Minimally Twisting Spectral Triples

to go beyond standard model

Devashish Singh, Pierre Martinetti

Dipartimento di Matematica
Università degli Studi di Genova, Italia

"International Conference on Noncommutative Geometry: Physical and Mathematical Aspects of Quantum Space-Time and Matter"
27-30 November, 2018

Minimal twist of a closed Riemannian manifold

- the flat case
- the curved case

Minimal twist of the two-point space of electrodynamics

- the non-twisted case
- the twisted case

The fermionic action of a twisted spectral triple

- generalities
- in case of the two-point space

Almost commutative geometry

An *almost commutative geometry* $M \times F$ is the product spectral triple of a canonical triple with a finite spectral triple (A_F, H_F, D_F), given by

$$(\mathcal{A} = C^\infty(M) \otimes A_F, \mathcal{H} = L^2(S, M) \otimes H_F, D = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma_M \otimes D_F).$$

- For standard model, we set $A_F = A_{SM} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, which acts on the Hilbert space $H_F = \mathbb{C}^{96}$ of elementary fermions.
- D_F is a 96×96 matrix, acting on H_F, with its entries being masses of elementary fermions, CKM matrix, and neutrino mixing parameters.
- Further, J_M, J_F, and $J := J_M \otimes J_F$, respectively, denote the real structures on the Hilbert spaces $L^2(S, M)$, H_F, and $\mathcal{H}$.
- And similarly, γ_M, γ_F, and $\gamma := \gamma_M \otimes \gamma_F$, respectively, denote the gradings on these Hilbert spaces.
- **Fermionic fields** are the (Grassmannian) elements of the Hilbert subspace

$$\mathcal{H}^+ := \{\psi \mid \gamma\psi = \psi, \ \psi \in \mathcal{H}\}.$$

- **Bosonic fields** arise as a connection 1-form ω, obtained via *fluctuations of the metric* given by $D \to D_\omega := D + \omega + J\omega J^{-1}$, where the *generalized 1-form* is

$$\omega = \omega^* \in \Omega^1_D(\mathcal{A}) := \{\sum_j a_j[D, b_j] : a_j, b_j \in \mathcal{A}\}.$$

- Particularly, for an almost commutative geometry, it takes the form

$$\omega = \gamma_M \otimes \phi + \gamma^\mu \otimes A_\mu, \text{ and } D_\omega = \partial\!\!\!/ \otimes \mathbb{I}_F + \gamma^\mu \otimes B_\mu + \gamma^5 \otimes \Phi,$$

where $\phi \in A_F$ is a scalar field on M and the 1-form $A_\mu \in \mathfrak{u}(A_F)$ are the gauge fields. Further, we define

$$B_\mu := \mathsf{ad}(A_\mu) = A_\mu - J_F A_\mu J_F^{-1} \quad \text{and} \quad \Phi := D_F + \phi + J_F \phi J_F^{-1}.$$

- The **spectral action** is a functional of the gauge field, given by

$$S[\omega] := \mathsf{Tr} f\left(\frac{D_\omega}{\Lambda}\right),$$

where f is a smooth approximation of the characteristic function of the interval $[0, 1]$ and Λ is a real cutoff parameter.

Twisting the geometry to build models beyond SM

- **Remarks:** The Higgs mass problem, vacuum instability, the desert hypothesis, and extending the Standard Model...
- *Twist* was introduced by Connes and Moscovici (2008),[1] for purely mathematical reasons, before the detection of Higgs (2012).
- One replaces the boundedness of the commutator $[D, a]$ with the boundedness of the twisted commutator, for some $\rho \in \mathsf{Aut}(\mathcal{A})$,

$$[D, a]_\rho := Da - \rho(a)D, \quad (\forall a \in \mathcal{A}).$$

- For an almost commutative geometry, the *twisted fluctuation*,

$$D \to D_\rho := D + \omega_\rho + J\omega_\rho J^{-1},$$

  generates a vector field X_μ and a scalar field σ.
- The interpretation of X_μ is yet to be understood well. σ cures the Higgs mass problem for NCG, and is also anticipated by particle physicists to resolve the vacuum instability of Higgs.

[1] A. Connes and H. Moscovici, *Type III and spectral triples*, Traces in number theory, geo. and quantum fields, Aspects Math. E38 (2008), no. Friedt. Vieweg, Wiesbaden, 57-71.

Other approaches:

- T. Brzezinski, N. Ciccoli, L. Dabrowski, and A. Sitarz, *Twisted reality condition for Dirac operators*, Math. Phys. Anal. Geo. (2016) 19.
- A.H. Chamseddine, A. Connes, W.D. van Suijlekom. *Inner fluctuations in noncommutative geometry without first order condition*. J. Geom. Phy. 73 (2013) 222?234.
- A. H. Chamseddine, A. Connes, W.D. van Suijlekom, *Beyond the spectral standard model: emergence of Pati-Salam unification*. JHEP 11 (2013) 132.

Minimal twist of a closed Riemannian manifold[2]

Minimal twist

A minimal twist of a given spectral triple $(\mathcal{A}, \mathcal{H}, D)$ by a unital C^*-algebra $\mathcal{B}$ is a twisted spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D)_\rho$ with $\rho \in \mathsf{Aut}(\mathcal{A} \otimes \mathcal{B})$ such that $\pi(a \otimes \mathbb{I}_\mathcal{B}) = \pi_0(a), \forall a \in \mathcal{A}$, where π and π_0 denote the representations for $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D)_\rho$ and $(\mathcal{A}, \mathcal{H}, D)$, respectively.

Minimal twist by grading

Given a graded spectral triple $(\mathcal{A}, \mathcal{H}, D; \gamma)$, one can decompose its Hilbert space into eigenspaces of the grading γ: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and furnish a representation for the algebra $\mathcal{A} \otimes \mathbb{C}^2 \ni (a, a')$ on $\mathcal{H}$ given by

$$\pi(a, a') := p_+ \pi_0(a) + p_- \pi_0(a') = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a') \end{pmatrix},$$

where $p_\pm := \frac{1}{2}(\mathbb{I} \pm \gamma)$ denote the projections on the eigenspaces of γ, and $\pi_\pm(a) := p_\pm \pi_0(a)|_{\mathcal{H}_\pm}$ denote the restrictions on $\mathcal{H}_\pm$ of the initial representation π_0 of $\mathcal{A}$ on $\mathcal{H}$.

* The term 'minimal' refers to the fact that fermionic structure of the theory is untouched.

---

[2] [Landi, Martinetti] *On twisting real spectral triples by algebra automorphisms* (2016), *Gauge transformations for twisted spectral triples* (2017)

The case of a flat Riemannian manifold[3]

- Minimal twist by grading is the only possible minimal twist (*by a finite-dimensional algebra with faithful representation*) of an even dimensional closed manifold M, given by $\left(C^\infty(M)\otimes\mathbb{C}^2, L^2(M,S), \partial\!\!\!/\right)_\rho$, with the automorphism $\rho(f,g)=(g,f), \forall(f,g)\in C^\infty(M)\otimes\mathbb{C}^2$, and the representation, $(\mathsf{dim}(M) =: 2m)$,

$$\pi(f,g)=\begin{pmatrix} f\mathbb{I}_{2^{m-1}} & 0 \\ 0 & g\mathbb{I}_{2^{m-1}}\end{pmatrix}, \quad \pi(\rho(f,g))=\begin{pmatrix} g\mathbb{I}_{2^{m-1}} & 0 \\ 0 & f\mathbb{I}_{2^{m-1}}\end{pmatrix}.$$

- The twisted 1-form ω_ρ, given by $\sum_j a_j[\partial\!\!\!/, b_j]_\rho$, is of the form:

$$\omega_\rho := -i\gamma^\mu Z_\mu, \quad \text{where} \quad Z_\mu := \begin{pmatrix} f'\partial_\mu g\mathbb{I}_2 & 0_2 \\ 0_2 & f\partial_\mu g'\mathbb{I}_2\end{pmatrix},$$

for some $(f,f'),(g,g')\in C^\infty(M)\otimes\mathbb{C}^2$. The *self-adjoint* twisted fluctuation of the free Dirac operator $\partial\!\!\!/$ associated to the commutative manifold M, is parametrized by a vector field $f_\mu\in C^\infty(M,\mathbb{R})$ as

$$\partial\!\!\!/ \mapsto \partial\!\!\!/_\rho := \partial\!\!\!/ + X\!\!\!/ = -i\gamma^\mu(\partial_\mu + f_\mu\gamma^5),$$

where $X\!\!\!/ := -i\gamma^\mu X_\mu$ with $X_\mu = f_\mu\gamma^5$.

---

[3] [Devastato, Martinetti] *Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry* (2014).

Theorem (Gilkey's theorem[4])

Given a differential operator P, acting on sections of a vector bundle V on a compact Riemannian manifold M of dimension m with metric g, with leading symbol given by the metric tensor. Thus, locally one has

$$P = -(g^{\mu\nu} I \partial_\mu \partial_\nu + A^\mu \partial_\mu + B),$$

where $g^{\mu\nu}$ plays the role of the inverse metric, I is the identity matrix, and A_μ and B are endomorphisms of the bundle V. It can be uniquely written in the form

$$P = \nabla^* \nabla - E,$$

where ∇ is a connection on V, with $\nabla^*\nabla$ the connection Laplacian, and E is an endomorphism of V. Let $\Gamma^\rho_{\mu\nu}$ be the Christoffel symbols of the Levi-Civita connection of the metric g.
We set $\Gamma^\rho := g^{\mu\nu}\Gamma^\rho_{\mu\nu}$. The explicit formulae for the connection ∇ and the endomorphism E are then of the form

$$\nabla_\mu = \partial_\mu + \omega'_\mu, \quad \omega'_\mu = \frac{1}{2} g_{\mu\nu}(A^\mu + \Gamma^\nu \cdot \mathsf{id}),$$

with id *the identity endomorphism of V, and*

$$E = B - g^{\mu\nu}(\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu - \Gamma^\rho_{\mu\nu}\omega'_\rho).$$

---

[4] P. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* (1984).

The case of a curved Riemannian manifold

- The free Dirac operator is $\partial\!\!\!/^S = -i\gamma^\mu \nabla^S_\mu$, where $\nabla^S_\mu = \partial_\mu + \omega^S_\mu$.
- Its twisted fluctuation is $\partial\!\!\!/^X = -i\gamma^\mu \nabla^X_\mu$, where $\nabla^X_\mu = \partial_\mu + \omega^X_\mu$ and $\omega^X_\mu := \omega^S_\mu + X_\mu$, with $X_\mu = f_\mu \gamma^5$ for some $f_\mu \in C^\infty(M)$.
- $$(\partial\!\!\!/^X)^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu + \alpha^\mu \partial_\mu + \beta),$$
  where $\alpha^\mu = \{iX\!\!\!/, \gamma^\mu\} + 2g^{\mu\nu}\omega^S_\nu - \Gamma^\mu$ and $\beta = -\partial\!\!\!/ \omega\!\!\!/^X - (\omega\!\!\!/^X)^2$.
- $$E = \beta - (g^{\mu\nu}\nabla_\mu - \Gamma^\nu)\omega_\nu,$$
  where $\nabla_\mu = \partial_\mu + \omega_\mu$ and
  $$\omega_\mu = \frac{1}{2} g_{\mu\nu}(\alpha^\nu + \Gamma^\nu) = \omega^S_\mu + \frac{1}{2} g_{\mu\nu}\{\gamma^\nu, iX\!\!\!/\}.$$
- Denoting $\Delta_\mu := X_\mu - \rho(X_\mu)$, one has
  $$\omega_\mu = \omega^X_\mu - \frac{1}{2}\gamma^\lambda\gamma_\mu\Delta_\lambda \quad \Leftrightarrow \quad \omega\!\!\!/ = \omega\!\!\!/^X + \Delta\!\!\!/.$$
- $$E = \frac{1}{2}\gamma^\mu\gamma^\nu \left(F^X_{\mu\nu} + \mathcal{D}^X_\nu \Delta^X_\mu + \Delta_\mu\Delta_\nu\right) - \frac{1}{2}\Gamma^\mu\Delta_\mu,$$
  where $F^X_{\mu\nu} = \nabla^X_\mu \omega^X_\nu - \nabla^X_\nu \omega^X_\mu$ and $\mathcal{D}^X_\mu = \partial_\mu + [\omega^X_\mu, \cdot]$.

$U(1)$-gauge theory from the two-point space[5]

The simplest nontrivial example of an almost commutative geometry is the two-point space, which describes the $U(1)$-gauge theory of electrodynamics,

$$M \times F_{ED} := \left(C^\infty(M) \otimes \mathbb{C}^2, L^2(S,M) \otimes \mathbb{C}^4, \partial\!\!\!/ \otimes \mathbb{I}_4 + \gamma^5 \otimes D_F\right),$$

where $A_F = \mathbb{C}^2$, $H_F = \mathbb{C}^4 = \mathsf{Span}\{e_L, \bar{e}_R, \bar{e}_L, e_R\}$, and

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix}, \quad J_F = \begin{pmatrix} 0_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0_2 \end{pmatrix} cc, \quad \gamma_F = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix}.$$

The algebra act via the representation

$$\pi_0(f,g) = \begin{pmatrix} \pi_M(f)\mathbb{I}_2 & 0_2 \\ 0_2 & \pi_M(g)\mathbb{I}_2 \end{pmatrix}, \quad \forall f, g \in C^\infty(M),$$

with $(\pi_M(f)\psi)(x) = f(x)\psi(x)$, $\forall \psi \in L^2(S,M)$. And, the fluctuation of the metric gives

$$D_\omega = D + \gamma^\mu Y_\mu \otimes \gamma_F, \quad \text{where} \quad Y_\mu \in C^\infty(M, \mathbb{R}).$$

[5] K. van den Dungen, W. van Suijlekom. *Electrodynamics from Noncommutative Geometry*, (2011).

Minimal twist of the 2-point space

- $\mathcal{A} = C^\infty(M) \otimes \mathbb{C}^2 = C^\infty(M) \oplus C^\infty(M)$, and the grading is

$$\gamma^5 \otimes \gamma_F = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix} \otimes \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix} \otimes \mathbb{I}_2 & 0_8 \\ 0_8 & \begin{pmatrix} -\mathbb{I}_2 & 0_2 \\ 0_2 & \mathbb{I}_2 \end{pmatrix} \otimes \mathbb{I}_2 \end{pmatrix}.$$

- The representation of minimal twist $\mathcal{A} \otimes \mathbb{C}^2 \ni (a, a')$ on $\mathcal{H}$ is given by

$$\pi(a, a') = \begin{pmatrix} \begin{pmatrix} f\mathbb{I}_2 & 0_2 \\ 0_2 & f'\mathbb{I}_2 \end{pmatrix} \otimes \mathbb{I}_2 & 0_8 \\ 0_8 & \begin{pmatrix} g'\mathbb{I}_2 & 0_2 \\ 0_2 & g\mathbb{I}_2 \end{pmatrix} \otimes \mathbb{I}_2 \end{pmatrix} = \begin{pmatrix} \pi_0(f, f') \otimes \mathbb{I}_2 & 0_8 \\ 0_8 & \pi_0(g', g) \otimes \mathbb{I}_2 \end{pmatrix}.$$

- **The free part**. The self-adjoint twisted inner fluctuation of $\partial\!\!\!/ \otimes \mathbb{I}_4$ of $M \times F_{ED}$ is of the form

$$\omega_M + J\omega_M J^{-1} = \gamma^\mu Y_\mu \otimes \gamma_F - i\gamma^\mu X_\mu \otimes \mathbb{I}_4,$$

parametrized by two real fields $Y_\mu, X_\mu \in C^\infty(M, \mathbb{R})$ defined as $Y_\mu := \Im(z_\mu), X_\mu := \Re(z_\mu)\gamma^5$, for some $z_\mu \in C^\infty(M, \mathbb{C})$.

- **The finite part** Self-adjoint $\omega_F + J\omega_F J^{-1}$ is of the form $\phi \otimes D_F$, where

$$\phi = \begin{pmatrix} \varphi_1 \mathbb{I}_2 & 0_2 \\ 0_2 & \varphi_2 \mathbb{I}_2 \end{pmatrix}, \quad \text{for some } \varphi_1, \varphi_2 \in C^\infty(M, \mathbb{R}).$$

The fermionic action[6]

- For the unitary $R = \gamma_E^0 \otimes \mathbb{I}_4 \in \mathcal{B}(\mathcal{H})$ implementing the twist ρ on $\mathcal{H} = L^2(M,S) \otimes \mathbb{C}^4$, a generic vector $\xi \in \mathcal{H}_R := \{\psi \in \mathsf{Dom}(D) : R\psi = \psi\}$ is of the form

$$\xi = \psi_1 \otimes e_L + \psi_2 \otimes \bar{e}_R + \psi_3 \otimes \bar{e}_L + \psi_4 \otimes e_R,$$

  where $\psi_k \in L^2(M,S)$ is a four-component Dirac spinor of the form $\psi_k = \begin{pmatrix} u_k \\ u_k \end{pmatrix}$ with a two-component Weyl spinor $u_k \in L^2(M,S)^{\pm}$ and $\{e_L, \bar{e}_R, \bar{e}_L, e_R\}$ is the orthonormal basis for the finite dimensional Hilbert space $\mathcal{H}_F = \mathbb{C}^4$.
- We restrict the fermionic action S^F to the Hilbert subspace $\mathcal{H}_R$, on which the bilinear form $(J\psi, D\phi)_\rho$ is necessarily anti-symmetric. We notice that

$$(J\psi, D\phi)_\rho := (J\psi, RD\phi) = (R^\dagger J\psi, D\phi) = \epsilon'''(JR^\dagger\psi, D\phi) = \epsilon'''(J\psi, D\phi).$$

  Thus, $(J\xi, D\xi)_\rho$ differs from $(J\xi, D\xi)$ by a sign factor of ϵ'''.

[6] [Devastato, Fransworth, Lizzi, Martinetti] *Lorentz signature and twisted spectral triples* (2018).

- One has

$$\frac{1}{2}\langle J_M\tilde{\psi}_k, D_X\tilde{\psi}_l\rangle = \bar{u}_k^\dagger\sigma_2\left(\sigma_j\partial_j - if_0\mathbb{I}_2\right)u_l, \quad \text{where } D_X := \partial\!\!\!/ + X\!\!\!/,$$

$$\frac{1}{2}\langle J_M\tilde{\psi}_k, \Phi\tilde{\psi}_l\rangle = -(1+\frac{\varphi_1-\varphi_2}{2})\bar{u}_k^\dagger\sigma_2 u_l, \quad \text{where } \Phi := \gamma_E^5 + \phi.$$

- The fermionic action S^F of the minimally twisted two-point space of electrodynamics is given by

$$S^F = \frac{1}{4}\langle J\tilde{\xi}, D_\rho\tilde{\xi}\rangle = \bar{u}_1^\dagger\sigma_2\left(\sigma_j\partial_j - if_0\mathbb{I}_2\right)u_3 + \bar{u}_2^\dagger\sigma_2\left(\sigma_j\partial_j - if_0\mathbb{I}_2\right)u_4$$
$$- (1+\frac{\varphi_1-\varphi_2}{2})(\bar{d}\bar{u}_1^\dagger\sigma_2 u_4 + d\bar{u}_2^\dagger\sigma_2 u_3).$$

- Comparing S^F with the Dirac Lagrangian (in Minkowski spacetime),

$$\mathcal{L}_M = i\Psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \Psi_L + i\Psi_R^\dagger \sigma^\mu \partial_\mu \Psi_R - m(\Psi_L^\dagger \Psi_R + \Psi_R^\dagger \Psi_L),$$

and identifying the spinors as

$$i\Psi_L^\dagger = \bar{u}_1^\dagger \sigma_2, \quad \Psi_L = u_3, \quad i\Psi_R^\dagger = \bar{u}_2^\dagger \sigma_2, \quad \Psi_R = -u_4,$$

that is $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} u_3 \\ -u_4 \end{pmatrix}$, and $\Psi^\dagger = (\Psi_L^\dagger, \Psi_R^\dagger) = -i(\bar{u}_1^\dagger \sigma_2, \bar{u}_2^\dagger \sigma_2)$, such identification gives us the following constraint:

$$\partial_0 \Psi = \partial_0 \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = -if_0 \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = -if_0 \begin{pmatrix} \Psi_L \\ -\Psi_R \end{pmatrix} = -if_0 \gamma_M^0 \Psi,$$

which is nothing but the Dirac equation for a free spin-1/2 particle at rest: $\partial_t \Psi = -i(E/\hbar)\gamma_M^0 \Psi$, where $f_0 = E/\hbar$, and, the solution is

$$\Psi(t) = \begin{pmatrix} \psi_L(0) e^{-iEt/\hbar} \\ \psi_R(0) e^{+iEt/\hbar} \end{pmatrix}.$$

Further, if we set $d := -ik$ for some $k \in \mathbb{R}$, then the mass term is

$$m = k\left(1 + \frac{\varphi_1 - \varphi_2}{2}\right).$$

Conclusions:

- The twisted fluctuation of a 4D-manifold parametrized by a real vector field $X_\mu = f_\mu \gamma^5$ is a purely geometric quantity.
- In case of a curved background, it couples with the curvature via the Christoffel symbols, i.e. $\Gamma^\mu(X_\mu - \rho(X_\mu))$.
- Starting with a Riemannian twisted spectral triple, requiring that the fermionic action obey twisted gauge transformations, one ends up naturally with a fermionic action that is Lorentzian.

THANKS FOR YOUR TIME AND ATTENTION!