Deformation Quantization of Symplectic Manifolds via Symmetries

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$\rightarrow$ Drinfel'd twist deformation quantization

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   Francesco D'Andrea, TW,
   Twist star products and Morita equivalence,
   C.R. Acad. Sci. Paris, Ser. I 355 (2017), 1178-1184.

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where $B_k : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ are bidifferential operators,

$$B_0(f, g) = fg \ , \qquad B_1(f, g) - B_1(g, f) = i\{f, g\},$$

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The algebra $(\mathscr{C}^\infty(M)[[\hbar]], \star)$ is called a *deformation quantization* of $(M, \{\cdot,\cdot\})$.

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iii.) $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\hbar)$.

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Example

Consider $M = \mathbb{R}^2$ with coordinates (x, y) and the standard Poisson bracket. The Moyal-Weyl star product

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The corresponding symmetry is $(T\mathbb{R}^2, [\cdot,\cdot] = 0)$ acting by the Lie derivative $\mathscr{L}$.

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Goal of the talk: Find obstructions for twist star products in the symplectic case!

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Theorem (Bieliavsky-Esposito-Waldmann-TW, 2016)

There are no twist star products

i.) *on the symplectic Riemann surfaces of genus* > 1.

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1. We can further assume that $r \in \Lambda^2\mathfrak{g}$ is non-degenerate.
2. All transitive Lie group actions on $\mathbb{S}^2$ (up to equivalence) are by semisimple Lie groups (see Onishchik 1967).
3. There are no non-degenerate r-matrices on semisimple Lie algebras.

■

3. Obstructions via Morita equivalence (I)

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Two star products $\star, \star'$ on a symplectic manifold $(M, \{\cdot,\cdot\})$ are Morita equivalent if there is L such that (1) is an isomorphism of $\mathbb{C}[[\hbar]]$-algebras.

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Remark

This coincides with the ring-theoretic definition of Morita equivalence on star product algebras.

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There are no twist star products on symplectic $\mathbb{CP}^{n-1}$ *based on* $\mathscr{U}(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$ *or any sub-bialgebra.*

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Proof of Corollary.

The tautological line bundle on $\mathbb{CP}^{n-1}$ has non-trivial Chern class and is $\mathrm{GL}_n(\mathbb{C})$-equivariant. ■

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4. $\Rightarrow$ there is an algebra isomorphism $\mathscr{C}^\infty(M)_\mathcal{F} \cong (\mathscr{C}^\infty(M)[[\hbar]], \star')$, i.e. $\Rightarrow\ c_1(L) = 0$.

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Thank you for your attention!