Quantum symmetries of the twisted tensor products of C*-algebras

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joint work with
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Background: Quantum symmetry groups

- **Wang**: quantum permutation groups and quantum symmetry groups of finite-dimensional C*-algebras equipped with reference states.
- **Banica, Bichon**: finite metric spaces and finite graphs.

  Raum, Schmidt, Speicher, Weber, Joardar, Mandal: *several interesting connections to combinatorics, representation theory and free probability*
- **Goswami**: quantum isometry groups associated to a given spectral triple á la Connes.

  Quantum isometry groups associated to the spectral triples for group C**-algebras of discrete groups*
- **Banica-Skalski**: a new framework of *quantum symmetry groups* based on *orthogonal filtrations* of unital C*-algebras.

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Orthogonal filtration of C-algebras*

Definition (Banica-Skalski, 2013; de Chanvalon, 2014)

Let A be a unital C*-algebra and let τ_A be a faithful state on A. An *orthogonal filtration* for the pair (A, τ_A) is a sequence of finite dimensional subspaces $\{A_i\}_{i\geq 0}$ such that $A_0 = \mathbb{C}1_A$, Span $\cup_{i\geq 0} A_i$ is dense in A and $\tau_A(a^*b) = 0$ if $a \in A_i$, $b \in A_j$ and $i \neq j$. We will usually write $\mathcal{A}$ for the triple $(A, \tau_A, \{A_i\}_{i\geq 0})$.

Example

Let Γ be a finitely generated discrete group endowed with a proper length function l. Then $B_n^l = \text{span}\{\lambda_g \mid l(g) = n\}$, $n \geq 0$, forms a filtration for the pair $(C_r^*(\Gamma), \tau_\Gamma)$ where τ_Γ is the canonical trace on $C_r^*(\Gamma)$. We denote $\mathcal{B} = (C_r^*(\Gamma), \tau_\Gamma, \{B_n^l\}_{n\geq 0})$.

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Compact quantum groups

Definition (Woronowicz, 1995)

A *compact quantum group* (CQG) is a pair $\mathbb{G} = (A, \Delta_A)$ consisting of a unital C^*-algebra A and a unital *-homomorphism $\Delta_A : A \to A \otimes A$ such that

1. Δ_A is coassociative: $(\Delta_A \otimes \mathrm{id}_A)\Delta_A = (\mathrm{id}_A \otimes \Delta_A)\Delta_A$,
2. Δ_A satisfies cancellation properties:
   $\Delta_A(A)(1_A \otimes A) = A \otimes A = \Delta_A(A)(A \otimes 1_A)$.

We denote A and Δ_A by $\mathrm{C}(\mathbb{G})$ and $\Delta_{\mathbb{G}}$ respectively.

- The dual of a CQG is a discrete group $\mathrm{C}_0(\widehat{\mathbb{G}})$.
- There is a unique $\mathrm{W}^{\mathbb{G}} \in \mathcal{U}(\mathrm{C}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}(\mathbb{G}))$ which encodes the pairing between $\mathbb{G}$ and $\widehat{\mathbb{G}}$.

Compact quantum groups

Example

- For any compact group G, the unital C^*-algebra $C(G)$ along with $\Delta : C(G) \to C(G \times G)$ defined by $(\Delta f)(x, y) = f(xy)$ is a CQG.
- For any discrete group Γ, the unital C^*-algebra $C^*_r(\Gamma)$ or $C^*(\Gamma)$ along with $\Delta : C^*_r(\Gamma) \to C^*_r(\Gamma \times \Gamma)$ or $\Delta : C^*(\Gamma) \to C^*(\Gamma \times \Gamma)$ defined by $\Delta(x) = x \otimes x$ is a CQG.

Coaction of compact quantum groups

Definition

A *(right) coaction* of $\mathrm{C}(\mathbb{G})$ (or an *action* of $\mathbb{G}$) on a unital C^*-algebra A is a unital *-homomorphism $\gamma\colon A \to A \otimes \mathrm{C}(\mathbb{G})$ with the following properties

1. γ is a comodule structure:
$$(\mathrm{id}_A \otimes \Delta_{\mathbb{G}})\gamma = (\gamma \otimes \mathrm{id}_C)\gamma;$$
2. γ satisfies the *Podleś condition*:
$$\gamma(A)(1_A \otimes C) = A \otimes C.$$

Similarly, one can consider coaction of $\mathrm{C}^{\mathrm{u}}(\mathbb{G})$ on unital C^*-algebras.

Theorem (Fischer, 2003)

In fact, every *injective* coaction γ of $\mathrm{C}(\mathbb{G})$ on A lifts to a unique universal coaction γ^{u} of $\mathrm{C}^{\mathrm{u}}(\mathbb{G})$ on A.

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CQG morphisms

- Let $\mathbb{G}$ and $\mathbb{H}$ be CQGs. A unital *-homomorphism $f\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{H})$ is said to be a *CQG morphism* if it satisfies the following condition:

$$\Delta_{\mathbb{H}} \circ f = (f \otimes f)\Delta_{\mathbb{G}}.$$

- Let A be a unital C*-algebra and let $\gamma_1 : A \to A \otimes \mathrm{C}(\mathbb{G})$ and $\gamma_2 : A \to A \otimes \mathrm{C}(\mathbb{H})$ be coactions.
  A CQG morphism $f\colon \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{H})$ *intertwines* the coactions γ_1 and γ_2 if

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Quantum symmetry groups

Let $\mathcal{A} = (A, \tau_A, \{A_i\}_{i\geq 0})$ be an orthogonal filtration.
Let $\mathcal{C}(\mathcal{A})$ be the category with objects as pairs $(\mathbb{G}, \alpha)$ where

- $\mathbb{G}$ is a compact quantum group
- α is an action of $\mathbb{G}$ on A such that $\alpha(A_i) \subseteq A_i \otimes_{\mathrm{alg}} \mathbb{C}(\mathbb{G})$ for each $i \geq 0$
- morphisms being CQG morphisms intertwining the respective actions.

Theorem (Banica-Skalski, 2013)

There exists a universal initial object in the category $\mathcal{C}(\mathcal{A})$ called the *quantum symmetry group* of the filtration $\mathcal{A}$ and denoted by $\mathrm{QISO}(\mathcal{A})$. Moreover the action of $\mathrm{QISO}(\mathcal{A})$ on A is *faithful*.

Remark

The condition $\alpha(A_i) \subseteq A_i \otimes_{\mathrm{alg}} \mathbb{C}(\mathbb{G})$ implies that the action α preserves the state τ_A. Converse is not true!

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The starting point

Suppose that

- $\{A_i\}_{i\geq 0}$ be an orthogonal filtration for a pair (A, τ_A).
- Γ be a discrete group with a length function.
- Γ acts on A. Equivalently, there is a coaction

$$\beta \colon A \to \mathcal{M}(A \otimes \mathrm{C}_0(\Gamma)).$$

- The reduced crossed product:

$$A \rtimes_{\beta,r} \Gamma := \{\beta(a)(1 \otimes \lambda_g) \mid a \in A, g \in \Gamma\}^{\mathrm{CLS}}.$$

Question

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Quantum symmetries of reduced crossed products

We write

- $a\lambda_g$ for $\beta(a)(1 \otimes \lambda_g)$, where $a \in A$, $g \in \Gamma$.
- $\tau := \tau_A \circ \tau' \in S(A \rtimes_{\beta,\mathrm{r}} \Gamma)$, where τ' is the canonical conditional expectation from $A \rtimes_{\beta,\mathrm{r}} \Gamma$ onto A defined by the continuous linear extension of the prescription $\tau'(\sum_g a_g \lambda_g) = a_e$.

Finally, given the data as above define for each $i, j \geq 0$

$$A_{ij} = \mathrm{span}\{a_i \lambda_{\gamma_j} \mid a_i \in A_i, l(\gamma_j) = j\}.$$

Proposition *(Bhowmick-Mandal-R.-Skalski, 2018)*

Suppose Γ is a finitely generated discrete group having an action β on A such that $\tau_A(\beta_g(a)) = \tau_A(a)$ for all a in A, $g \in \Gamma$. Then the triplet $\mathcal{A} \rtimes_\beta \mathcal{B} = (A \rtimes_{\beta,\mathrm{r}} \Gamma, \tau, (A_{ij})_{i,j\geq 0})$ defines an orthogonal filtration of the C*-algebra $A \rtimes_{\beta,\mathrm{r}} \Gamma$.

An example: relations with the QISOs of the factors

- Consider $A = \mathrm{C}^*(\mathbb{Z}_9)$, and $\Gamma = \mathbb{Z}_3$.
- Let ϕ be an automorphism of $\mathbb{Z}_9$ of order 3, given by the formula $\phi(n) = 4n$ for $n \in \mathbb{Z}_9$.
- ϕ induces an action $\beta \in \mathrm{Mor}(A, A \otimes \mathrm{C}(\mathbb{Z}_3))$ of $\mathbb{Z}_3$ on $A = \mathrm{C}^*(\mathbb{Z}_9)$ defined by

$$\beta(\lambda_n) = \lambda_n \otimes \delta_{\overline{0}} + \lambda_{\phi(n)} \otimes \delta_{\overline{1}} + \lambda_{\phi^2(n)} \otimes \delta_{\overline{2}}.$$

- $\mathrm{C}(\mathrm{QISO}(\mathrm{C}^*(\mathbb{Z}_9) \rtimes_\beta \mathbb{Z}_3, \tau, \{U_n\}_{n\geq 0}))$ is isomorphic to $\mathrm{C}^*(\mathbb{Z}_9 \rtimes_\beta \mathbb{Z}_3) \oplus \mathrm{C}^*(\mathbb{Z}_9 \rtimes_\beta \mathbb{Z}_3)$, so it has the vector space dimension equal $27 + 27 = 54$.

On the other hand$\cdots$

$\cdots$ $\mathrm{C}(\mathrm{QISO}(\mathrm{C}^*(\mathbb{Z}_n))) \cong \mathrm{C}^*(\mathbb{Z}_n) \oplus \mathrm{C}^*(\mathbb{Z}_n) (n \neq 4)$. Hence, the vector space dimension of $\mathrm{C}(\mathrm{QISO}(\mathrm{C}^*(\mathbb{Z}_9))) \otimes \mathrm{C}(\mathrm{QISO}(\mathrm{C}^*(\mathbb{Z}_3)))$ equals $(9+9)(3+3) = 108$.

An example: relations with the QISOs of the factors

- Consider $A = C^*(\mathbb{Z}_9)$, and $\Gamma = \mathbb{Z}_3$.
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$\mathrm{QISO}(C^*(\mathbb{Z}_9) \rtimes_\beta \mathbb{Z}_3, \tau, \{U_n\}_{n\geq 0})$ is *much smaller* than $\mathrm{QISO}(C^*(\mathbb{Z}_9)) \otimes \mathrm{QISO}(C^*(\mathbb{Z}_3))$

Quantum symmetry of twisted tensor products: set-up

Suppose,

- $\mathcal{A} := (A, \tau_A, \{A_i\}_{i\geq 0})$ and $\mathcal{B} := (B, \tau_B, \{B_j\}_{j\geq 0})$ will denote orthogonal filtrations of unital C*-algebras A and B.
- γ_A and γ_B will denote the canonical coactions of $\mathrm{C}(\mathrm{QISO}(\mathcal{A}))$ on A and $\mathrm{C}(\mathrm{QISO}(\mathcal{B}))$ on B, respectively.
- $\chi \in \mathcal{UM}(\mathrm{C}_0(\widehat{\mathrm{QISO}(\mathcal{A})}) \otimes \mathrm{C}_0(\widehat{\mathrm{QISO}(\mathcal{B})}))$ is a bicharacter.

Thoerem (Meyer-R.-Woronowicz, 2012)

χ is equivalent to

- is a Hopf *- (quantum group) homomorphism $f: \mathrm{C}(\mathrm{QISO}(\mathcal{A})) \to \mathcal{M}(\mathrm{C}_0(\widehat{\mathrm{QISO}(\mathcal{B})}))$
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The twisted tensor product $A \boxtimes_\chi B$

Theorem (Meyer-R.-Woronowicz, 2014)

- The bicharacter χ gives a rise to a faithful *χ-Heisenberg*: a pair (of non-degenerate representations) (π_1, π_2) of $\mathrm{C}(\mathrm{QISO}(\mathcal{A}))$ and $\mathrm{C}(\mathrm{QISO}(\mathcal{B}))$ on a suitable Hilbert space $\mathcal{H}$ satisfying the following commutation relation

$$\big((\mathrm{id} \otimes \pi_1)\mathrm{W}^{\mathrm{QISO}(\mathcal{A})}\big)_{13}\big((\mathrm{id} \otimes \pi_2)\mathrm{W}^{\mathrm{QISO}(\mathcal{B})}\big)_{23} = \big((\mathrm{id} \otimes \pi_2)\mathrm{W}^{\mathrm{QISO}(\mathcal{B})}\big)_{23}\big((\mathrm{id} \otimes \pi_1)\mathrm{W}^{\mathrm{QISO}(\mathcal{A})}\big)_{13}\chi_{12}.$$

- Define

$$j_A\colon A \xrightarrow{\gamma_A} A \otimes \mathrm{C}(\mathrm{QISO}(\mathcal{A})) \xrightarrow{(\mathrm{id}_A \otimes \pi_1)_{13}} \mathcal{M}(A \otimes B \otimes \mathbb{K}(\mathcal{H}))$$

$$j_B\colon B \xrightarrow{\gamma_B} B \otimes \mathrm{C}(\mathrm{QISO}(\mathcal{B})) \xrightarrow{(\mathrm{id}_B \otimes \pi_1)_{23}} \mathcal{M}(A \otimes B \otimes \mathbb{K}(\mathcal{H}))$$

- $A \boxtimes_\chi B := j_A(A) j_B(B)$ is a unital C^*-algebra and does not dependent on π_1 and π_2. $A \boxtimes_\chi B$ is called the *twisted tensor product* of A and B.

Orthogonal filtration on $A \boxtimes_\chi B$

- The canonical coactions γ_A and γ_B preserves the states τ_A and τ_B.
- This allows to define a functional

$$\tau_A \boxtimes_\chi \tau_B \colon A \boxtimes_\chi B \to \mathbb{C}$$

by restricting $\tau_A \otimes \tau_B \otimes \mathrm{id}_{\mathbb{B}(\mathcal{H})}$ on $A \boxtimes_\chi B$.

Proposition (Bhowmick-Mandal-R.-Skalski, 2018)

The functional $\tau_A \boxtimes_\chi \tau_B$ is a faithful state on $A \boxtimes_\chi B$ and the triple $\mathcal{A} \boxtimes_\chi \mathcal{B} := (A \boxtimes_\chi B, \tau_A \boxtimes_\chi \tau_B, \{j_A(A_i) j_B(B_j)\}_{i,j \geq 0})$ is an orthogonal filtration of $A \boxtimes_\chi B$.

Generalised Drinfeld's double

- There exists a CQG, denoted by $\mathfrak{D}_\chi$, with canonical injective CQG homomorphisms

$$\rho\colon \mathrm{C}(\mathrm{QISO}(\mathcal{A})) \to \mathrm{C}(\mathfrak{D}_\chi)$$
$$\theta\colon \mathrm{C}(\mathrm{QISO}(\mathcal{B})) \to \mathrm{C}(\mathfrak{D}_\chi)$$

such that $\mathrm{C}(\mathfrak{D}_\chi) = \rho(\mathrm{C}(\mathrm{QISO}(\mathcal{A})))\theta(\mathrm{C}(\mathrm{QISO}(\mathcal{B})))$ is a unital C^*-algebra with the comultiplication map defined by

$$\Delta_{\mathfrak{D}_\chi}(\rho(x)) := (\rho \otimes \rho)\Delta_{\mathrm{QISO}(\mathcal{A})}(x) \qquad \text{for all } x \in \mathrm{C}(\mathrm{QISO}(\mathcal{A})),$$
$$\Delta_{\mathfrak{D}_\chi}(\theta(y)) := (\theta \otimes \theta)\Delta_{\mathrm{QISO}(\mathcal{B})}(y) \qquad \text{for all } y \in \mathrm{C}(\mathrm{QISO}(\mathcal{B})).$$

Drinfeld Pairs

The pair (ρ, θ) is called the *(canonical) χ-Drinfeld pair*:

$$\chi_{12}\big((\mathrm{id}\otimes\rho)\mathrm{W}^{\mathrm{QISO}(\mathcal{A})}\big)_{13}\big((\mathrm{id}\otimes\theta)\mathrm{W}^{\mathrm{QISO}(\mathcal{B})}\big)_{23}$$
$$=\big((\mathrm{id}\otimes\theta)\mathrm{W}^{\mathrm{QISO}(\mathcal{B})}\big)_{23}\big((\mathrm{id}\otimes\rho)\mathrm{W}^{\mathrm{QISO}(\mathcal{A})}\big)_{13}\chi_{12}.$$

There exists the *universal χ-Drinfeld pair* $(\rho^{\mathrm{u}}, \theta^{\mathrm{u}})$ such that

$$\mathrm{C}^{\mathrm{u}}(\mathfrak{D}_\chi) = \rho^{\mathrm{u}}(\mathrm{C}^{\mathrm{u}}(\mathrm{QISO}(\mathcal{A})))\theta^{\mathrm{u}}(\mathrm{C}(\mathrm{QISO}(\mathcal{B}))).$$

Actions of $\mathfrak{D}_\chi$ on $A \boxtimes_\chi B$

Theorem (R., 2015)

There is a canonical *injective* coaction

$$\gamma_A \bowtie_\chi \gamma_B \colon A \boxtimes_\chi B \to A \boxtimes_\chi B \otimes \mathrm{C}(\mathfrak{D}_\chi)$$

defined by

$$\begin{aligned} \gamma_A \boxtimes_\chi \gamma_B(j_A(a)) &:= (j_A \otimes \rho)\gamma_A(a) \qquad \text{for all } a \in A, \\ \gamma_A \boxtimes_\chi \gamma_B(j_B(b)) &:= (j_B \otimes \theta)\gamma_B(b) \qquad \text{for all } b \in B. \end{aligned}$$

Quantum Symmetry of the twisted tensor products

Theorem (Bhowmick-Mandal-R.-Skalski, 2018)

- There is a coction $\gamma^{\mathrm{u}} : A \boxtimes_\chi B \to A \boxtimes_\chi B \otimes \mathrm{C}^{\mathrm{u}}(\mathfrak{D}_\chi)$ of $\mathrm{C}^{\mathrm{u}}(\mathfrak{D}_\chi)$ on $A \boxtimes_\chi B$ such that

$$\gamma^{\mathrm{u}}(j_A(a)) := (j_A \otimes \rho^{\mathrm{u}})\gamma^{\mathrm{u}}_A(a) \qquad \text{for all } a \in A,$$
$$\gamma^{\mathrm{u}}(j_B(b)) := (j_B \otimes \theta^{\mathrm{u}})\gamma^{\mathrm{u}}_B(b) \qquad \text{for all } b \in B.$$

- Moreover, the quantum symmetry group $\mathrm{QISO}(\mathcal{A} \boxtimes_\chi \mathcal{B})$ is isomorphic to $\mathfrak{D}_\chi$.

Corollary

The quantum symmetry group $\mathrm{QISO}(\mathcal{A} \otimes \mathcal{B})$ *is isomorphic to* $\mathrm{QISO}(\mathcal{A}) \times \mathrm{QISO}(\mathcal{B})$.

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Examples coming from the Rieffel deformation

- Let A and B be unital C*-algebras equipped with orthogonal filtrations.
- Assume that G and H are compact abelian groups acting respectively on C and on D in the filtration preserving way (so that they are objects of respective categories).
- Moreover, let $\chi : \hat{G} \times \hat{H} \to \mathbb{T}$ be a bicharacter.
- The coactions $\alpha_A : A \to A \otimes \mathrm{C}(G)$ and $\alpha_B : B \to B \otimes \mathrm{C}(H)$ define a canonical coaction γ of $\mathrm{C}(K) := \mathrm{C}(G \times H)$ on $E := A \otimes B$.
- Furthermore χ defines a bicharacter ψ on $\hat{K}$ via the formula

$$\psi : \hat{K} \times \hat{K} \to \mathbb{T},\ \psi((g_1, h_1),\ (g_2, h_2)) = \chi(g_2, h_1)^{-1}.$$

- It defines a 2-cocycle on the group $\hat{K}$. The Rieffel deformation of the data (E, γ, ψ) yields a new unital C*-algebra E_ψ.
- **Meyer-R.-Woronowicz**, 2014: E_ψ is isomorphic to $A \boxtimes_\psi B$.

Reduced crossed product revisited

- Let the triple $\mathcal{A} = (A, \tau_A, \{A_i\}_{i\geq 0})$ will denote an orthogonal filtration of a unital C*-algebra A.
- Let Γ is a finitely generated discrete group having an coaction $\beta\colon A \to \mathcal{M}(A \otimes \mathrm{C}_0(\Gamma))$ of $\mathrm{C}_0(\Gamma)$ on A. Denote $\mathcal{B}$ be the orthogonal filtration on $\mathrm{C}^*_\mathrm{r}(\Gamma)$.

Recall

- $a\lambda_g$ for $\beta(a)(1 \otimes \lambda_g)$, where $a \in A$, $g \in \Gamma$.
- $\tau := \tau_A \circ \tau' \in S(A \rtimes_{\beta,\mathrm{r}} \Gamma)$, where τ' is the canonical conditional expectation from $A \rtimes_{\beta,\mathrm{r}} \Gamma$ onto A defined by the continuous linear extension of the prescription $\tau'(\sum_g a_g\lambda_g) = a_e$.

Finally, given the data as above define for each $i, j \geq 0$

$$A_{ij} = \mathrm{span}\{a_i\lambda_{\gamma_j} \mid a_i \in A_i, l(\gamma_j) = j\}.$$

Quantum symmetry of the reduced crossed products

"The hypothesis"

There is a quantum group homomorphism

$$f \colon \mathrm{C}^{\mathrm{u}}(\mathrm{QISO}(\mathcal{A})) \to \mathrm{C}_{\mathrm{b}}(\Gamma).$$

Theorem (Bhowmick-Mandal-R.-Skalski, 2018)

Define $\beta = (\mathrm{id}_A \otimes f)\gamma_A^{\mathrm{u}}$, where γ_A^{u} is the canonical universal coaction of $\mathrm{C}^{\mathrm{u}}(\mathrm{QISO}(\mathcal{A}))$ on A.

- The triplet $(A \rtimes_{\beta,\mathrm{r}} \Gamma, \tau, (A_{ij})_{i,j\geq 0})$ is an orthogonal filtration, denoted $\mathcal{A} \rtimes_\beta \mathcal{B}$.
- There exists a canonical bicharacter $\chi \in \mathcal{U}(\mathrm{C}_0(\widehat{\mathrm{QISO}(\mathcal{A})}) \otimes \mathrm{C}_0(\widehat{\mathrm{QISO}(\mathcal{B})}))$ induced by the dual of the quantum group homomorphisms f and $\mathrm{C}_0(\mathrm{QISO}(\mathcal{B}))) \to \mathrm{C}_{\mathrm{r}}^*(\Gamma)$.
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- $\mathrm{QISO}(\mathcal{A} \rtimes_\beta \mathcal{B})$ is isomorphic to $\mathfrak{D}_\chi$.

How practical the "hypothesis" is?

For a fixed natural number n we define a group homomorphism $g\colon \mathbb{Z}^n \to \mathbb{T}^n$ by

$$g(m_1, m_2, \cdots, m_n) := (\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_n^{m_n}).$$

This defines a quantum group homomorphism $\hat{g}\colon \mathrm{C}(\mathbb{T}^n) \to \mathrm{C_b}(\mathbb{Z}^n)$.

- Suppose that there is a quantum group homomorphism $h\colon \mathrm{C^u}(\mathrm{QISO}(\mathcal{A})) \to \mathrm{C}(\mathbb{T}^n)$.
- The composition $f = \hat{g} \circ h\colon \mathrm{C^u}(\mathrm{QISO}(\mathcal{A})) \to \mathrm{C_b}(\mathbb{Z}^n)$ is a quantum group homomorphism.
- Moreover, we get a coaction $\beta\colon A \to \mathcal{M}(A \otimes \mathrm{C_0}(\mathbb{Z}^n))$ of $\mathrm{C_0}(\mathbb{Z}^n)$ on A defined by $\beta := (\mathrm{id}_A \otimes f)\gamma_A^{\mathrm{u}}$.

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Class of examples

1. $A = \mathrm{C}(M)$ for a compact Riemannian manifold M, $\tau := \int d\mathrm{vol}$, orthogonal filtration comes from the eigen spaces of Hodge-Dirac operator $d + d^*$.
2. For $q \in (0,1)$, $A = \mathrm{C}(G_q)$ for a q-deformation of a compact semisimple Lie group, $\tau =$ Haar state, orthogonal filtration comes from the matrix coefficients of irreducible representations of G_q.
3. $A = \mathrm{C}(\mathbb{G}_q/\mathbb{H})$ (quantum homogeneous spaces of G_q by a closed quantum subgroup $\mathbb{H}$).
4. $A = \mathcal{O}_N$, orthogonal filtration was constructed by Banica-Skalski.
5. Twisted crossed products $A \rtimes_{\beta,r,\Omega} \Gamma$ (provided β satisfies the condition).
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A counterexample: $C^*(\mathbb{Z}_9) \rtimes_{\beta,r} \mathbb{Z}_3$ *revisited*

- Recall that the vector space dimension of $C(QISO(C^*(\mathbb{Z}_9) \rtimes_{\beta,r} \mathbb{Z}_3))$ is 54, whereas vector space dimension of $C(QISO(C^*(\mathbb{Z}_9))) \otimes C(QISO(C^*(\mathbb{Z}_3)))$ equals $(9+9)(3+3) = 108$.
- Hence, $QISO(C^*(\mathbb{Z}_9) \rtimes_{\beta,r} \mathbb{Z}_3)$ is not isomorphic to the Drinfeld double of $QISO(C^*(\mathbb{Z}_9))$ and $QISO(C^*(\mathbb{Z}_3))$ with respect to any bicharacter.

Final remark

It seems that "*the hypothesis*" is necessary to have $QISO(A \rtimes_{r,\beta} \Gamma)$ to be isomorphic to the Drinfeld's double of the quantum symmetry group of the factors with respect to a bicharacter.

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Thank You!