Noncommutative Quasi-normal Modes & Holography

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Plan

- The first direct detection of gravitational waves from black hole mergers took place in 2015.
- The frequencies associated with the gravitational waves arising from an astrophysical event approximately 1.4 billion years ago were detected on earth.
- This discovery has ushered in a new field of gravitational wave astronomy.
- It is now possible to think of the of detection of gravitational waves from the primordial sources.
- These waves can in principle carry signatures of quantum structure of space-time.
- It is thus important to ask how quantum structure of space-time can affect the gravitational wave spectrum.

Introduction

- How to describe quatum structure of space-time? In particular:
  - Can space-time coordinates be measured with arbitrary precision ?
  - Is there a fundamental and elementary length scale in nature ?
- These issues are related to the quantum structure of space-time relevant at the Planck scale.
- Noncommutative Geometry is one of the candidates for describing physics at that regime.

Space-time UR

Heisenberg's Principle
+ $\Longrightarrow$ Space-time uncertainty relations
Einstein's Theory

- Measuring a space-time coordinate with an accuracy δ causes and uncertainty in the momentum $\sim \frac{1}{\delta}$.
- Neglecting rest mass, an energy of the order $\frac{1}{\delta}$ is transmitted to the system and concentrated for some time in the localization region. The associated energy-momentum tensor generates a gravitational field.
- The smaller the uncertainties in the measurement of coordinates, the stronger will be the gravitational field generated by the measurement.

Space-time UR

- To probe physics at Planck Scale l_p, the Compton wavelength $\frac{1}{M}$ of the probe must be less than l_p, hence $M > \frac{1}{l_p}$, i.e. Planck mass.
- When this field becomes so strong as to prevent light or other signals from leaving the region in question, an operational meaning can no longer be attached to the localization.
- Similarly, observations of very short time scales also require very high energies. Such observations can also form black holes and limit spatial resolutions leading to a relation of the form

$$\Delta t \Delta x \geq L^2, \quad L = \text{fundamental length}$$

Space-time UR

- Based on these arguments, Doplicher, Fredenhagen and Roberts (1994) arrived at uncertainty relations between the coordinates, which they showed could be deduced from a commutation relation of the type

$$[q_\mu, q_\nu] = iQ_{\mu\nu}$$

  where q_μ are self-adjoint coordinate operators, μ, ν run over space-time coordinates and $Q_{\mu\nu}$ is an antisymmetric tensor, with the simplest possibility that it commutes with the coordinate operators.
- We take NC geometry as a model of a quantum space-time.

Model

To study gravitational waves in NC spacetime, we have three possibilities :

- NC geometry probed by a NC field
- NC geometry probed by commutative field
- Commutative geometry probed by a NC field

We have chosen the third possibility for our analysis.

This is in the similar spirit of a given classical geometry being probed by different type of fields to extract physical information.

Model

- We consider the action

$$\mathcal{S} = \int \mathrm{d}^4x \sqrt{-g}\;\; g^{\mu\nu} \left(\partial_\mu \phi \star \partial_\nu \phi\right)$$

- The noncommutativity is chosen to be given by the κ-deformed algebra

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu)$$

where we shall choose $a_0 = \frac{1}{\kappa} \equiv a$ and $a_i = 0$.

- The κ-deformed algebra naturally appears in the NC description of a large class of black holes and in certain NC versions of cosmology. This motivates our choice.

Model

- Operators $\hat{x}_\mu$ can be realized in terms of the operators x_μ and $p_\mu (= i\partial_\mu)$ defined as

$$\hat{x}_\mu = x_\alpha \varphi^\alpha{}_\mu(p)$$

- From now on, we work upto the first order in the deformation parameter a. Demanding consistency of the realization with the algebra gives

$$\varphi^\alpha{}_\mu = \delta^\alpha_\mu [1 + \alpha(a \cdot p)] + \beta a^\alpha p_\mu + \gamma p^\alpha a_\mu, \qquad \alpha, \beta, \gamma \in \mathbb{R}, \qquad \gamma - \alpha = 1$$

- Upto the first order, the action is given by

$$\mathcal{S} = \mathcal{S}_0 + \int \mathrm{d}^4 x \left(\mathcal{A}_{\alpha\beta\gamma\delta} \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 \phi}{\partial x_\gamma \partial x_\delta} \right),$$

where $\mathcal{S}_0$ is the commutative action and

$$\mathcal{A}_{\alpha\beta\gamma\delta} = i\sqrt{-g}\, g_{\beta\delta} \left(\alpha x_\alpha a_\gamma + \beta (a \cdot x) \eta_{\alpha\gamma} + \gamma a_\alpha x_\gamma \right)$$

Model

- We shall next find the equation of motion to the first order in a. This is in general very complicated and to simplify it, we use the long wavelength approximation and keep terms only upto the lowest order in derivatives.
- We choose the classical geometry to be given by a massive spinless BTZ black hole. This is a simple background for which the quasi-normal modes can be studied analytically in the commutative case. This motivates the choice of the background.

EOM- Scalar

The massive spinless BTZ black hole is described by the metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{r^2}{l^2} - M & 0 & 0 \\ 0 & -\frac{1}{\frac{r^2}{l^2}-M} & 0 \\ 0 & 0 & -r^2 \end{pmatrix},$$

We start with a massive spinless BTZ black hole and a κ-type NC scalar field. The equations of motion are derived from

$$\hat{\mathcal{S}} = \int \mathrm{d}^4x\sqrt{-g}\;\; g^{\mu\nu}\left(\partial_\mu\phi \star \partial_\nu\phi\right).$$

where the NC star product has been defined before.

EOM - Scalar

Using the decomposition

$$\phi(r,\theta,t) = R(r)e^{-i\omega t}e^{im\theta}$$

the radial equation of motion upto first order in the NC parameter is

$$r\left(8GM - \frac{r^2}{l^2}\right)\frac{\partial^2 R}{\partial r^2} + \left(8GM - \frac{3r^2}{l^2}\right)\frac{\partial R}{\partial r}$$

$$+\left(\frac{m^2}{r} - \omega^2 \frac{r}{\frac{r^2}{l^2} - 8GM} - a\beta\omega\frac{8r}{l^2}\frac{\frac{3r^2}{2l^2} - 8GM}{\frac{r^2}{l^2} - 8GM}\right)R = 0$$

EOM - Scalar

Using

$$z = 1 - \frac{Ml^2}{r^2},$$

we get

$$z(1-z)\frac{d^2R}{dz^2} + (1-z)\frac{dR}{dz} + \left(\frac{A}{z} + B + \frac{C}{1-z}\right)R = 0,$$

$$A = \frac{\omega^2 l^2}{4M} + a\beta\omega, \quad B = -\frac{m^2}{4M}, \quad C = 3a\beta\omega.$$

These equations have very special features and we shall discuss those shortly.

QNM - Scalar

Using the ansatz

$$R(z) = z^{\lambda_1}(1-z)^{\lambda_2} F(a,b,c,z)$$

we get

$$z(1-z)\frac{d^2F}{dz^2} + [c-(1+a+b)z]\,\frac{dF}{dz} - abF = 0.$$

where

$$a = \lambda_1 + \lambda_2 + i\sqrt{-B}, \quad b = \lambda_1 + \lambda_2 - i\sqrt{-B} \quad c = 2\lambda_1 + 1$$

$$\lambda_1 = -i\sqrt{A} \quad \lambda_2 = \frac{1}{2}(1 - \sqrt{1-4C})$$

QNM - Scalar

The quasinormal modes are defined as solutions which are purely ingoing at the horizon, and which vanish at infinity. We have two linearly independent solutions $F(a,b,c,z)$ and $z^{1-c}F(a-c+1,b-c+1,2-c,z)$ near the horizon $z=0$. Thus, the solution which has ingoing flux at the horizon is given by

$$R(z) = z^{\lambda_1}(1-z)^{\lambda_2}F(a,b,c,z)$$

This is valid only in some neighborhood of the horizon, for the infinity, $z=1$, we use analytic continuation

$$R(z) = z^{\lambda_1}(1-z)^{\lambda_2+c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1,1-z)$$

$$+z^{\lambda_1}(1-z)^{\lambda_2}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z),$$

Using the QNM boundary conditions we get

$$c-a=-n, \quad \text{or} \quad c-b=-n,$$

and $n=0,1,2...$ These conditions determine the frequencies of the quasinormal modes.

QNM - Scalar

- The NC QNM frequencies are given by

$$\omega_{L,R} = \pm \frac{m}{l} + a\beta \frac{2M}{l^2}(6n+5) - 2i \left[\frac{\sqrt{M}}{l}(n+1) \mp 3a\beta \frac{m}{l^2}\sqrt{M} \right]$$

$n = 0, 1, 2...$

- The gravity waves in principle provide an opportunity to observe quantum gravity effects as described by NC physics.

NC Duality

- We started with a massive spinless BTZ black hole probed by a massless NC scalar field.
- The equation of motion that we got corresponds to that of a massive spinning BTZ black hole probed by a massive commutative scalar field.
- Within our scheme of approximation, this is a new kind of duality.
- Thus we have
  $$M^f = M^f(a, M), \quad J^f = J^f(a, M), \quad \mu^f = \mu^f(a, M)$$
  where M^f and J^f are the mass and spin of the dual black hole and μ^f is the mass of the scalar field.

NC Duality

- Now we determine the parameters of the dual black hole?
- For that, we first calculate the entropy of the original black hole with mass M using the brick wall method and get

$$S^{NC} = \frac{2\pi l\sqrt{M}}{4G}\left(1 + a\beta\sqrt{M}\frac{8\pi\zeta(2)}{3l\zeta(3)}\right)$$

- For the dual spining black hole

$$S^d = \frac{2\pi r_+}{4G}, \quad r_+ = \frac{l\sqrt{M}}{\sqrt{2}}\sqrt{1+\sqrt{1-\frac{(J^d)^2}{M^2 l^2}}}$$

- We now demand $S^{NC} = S^d$

NC Duality

- This leads to

$$(J^d(a))^2 = \lambda \frac{64}{3} \pi \frac{\zeta(2)}{\zeta(3)} l M^{5/2} + O(a^2) \quad \lambda = -a\beta$$

  This implies that $\beta < 0$.

- Thus the dual black hole has mass M and spin J^d.
- The scalar field probe also picks up an effective mass proportional to $-a\beta$. This is consistent with the restriction that $\beta < 0$.
- We could have taken taken M to depend on the parameter a. In that case we can show that M picks up a correction to higher order than that for J^d.

QNM - Fermionic

- We would like to probe the black hole with a fermionic field
- This is a very involved procedure. To simplify the method, we use the dual black hole picture discussed above.
- Instead of probing a massless spinning BTZ with e NC fermionic probe, we probe the dual black hole with a commutative fermionic probe.
- We ensure that $D^2 = \Box_g$, where $\Box_g$ corresponds to the KG operator for the dual black hole with mass M and $J \propto \sqrt{a\beta}$.

QNM - Fermionic

We start with the metric

$$ds^2 = -\left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}} + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2.$$

In terms of coordinates

$$x^+ = \frac{1}{l}r_+ t - r_-\phi, \qquad x^- = r_+\phi - \frac{1}{l}r_- t \qquad \text{and} \qquad \tanh\rho = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}$$

$$ds^2 = -\sinh^2\rho\,(dx^+)^2 + l^2 d\rho^2 + \cosh^2\rho\,(dx^-)^2.$$

We choose

$$\gamma^0 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

QNM - Fermionic

The Dirac equation is given by

$$\left[\gamma^a e_a{}^\mu \left(\partial_\mu - \frac{i}{2}\omega_\mu{}^{cd}\Sigma_{cd}\right) + m\right]\Psi = 0$$

where $e^0{}_{x_+} = \sinh\rho,\ e^1{}_\rho = l,\ e^2{}_{x_-} = \cosh\rho$ and others being 0 and

$$\omega_0{}^{01} \equiv \omega_{x^+}{}^{01} = -\omega_0{}^{10} = \frac{1}{l}\cosh\rho,$$
$$\omega_2{}^{12} \equiv \omega_{x^-}{}^{12} = -\omega_2{}^{21} = -\frac{1}{l}\sinh\rho,$$

This leads to the Dirac equation

$$\left[\frac{1}{l}\gamma^1\left(\frac{\partial}{\partial\rho} + \frac{\cosh\rho}{2\sinh\rho} + \frac{\sinh\rho}{2\cosh\rho}\right) + \gamma^0\frac{1}{\sinh\rho}\frac{\partial}{\partial x^+} + \gamma^2\frac{1}{\cosh\rho}\frac{\partial}{\partial x^-} + m\right]\Psi = 0$$

QNM - Fermionic

We choose the ansatz

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1(r) \\ \psi_2(r) \end{pmatrix} \text{exp}\left[-\frac{i}{l}(\omega t - j\phi)\right] = \begin{pmatrix} \psi_1(\rho) \\ \psi_2(\rho) \end{pmatrix} \text{exp}\left[-i(k_+ x^+ + k_- x^-)\right]$$

$$= \frac{1}{2}\begin{pmatrix} P(\rho) + Q(\rho) \\ P(\rho) - Q(\rho) \end{pmatrix} \text{exp}\left[-i(k_+ x^+ + k_- x^-)\right],$$

where ω and j are respectively the energy and angular momentum of the spin $1/2$ particle and they are related to k_+ and k_- as

$$k_+ = \frac{l\omega r_+ - j r_-}{l(r_+^2 - r_-^2)}, \quad k_- = \frac{l\omega r_- - j r_+}{l(r_+^2 - r_-^2)}.$$

Now defining

$$P(\rho) = \sqrt{\frac{\cosh\rho + \sinh\rho}{\cosh\rho\ \sinh\rho}}\, P'(\rho), \quad Q(\rho) = \sqrt{\frac{\cosh\rho - \sinh\rho}{\cosh\rho\ \sinh\rho}}\, Q'(\rho), \quad z = \tanh^2\rho$$

and further putting $P' = \psi_1' + \psi_2'$ and $Q' = \psi_1' - \psi_2'$

QNM - Fermionic

We get

$$2\sqrt{z}(1-z)\frac{d}{dz}\psi_1' + il\left(k_+\frac{1}{\sqrt{z}} + k_-\sqrt{z}\right)\psi_1' + \left[il(k_+ + k_-) + lm + \frac{1}{2}\right]\psi_2' = 0,$$
$$2\sqrt{z}(1-z)\frac{d}{dz}\psi_2' - il\left(k_+\frac{1}{\sqrt{z}} + k_-\sqrt{z}\right)\psi_2' - \left[il(k_+ + k_-) - lm - \frac{1}{2}\right]\psi_1' = 0.$$

which can be combined to give

$$z(1-z)\frac{d^2}{dz^2}\psi_1' + \frac{1-3z}{2}\frac{d}{dz}\psi_1' + \frac{1}{4}\left[\frac{l^2k_+^2 - ilk_+}{z} + ilk_- - l^2k_-^2 - \frac{(lm+\frac{1}{2})^2}{1-z}\right]\psi_1' = 0.$$

The solutions with purely ingoing flux at the horizon are given by

$$\psi_1' = z^\alpha(1-z)^\beta F(a,b,c;z),$$
$$\psi_2' = \left(\frac{a-c}{c}\right) z^{\alpha+\frac{1}{2}}(1-z)^\beta F(a,b+1,c+1;z),$$

QNM - Fermionic

where

$$\alpha = -\frac{ilk_+}{2}, \quad \beta = -\frac{1}{2}\left(lm + \frac{1}{2}\right), \quad c = 2\alpha + \frac{1}{2},$$

$$a = \alpha + \beta + \frac{ilk_-}{2} + \frac{1}{2} = \frac{l(k_+ - k_-)}{2i} + \beta + \frac{1}{2},$$

$$b = \alpha + \beta - \frac{ilk_-}{2} = \frac{l(k_+ + k_-)}{2i} + \beta.$$

QNM boundary conditions also require vanishing flux at infinity. The flux in the radial direction is given by

$$\mathcal{J}_\rho = \frac{l^2}{\sqrt{1-z}}\left[\psi_1'^{\,*}\psi_2' + \psi_2'^{\,*}\psi_1' + \sqrt{z}\left(\psi_1'^{\,*}\psi_1' + \psi_2'^{\,*}\psi_2'\right)\right]$$

Demanding that the outgoing flux vanishes at infinity requires that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = 0.$$

This requires either $c - a = -n$ or $c - b = -n,$ with $n = 0, 1, 2, \ldots$

QNM - Fermionic

These equations lead to the conditions

$$\omega_L = \frac{j}{l} - 2i\frac{r_+ - r_-}{l}\left(n + \frac{lm}{2} + \frac{1}{4}\right),$$

$$\omega_R = -\frac{j}{l} - 2i\frac{r_+ + r_-}{l}\left(n + \frac{lm}{2} + \frac{3}{4}\right).$$

The QNM frequencies upto the first order in the deformation parameter are given by

$$\omega_L = \frac{j}{l} - 2i\sqrt{M}\left(1 - \frac{J(a)}{2lM} - \frac{1}{8}\frac{J^2(a)}{l^2M^2}\right)\left(n + \frac{1}{4} + \frac{lm}{2}\right) + O(a^{3/2})$$

$$\omega_R = -\frac{j}{l} - 2i\sqrt{M}\left(1 + \frac{J(a)}{2lM} - \frac{1}{8}\frac{J^2(a)}{l^2M^2}\right)\left(n + \frac{3}{4} + \frac{lm}{2}\right) + O(a^{3/2})$$

Holography

- Holography and AdS/CFT duality are fundamental aspects in certain quantum theories of gravity, such as string theory.
- NC effects are also relevant at the Planck scale.
- Thus it is natural to investigate holography within the NC framework.
- One approach is through QNM’s, which play an important role in holography, especially for the BTZ.
- Second approach is via Sullivan’s theorem, which is relevant for the BTZ.

Holography

- For the BTZ black hole, the poles of the retarded Green's function in the boundary CFT are in exact correspondence with the QNM frequencies in the bulk
- This provides a demonstration for the AdS/CFT conjecture for the BTZ
- The NC duality allows us to discuss our problem in the framework of a commutative BTZ, for which the AdS/CFT conjecture holds
- We can thus predict that the poles of the retarded Green's function would pick up NC corrections as

$$2i\sqrt{M}\left(\frac{J(a)}{2lM}+\frac{J^2(a)}{8l^2M^2}\right) \quad -2i\sqrt{M}\left(\frac{J(a)}{2lM}-\frac{J^2(a)}{8l^2M^2}\right)$$

Holography

- Sullivan's theorem says that for a certain class of manifolds, there is a 1-1 correspondence of the hyperbolic structure as encoded in the metric and the conformal structure of the boundary.
- It has been shown to be valid for the BTZ.
- Sullivan's theorem implies that the monodromis of the solutions of the wave equation around the two horizons of the BTZ satisfy

$$\mathcal{M}(r_+)\mathcal{M}^+(r_-) = 1 \qquad \mathcal{M}(r_+)\mathcal{M}^-(r_-) = 1$$

Holography

- These conditions lead to

$$\omega_L = \frac{j}{l} - 2i\frac{r_+ - r_-}{l}\left(n + \frac{1}{4}\right),$$

$$\omega_R = -\frac{j}{l} - 2i\frac{r_+ + r_-}{l}\left(n + \frac{3}{4}\right).$$

  which are the same equations that we got by direct calculation restricted to the massless probe.

- These modes are obtained without using any condition at infinity and hence are also known as non-QNM's.
- They are obtained purely from holographic considerations as applied to the BTZ.

Entropy

- We have analyzed black hole entropy within our framework in two different ways.
- The first one involves the brick wall method introduced by 't Hooft.
- The second one involves the study of the area operator and the quantization of entropy.
- Our results lead to a NC correction to the usual BTZ entropy.
- Our analysis predicts a renormalization of the Newton's constant due to NC corrections.

Entropy - Brick Wall Method

In the WKB approximation, the r-dependent radial wavefunction has the form $R(r) = \mathrm{e}^{i\int k(r)\mathrm{d}r}$ where

$$k^2(r,m,\omega) = -\frac{m^2}{r^2\left(\frac{r^2}{l^2} - 8GM\right)} + \omega^2 \frac{1}{\left(\frac{r^2}{l^2} - 8GM\right)^2} + a\beta\omega \frac{8}{l^2} \frac{\frac{3r^2}{2l^2} - 8GM}{\left(\frac{r^2}{l^2} - 8GM\right)^2}$$

According to the semi-classical quantization rule, the radial wave number is quantized as

$$\pi n = \int_{r_+ + h}^{L} k(r,m,\omega)\mathrm{d}r$$

where the quantum number $n > 0$, m should be fixed such that $k(r,m,\omega)$ is real and h is the brick wall cutoff (UV regulator) and L is the infrared regulator. The total number ν of solutions with energy not exceeding ω is given by

$$\nu = \sum_{-m_0}^{m_0} n = \int_{-m_0}^{m_0} \mathrm{d}m\, n = \frac{1}{\pi}\int_{-m_0}^{m_0} \mathrm{d}m \int_{r_+ + h}^{L} k(r,m,\omega)\mathrm{d}r$$

Entropy - Brick Wall Method

The free energy at inverse temperature β_T of the black hole is

$$\mathrm{e}^{-\beta_T F} = \sum_\nu \mathrm{e}^{-\beta_T E} = \prod_\nu \frac{1}{1-\mathrm{e}^{-\beta_T E}} \qquad /\ \ln$$

$$\beta_T F = \sum_\nu \ln\left(1-\mathrm{e}^{-\beta_T E}\right) = \int \mathrm{d}\nu \ln\left(1-\mathrm{e}^{-\beta_T E}\right) \qquad /\ \text{part. integ.}$$

$$= -\int_0^\infty \mathrm{d}E \frac{\beta_T \nu(E)}{\mathrm{e}^{\beta_T E}-1}$$

where $\beta_T = \frac{2\pi l^2}{r_+}$. For this, we find the free energy F as

$$F = -\frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T \omega}-1}\int_{r_++h}^{L} \mathrm{d}r \int_{-m_0}^{m_0} \mathrm{d}m \;\; k(r,m,\omega)$$

Keeping the most divergent terms in h, we get

$$F = -\frac{l^{\frac{5}{2}}}{(8GM)^{\frac{1}{4}}}\frac{\zeta(3)}{\beta_T^3}\frac{1}{\sqrt{2h}} - 2a\beta\frac{(8GM)^{\frac{3}{4}}\sqrt{l}}{\sqrt{2h}}\frac{\zeta(2)}{\beta_T^2}$$

Entropy - Brick Wall Method

Using $S = \beta_T^2 \frac{\partial F}{\partial \beta_T}$, we get

$$S = 3\frac{l^{\frac{5}{2}}}{(8GM)^{\frac{1}{4}}}\frac{\zeta(3)}{\beta_T^2}\frac{1}{\sqrt{2h}} + 4a\beta\frac{(8GM)^{\frac{3}{4}}\sqrt{l}}{\sqrt{2h}}\frac{\zeta(2)}{\beta_T}$$
$$= S_0\left(1 + \frac{4}{3}a\beta\frac{8GM}{l^2}\frac{\zeta(2)}{\zeta(3)}\beta_T\right)$$

where S_0 is the undeformed entropy for BTZ and

$$h = \frac{9G^2\zeta^2(3)\sqrt{8GM}}{8l\pi^6} \tag{1}$$

The cutoff h is fixed by demanding that $S_0 = \frac{A}{4G} = \frac{2\pi r_+}{4G}$.

Entropy - Renormalization of G

- Now writing $S = \frac{A}{4G^*}$, we find that

$$\frac{1}{G^*} = \frac{1}{G}\left(1 + \frac{8}{3}\frac{a\beta\pi}{l}\frac{\zeta(2)}{\zeta(3)}\sqrt{8GM}\right)$$

- This gives a renormalization of the Newton's constant due to NC effects

Entropy - Quantization

- Given a system with energy E and vibrational frequency $\Delta\omega(E)$, it can be shown that

$$\mathcal{I} = \int \frac{\delta E}{\Delta\omega(E)}$$

  is an adibatic invariant.

- By Bohr-Sommerfeld semi-classical quantization,

$$\mathcal{I} \approx n\hbar$$

- We identify the energy E with the black hole mass M.
- The frequency is identified with the absolute value of QNM frequency.

Entropy - Quantization

Using the formulae

$$\Delta M = \hbar \Delta\omega = \hbar(\left|\omega_{L,R}\right|_n - \left|\omega_{L,R}\right|_{n-1})$$

$$\Delta\omega = \left|\omega_{L,R}\right|_n - \left|\omega_{L,R}\right|_{n-1} = \frac{2\sqrt{M}}{l}\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)}\right)$$

$$\Delta\omega = \frac{2\sqrt{M}}{l}\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)}\right).$$

$$A = 2\pi r_+ = 2\pi l\sqrt{M},$$

$$\Delta A = 2\pi\hbar\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)}\right).$$

we can show that

$$\mathcal{I} = \int \frac{\delta M}{\Delta\omega} \approx l\sqrt{M}\left(1 \mp \frac{a\beta}{2l}\frac{m}{n(n+1)}\right) = N\epsilon, \quad N \in \mathbb{N}$$

Entropy - Quantization

Using these equations we find

$$A = 2\pi\mathcal{I}\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)}\right) \Rightarrow A_N = 2\pi N\epsilon\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)}\right).$$

Using the Bekenstein-Hawking relation with a renormalized G, we have

$$S_N = \frac{A_N}{4G^*} = \frac{A_N}{4G}\left(1 + a\beta\frac{8}{3}\frac{\pi}{l}\frac{\zeta(2)}{\zeta(3)}\sqrt{8GM}\right)$$

we are led to a quantized entropy

$$S_N = N\epsilon\frac{\pi}{2G}\left(1 \pm \frac{a\beta}{2l}\frac{m}{n(n+1)} + a\beta\frac{8}{3}\frac{\pi}{l}\frac{\zeta(2)}{\zeta(3)}\sqrt{8GM}\right)$$

Concluding Remarks

- We have investigated a toy model which illustrates the possibility of capturing Planck scale effects through gravitational waves.
- The QNM's explicitly depend on the NC parameter, which can be used put contraints on such parameters.
- Upto the first order in the deformation parameter, a new kind of black hole duality in AdS_3 has been found.
- We have a prediction for the retarded pole of Green's function in a NC boundary field theory.
- Analysis of various other physical effects and more realistic backgrounds are open areas.

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