Quantum Symmetry of Classical and noncommutative geometry

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A tribute to Professor S N Bose on his 125th birth anniversary

contents

Introduction

- Two important branches of 'noncommutative mathematics' : **Quantum groups** a la Woronowicz, Drinfeld, Jimbo and others, and **Noncommutative Geometry** a la Connes.
- However, a successful marriage of the two is still not fully achieved...but there are several class of representative examples by now.
- More precisely, spectral triples which are equivariant w.r.t. quantum group actions have been constructed and studied by a number of mathematicians (Chakraborty, Pal, Landi, Dabrwoski, Sitarz, Hajac, Neshvyev, Tuset...just to mention a few). Also, Majid, Woronowicz and many others gave general theories for quantum group equivariant noncommutative differential structures.
- One may turn around and ask: given a spectral triple, what are all (compact) quantum group actions on the underlying C^* algebra for which the given spectral triple is equivariant? This leads to the notion of **quantum isometry group**, which (if exists) should be the 'biggest' such quantum group.

- As in classical geometry, quantum isometry groups should play an important role in understanding noncommutative Riemannian geometry and more generally, noncommutative (quantum) metric space in the sense of Rieffel.
- Study of such quantum groups may also enrich quantum group theory.

We shall present a brief sketch of development of the theory of quantum isometry groups which is an outcome of collaboration with J. Bhowmick (JB), A. Skalski (AS), T Banica (TB), B. Das (BD) and S. Joardar (SJ). This will include formulation in various frameworks, e.g. in terms of Laplacian (DG-CMP), formulation in terms of Dirac operator (JB+DG-JFA), and also results about deformation (JB+DG-JFA), and computations for AF algebras (JB+DG+AS- Trans. AMS) Podles spheres (JB+DG- JFA), free and half liberated spheres (TB+DG- CMP) and finally, non-existence of genuine quantum isometries of compact connected (classical) manifolds (DG+SJ), and even non existence of smooth actions (DG).

Quick review of basic concepts

Definition

a **compact quantum group** *(CQG for short) a la Woronowicz is a pair* $(\mathcal{A}, \Delta)$ *where* $\mathcal{A}$ *is a unital separable* C^**-algebra,* Δ *is a coassociative comultiplication, i.e. a unital* C^**-homomorphism from* $\mathcal{A}$ *to* $\mathcal{A} \otimes \mathcal{A}$ *(minimal tensor product) satisfying* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*, and each of the sets* $\{(b \otimes 1)\Delta(c) \colon b, c \in \mathcal{A}\}$ *and* $\{(1 \otimes b)\Delta(c) \colon b, c \in \mathcal{A}\}$ *generates dense linear subspace of* $\mathcal{A} \otimes \mathcal{A}$

There is a natural generalisation of group action on spaces in this noncommutative set-up, which is given below :

Definition

We say that a CQG $(\mathcal{A}, \Delta)$ *acts on a (unital)* C^**-algebra* $\mathcal{C}$ *is there is a unital* $*$*-homomorphism* $\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$ *such that* $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$*, and the linear span of* $\alpha(\mathcal{C})(1 \otimes \mathcal{A})$ *is norm-dense in* $\mathcal{C} \otimes \mathcal{A}$*.*

Definition

A **spectral triple** *or* **spectral data** *is a tuple* $(\mathcal{A}, \mathcal{H}, D)$ *where* $\mathcal{H}$ *is a separable Hilbert space,* $\mathcal{A}$ *is a* $*$*-subalgebra of* $\mathcal{B}(\mathcal{H})$ *(not necessarily norm-closed) and* D *is a self-adjoint (typically unbounded) operator such that for each* $a \in \mathcal{A}$*, the operator* $[D, a]$ *admits bounded extension. Such a spectral triple is also called an* **odd spectral triple**. *If in addition, we have* $\gamma \in \mathcal{B}(\mathcal{H})$ *satisfying* $\gamma = \gamma^* = \gamma^{-1}$, $D\gamma = -\gamma D$ *and* $[a, \gamma] = 0$ *for all* $a \in \mathcal{A}$*, then we say that the quadruplet* $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *is an* **even spectral triple** *or* **even spectral data**. *The operator* D *is called the* **Dirac operator** *corresponding to the spectral triple.*
We say that the spectral triple is of **compact type** *if* D *has compact resolvents. It is* Θ*-summable if* $Tr(e^{-tD^2}) < \infty$ *for* $t > 0$.

The motivation of this formulation comes from the typical classical examples of spectral triple associated with a Riemannian spin manifold M, where $\mathcal{H}$ can be chosen as the Hilbert space of square integrable sections of the spinor bundle, D as the Dirac operator, and $\mathcal{A}$ as $C^\infty(M)$ acting by multiplication on the sections of spinor bundle. In this case, the spectral triple contains full information about the underlying topology and the Riemannian metric.

Background

- Early work : formulation of quantum automorphism and quantum permutation groups by Wang, and follow-up work by Banica, Bichon and others.
- Basic principle: For some given mathematical structure (e.g., a finite set, a graph, a C^* or von Neumann algebra) identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type.
- However, most of the earlier work done concerned some kind of quantum automorphism groups of a 'finite' structure. So, one should extend these to the 'continuous'/ 'geometric' set-up. This motivated my definition of quantum isometry group in [5].

Wang's quantum permutation and quantum automorphism groups

Quantum permutation group (Wang):
Let $X = \{1, 2, ..., n\}$, G group of permutations of X. G can be identified as the universal object in the category of groups acting on X. For a simiar (bigger) category of compact quantum groups acting on $C(X)$, Wang obtained the following universal object:

$$\mathcal{Q} := C^* \left(q_{ij}, i, j = 1, ..., n; \mid q_{ij} = q_{ij}^* = q_{ij}^2, \ \sum_i q_{ij} = 1 = \sum_j q_{ij} \right).$$

The co product is given by $\Delta(q_{ij}) = \sum_k q_{ik} \otimes q_{kj}$, and the action on $C(X)$ is given by $\alpha(\chi_i) = \sum_j \chi_j \otimes q_{ji}$.
This CQG is naturally called 'quantum permutation group' of n objects. This is **infinite dimensional** as a vector space for $n \geq 4$, i,e, a finite set of 4 or more points can have infinitely many quantum symmetries (very rich).

A similar question can be asked for finite dimensional matrix algebras. However, the answer is negative, i.e. the cateogory of CQG acting on M_n does NOT have a universal object!

Remedy (due to Wang): consider the subcategory of actions which preserves a given faithful state.

More precisely: For an $n \times n$ positive invertible matrix $Q = (Q_{ij})$, let $A_u(Q)$ be the universal C^*-algebra generated by $\{u_{kj}, k, j = 1, ..., d_i\}$ such that $u := ((u_{kj}))$ satisfies

$$uu^* = I_n = u^* u, \quad u' Q \overline{u} Q^{-1} = I_n = Q \overline{u} Q^{-1} u'.$$

Here $u' = ((u_{ji}))$ and $\overline{u} = ((u^*_{ij}))$. This is made into a CQG by the coproduct given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Proposition

$A_u(Q)$ is the universal object in the category of compact quantum groups which admit a unitary representation, say U, on the finite dimensional Hilbert space $\mathbb{C}^n$ such that ad_U preserves the functional $M_n \ni x \mapsto \mathrm{Tr}(Q^T x)$.

Quantum symmetry of finite graphs

This is due to Bichon and Banica. Given a finite graph with the set of vertices V and the set of edges E, there is a maximal quotient ('quantum subgroup') $\mathcal{Q}(V, E)$ of the quantum permutation group of V which 'preserves ' the set of edges E in a natural sense. For certain graphs, $\mathcal{Q}(V, E)$ coincides with (set of functions on) the group of automorphisms of the graph, ie there is no genuine 'quantum automorphism', but for many interesting graphs there are (often infinitely many) genuine quantum symmetries. For example, for the square, i.e. 4-cycle, we have a genuine (infinite dimensional) quantum group of symmetries. One can give examples of two graphs having isomorphic automorphism groups and non-isomorphic quantum automorphism groups.

Quantum symmetry in physical models?

- $\mathcal{Q}(V, E)$ should naturally act as symmetry object on Potts /Ising models or other suitable models on graphs. In some cases, one may get a much bigger class of irreducibles (prehaps representing elementary oscillations /quasi particles etc.) than those coming from classical group symmetries.
- The Wang quantum automorohism groups $A_u(Q)$ for suitable matrices Q can produce ergodic actions on all types of (von Neumann algebraic) factors, in particular the so-called type III_1 factor , which are important in quantum Field theory.
- Quantum summetry/automorphism groups (if exist) of the noncommutative space-time manifolds modeling qantum gravity should give information about the fundamental particles as in the Wigner's work about the classical Poincare group.

As a general principle, given a physcial model given by some Hamiltonian on a Hilbert space, we seek the universal quantum group (if exists) which acts by unitary representation on the Hilbert space commuting with the Hamiltonian (and preserving some von Neumann algebra of observables). Often we have to add the condition that it also preserves certain canonical state to get existence. This gives the quantum automorphism/symmetry group of the model...its irreducible representations should physcially represent some fundamental 'particles' of the model; change of this quantum symmetry group by changing the reference state which it preserves should signify a change of state in the sense of Landau and so on... In the case of a (possibly noncommutative) space-time manifold and a theory of (quantum) gravity, this universal quantum group should be nothing but an analogue of the isometry group for the space-time metric. This brings us to the formulation of 'quantum isometry group' in the context of a noncommutative manifold given by spectral triples. For physcial applications, it is imperative to extend this to Lorentzian spectral triples, which is yet to be done.

Quantum isometry in terms of 'Laplacian' (Goswami 2009)

- Classical isometries: the group of Riemannian isometries of a compact Riemannian manifold M is the universal object in the category of all compact metrizable groups acting on M, with smooth and isometric action.
- Moreover, a smooth map γ on M is a Riemannian isometry if and only if the induced map $f \mapsto f \circ \gamma$ on $C^\infty(M)$ commutes with the Laplacian $-d^*d$.

Under reasonable regularity conditions on a (compact type, Θ-summable) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, one has analogues of Hilbert space of forms $\mathcal{H}_i^D$, say, $i = 0, 1, \ldots$. The map $d(a) := [D, a]$ then extends to a (closable, densely defined) map from $\mathcal{H}_0^D$ (space of 0-forms) to $\mathcal{H}_1^D$ (space of one-forms). The self-adjoint negative map $-d^*d$ is the noncommutative analogue of Laplacian $\mathcal{L} \equiv \mathcal{L}_D$, and we additionally assume that

(a) $\mathcal{L}$ maps $\mathcal{A}^\infty$ into itself;

(b) $\mathcal{L}$ has compact resolvents and its eigenvectors belong to $\mathcal{A}^\infty$ and form a norm-total subset of $\mathcal{A}$;

(c) the kernel of $\mathcal{L}$ is one dimensional ("connectedness").

It is then natural to call an action α of some CQG $\mathcal{Q}$ on the C^*-completion of $\mathcal{A}^\infty$ to be *smooth and isometric* if for every bounded linear functional ϕ on $\mathcal{Q}$, one has $(\mathrm{id} \otimes \phi) \circ \alpha$ maps $\mathcal{A}^\infty$ into itself and commutes with $\mathcal{L}_D$.

Theorem

Under assumptions (a)-(c), there exists a universal object (denoted by $QISO^{\mathcal{L}}$) in the category of CQG acting smoothly and isometrically on the given spectral triple.

- The assumption (c) can be relaxed for classical spectral triples and their Rieffel-deformations, i.e. the above existence theorem applies to arbitrary compact Riemannian manifolds (not necessarily connected) and their Rieffel deformations.

Quantum isometry in terms of the Dirac operator (Bhowmick-Goswami 2009)

- From the NCG perspective, it is more appropriate to have a formulation in terms of the Dirac operator directly.
- Classical fact: an action by a compact group G on a Riemannian spin manifold is an orientation-preserving isometry if and only if lifts to a unitary representation of a 2-covering group of G on the Hilbert space of square integrable spinors which commutes with the Dirac operator.

For a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ of compact type, it is thus reasonable to consider a category $\mathbf{Q}'$ of CQG $(\mathcal{Q}, \Delta)$ having unitary (co-) representation, say U, on $\mathcal{H}$, (i.e. U is a unitary in $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{Q})$ such that $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$) which commutes with $D \otimes 1_{\mathcal{Q}}$, and for every bounded functional ϕ on $\mathcal{Q}$, $(\mathrm{id} \otimes \phi) \circ \mathrm{ad}_U$ maps $\mathcal{A}^\infty$ into its weak closure. Objects of this category will be called 'orientation preserving quantum isometries'.

- If $\mathbf{Q}'$ has a universal object, we denote it by $\widetilde{QISO^{+}}(D)$. In general, however, $\mathbf{Q}'$ may fail to have a universal object.
- We discussed in [3] some sufficient conditions, such as the existence of a suitable cyclic separating eigenvector of D, to ensure that a universal object exists in $\mathbf{Q}'$.

In general, we do get a universal object in suitable subcategories by fixing a 'volume form'...

Theorem

Let R be a positive, possibly unbounded, operator on $\mathcal{H}$ commuting with D and consider the functional (defined on a weakly dense domain) $\tau_R(x) = Tr(Rx)$. Then there is a universal object (denoted by $\widetilde{QISO^{+}}_R(D)$) in the subcategory of $\mathbf{Q}'$ consisting of those $(\mathcal{Q}, \Delta, U)$, for which $(\tau_R \otimes \mathrm{id})(\mathrm{ad}_U(\cdot)) = \tau_R(\cdot)1_{\mathcal{Q}}$.

Given such a choice of R, we shall call the spectral triple to be R-twisted.

- The C^*-subalgebra $QISO_R^+(D)$ of $\widetilde{QISO}_R^+(D)$ generated by elements of the form $\{< (\xi \otimes 1), \mathrm{ad}_{U_0}(a)(\eta \otimes 1) >, \ a \in \mathcal{A}^\infty\}$, where U_0 is the unitary representation of $\widetilde{QISO^+}_R(D)$ on $\mathcal{H}$ and $< \cdot, \cdot >$ denotes the $\widetilde{QISO}_R^+(D)$-valued inner product of the Hilbert module $\mathcal{H} \otimes \widetilde{QISO}_R^+(D)$, will be called the quantum group of orientation and (R-twisted) volume preserving isometries. A similar C^*-subalgebra of $\widetilde{QISO^+}(D)$, if it exists, will be denoted by $QISO^+(D)$.
- However, $QISO_R^+(D)$ may not have C^* action for noncommutative manifolds, and the subcatogory of $\mathbf{Q}'_R$ with those objects for which ad_U gives a C^* action, or even it maps into the C^* algebra, does not in general admit a universal object.
- Under mild conditions (valid for classical manifolds and their deformations), $QISO^{\mathcal{L}}$ coincides with $QISO_I^+(d + d^*)$, where $d + d^*$ is the 'Hodge Dirac operator' on the space of all (noncommutative) forms.

The construction and existence proofs for the above universal quantum groups can be adapted verbatim to any physical model with Hamiltonian with compact resplvents (discrete spectrum) and it need not come from a geometric set-up. Such a framework is considered by Banica and Skalski in the theory of quantum symmetry groups of orthogonal filtrations...won't go into more details.

Two computational techniques for finding out $QISO$ (both in Laplacian and Dirac approaches):

- $QISO$ of cocycle-deformation of spectral triple is isomorphic with similar deformation of $QISO$ of the undeformed spectral triple. This gives $QISO$ of the noncommutative tori, for example.
- Under some mild conditions, $QISO$ of inductive limit of spectral triples is isomorphic with the inductive limit of the corresponding $QISO$. This helps compute $QISO$ of many models on AF algebras, Cantor set, infinite graph etc.

Classical spaces: no-go theorem (DG+SJ)

Theorem

There is no genuine quantum isometry of a compact connected Riemannian manifold, i.e. $QISO^{\mathcal{L}}$ *is* $C(ISO(M))$ *for such a manifold* M.

Even more generally, we have

Theorem

If a CQG $\mathcal{Q}$ *acts faithfully on* $C(M)$ *where* M *is a compact, connected, smooth manifold and the action maps* $C^\infty(M)$ *to* $C^\infty(M, \mathcal{Q})$, *then* $\mathcal{Q} \cong C(G)$ *for a compact group* G *acting on* M *by smooth diffeomorphism.*

This, coupled with the fact that *QISO* functor commutes with the deformation procedure, gives us *QISO* of noncommutative manifolds obtained by deforming classical (compact connected) manifolds, to be the deformations of the classical *ISO* group.

Genuine quantum symetries of classical spaces: non-smooth or non-compact

- In the no-go theorem for smooth CQG action on classical manifolds, three things are crucial: connectedness, compactness of the quantum group and smoothness of the action/space.
- For connected compact metric (not smooth manifold) spaces one can have genuine CQG acting faithfully (Huang).
- For non-smooth varieties, e.g $\{xy = 0\}$, one can have genuine CQG action (algebraic).
- Holf algebras corresponding to non-compact quantum groups, e.g. quantum $ax + b$, can have faithful smooth action on nice smooth algebraic varieties or smooth manifolds.
- There is a purely algebraic no-go result due to Etingof and Walton analogous to our geometric no-go result.

Noncommutative Tori (Bhowmick-Goswami 2009)

Consider the noncommutative two-torus $\mathcal{A}_\theta$ (θ irrational) generated by two unitaries U, V satisfying $UV = e^{2\pi i\theta} VU$, and the standard spectral triple on it described by Connes. Here, $\mathcal{A}^\infty$ is the unital $*$-algebra spanned by U, V; $\mathcal{H} = L^2(\tau) \oplus L^2(\tau)$ (where τ is the unique faithful trace on $\mathcal{A}_\theta$) and D is given by

$D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$, where d_1 and d_2 are closed unbounded linear maps on $L^2(\tau)$ given by $d_1(U^m V^n) = mU^m V^n$, $d_2(U^m V^n) = nU^m V^n$. For this, there is a nice 'Laplacian' $\mathcal{L}$ given by $\mathcal{L}(U^m V^n) = (-m^2 + n^2)U^m V^n$.

Theorem

(i) $QISO^{\mathcal{L}} = \oplus_{k=1}^{8} C^*(U_{k1}, U_{k2})$ (as a C^* algebra), where for odd k, U_{k1}, U_{k2} are the two commuting unitary generators of $C(\mathbb{T}^2)$, and for even k, $U_{k1}U_{k2} = \exp(4\pi i\theta)U_{k2}U_{k1}$, i.e. they generate $\mathcal{A}_{2\theta}$.
(ii) $QISO^+(D)$ exists and coincides with $C(\mathbb{T}^2)$, i.e. there are no quantum orientation preserving isometries, although there are genuine quantum isometries in the Laplacian-based approach.

$U_\mu(2)$ as $QISO^+$ of $SU_\mu(2)$ (Bhowmick-Goswami 2009)

- The CQG $SU_\mu(2)$ $\mu \in [-1,1]$ is the universal unital C^* algebra generated by $\alpha,\ \gamma$ satisfying: $\alpha^*\alpha + \gamma^*\gamma = 1$, $\alpha\alpha^* + \mu^2\gamma\gamma^* = 1$, $\gamma\gamma^* = \gamma^*\gamma$, $\mu\gamma\alpha = \alpha\gamma$, $\mu\gamma^*\alpha = \alpha\gamma^*$., and the coproduct given by : $\Delta(\alpha) = \alpha \otimes \alpha - \mu\gamma^* \otimes \gamma$, $\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$.
- On the Hilbert space $L^2(h)$ (h Haar state), Chakraborty-Pal described a natural spectral triple with the D given by $D(e_{ij}^{(n)}) = (2n+1)e_{ij}^{(n)}$ if $n \neq i$, and $-(2n+1)e_{ij}^{(n)}$ for $n = i$, where $e_{ij}^{(n)}$ are normalised matrix elements of the $2n+1$ dimensional irreducible representation, n being half-integers.

Theorem

$QISO^+(D)$ is the CQG $U_\mu(2)$ which is the universal C^* algebra generated by $u_{11}, u_{12}, u_{21}, u_{22}$ satisfying:
$u_{11}u_{12} = \mu u_{12}u_{11}$, $u_{11}u_{21} = \mu u_{21}u_{11}$, $u_{12}u_{22} = \mu u_{22}u_{12}$, $u_{21}u_{22} = \mu u_{22}u_{21}$, $u_{12}u_{21} = u_{21}u_{12}$, $u_{11}u_{22} - u_{22}u_{11} = (\mu - \mu^{-1})u_{12}u_{21}$ *and* $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ *is a unitary.*

Podles spheres (Bhowmick-Goswami 2010)

- The Podles sphere $S^2_{\mu,c}$ is the universal C^* algebra generated by A, B satisfying

$$AB = \mu^{-2}BA, A = A^* = B^*B + A^2 - cI = \mu^{-2}BB^* + \mu^2 A^2 - c\mu^{-2}I.$$

- $S^2_{\mu,c}$ can also be identified as a suitable C^* subalgebra of $SU_\mu(2)$ and leaves invariant the subspace $\mathcal{K} = \overline{\mathrm{Span}\{e^{(l)}_{\pm\frac{1}{2},\ m} : l = \frac{1}{2}, \frac{3}{2}, \ldots,\ m = -l, -l+1, \ldots l\}}$ of $L^2(SU_\mu(2), h)$.
- R-twisted spectral triple given by: $D(e^{(l)}_{\pm\frac{1}{2},\ m}) = (c_1 l + c_2)e^{(l)}_{\mp\frac{1}{2},\ m}$, (where $c_1, c_2 \in \mathbb{R},\ c_1 \neq 0$), $R(e^{(n)}_{\pm\frac{1}{2},\ i}) = \mu^{-2i}e^{(n)}_{\pm\frac{1}{2},\ i}$.
- $$QISO^+_R(D) = SO_\mu(3) \equiv C^*\left(e^{(1)}_{ij},\ i,j = -1,0,1\right).$$
- There is also a real structure on this noncommutative manifold for which $QISO_{\mathrm{real}}$ turns out to be $SO_\mu(3)$.

Free and half liberated spheres (Banica-Goswami)

- Free sphere: $A_n^+ = C^*\left(x_1, \ldots, x_n \middle| x_i = x_i^*, \sum x_i^2 = 1\right)$.
- It has a faithful trace, and in the corresponding GNS space we can construct a spectral triple for which the quantum isometry group is the free orthogonal group

$$O_n^+ = C^*\left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u^t = u^{-1}\right).$$

- Similarly, consider the half-liberated sphere: $A_n^* = C^*\left(x_1, \ldots, x_n \middle| x_i = x_i^*, x_i x_j x_k = x_k x_j x_i, \sum x_i^2 = 1\right)$.
- Again, for a natural spectral triple on this, we get the following CQG (half-liberated quantum orthogonal group) as the quantum isometry group:

$$O_n^* = C^*\left(u_{11}, \ldots, u_{nn} \middle| u_{ij} = u_{ij}^*, u_{ij} u_{kl} u_{st} = u_{st} u_{kl} u_{ij}, u^t = u^{-1}\right).$$

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Sketch of proof for existence of $QISO^{\mathcal{L}}$

Let us give some ideas of a typical construction of quantum isometry groups. Consider the approach based on Laplacian.

- Let $\{e_{ij}, j = 1, \ldots, d_i; i = 1, 2, \ldots\}$ be the complete list of eigenvectors of the Laplacian $\mathcal{L}$, $\{e_{ij} j = 1, \ldots, d_i$ being the (orthonormal) basis for i-th eigenspace. recall that these are actually elements of $\mathcal{A}^\infty$, and let $\mathcal{A}_0^\infty$ be the span of these elements which is norm-dense in $\mathcal{A}$ by assumption
- We have to use the formalism of isometric quantum family. Call $(\mathcal{S}, \alpha)$ be such a family if $\mathcal{S}$ is a unital C^* algebra and $\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{S}$ is a $*$-homomorphism which commutes with $\mathcal{L}$, ie isometric, and also the linear span of $\alpha(\mathcal{A})(1 \otimes \mathcal{S})$ is norm dense in $\mathcal{A} \otimes \mathcal{S}$.
- We first claim that the 'connectedness assumption' that $\ker(\mathcal{L}) = \mathbb{C}1$ implies α preserves the volume form τ.

- Proof of claim: for any state ϕ on $\mathcal{S}$, consider the linear map $C := \alpha_\phi = (\mathrm{id} \otimes \phi) \circ \alpha$ on $\mathcal{A}_0^\infty$ which commutes with the self-adjoint operator $\mathcal{L}$, so leaves invariant each eigenspace, in particular maps the vector 1 to itself, and its orthocomplement (which is the direct sum of eigenspaces of $\mathcal{L}$) to itself too. For $a \in \mathcal{A}_0^\infty$, $< 1, (a - \tau(a)1) >= 0$, so $\tau(C(a)) - \tau(a) =< 1, C(a - \tau(a)1) >= 0$.
- Thus, α extends to a unitary operator from $\mathcal{H} \otimes \mathcal{S} = L^2(\mathcal{A}, \tau) \otimes \mathcal{S}$ to , which maps $e_{ij} \otimes 1$ to say $\sum_k e_{ik} \otimes q_{kj}^{(i)}$, and tracial property of τ implies that $(q_{kj}^{(i)})$ give a copy of $A_u(I_{d_i})$. This identifies $\mathcal{S}$ as a quotient of $*_i A_u(I_{d_i})$, say w.r.t. the ideal $\mathcal{I}_\mathcal{S}$.
- Now consider all the ideals of the form $\mathcal{I}_\mathcal{S}$ as above and take their intersection, say $\mathcal{I}$. One can prove that $(*_i A_u(I_{d_i}))/\mathcal{I}$ is the universal quantum family of isometries and is also a CQG, which is indeed the desired QISO.

QISO of deformed noncommutative manifolds (Bhowmick-Goswami 2009)

Recall Rieffel deformation of C^* algebras and Rieffel-Wang deformation of CQG. we give a general scheme for computing quantum isometry groups by proving that $\widetilde{QISO_R^+}$ of a deformed noncommutative manifold coincides with (under reasonable assumptions) a similar (Rieffel-Wang) deformation of the $\widetilde{QISO_R^+}$ of the original manifold.

Let $(\mathcal{A}, \mathbb{T}^n, \beta)$ be a C^* dynamical system, $\mathcal{A}^\infty$ be the algebra of smooth (C^∞) elements for the action β., and D be a self-adjoint operator on $\mathcal{H}$ such that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is an R-twisted, θ-summable spectral triple of compact type. Assume that there exists a compact abelian group $\widetilde{\mathbb{T}^n}$ with a covering map $\gamma : \widetilde{\mathbb{T}^n} \to \mathbb{T}^n$, and a strongly continuous unitary representation $V_{\tilde{g}}$ of $\widetilde{\mathbb{T}^n}$ on $\mathcal{H}$ such that

$$V_{\tilde{g}} D = D V_{\tilde{g}}, \;\; V_{\tilde{g}} a V_{\tilde{g}}^{-1} = \beta_g(a), \;\; g = \gamma(\tilde{g}).$$

Theorem

(i) For each skew symmetric $n \times n$ real matrix J, there is a natural representation of the Rieffel-deformed C^* algebra $\mathcal{A}_J$ in $\mathcal{H}$, and $(\mathcal{A}_J^\infty = (\mathcal{A}^\infty)_J, \mathcal{H}, D)$ is an R-twisted spctral triple of compact type.
(ii) If $QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D)$ and $(QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}$ have C^* actions on $\mathcal{A}$ and $\mathcal{A}_J$ respectively, where $\tilde{J} = J \oplus (-J)$, we have

$$QISO_R^+(\mathcal{A}_J^\infty, \mathcal{H}, D) \cong (QISO_R^+(\mathcal{A}^\infty, \mathcal{H}, D))_{\tilde{J}}.$$

(iii) A similar conclusion holds for $QISO^+(\mathcal{A}^\infty)$, $QISO^+(\mathcal{A}_J^\infty)$ provided they exist.
(iv) In particular, for deformations of classical spectral triples, the C^* action hypothesis of (ii) or (iii) hold, and hence the above conclusions hold too.

- Similar deformation results hold for $QISO^{\mathcal{L}}$ under reasonable conditions, e.g. faithfulness of haar state of $QISO^{\mathcal{L}}$.

QISO for AF algebras (Bhowmick, Goswami, Skalski)

We have shown the $QISO^+$ is well-behaved w.r.t. the inductive limit construction, and used this principle to compute $QISO^+$ of many interesting spectral triples on AF algebras. A more precise result is the following:

Theorem

Suppose that $(\mathcal{A}^\infty, \mathcal{H}, D)$ is a spectral triple of compact type such that
(a) D has a one-dimensional eigenspace spanned by a vector ξ which is cyclic and separating for $\mathcal{A}$.
(b) There is an increasing sequence $(\mathcal{A}_n^\infty)_{n \in \mathbb{N}}$ of unital *-subalgebras of $\mathcal{A}^\infty$ whose union is $\mathcal{A}^\infty$, and D commutes with the projection P_n onto the closed subspace $\mathcal{H}_n$ generated by $\mathcal{A}_n^\infty \xi$ for each n.
Then each $(\mathcal{A}_n^\infty, \mathcal{H}_n, D_n := D|_{\mathcal{H}_n})$ is a spectral triple for which $QISO^+$ exists, and there exist natural compatible CQG morphisms $\pi_{m,n} : QISO^+(D_m) \to QISO^+(D_n)$, $m \leq n$ such that

$$QISO^+(D) = \lim_{n \in \mathbb{N}} QISO^+(D_n).$$

Open problems to be investigated

- Proving some general results about the structure and representation theory of such quantum isometry groups.
- Extending the formulation of quantum isometry groups to the set-up of possibly noncompact manifolds (both classical and noncommutative), where one has to work in the category of locally compact quantum groups.
- Formulating a definition (and proving existence) of a quantum group of isometry for compact metric spaces , and more generally, for quantum metric spaces in the sense of Rieffel. Some wrok in this direction is done by Sabbe and Quaegebeur recently.