Weyl algebra on Hyperbolic plane

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Weyl Relation

- One parameter groups of unitaries corresponding to the position and momentum operators on $L^2(\mathbb{R})$ are given by

$$U_t f(x) = e^{itx} f(x), V_s f(x) = f(x - s),$$

respectively where $t, s \in \mathbb{R}$.

- The canonical commutation relation in the Weyl form:

$$U_t V_s = e^{its} V_s U_t, t, s \in \mathbb{R}.$$

- For $(t, s) \in \mathbb{R}^2$, we define a unitary $W_{(t,s)}$ acting on $L^2(\mathbb{R})$ by

$$W_{(t,s)} = e^{\{-\frac{i}{2}ts\}} U_t V_s.$$

Weyl Algebra on $\mathbb{R}$

- For $\phi \in \mathbb{S}(\mathbb{R}^2)$, define

$$b(\phi) := \int_{\mathbb{R}^2} \hat{\phi}(t,s) W_{(t,s)} dtds.$$

- With this definition of $b(\phi)$, we have the following:

$$||b(\phi)|| \leq ||\hat{\phi}||_1 \qquad (1)$$

$$b(\phi)b(\psi) = b(\phi \star \psi),$$

$$\widehat{(\phi \star \psi)}(t,s) = \int \hat{\phi}(t-t', s-s')\hat{\psi}(t',s')$$

$$\exp\{-\frac{i}{2}(t's - ts')\} \qquad (2)$$

$$b(\phi)^* = b(\phi^\sharp), \widehat{\phi^\sharp}(t,s) = \overline{\hat{\phi}(-t,-s)}. \qquad (3)$$

Weyl algebra on $\mathbb{R}$

- Define Weyl algebra as

$$\mathcal{A}^{\infty} = \{b(\phi) : \phi \in \mathbb{S}(\mathbb{R}^2)\}$$

- The previous slide shows that $\mathcal{A}^{\infty}$ in deed is a $*$-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^2))$. The C^* completion generate the compact operators.

- There is a unique trace on $\mathcal{A}^{\infty}$ denoted by τ given by

$$\nu(b(\phi)) = \hat{\phi}(0) = \int_{\mathbb{R}^2} \phi(u_1, u_2) du_1 du_2.$$

- The GNS Hilbert space $L^2(\mathcal{A}^{\infty}, \nu)$ is unitarily isomorphic to $L^2(\mathbb{R}^2)$.
- $\mathcal{A}^{\infty}$ has a representation on $L^2(\mathcal{A}^{\infty}, \nu)$ by left multiplication.

Upperhalf plane model of 2-dimensional Hyperbolic space

- The upperhalf plane model of hyperbolic space in 2 dimension is given by

$$\mathbb{H}^2 = \{x, y : (x, y) \in \mathbb{R} \times \mathbb{R}^+\}.$$

- The metric is given on the tangent plane by $\begin{bmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{bmatrix}$
- The hyperbolic volume measure $d\mu = \frac{dxdy}{y^2}$.
- We have a global change of coordinates given by $\tau_1 = \frac{x}{y}, \tau_2 = \ln y$.
- Easy to see that this transformation maps the hyperbolic measure to Labesgue measure and also that we have a Hilbert space isomorphism between $L^2(\mathbb{H}^2, d\mu)$ and $L^2(\mathbb{R}^2)$.

Hyperbolic Laplacian

- The Laplace Beltrami operator in the hyperbolic coordinate system is given by

$$\Delta_{\mathbb{H}^2} = y^2(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}).$$

- With the change of coordinates the Laplace-Beltrami operator in the coordinates is given by

$$\Delta_{\mathbb{H}^2} = \underline{\delta}^T((G))\underline{\delta},$$

where $\underline{\delta} = \begin{bmatrix} \frac{\partial}{\partial \tau_1} \\ \frac{\partial}{\partial \tau_2} \end{bmatrix}$. and $G = \begin{bmatrix} 1+\tau_1^2 & -\tau_1 \\ -\tau_1 & 1 \end{bmatrix}$

Weyl algebra on Hyperbolic plane

- For $\underline{t} = (t_1, t_2), \underline{s} = (s_1, s_2)$, we can define four one parameter unitary groups acting on $L^2(\mathbb{H}^2, d\mu)$ by

$$U^{(1)}_{t_1} f(x,y) = f(x + t_1 y, y), U^{(2)}_{t_2} f(x,y) = f(xe^{t_2}, ye^{t_2})$$
$$V^{(1)}_{s_1} f(x,y) = e^{is_1 \frac{x}{y}} f(x,y), V^{(2)}_{s_2} f(x,y) = y^{is_2} f(x,y).$$

- Then the Weyl commutation relations are given by

$$U^{(1)}_{t_1} V^{(1)}_{s_1} = e^{-it_1 s_1} V^{(1)}_{s_1} U^{(1)}_{t_1}$$
$$U^{(2)}_{t_2} V^{(2)}_{s_2} = e^{-it_2 s_2} V^{(2)}_{s_2} U^{(2)}_{t_2}$$
$$[U^{(1)}_{t_1}, U^{(2)}_{t_2}] = [V^{(1)}_{s_1}, V^{(2)}_{s_2}] = 0$$
$$[U^{(1)}_{t_1} V^{(2)}_{s_2}] = [U^{(2)}_{t_2}, V^{(1)}_{s_1}] = 0.$$

Weyl algebra on Hyperbolic plane

- Let $(\underline{t}.\underline{s}) = (t_1 s_1 + t_2 s_2)$, $U_{\underline{t}} = U^{(1)}_{t_1} U^{(2)}_{t_2}$, $V_{\underline{s}} = V^{(1)}_{s_1} V^{(2)}_{s_2}$. Then we can define

$$b(\phi) := \int_{\mathbb{R}^4} \hat{\phi}(\underline{t}, \underline{s}) U_{\underline{t}} V_{\underline{s}} e^{-\frac{i}{2}(\underline{t}.\underline{s})} d\underline{t} d\underline{s}.$$

- With this definition very similar properties of the algebra $\mathcal{A}^\infty := \{b(\phi) : \phi \in \mathbb{S}(\mathbb{R}^4)\}$ as discussed in the beginning can be proved.
- This is a non unital noncommutative $*$-subalgebra of $\mathcal{B}(L^2(\mathbb{H}^2, d\mu))$.
- It has a representation on the GNS space of $\mathcal{A}^\infty$ with respect to the canonical trace ν by left multiplication. The trace ν is given by $\nu(b(\phi)) = \hat{\phi}(0) = \int_{\mathbb{R}^4} \phi(\underline{x}) dx$.

Component planes of the Weyl algebra

- For $\tilde{\phi} \in \mathbb{S}(\mathbb{R}^2)$ define $\phi(u_1, u_2, v_1, v_2) = \tilde{\phi}(u_1, u_2)\delta_0(v_1, v_2)$.
- Then

$$b(\phi) = \int_{\mathbb{R}^2} \hat{\tilde{\phi}}(t_1, t_2) U^{(1)}_{t_1} U^{(2)}_{t_2} dt_1 dt_2.$$

- We define

$$P_{u_1 - u_2} = \{b(\phi) : \phi \text{ as above}\}$$

- $P_{u_1-u_2} \subset \mathcal{B}(L^2(\mathbb{R}^4))$ is a commutative algebra, but it is **NOT** a subalgebra of $\mathcal{A}^\infty$ as ϕ as above do not belong to $\mathbb{S}(\mathbb{R}^4)$.
- Similarly $P_{u_1-v_1}, P_{u_2-v_2}, P_{v_1-v_2}$ can be defined and we call them **component planes** of $\mathcal{A}^\infty$.
- While $P_{u_1-u_2}$ and $P_{v_1-v_2}$ are each commutative, $P_{u_1-v_1}$ and $P_{u_2-v_2}$ are not.

Derivations on the Weyl algebra

- We denote the direct variables of $\mathbb{R}^4$ by (u_1, u_2, v_1, v_2) and the conjugate Fourier variables by (t_1, t_2, s_1, s_2).
- There are 4 canonical derivations on the algebra $\mathcal{A}^\infty$ coming from actions of $\mathbb{R}^4$ are given by

$$\delta_1(b(\phi)) = b(\frac{\partial\phi}{\partial u_1})$$
$$\delta_2(b(\phi)) = b(\frac{\partial\phi}{\partial u_2})$$
$$\delta_3(b(\phi)) = b(\frac{\partial\phi}{\partial v_1})$$
$$\delta_4(b(\phi)) = b(\frac{\partial\phi}{\partial v_2}),$$

- We call the above 4 derivations as canonical derivations.

Derivations on the Weyl algebra

- The symplectic group $Sp(4)$ is a part of the symmetry group of the algebra $\mathcal{A}^{\infty}$ as well. For $g \in Sp(4)$ The action is given by $g.b(\phi) = b(g.\phi)$, where $g.\phi(u_1, u_2, v_1, v_2) = \phi(g(u_1, u_2, v_1, v_2))$.
- Consider the one parameter subgroup of $Sp(4)$ given by $\begin{bmatrix} e^{\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & e^{-\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ for $\lambda \in \mathbb{R}$.
- The derivation corresponding to the above one parameter group of automorphisms is given by

$$\mathcal{R}(b(\phi)) = b(u_1 \frac{\partial \phi}{\partial u_1} - v_1 \frac{\partial \phi}{\partial v_1}).$$

Lindbladian on the Weyl algebra: The metric matrix

- We denote the space of linear maps on a $*$-algebra A by $L(A)$ and the linear subspace of derivations by $\mathrm{Der}(A)$.
- We call a matrix $\mathcal{M} \in M_n(L(A))$ a **real metric matrix** on the algebra A with a choice of canonical derivations $\{\delta_j\}_{1 \leq j \leq n}$ if it has all entries $*$-preserving linear maps and it can be written as $\Sigma^T \Sigma$ for some matrix $\Sigma \in M_n(L(A))$ such that

$$\Sigma \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{bmatrix} \in \mathrm{Der}(A) \underbrace{\oplus \ldots \oplus}_{n-times} \mathrm{Der}(A).$$

Lindbladian on the Weyl algebra: The metric matrix

- If we denote $\begin{bmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_n\end{bmatrix}$ by $\underline{\delta}$ and the ij-th entry of Σ by c_{ij}, $(\Sigma\underline{\delta})_j = \sum_{k=1}^{n} c_{jk}\delta_k$ where $c_{jk}\delta_k$ means composition of the maps c_{jk} and δ_k. Σ^T means the usual transpose of Σ that is, $((\Sigma^T))_{ij} = ((\Sigma))_{ji}$.
- An example of a real metric matrix is given by $\begin{bmatrix}1+\tau_1^2 & -\tau_1\\ -\tau_1 & 1\end{bmatrix}$ which comes from the hyperbolic plane already discussed. The factorization matrix Σ is given by $\begin{bmatrix}\tau_1 & -1\\ 1 & 0\end{bmatrix}$.

Lindbladian on the Weyl algebra

- On an algebra A with n canonical $*$-preserving derivations $\{\delta_j\}_{1\leq j\leq n}$ a Laplacian or a **Lindbladian** will be given by $\frac{1}{2}[\delta_1\ \delta_2\ldots\delta_n]((\mathcal{M}))\begin{bmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_n\end{bmatrix}$, where $\mathcal{M}$ is a real metric matrix.
- With the choice of canonical derivations $(\frac{\partial}{\partial\tau_1},\frac{\partial}{\partial\tau_2})$ and the real metric matrix $\begin{bmatrix}1+\tau_1^2 & -\tau_1\\ -\tau_1 & 1\end{bmatrix}$, we get the Laplace Beltrami operator on the hyperbolic plane in real coordinates (see slide 6).

Lindbladian on the Weyl algebra

- Define two $*$-preserving linear maps on $\mathcal{A}^\infty$ by

$$\gamma_1 b(\phi) = b(u_1\phi), \eta_1(b(\phi)) = b(v_1\phi).$$

- One choice of the metric matrix is given by
$$\begin{bmatrix} 1+\gamma_1^2 & -\gamma_1 & 0 & 0 \\ -\gamma_1 & 1 & 0 & 0 \\ 0 & 0 & 1+\eta_1^2 & -\eta_1 \\ 0 & 0 & -\eta_1 & 1 \end{bmatrix}.$$
- With the choice of canonical derivations it produces two classical Hyperbolic Laplacians on the commutative component planes $P_{u_1-u_2}$ and $P_{v_1-v_2}$.

Lindbladian on the Weyl algebra

- In order to study the cases of non commuting component planes, we need to have some non zero elements in the off diagonal corners.
- One particular choice of real metric matrix on $\mathcal{A}^\infty$ with respect to the ordered choice of canonical derivations $\{\delta_j\}_{1\leq j\leq 4}$ is given by

$$\mathcal{M} = \begin{bmatrix} \gamma_1^2+1 & -\gamma_1 & -\gamma_1\eta_1 & 0 \\ -\gamma_1 & 1 & \eta_1 & 0 \\ -\gamma_1\eta_1 & \eta_1 & 1+\eta_1^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The factorization matrix Σ is given by

$$\Sigma = \begin{bmatrix} \gamma_1 & -1 & -\eta_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Lindbladian on the Weyl algebra

- To see that $\mathcal{M}$ is a real metric matrix for the choice of the canonical derivations, note that $\Sigma \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} \gamma_1\delta_1 - \delta_2 - \eta_1\delta_3 \\ \delta_1 \\ -\delta_4 \\ \delta_3 \end{bmatrix}$. Also note that $\gamma_1\delta_1 - \eta_1\delta_3$ is nothing but the derivation D already discussed earlier.
- With the above choice of real metric matrix the Lindbladian on the Weyl algebra is given by

$$\mathcal{L}(b(\phi)) = \frac{1}{2}((\mathcal{R} - \delta_2)^2 + \delta_1^2 + \delta_4^2 + \delta_3^2)(b(\phi))$$

- Note that the upper diagonal corner of Σ is the factorization matrix for the metric on the Hyperbolic plane and $\mathcal{L}$ satisfies the 2nd order cocycle condition.

Lindbladian on component planes

- The restrictions of the Lindbladian on Weyl algebra on various component planes are given by

$$\mathcal{L}_{u_1-u_2}(b(\phi)) = \frac{1}{2}[\delta_1(1+\gamma_1^2)\delta_1 - \delta_1(\gamma_1\delta_2) - \delta_2(\gamma_1\delta_1) + \delta_2^2$$
$$-\gamma_1\delta_1 + \delta_2](b(\phi)) = \frac{1}{2}b((\Delta_{\mathbb{H}^2} - u_1\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2})(\phi)) \quad (4)$$
$$\mathcal{L}_{u_1-v_1}(b(\phi)) = \frac{1}{2}(\mathcal{R}^2 + \delta_1^2 + \delta_3^2)(b(\phi)) \quad (5)$$
$$\mathcal{L}_{v_1-v_2}(b(\phi)) = \frac{1}{2}[(\eta_1\delta_3)^2 + \delta_4^2](b(\phi)) \quad (6)$$
$$\mathcal{L}_{u_2-v_2}(b(\phi)) = \frac{1}{2}[\delta_2^2 + \delta_4^2](b(\phi)) \quad (7)$$

- $\mathcal{L}_{u_1-u_2}$ is a first order perturbation of the classical hyperbolic Laplacian in the $u_1 - u_2$ coordinates whereas the Lindbladian $\mathcal{L}_{u_2-v_2}$ corresponds to the case Weyl algebra on flat $\mathbb{R}^2$.

Stochastic Process on the component plane $P_{u_1-v_1}$

- The automorphisms generated by the derivations $\mathcal{R}, \delta_1, \delta_3$ are given by

$$\Lambda_\alpha(\tilde{\phi})(u_1, v_1) = \tilde{\phi}(e^\alpha u_1, e^{-\alpha} v_1)$$
$$\tau_\beta^{(1)}(\tilde{\phi})(u_1, v_1) = \tilde{\phi}(u_1 + \beta, v_1)$$
$$\tau_\beta^{(3)}(\tilde{\phi})(u_1, v_1) = \tilde{\phi}(u_1, v_1 + \beta)$$

- The commutations among the automorphisms are given by

$$\Lambda_\alpha \tau_\beta^{(1)} \Lambda_\alpha = \tau_{\beta e^{-\alpha}}^{(1)}$$
$$\Lambda_\alpha \tau_\beta^{(3)} \Lambda_\alpha = \tau_{\beta e^{\alpha}}^{(3)}.$$

- $\tau_\beta^{(1)}$ and $\tau_\beta^{(3)}$ commute.

Stochastic Process on the component plane $P_{u_1-v_1}$

- The stochastic process $j_t : P_{u_1-v_1} \to P_{u_1-v_1} \otimes \mathcal{B}(\mathrm{Fock})$ on the noncommuting component plane $P_{u_1-v_1}$ associated with the above 3 automorphisms needs 3 independent SBM's and satisfies the following SDE:

$$\begin{aligned} dj_t(b(\tilde{\phi})) &= j_t(D(b(\tilde{\phi})))dw_2(t) + j_t(\delta_1(b(\tilde{\phi})))dw_1(t) \\ &+ j_t(\delta_3(b(\tilde{\phi})))dw_3(t) + j_t(\mathcal{L}_{u_1-v_1}(b(\tilde{\phi})))dt. \end{aligned}$$

- The solution is given by

$$j_t(b(\tilde{\phi})) = b(C_{\underline{w}(t)}\tilde{\phi}), \underline{w}(t) = (w_1(t), w_2(t), w_3(t)).$$

where

$$(C_{\underline{w}(t)}(\tilde{\phi})(u_1, v_1)) = \tilde{\phi}((u_1+w_1(t))e^{w_2(t)}, (u_2+w_3(t))e^{-w_2(t)}).$$

Geometry of the component plane $P_{u_1 - v_1}$

- Consider the following quadratic forms on $L^2(\mathbb{R}^2)$:
  $a[f, g] = \langle -i\delta_1(f), -i\delta_1(g)\rangle$
  $b[f, g] = \langle -i\delta_3(f), -i\delta_3(g)\rangle$
  $c[f, g] = \langle -i\mathcal{R}(f), -i\mathcal{R}(g)\rangle$.
- If we denote the generators of the quadratic forms by A, B, C respectively, then we denote their formal sum by $\mathcal{L}$. It agrees with $\mathcal{L}_{u_1 - v_1}$ on $b(f)$ for $f \in C_c^\infty(\mathbb{R}^2)$.
- We want to study the Asymptotic behaviour of the trace of the operator $b(\phi)e^{T\mathcal{L}}$ acting on $L^2(\mathbb{R}^2)$ for small $T > 0$.

Geometry of the component plane $P_{u_1 - v_1}$

- Let ω_i be three independent Standard Brownian motions and $\mathcal{P} \equiv \{0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition of $[0, T]$. Then define

$$I_{\mathcal{P}} = \overrightarrow{\Pi}_{\mathcal{P}} \tau^{(1)}_{\omega_1(\Delta_j)} \tau^{(3)}_{\omega_3(\Delta_j)} \Lambda_{\omega_2(\Delta_j)}.$$

- Define $\mathbb{E}_k(F(.)) = \int F(\omega_k)\mathbb{P}(d\omega_k)$ for $k = 1, 2, 3$, where $\mathbb{P}(\omega_k)$ are Wiener measures on the path space.
- Then using the independence among ω_k's for different k's and $\omega_k(\Delta_j)$ and $\omega_k(\Delta_l)$ for $j \neq l$, one can show that

$$\mathbb{E}_1\mathbb{E}_2\mathbb{E}_3(I_{\mathcal{P}}) = (e^{\frac{1}{2}(A+B)\frac{T}{n}} e^{\frac{1}{2}C\frac{T}{n}})^n.$$

Geometry of the component plane $P_{u_1 - v_1}$

- By Trotter-Kato product formula,

$$e^{T\mathcal{L}} = \mathrm{s}-\lim_{n\to\infty}(e^{\frac{1}{2}(A+B)\frac{T}{n}} e^{\frac{1}{2}C\frac{T}{n}})^n = \mathrm{s}-\lim_{n\to\infty}\mathbb{E}_1\mathbb{E}_2\mathbb{E}_3(I_{\mathcal{P}}).$$

- Using the commutation relations among $\tau_\alpha^{(1)}$, $\tau_\alpha^{(3)}$ and Λ_α, we get

$$\mathrm{s}-\lim_{n\to\infty}\mathbb{E}_1\mathbb{E}_2\mathbb{E}_3(I_{\mathcal{P}}) = \mathbb{E}_2(e^{\frac{1}{2}A\int_0^T e^{2\omega_2(\tau)}d\tau} e^{\frac{1}{2}B\int_0^T e^{-2\omega_2(\tau)}d\tau} \Lambda_{\omega_2(T)})$$

Hence we have the following

Lemma

$$e^{T\mathcal{L}} = \mathbb{E}_2(e^{\frac{1}{2}A\int_0^T e^{2\omega_2(\tau)}d\tau} e^{\frac{1}{2}B\int_0^T e^{-2\omega_2(\tau)}d\tau} \Lambda_{\omega_2(T)}).$$

Weyl Asymptotics

- For $\phi, f \in \mathbb{S}(\mathbb{R}^2)$, $b(\phi)e^{T\mathcal{L}}(b(f)) = b(\phi \star e^{TL}(f))$.
-

$$\begin{aligned} & \widehat{\phi \star e^{TL}}(f)(t,s) \\ = \; & \mathbb{E}_2\Big(\int \hat{\phi}(t-t_1, s-s_1) e^{-\frac{1}{2}t_1^2\int_0^T e^{2\omega_2(\tau)}d\tau} e^{-\frac{1}{2}s_1^2\int_0^T e^{-2\omega_2(\tau)}d\tau} \\ & \hat{f}(e^{-\omega_2(T)}t_1, e^{\omega_2(T)}s_1) p(t,s,t_1,s_1) dt_1 ds_1\Big). \end{aligned}$$

- After a change of variables, the last integral becomes

$$\begin{aligned} & \mathbb{E}_2\Big(\int \hat{\phi}(t - e^{\omega_2(T)}t_1, s - e^{-\omega_2(T)}s_1) \\ & e^{-\frac{1}{2}t_1^2\int_0^T e^{2\omega_2(\tau)+2\omega_2(T)}d\tau} e^{-\frac{1}{2}s_1^2\int_0^T e^{-(2\omega_2(\tau)+2\omega_2(T))}d\tau} \\ & \hat{f}(t_1, s_1) p(t,s,e^{\omega_2(T)}t_1, e^{-\omega_2(T)}s_1) dt_1 ds_1\Big). \end{aligned}$$

Weyl Asymptotics

- Hence the operator $b(\phi)e^{T\mathcal{L}}$ is an integral operator with kernel $K(t,s,t_1,s_1)$ given by

$$
\begin{aligned}
& K(t,s,t_1,s_1) \\
= \; & \mathbb{E}_2\{\hat{\phi}(t - e^{\omega_2(T)}t_1, s - e^{-\omega_2(T)}s_1) \\
& \exp[-\frac{1}{2}t_1^2 \int_0^T e^{2\omega_2(\tau)+2\omega_2(T)} d\tau] \\
& \exp[-\frac{1}{2}s_1^2 \int_0^T e^{-(2\omega_2(\tau)+2\omega_2(T))} d\tau] \\
& p(t,s,e^{\omega_2(T)}t_1, e^{-\omega_2(T)}s_1)\}
\end{aligned}
$$

- It can be shown that the operator $b(\phi)e^{T\mathcal{L}}$ is trace class operator for all $T > 0$ and the kernel $K(.,.,.,.)$ is continuous in each of the four variables.

Weyl Asymptotics

- Hence
$$tr(b(\phi)e^{T\mathcal{L}}) = \int\int K(t,s,t,s)dtds$$
- If we denote $\frac{1}{T}\int_0^T e^{\pm\omega_2(\tau)}d\tau$ by $F^{\pm}(\omega_2;T)$, then we can prove the following

Lemma

Let $\{\omega(t)\}_{t\geq 0}$ be a SBM and let $\alpha \in \mathbb{R}$, $T > 0$. Then
$$\int \mathbb{P}(d\omega)|F^{\pm}(\omega_2;T) - 1|^2 \to 0, \text{ as } T \to 0^+.$$
In fact, more precisely, $\mathbb{E}|F^{\pm}(.,T) - 1|^2 = \frac{T}{3} + O(T^2)$. Consequently along a subsequence as $T \to 0^+$, $F^{\pm}(\omega_2;T) \to 1$ pointwise almost everywhere.

Weyl Asymptotics

- Using the previous Lemma, we can prove the following

Lemma

For $\phi \in \mathbb{S}(\mathbb{R}^2)$ and for the fixed subsequence of T in a sufficiently small interval,

$$
\begin{aligned}
& tr(b(\phi)e^{T\mathcal{L}}) \\
= \quad & \mathbb{E}_2(\{\frac{1}{T\sqrt{F^+F^-}}\}\int \hat{\phi}(\{\frac{(e^{-\omega_2(T)}-1)}{\sqrt{T}\sqrt{F^+}}\}t, \{\frac{(e^{\omega_2(T)}-1)}{\sqrt{T}\sqrt{F^-}}\}s) \\
& e^{-\frac{1}{2}(t^2+s^2)}e^{its\{\frac{sinh\omega_2(T)}{T\sqrt{F^+F^-}}\}}dtds)
\end{aligned}
$$

Weyl Asymptotics

- To compute the required trace, we use the following

Lemma

(i) The random variables $\frac{e^{\pm\omega(T)}-1}{\sqrt{T}}$ *converges weakly to* $\mathrm{N}(0,1)$ *as* $T \to 0^+$.
(ii) For fixed $y, t, s \in \mathbb{R}$ *and* $T \in \mathbb{R}^+$, *define*

$$H(y,t,s;T) = \exp\{its\frac{sinhy\sqrt{T}}{T}\} - \exp\{its\frac{y}{\sqrt{T}}\}.$$

Then $\lim_{T\to 0+} H(y,t,s;T) = 0$

Weyl Asymptotics

- It can be shown that

$$
\begin{aligned}
& tr(\phi \star e^{T\mathcal{L}}) \\
= & \frac{1}{T}\int \frac{dy}{2\pi} e^{-\frac{y^2}{2}} \int \hat{\phi}(-yt, ys) e^{-\frac{t^2+s^2}{2}} \exp\{its\frac{y}{\sqrt{T}}\} dtds \\
= & \frac{1}{T}\int\int\int dudvdt\phi(u,v)e^{-\frac{t^2}{2}} \int e^{-\frac{1}{2}y^2(1+(v-\frac{t}{\sqrt{T}})^2)} e^{iytu}\frac{dy}{2\pi}.
\end{aligned}
$$

- With the change of variable $z = y\sqrt{(1+(v-\frac{t}{\sqrt{T}})^2)}$,

$$
\begin{aligned}
& tr(b(\phi)e^{T\mathcal{L}}) \\
= & \frac{1}{T}\int du \int\int \phi(u,v) \frac{1}{\sqrt{(1+(v-\frac{t}{\sqrt{T}})^2)}} \\
& e^{-\frac{t^2}{2}(1+\frac{u^2}{(1+(v-\frac{t}{\sqrt{T}})^2)})} dvdt
\end{aligned}
$$

Weyl Asymptotics

- We have the following

Theorem

For every $\delta > 0$ *and every* $\phi \in \mathbb{S}(\mathbb{R}^2)$*,*
$\lim_{T \to 0^+} T^{\frac{1}{2}+\delta} tr(b(\phi)e^{T\mathcal{L}}) = 0.$

- To prove the theorem observe that

$$T^{\frac{1}{2}} tr(b(\phi)e^{T\mathcal{L}}) = \int du \int \int \phi(u,v) \frac{1}{\sqrt{(T + (v\sqrt{T} - t)^2)}} e^{-\frac{t^2}{2}(1+\frac{u^2}{(1+(v-\frac{t}{\sqrt{T}})^2)})} dvdt.$$

Weyl Asymptotics

- Breaking the above integral in two regions $|t| < 2(1+|v|)$ and $|t| \geq 2(1+|v|)$, we write the integral as $I_1 + I_2$, where

$$I_1 = \int du \int \int_{|t|<2(1+|v|)} \phi(u,v) \frac{1}{\sqrt{(T + (v\sqrt{T} - t)^2)}} e^{-\frac{t^2}{2}(1+\frac{u^2}{(1+(v-\frac{t}{\sqrt{T}})^2)})} dvdt,$$

and

$$I_2 = \int du \int \int_{|t|\geq 2(1+|v|)} \phi(u,v) \frac{1}{\sqrt{(T + (v\sqrt{T} - t)^2)}} e^{-\frac{t^2}{2}(1+\frac{u^2}{(1+(v-\frac{t}{\sqrt{T}})^2)})} dvdt$$

Weyl Asymptotics

- We can show the following estimates for the integrals I_1 and I_2:

$$
\begin{aligned}
|I_1| \leq\ & 2\int\int dudv|\phi(u,v)| \\
& (ln(3(1+|v|)+\sqrt{1+9(1+|v|)^2}) - \frac{1}{2}lnT),
\end{aligned}
$$

and

$$|I_2| \leq \frac{\pi}{2}\int\int|\phi(u,v)|dudv.$$

- Using the fact that $\phi \in \mathbb{S}(\mathbb{R}^2)$, one has constants $C_1, C_2 \geq 0$, such that

$$|T^{\frac{1}{2}+\delta}tr(b(\phi)e^{T\mathcal{L}})| < T^{\delta}\frac{\pi}{2}C_1 + 2T^{\delta}C_2 - C_1T^{\delta}lnT.$$

Weyl Asymptotics

- Using similar estimates, one can prove the following

Theorem

For $\phi \in \mathbb{S}(\mathbb{R}^2)$ such that $\phi(u,v) > 0$ for all u, v,
$\lim_{T \to 0^+} T^{\frac{1}{2}} tr(b(\phi) e^{T\mathcal{L}}) \neq 0$

- Consider the set
$\mathcal{D} = \{r \in \mathbb{R} : \lim_{T \to 0^+} T^r tr(\phi \star e^{T\mathcal{L}}) = 0 \forall \phi \in \mathbb{S}(\mathbb{R}^2)\}$.

Definition

If infimum of the set $\mathcal{D}$ exists, then the dimension of the plane $P_{u_1 - v_1}$ is defined to be $2\inf(\mathcal{D})$.

Theorem

$\dim P_{u_1 - v_1} = 1$.

Conclusions

- If we agree with the definition of dimension , then the dimension of such an “Noncommutative” plane drops. Following the examples of classical plane and noncommutative Euclidean plane, it is expected that the component plane of the hyperbolic algebra discussed here would be of 2-dimension. But due to the noncommutativity of the algebra as well as the presence of the derivation $\mathcal{R}$, the dimension actually drops which can be attributed to the hyperbolicity of the plane.
- Although we do not have a closed form for the ‘volume form’ of the palne, we can take non-zero limit to be the ‘volume form’.

Questions to be addressed

- How big is the linear space of derivations?
- How much freedom do we have in the choice of the metric matrix?
- Can one formulate some coordinate free definition of the metric in this set up?

Thank You!