# Levi-Civita connections in noncommutative geometry

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## Levi-Civita's theorem

Suppose $(M, g)$ is a pseudo-Riemannian manifold. Then there exists a unique connection

$$\nabla : \Omega^1(M) \to \Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$$

such that $\nabla$ is torsionless and compatible with $g$.

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Suppose $\mathcal{A}$ be a unital ( possibly noncommutative ) algebra and $(\Omega^\cdot(\mathcal{A}), d)$ be a differential calculus on $\mathcal{A}$ and $g$ a pseudo-Riemannian metric on $\Omega^1(\mathcal{A})$.

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## Question

Suppose $\mathcal{A}$ be a unital ( possibly noncommutative ) algebra and $(\Omega^\cdot(\mathcal{A}), d)$ be a differential calculus on $\mathcal{A}$ and $g$ a pseudo-Riemannian metric on $\Omega^1(\mathcal{A})$. Does there exist a connection

$$\nabla : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

such that $\nabla$ is torsionless and compatible with $g$.

1. Frolich, Grandjean and Recknagel ( Hermitian metric on spectral triples )
2. Connes, Tretkoff, Moscovici, Khalkhali, Fatizadeh, Dabrowski, Sitarz, Yang Liu, Mathias Lesch etc ( curvature for conformal perturbation from spectral asymptotics of the Laplacian of the noncommutative torus )
3. Beggs, Majid and collaborators ( bimodule connections, or zero co-torsion replacing metric compatibility )

### Aim

Identify a set of sufficient conditions on the differential calculus which ensures the existence ( and uniqueness ) of the Levi-Civita connection.

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## Theorem ( Goswami )

Suppose $(\Omega^{\cdot}(\mathcal{A}), d)$ is a differential calculus such that $\mathcal{E} := \Omega^1(\mathcal{A})$ satisfies **Assumption I - III**. Moreover assume that there exists a nondegenerate pseudo-Riemannian bilinear metric $g_0$ on $\mathcal{E}$. Then there exists a unique connection on $\mathcal{E}$ which is torsionless and compatible with $g$.

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## Definition

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as $\mathcal{A} - \mathcal{A}$-bimodules, where $\mathcal{F} \cong \mathrm{Im}(m)$.

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Let us set up the following notations:

1. $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} := \mathrm{Ker}(m)$.
2. $P_{\mathrm{sym}}$ = unique idempotent on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ with range $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ and kernel $\mathcal{F}$.
3. $\sigma = 2P_{\mathrm{sym}} - 1$.

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A pseudo-Riemannian metric $g$ on $\mathcal{E}$ is an element of $\text{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ such that
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(i) $g$ is symmetric, i.e. $g\sigma = g$,
(ii) $g$ is non-degenerate, i.e, the right $\mathcal{A}$-linear map $V_g : \mathcal{E} \to \mathcal{E}^*$ defined by
$V_g(\omega)(\eta) = g(\omega \otimes_{\mathcal{A}} \eta)$
is an isomorphism of right $\mathcal{A}$-modules.

## Connection on $\mathcal{E}$

A $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that

$$\nabla(ea) = \nabla(e)a + e \otimes_{\mathcal{A}} da.$$

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## Torsion of $\nabla$

$$T_{\nabla} := m \circ \nabla + d : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}).$$

$T_{\nabla}$ is right $\mathcal{A}$-linear. $\nabla$ is called torsionless if $T_{\nabla} = 0$.

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Need to make some assumptions ( **Assumption I, II, III** ) for defining the metric compatibility of a connection. Recall that $\mathcal{Z}(\mathcal{E}) = \{e \in \mathcal{E} : ea = ae \ \forall a \in \mathcal{E}\}$.

## Assumption I

$\mathcal{E}$ is finitely generated and projective as a right $\mathcal{A}$-module. Moreover, the map

$$u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$$

$$\sum_i e_i' \otimes_{\mathcal{Z}(\mathcal{A})} a_i \mapsto \sum_i e_i' a_i$$

is an isomorphism of vector spaces.

## Classical case: $\mathcal{E} = \Omega^1(M)$

$\mathcal{Z}(\mathcal{A}) = \mathcal{A}, \mathcal{Z}(\mathcal{E}) = \mathcal{E}$.

$$m : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$$

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**Assumption II:**

The $\mathcal{A} - \mathcal{A}$ bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ admits a splitting as a direct sum of right $\mathcal{A}$ modules : $\mathrm{Ker}(m) \oplus \mathcal{F}$, where $\mathcal{F} \cong \mathrm{Im}(m)$.

**Assumption III:**

If $\omega, \eta$ are in $\mathcal{Z}(\mathcal{E})$, then $\sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega$.

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Consequences of **Assumption I - III**

$\mathcal{E}$ is a centered bimodule.

# Definition of metric compatibility

Fix a pseudo Riemannian metric $g$ on $\mathcal{E}$.

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**Lemma**

For a pseudo-Riemannian metric on $\mathcal{E}$, let us define $\Pi^0_g(\nabla) : \mathcal{Z}(\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{Z}(\mathcal{E}) \to \mathcal{E}$ as the map given by

$$\Pi^0_g(\nabla)(\omega \otimes_{\mathbb{C}} \eta) = (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega).$$

Then $\Pi^0_g$ extends to a well defined map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{E}$ to be denoted by $\Pi_g(\nabla)$.

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## The definition

A connection $\nabla$ on $\mathcal{E}$ is said to be compatible with $g$ if $\Pi_g(\nabla) = dg$.

Suppose $(\Omega^{\cdot}(\mathcal{A}), d)$ is a differential calculus such that $\mathcal{E} := \Omega^1(\mathcal{A})$ satisfies

**Assumption I**

$\mathcal{E}$ fgp as a right $\mathcal{A}$-module.
$u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}, \sum_i e'_i \otimes_{\mathcal{Z}(\mathcal{A})} a_i \mapsto \sum_i e'_i a_i$
is an isomorphism of vector spaces.

**Assumption II:**

$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(m) \oplus \mathcal{F}$, where $\mathcal{F} \cong \mathrm{Im}(m)$.

**Assumption III:**

If $\omega, \eta$ are in $\mathcal{Z}(\mathcal{E})$, then $\sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega$.

Moreover assume that there exists a nondegenerate pseudo-Riemannian bilinear metric $g_0$ on $\mathcal{E}$. Then there exists a unique connection on $\mathcal{E}$ which is torsionless and compatible with $g$.

## Theorem ( Goswami )

$$\text{Let } \Phi_g : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E}) \text{ be defined by :}$$

$$\Phi_g(L) = (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma)|_{\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}}.$$

Then $\Phi_g$ is right $\mathcal{A}$-linear.

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$$\Phi_g(L) = (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma)|_{\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}}.$$

Then $\Phi_g$ is right $\mathcal{A}$-linear.
Moreover, if $\Phi_g$ is an isomorphism of right $\mathcal{A}$-modules,
then there exists a unique connection $\nabla$ on $\mathcal{E}$ which is torsion-less and compatible with $g$.
Moreover, $\nabla$ is given by the following equation:

$$\nabla = \nabla_0 + \Phi_g^{-1}(dg - \Pi_g(\nabla_0)). \quad (1)$$

A spectral triple over a unital $*$-algebra $A^\infty \subseteq B(H)$, ( $H$ is a Hilbert space ) is a triple $(A^\infty, H, D)$ such that $D$ is a self adjoint ( possibly unbounded ) operator on $H$, For all $a \in A^\infty$, $[D, a]$ extends to a bounded operator on $H$, $D$ has compact resolvents.

## The space of forms from a spectral triple

Define $d_D(\cdot) = \sqrt{-1}[D, \cdot]$. $\Omega^1(\mathcal{A}) := \mathrm{Span}\{a[D, b] : a, b \in \mathcal{A}\}$.
Have a natural multiplication map
$m_0 : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{B}(H); (\omega \otimes_{\mathcal{A}} \eta) \mapsto \omega\eta \in \mathcal{B}(\mathcal{H})$.
Let $\mathcal{J}$ = right $\mathcal{A}$-submodule of the $\mathrm{Im}(m_0)$ spanned by elements of the form $\sum_i [D, a_i][D, b_i]$ (finite sum) such that $\sum_i a_i[D, b_i] = 0$ $(a_i, b_i \in \mathcal{A})$.
We define $\Omega^2(\mathcal{A}) = \mathrm{Im}(m_0)/\mathcal{J}$ and let $m : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$ be the composition of $m_0$ and the quotient map from $\mathrm{Im}(m_0)$ to $\Omega^2(\mathcal{A})$.

Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a $p$-summable spectral triple. Consider the positive linear functional $\tau$ on $\mathcal{B}(\mathcal{H})$ given by

$$\tau(X) = \mathrm{Lim}_{\omega} \frac{\mathrm{Tr}(X\,|D|^{-p})}{\mathrm{Tr}(|D|^{-p})},$$

$\mathrm{Lim}_{\omega}$ being the Dixmier trace. We will assume that $\tau$ is a faithful normal trace on the von Neumann algebra generated by the $*$-subalgebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ in $\mathcal{B}(\mathcal{H})$.

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### The $\mathcal{A}''$-valued bilinear form ( after Frolich et al )

Let $g : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{A}''$ be given by

$$g(\omega \otimes_{\mathbb{C}} \eta) = \langle\langle \omega^*, \eta \rangle\rangle \,.$$

Then $g$ descends to an $\mathcal{A}$-bilinear, $\mathcal{A}''$-valued map, to be denoted by $g$ again.

## Proposition ( Goswami )

Suppose that the map

$$\mathbb{R} \to \mathcal{B}(\mathcal{H}) \text{ defined by } t \mapsto e^{itD} X e^{-itD}$$

is differentiable at $t = 0$ in the norm topology of $\mathcal{B}(\mathcal{H})$, so that the map $\mathcal{L} := -d^* d$ makes sense.
If $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, then

$$g(\omega \otimes_{\mathcal{A}} \eta) \in \mathcal{A} \text{ for all } \omega, \eta \in \Omega^1(\mathcal{A}).$$

## Definition

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple so that **Assumption I - III** are satisfied. If the $\mathcal{A}$-$\mathcal{A}$ bilinear map $g$ above is $\mathcal{A}$-valued, $V_g : \mathcal{E} \to \mathcal{E}^*$ is nondegenerate and $g \circ \sigma = g$, then we call $g$ to be the canonical Riemannian bilinear metric for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

1. The matrix geometry of the Fuzzy sphere
2. The quantum Heisenberg manifold
3. Isospectral deformations of classical manifolds by free and isometric toral action.

## Spectral triple for the Fuzzy sphere ( Frolich et al)

$G = SU(2)$ and $V_j, j \in \frac{1}{2}\mathbb{N}_0$, denote the $(2j+1)$ dimensional irreducible representation of $SU(2)$.
$\mathcal{H}_0 := \bigoplus_{j=0,\frac{1}{2},\ldots,\frac{k}{2}} V_j^* \otimes_{\mathbb{C}} V_j$
$\mathcal{A} := \mathcal{B}(\mathcal{H}_0)$.
Let $\mathcal{H} := \mathcal{H}_0 \otimes W$.
$W =$ the canonical irreducible representation space of the Clifford algebra generated by the vector space $T_e G$ with respect to the Killing form on $G$.
There exists a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

## The space of forms

1. $\mathcal{E} \cong \mathrm{Span}\{a_i \otimes_{\mathbb{C}} e_i : i = 1, 2, 3\}$ and thus is a free right $\mathcal{A}$ module of rank three.
2. The bimodule structure for $\mathcal{E} := \Omega^1(\mathcal{A})$ is given by $a(b \otimes_{\mathbb{C}} e_i)c = abc \otimes_{\mathbb{C}} e_i$.
3. $\Omega^2(\mathcal{A}) \cong \mathrm{Span}\{a_{ij} \otimes_{\mathbb{C}} e_i e_j : a_{ij} = -a_{ji}\}$ is a free right $\mathcal{A}$ module of rank three.
4. $\mathrm{Ker}(m)$ is generated ( as a right $\mathcal{A}$ module ) by the set $\{e_i \otimes_{\mathcal{A}} e_i, e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i : i = 1, 2, 3\}$.
5. The space of three-forms is a free rank one module and all the higher forms are zero.

**Theorem ( Frolich et al )**

Frolich et al proves ( with their definition of metric compatibility ) that there exists a family of torsionless and metric compatible connections.

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**Proposition ( B. + Goswami + Mukhopadhyay )**

With our definition of metric compatibility of a connection, there exists a unique Levi-Civita connection and this connection coincides with the real unitary and torsion-less connection of Frolich et al.

## The Heisenberg manifold and its quantization

Suppose $G$ is the Heisenberg group and $\mathbb{Z}$ the corresponding discrete subgroup. The classical Heisenberg manifold is the homogeneous space $G/\mathcal{Z}$. Rieffel constructed a 2-parameter strict deformation quantization of $C(G/\mathbb{Z})$.

## Proposition ( Chakraborty and Sinha )

There exists a family of spectral triples on the quantum Heisenberg manifold. The module of one forms $\mathcal{E} := \Omega^1(\mathcal{A})$ is a free module generated by three central elements. The space of two forms $\Omega^2(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$.
If we take the metric compatibility condition of Frolich et al, then there exists no torsionless connection which is compatible with the metric.

## Proposition( B. + Goswami + Joardar )

Levi-Civita connection exists. Moreover, this connection has a constant negative scalar curvature $\text{Scal} = -0.125$.

## The spectral triple

Let $e_1, e_2, e_3$ be Pauli's spin matrices. $\mathcal{A} =$ the algebra of smooth functions on the quantum Heisenberg manifold. The algebra $\mathcal{A}$ admits an action of the Heisenberg group. $\tau$ will denote a certain state on $\mathcal{A}$ invariant under the action of the Heisenberg group. Let $X_1, X_2, X_3$ denote the canonical basis of the Lie algebra of the Heisenberg group so that we have associated self-adjoint operators $d_{X_i}$ on $L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^3$ in the natural way. Then the triple $(\mathcal{A}, L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2, D)$ defines a spectral triple on $\mathcal{A}$ where $\mathcal{A}$ is represented on $L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2$ diagonally and the Dirac operator $D$ is defined as $D = \sum_j d_{X_j} \otimes_{\mathbb{C}} \gamma_j$, where $\{\gamma_j : j = 1, 2, 3\}$ are self-adjoint $3 \times 3$ matrices satisfying $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$.

## Theorem ( B. + Goswami + Mukhopadhyay )

Suppose $M$ is a compact Riemannian manifold equipped with a free and isometric action of $\mathbb{T}^n$.
Let $\mathcal{A}_\theta$ denote the Connes-Dubois-Violette-Rieffel deformation of $C(M)$ with respect to the action of $\mathbb{T}^n$.
Let $\mathcal{E} := \Omega^1(M)$ denote the space of one forms of the spectral triple $(C^\infty(M), \bigoplus_k L^2(\Omega^k(M), d + d^*)$.
Then we have the Connes-Landi isospectral deformation of the above spectral triple. Let $\mathcal{E}_\theta$ be the space of one forms for the isospectral deformation of $\mathcal{E}$.
Then for any bilinear Riemannian metric on $\mathcal{E}_\theta$ there exists a unique Levi-Civita connection on the bimodule $\mathcal{E}_\theta$.
The Levi-Civita connection $\nabla$ on the bimodule $\mathcal{E}$ deforms to the Levi-Civita connection $\nabla_\theta$ on $\mathcal{E}_\theta$.

## Proposition

Suppose that there exists a unital subalgebra $\mathcal{A}'$ of $\mathcal{Z}(\mathcal{A})$ and an $\mathcal{A}'$-submodule $\mathcal{E}'$ of $\mathcal{Z}(\mathcal{E})$ such that $\mathcal{E}'$ is projective and finitely generated over $\mathcal{A}'$.
If the map

$$u^{\mathcal{E}}_{\mathcal{E}'} : \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \to \mathcal{E},$$

defined by

$$u^{\mathcal{E}}_{\mathcal{E}'}\left(\sum_i e'_i \otimes_{\mathcal{A}'} a_i\right) = \sum_i e'_i a_i$$

is an isomorphism of vector spaces,
then $u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$ is an isomorphism.
Moreover, if $\mathcal{Z}(\mathcal{E})$ is a finitely generated projective module over $\mathcal{Z}(\mathcal{A})$, then $u^{\mathcal{E}}$ is an isomorphism if and only if there exists $\mathcal{E}'$ and $\mathcal{A}'$ such that $u^{\mathcal{E}}_{\mathcal{E}'}$ is an isomorphism.

## Choice of $\mathcal{E}'$ and $\mathcal{A}'$

Suppose $M$ is a compact Riemannian manifold equipped with a free and isometric action of $\mathbb{T}^n$. Let $\mathcal{E} = \Omega^1(M)$ and $\mathcal{A} = C^\infty(M)$. Take $\mathcal{E}' =$ fixed point submodule ( of $\mathcal{E}$ ) and $\mathcal{A}' =$ fixed point subalgebra of $\mathcal{A}$. Then the Proposition holds.

Let $Q : \mathrm{Im}(1 - P_{\mathrm{sym}}) \to \mathrm{Im}(m) \cong \Omega^2(\mathcal{A})$ be the isomorphism from **Assumption II**.

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## $\nabla^2$

The map $H : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$

$$\omega \otimes_{\mathbb{C}} \eta \mapsto (1 - P_{\mathrm{sym}})_{23}(\nabla\omega \otimes_{\mathcal{A}} \eta) + \omega \otimes_{\mathcal{A}} Q^{-1}(d\eta)$$

descends to a map $H : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$.

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Define $R(\nabla) := H \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$.
Then $R(\nabla)$ is a right $\mathcal{A}$-linear map.

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## The curvature operator

$\Theta := (\sigma_{23} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}^*})\zeta^{-1}_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} R(\nabla) \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^*$.

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Define

$$T^R_{\mathcal{E},\mathcal{E}^*} : \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}^* \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}),$$
$$T^L_{\mathcal{E},\mathcal{E}^*} := \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^* \to \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{E}^*,$$

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Then $T^R_{\mathcal{E},\mathcal{E}^*}$ defines a left $\mathcal{A}$, right $\mathcal{Z}(\mathcal{A})$-linear isomorphism, $T^L_{\mathcal{E},\mathcal{E}^*}$ defines a right $\mathcal{A}$-module isomorphism left $\mathcal{Z}(\mathcal{A})$-linear isomorphism.

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where flip : $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{E}^* \to \mathcal{E}^* \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ is the map given by

$$\text{flip } (e' \otimes_{\mathcal{Z}(\mathcal{A})} \phi) = \phi \otimes_{\mathcal{Z}(\mathcal{A})} e'$$

which is well defined and a right $\mathcal{Z}(\mathcal{A})$-linear isomorphism.

## The Ricci curvature

The Ricci curvature Ric is defined as the element in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ given by

$$\mathrm{Ric} := (\mathrm{id}_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} \otimes_{\mathcal{A}} \mathrm{ev} \circ \rho)(\Theta), \quad (2)$$

where $\mathrm{ev} : \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ is the $\mathcal{A} - \mathcal{A}$-bilinear map sending $e^* \otimes_{\mathcal{A}} f$ to $e^*(f)$ for all $e^* \in \mathcal{E}^*$ and $f \in \mathcal{E}$.

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## The scalar curvature

The scalar curvature Scal is defined as:

$$\mathrm{Scal} := \mathrm{ev}(V_g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}})(\mathrm{Ric}) \;\in\; \mathcal{A}.$$

### Theorem ( B. + Goswami + Joardar )

Suppose $\mathcal{E}$ satisfies **Assumption I - III**. If $g_0$ is a pseudo-Riemannian bilinear metric on $\mathcal{E}$ and $k$ is an invertible element of $\mathcal{A}$, then the Levi-Civita connection on $\mathcal{E}$ is given by

$$\nabla(\omega) = \nabla_0(\omega) + k^{-1}P_{\rm sym}(dk \otimes_{\mathcal{A}} \omega) - \frac{1}{2}k^{-1}\Omega_{g_0}g_0(dk \otimes_{\mathcal{A}} \omega),$$

where $\Omega_{g_0} \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is defined by

$$\Omega_{g_0} = ({\rm id}_{\mathcal{E}} \otimes_{\mathcal{A}} V_{g_0}^{-1})\zeta_{\mathcal{E},\mathcal{E}}^{-1}({\rm id}_{\mathcal{E}}).$$

## Proposition ( B. + Goswami + Joardar )

If $\mathcal{E}$ is a free module ( satisfying **Assumption I - III** ) with a basis $\{e_1, e_2, \cdots e_n\}$ such that

1. $e_i \in \mathcal{Z}(\mathcal{E})$
2. $d(e_i) = 0$ for all $i = 1, 2, \cdots n$,
3. There exists a torsionless connection $\nabla_0(e_i) = 0$ for all $i = 1, 2, \cdots n$.
4. Suppose that $g_0$ is a pseudo-Riemannian bilinear metric on $\mathcal{E}$ such that $g_0(e_i \otimes_{\mathcal{A}} e_j) = \delta_{ij} 1_{\mathcal{A}}$.

Consider the conformally deformed metric $g := kg_0$ where $k$ is an invertible element in $\mathcal{A}$.

Then the Christoffel symbols of the Levi-Civita connection are given by:

$$\Gamma^i_{jl} = \frac{1}{2}(\delta_{il} k^{-1} \partial_j(k) + \delta_{ij} k^{-1} \partial_l(k) - \delta_{jl} k^{-1} \partial_i(k)). \quad (3)$$

The module $\mathcal{E}$ is freely generated by the central elements

$$e_1 = 1 \otimes_{\mathbb{C}} \gamma_1, \; e_2 = 1 \otimes_{\mathbb{C}} \gamma_2, d(e_1) = d(e_2) = 0.$$

The space of two forms is a rank one free module generated by $e_1.e_2$.

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The space of two forms is a rank one free module generated by $e_1.e_2$.

**Proposition ( B. + Goswami + Joardar )**

Let $g_0$ be the metric on $\mathcal{E}$ defined by $g_0(\omega \otimes_{\mathcal{A}} \eta) = (\tau \otimes_{\mathbb{C}} \mathrm{Tr}_{M_2(\mathbb{C})})(\omega\eta)$. Let $k$ be an invertible element of $\mathcal{A}$.

$$\mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) =$$

$$-\frac{1}{2}(k^{-1}(\partial_1^2 + \partial_2^2)(k) + \partial_1(k^{-1})\partial_1(k) + \partial_2(k^{-1})\partial_2(k)).$$

$$\mathrm{Ric}(e_1, e_2) = -\mathrm{Ric}(e_2, e_1) = \frac{1}{2}(\partial_1(k^{-1})\partial_2(k) - \partial_2(k^{-1})\partial_1(k)).$$

$$\mathrm{Scal} = -(\partial_1^2 + \partial_2^2)(k) - k(\partial_2(k^{-1})\partial_2(k) - k\partial_1(k^{-1})\partial_1(k)).$$

Suppose $\mathcal{A}$ is an algebra over $\mathbb{C}$. A differential calculus on $\mathcal{A}$ is a pair $(\Omega(\mathcal{A}), d)$ such that:

Suppose $\mathcal{A}$ is an algebra over $\mathbb{C}$. A differential calculus on $\mathcal{A}$ is a pair $(\Omega(\mathcal{A}), d)$ such that:

1. $\Omega(\mathcal{A})$ is an $\mathcal{A} - \mathcal{A}$-bimodule,
2. $\Omega(\mathcal{A}) = \oplus_{i \geq 0} \Omega^i(\mathcal{A})$, where $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^i(\mathcal{A})$ are $\mathcal{A} - \mathcal{A}$-bimodules.
3. We have a bimodule map $m : \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \to \Omega(\mathcal{A})$ such that $m(\Omega^i(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^j(\mathcal{A})) \subseteq \Omega^{i+j}(\mathcal{A})$,
4. We have a map $d : \Omega^i(\mathcal{A}) \to \Omega^{i+1}(\mathcal{A})$ such that $d^2 = 0$ and $d(\omega.\eta) = d\omega.\eta + (-1)^{\deg(\omega)}\omega.d\eta$,
5. $\Omega^i(\mathcal{A})$ is spanned by elements of the form $da_0 da_1 \cdots da_i a_{i+1}$.

We recall that the noncommutative 2-torus $C(\mathbb{T}^2_\theta)$ is the universal $C^*$ algebra generated by two unitaries $U$ and $V$ satisfying $UV = e^{2\pi i\theta} VU$ where $\theta$ is a number in $[0, 1]$. The $*$- subalgebra $\mathcal{A}(\mathbb{T}^2_\theta)$ of $C(\mathbb{T}^2_\theta)$ generated by $U$ and $V$ will be denoted by $\mathcal{A}$.
We have the following concrete description of the spectral geometry of $\mathcal{A}$: there are two derivations $d_1$ and $d_2$ on $\mathcal{A}$ obtained by extending linearly the rule:

$$d_1(U) = U,\ d_1(V) = 0,\ d_2(U) = 0,\ d_2(V) = V.$$

There is a faithful trace on $\mathcal{A}$ defined as follows:
$\tau(\sum_{m,n} a_{mn} U^m V^n) = a_{00},$ where the sum runs over a finite subset of $\mathbb{Z} \times \mathbb{Z}$.
Let $\mathcal{H} = L^2(C(\mathbb{T}^2_\theta), \tau) \oplus L^2(C(\mathbb{T}^2_\theta), \tau)$ where $L^2(C(\mathbb{T}^2_\theta), \tau)$ denotes the GNS Hilbert space of $\mathcal{A}$ with respect to the state $\tau$. We note that $\mathcal{A}$ is embedded as a subalgebra of $\mathcal{B}(\mathcal{H})$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. The Dirac operator on $\mathcal{H}$ is defined by $D = \begin{pmatrix} 0 & d_1 + \sqrt{-1} d_2 \\ d_1 - \sqrt{-1} d_2 & 0 \end{pmatrix}$. Let $\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$\Big( \quad 0 \qquad \sqrt{-1} \quad \Big)$

## Set up

1. A strongly continuous action $\alpha$ by automorphisms on a unital $C^*$ algebra $A$.
2. Let $A^\infty$ denote the dense $*$-subalgebra of smooth vectors in $A$.
3. Fix an $n \times n$ skew symmetric matrix $J$.

## The deformation

The deformation $A_J^\infty$ of $A^\infty$ is given by the osillatory integral

$$a \times_J b = \int \int \alpha_{Ju}(a)\alpha_v(b)e(u.v)dudv.$$

$A_J^\infty$ can be completed to a unital $C^*$ algebra $A_J$ called the Rieffel deformation of $A$.