to the limit $g \to 1$ of

$$L_{I,\mathrm{eff}}(\varphi) = \frac{1}{(\sqrt{2\pi}\lambda)^{4n}} \int_M dx\, g(x) : \left[\int_M dy\, e^{-\frac{1}{2\lambda^2}|y-x|^2} \varphi(y) \right]^n :$$

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Further deformation of fields $*$-product

With

$$G_\lambda(x) := \frac{1}{(\sqrt{2\pi}\lambda)^4} e^{-\frac{|x|^2}{2\lambda^2}}$$

define the following:

- $\iota_\lambda : \mathcal{C} \to \mathcal{C}$, $\iota_\lambda \phi(x) = \int_M G_\lambda(x-y)\phi(y)$
- $r_\lambda : \mathcal{F} \to \mathcal{F}$, $(r_\lambda F)(\phi) = F(\iota_\lambda \phi)$
- $\Delta_\lambda(x) = \int_{M^2} G_\lambda(x-y)\Delta(y-z)G_\lambda(z) \Leftrightarrow \hat{\Delta}_\lambda(p) = e^{-\lambda^2|p|^2}\hat{\Delta}(p)$

and further deform the product on $\mathcal{F}_{\text{reg}}$ as

$$F *_{\frac{i}{2}\Delta_\lambda} G = M \circ e^{\int_{M^2} \frac{i}{2}\Delta_\lambda(x-y)\frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

Then $r_\lambda : (\mathcal{F}_{\text{reg}}, *_{\frac{i}{2}\Delta_\lambda}) \to (\mathcal{F}_{\text{reg}}, *_{\frac{i}{2}\Delta})$ is a *-homomorphism and

$$L_{I,\text{eff}}(\phi) = \int_M dx\, g(x)\alpha_H^{-1}(\iota_\lambda\phi(x)^n) = (r_\lambda L_I)(\phi)$$

$$L_I(\phi) = \int_M dx\, g(x)\alpha_{H_\lambda}^{-1}(\phi(x)^n) \Rightarrow \alpha_{H_\lambda} L_I \in \mathcal{F}_{\text{loc}}$$

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Wick polynomials and $*_{\Delta_{+,\lambda}}$ product

In order to include Wick polynomials we can also deform the $*_{\Delta_+}$ product:

$$F *_{\Delta_{+,\lambda}} G = M \circ e^{\int_{M^2} \Delta_{+,\lambda}(x-y) \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)}} (F \otimes G)$$

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so that $r_\lambda : (\mathcal{F}, *_{\Delta_{+,\lambda}}) \to (\mathcal{F}, *_{\Delta_+})$ is a *-homomorphism

There holds also

$$i\Delta_\lambda(x-y) = [\mathrm{id} \otimes \omega_x(\varphi(q)), \mathrm{id} \otimes \omega_y(\varphi(q))]$$
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$\cdot_{T_\lambda}$ product

Proposition

The modified Feynmann propagator

$$\Delta_{F,\lambda}(x) := \theta(x^0)\Delta_{+,\lambda}(x) + \theta(-x^0)\Delta_{+,\lambda}(-x)$$

is a continuous bounded function and

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S-matrix and interacting product

Theorem

With $H_\lambda := \Delta_{+,\lambda} - \frac{i}{2}\Delta_\lambda$, the S-matrix

$$S(L_I) := \sum_{n=0}^{+\infty} \frac{i^n}{n!} \alpha_{H_\lambda}^{-1}(\alpha_{H_\lambda} L_I \cdot_{T_\lambda} \cdots \cdot_{T_\lambda} \alpha_{H_\lambda} L_I)$$

is unitary in $(\alpha_{H_\lambda}^{-1}(\mathcal{F}), *_{\frac{i}{2}\Delta_\lambda})$: $S(L_I) *_{\frac{i}{2}\Delta_\lambda} \overline{S(L_I)} = 1$

Perturbative expansion given by usual Feynmann diagrams with $\Delta_{F,\lambda}$ propagators

Moreover, R_{L_I} can be perturbatively inverted, and we can define the interacting algebra as $(\mathcal{F}, *_{L_I})$ with interacting product:

$$F *_{L_I} G := R_{L_I}^{-1}(R_{L_I}(F) *_{\frac{i}{2}\Delta_\lambda} R_{L_I}(G))$$

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Outline

Localizability in a spherically symmetric spacetime

Aim: produce a rigorous version of DFR argument on curved spacetime

Strategy:

1. consider a (scalar massless) free quantum field ϕ on a background $(M, g_{\mu\nu})$ in a (Hadamard) state such that
$$\Box\phi = 0, \qquad G_{\mu\nu} = 8\pi\omega(T_{\mu\nu})$$
2. prepare a localized state: for $f \in C_c^\infty(M)$
$$\omega_f(A) := \frac{\omega(\phi(f)A\phi(f))}{\omega(\phi(f)\phi(f))}, \qquad A \in \mathcal{A}$$
3. evaluate change to expectation value of $T_{\mu\nu}$ after localization
4. estimate backreaction on metric and formation of TS by Raychauduri equation (no linearization of gravity)
5. impose principle of gravitational stability

Step 4 (and 5) only under assumption of spherical symmetry of background metric

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Spherical symmetry

To evaluate backreaction, we should solve

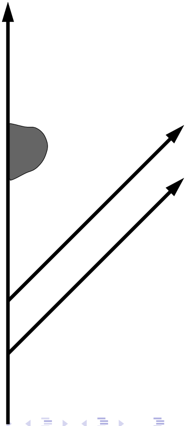
$$G_{\mu\nu} = 8\pi\, \omega_f(T_{\mu\nu})$$

It is **very** difficult. Assume spherical symmetry

- Spacetime is $I \times \mathbb{R}_+ \times \mathbb{S}^2$, retarded coordinates:
- spanned by future null geodesic emanated from the center of the sphere
  - u proper time on the worldline γ of center
  - s retarded distance: affine parameter along the null geodesics with $s(0) = 0$ and $\dot{s}(0) = 1$
- The general spherically symmetric metric is

$$ds^2 := -A(u,s)du^2 - 2ds\,du + r(u,s)^2 d\Omega^2$$

- Fix u, the family of null geodesics forms a cone $\mathcal{C}_u$



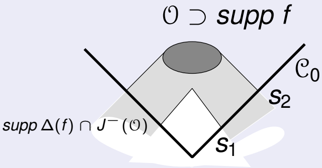
Backreaction and trapped surfaces

Theorem ([Doplicher, M., Pinamonti '13])

For a large class of spherically symmetric $(M, g_{\mu\nu})$ and ω (including cosmological ones), and for $f \in C_c^\infty(M)$ as in figure with

$$s_1 < s_2 < \frac{3}{2} s_1, \quad (s_2)^2 < \bar{s}^2 , \quad \bar{s}^2 := \frac{1}{6\,C}$$

the future of $\mathcal{C}_0$ contains a trapped surface.



For a flat Friedmann-Robertson-Walker spacetime with metric

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2 d\Omega^2]$$

the limitation becomes $r \gtrsim \frac{\lambda}{a(t)} \Rightarrow$ effective Planck length diverges near the singularity, as argued by [Doplicher, '01]

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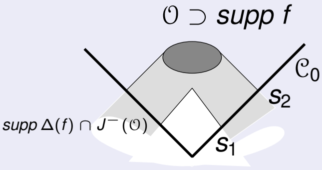

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The $\lambda \to +\infty$ limit of the S-matrix

We expect that near the Big Bang the effective Planck length diverges, so we prove

Theorem

To all perturbative orders

$$\lim_{\lambda \to \infty} S(V) = e^{iV}, \qquad \lim_{\lambda \to \infty} R_V(F) = F$$

(e^{iV} defined by pointwise product)

This suggest that:

- near the Big Bang interactions should disappear
- and correlations of free fields diverge

Thus there should remain no degrees of freedom at initial times.

Similar indications obtained in the Yang-Feldman approach

This could provide an alternative solution to the initial conditions problem

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Conclusions and outlook

Summary:

- pAQFT is an effective approach to the perturbative construction of interacting observables in QFT
- pAQFT can be modified to treat QFT on QST (or suitable non-local QFT on ordinary spacetime) yielding unitary and UV-finite S-matrix without renormalization
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Outlook:

- There are indications that perturbative series for S-matrix is Borel summable (in $d = 4$)
- pAQFT is naturally adapted to curved spacetimes, yielding generally covariant interacting theories, so it is natural to look for generally covariant QFT on curved QST (free as a first step)
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