Coarse-graining of measurement and quantum-to-classical transition in the bipartite scenario

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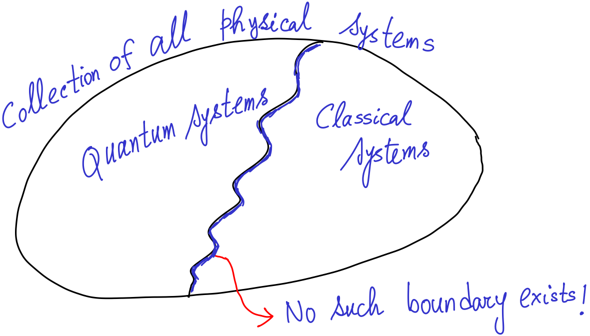
[Based on the joint work with Madhav Krishnan V. and Tanmoy Biswas (arXiv:1703.00502 (quant-ph))]

Motivation

Quantum-to-classical transition

- We do believe that every physical world can be effectively described using laws of Quantum Mechanics.
- But, in practice, we don't need that always – laws of classical physics are sufficient, in plenty of such cases.
- A quantum mechanical system **may** start behaving classically (i) with increase in no. of sub-systems, (ii) with increase in system dimension, (iii) change in interaction under external fields and/or among the constituent sub-systems, (iv) change in observation of the system, etc.
- A pertinent question therefore is: **When does a quantum system start behaving classically?**



But System-specific boundary may exist

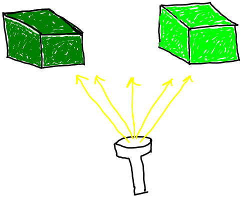
Quantum-to-classical transition via coarse-graining

- **Microscopic** systems are *generally* described by Quantum Physics while **macroscopic** systems are *generally* described by Classical Physics.

- **Coarse-graining** of measurements may lead to distinguish states of macroscopic systems while we may need **fine-grained** measurements for the corresponding microscopic scenario.

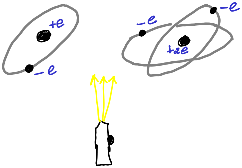
- Here we will study properties of specific states of continuous variable bipartite systems under the action of different coarse-graining measurements to identify quantum-to-classical transition.

Distingushing macroscopic objects



Normal light is enough to distinguish the colours

Distinguishing microscopic objects



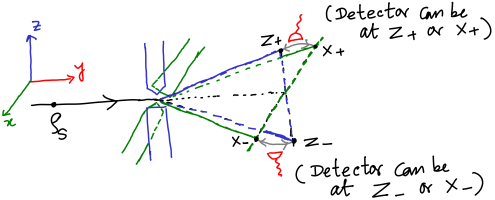
Laser light is required to distinguish the energy transitions

Coarse-graining vs. fine-graining of measurement

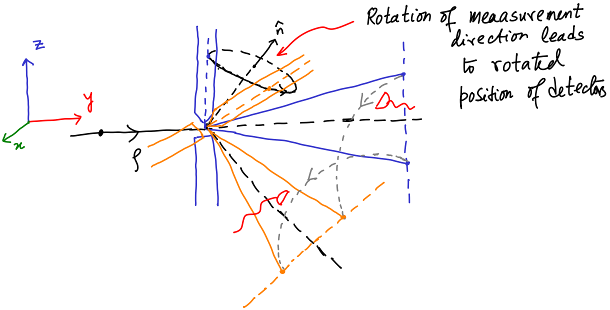
Outline

- Two types of coarse-graining measurements: resolution and reference
- A brief history about related works
- Non-classicality of states in Quantum Optics: our approach
- **Case 1:** On quantum-to-classical transition for $|\psi_\alpha\rangle = (1/\sqrt{2}) \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle)$
- **Case 2:** On quantum-to-classical transition for $|\psi_N\rangle = (1/\sqrt{2}) \times (|0\rangle \otimes |N\rangle + |N\rangle \otimes |0\rangle)$
- **Case 3:** On quantum-to-classical transition for $|\psi_r\rangle = (1/\sqrt{2}) \times (|\Psi^r_+\rangle \otimes |\Psi^r_-\rangle + |\Psi^r_-\rangle \otimes |\Psi^r_+\rangle)$
- Discussion (with speculations on non-commutative extensions)

Two types of coarse-graining measurements: resolution and reference



Resolution Coarse-graining



Reference Coarse-graining

Resolution and reference coarse-graining

- Coarse-grained measurements are of **two** types, according to Jeong et al. [*Phys. Rev. Lett.* **112**, 010402 (2014)]:-

  • **Resolution coarse-graining:** Measurement of POVM $\mathcal{M}_k = \{E_{k\alpha} : E_{k\alpha} \geq 0, \ \sum_{\alpha=1}^{N} E_{k\alpha} = I\}$ is performed with apriori probability p_k.

  • **Reference coarse-graining:** Measurement set-up (*i.e.*, direction) is chosen from a given set with apriori probability.

Resolution coarse-graining

- The α-th outcome of the **resolution** coarse-grained POVM occurs $\Leftrightarrow$ α-th outcome of the POVM $\mathcal{M}_1$ occurs (with probability p_1) or, α-th outcome of the POVM $\mathcal{M}_2$ occurs (with probability p_2), ...
- The resolution coarse-grained POVM is: $\{F_\alpha \equiv \sum_k p_k E_{k\alpha} : \alpha = 1, 2, \ldots, N\}$

Example of resolution coarse-graining (Jeong et al.)

- **Example:**

• $\{|o_n\rangle : n = 0, \pm 1, \pm 2, \ldots\}$ be an ONB for an infinite dimensional quantum system.

• For each $k \in \{0, \pm 1, \pm 2, \ldots\}$, consider the two-outcome observable: $O^k = O_+^k - O_-^k$ where $O_+^k = \sum_{n=k}^{+\infty} |o_n\rangle\langle o_n|$ and $O_-^k = \sum_{n=-\infty}^{k-1} |o_n\rangle\langle o_n|$.

• Choose now measurement of O^k with the apriori probability $P_\delta(k)$, where $P_\delta(k) = N_\delta \exp[-k^2/(2\delta)]$ – the discrete Gaussian distribution with variance δ^2.

• The resolution coarse-grained observable: $O_\delta = \sum_{k=-\infty}^{+\infty} P_\delta(k) O^k$.

Reference coarse-graining

- Original POVM: $\{E_\alpha : \alpha = 1, 2, \ldots N\}$.
- Choose an element U from a given set $\mathcal{S} = \{U_1, U_2, \ldots\}$ of unitaries with probability $P(U)$.
- The α-th outcome of the **reference** coarse-grained POVM occurs $\Leftrightarrow$ the α-th outcome of the POVM $\{U_1^\dagger E_\alpha U_1 : \alpha = 1, 2, \ldots N\}$ occurs (with probability $P(U_1)$) or, the α-th outcome of the POVM $\{U_2^\dagger E_\alpha U_2 : \alpha = 1, 2, \ldots N\}$ occurs (with probability $P(U_2)$), ...
- The reference coarse-grained POVM is: $\{F_\alpha \equiv \sum_k P(U_k) U_k^\dagger E_\alpha U_k : \alpha = 1, 2, \ldots, N\}$

Example of reference coarse-graining (Jeong et al.)

- **Example:**

• Consider the two-outcome observable: $O^0 = O^0_+ - O^0_-$.

• $U(\theta)$ be the unitary acting on the pair $\{|o_n\rangle, |o_{-n}\rangle\}$ (for $n = 1, 2, \ldots$) as:

$$U(\theta)|o_n\rangle = \cos\theta|o_n\rangle + \sin\theta|o_{-n}\rangle,$$
$$U(\theta)|o_{-n}\rangle = \sin\theta|o_n\rangle - \cos\theta|o_n\rangle.$$

• θ is chosen from $[0, 2\pi]$ with probability $P_\Delta(\theta - \theta_0) \equiv N_\Delta \exp[-(\theta - \theta_0)^2/(2\Delta)]$, the Gaussian probability distribution with mean θ_0 and variance Δ^2.

Example of reference coarse-graining (continued)

- **Example (continued):**

  • Choose now measurement of the two-outcome observable $U(\theta)^{\dagger} O^{0} U(\theta)$ with the apriori probability $P_{\Delta}(\theta - \theta_0)$.

  • The reference coarse-grained observable: $O_{\Delta}(\theta_0) = \int_0^{2\pi} P_{\Delta}(\theta - \theta_0) U(\theta)^{\dagger} O^{0} U(\theta) d\theta$.

A brief history about related works

Different ways of testing quantumness

- **Quantumness** (more generally, **non-classicality**) of a system may be revealed in different facets:

  • Contextuality

  • Existence of indistinguishable states

  • Violation of some temporal Bell-type inequality

  • Steerability (for bi-partite case)

  • Violation of some local-realistic inequality (for bi-partite/multi-partite case)

  • Singularity and/or negativity of phase-space distribution function

  • . . . . . .

Testing quantumness via violation of local-realistic inequality

- Violation of any local-realistic inequality by a bi-partite/multi-partite state *necessarily* implies 'non-classicality' of the state.
- Even in the 'macroscopic' limit $J \to +\infty$, Mermin [*Phys. Rev. D* **22**, 356 (1980)] showed violation of a local-realistic inequality by some states of a two spin-J system.
- Unfortunately, this violation demands measurement precision of the order of J.
- So, non-classicality of **no** state of the system will be revealed through violation of the inequality if measurements are sufficiently coarse-grained.

Jeong et al.'s work

- Jeong et al. [*Phys. Rev. Lett.* **112**, 010402 (2014)] used the coarse-grained observables O_δ and $O_\Delta(\theta_0)$ (separately) and showed the violation amounts – in the case of the Bell-CHSH inequality – decrease to zero with the increase in the coares-graining parameters δ and Δ.

- But this – **in no way** – indicates approach to 'classicality', as non-violation of the BCHSH inequality does not **guarantee** existence of a local-realistic description of the measurement statistics of the initial state due to coarse-graining of the measurements.

- Is there a way out?

Non-classicality of states in Quantum Optics: our approach

Phase-space distribution functions of quantum optical states

- Corresponding to any classical Hamiltonian system, one can always find its phase-space probability distribution function.

- One can similarly associate different phase-space distribution functions to any state of a quantized electromagnetic field (an infinite dim. quantum system) – depending upon different orderings of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$.

Phase-space distribution functions (continued)

- For normal ordering, any state ρ of the system has the Glauber-Sudarshan P-distribution:

$$P_\rho(\gamma) \equiv \frac{1}{\pi^2} \int_{\lambda \in \mathbb{C}} \mathrm{Tr}\left[\rho e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}\right] e^{\lambda^* \gamma - \lambda \gamma^*} d^2\lambda$$

- For symmetric ordering, we have the Wigner distribution function for ρ:

$$W_\rho(\gamma) \equiv \frac{1}{\pi^2} \int_{\lambda \in \mathbb{C}} \mathrm{Tr}\left[\rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}\right] e^{\lambda^* \gamma - \lambda \gamma^*} d^2\lambda$$

Phase-space distribution functions (continued)

- **Diagonal representation of ρ:**

$$\rho = \int_{\gamma \in \mathbb{C}} P_\rho(\gamma) |\gamma\rangle\langle\gamma| d^2\gamma,$$

$|\gamma\rangle$ being a coherent (classical) state.

- ρ is **classical** iff $P_\rho(\gamma) \geq 0$ for **all** phase-space points $(x = \sqrt{2}\ \mathrm{Re}(\gamma), p = \sqrt{2}\ \mathrm{Im}(\gamma))$.
- ρ is **non-classical** if $W_\rho(\gamma)$ is negative at least at one phase-space point.
- Generalization to multi-mode states is straight-forward.

Our approach (with P-distribution)

- We applied measurement of the resolution coarse-grained observable O_δ on certain types of single-mode as well as two-mode states and looked for positivity of the P-distributions of the post-measurement states (non-selective case) – to certify classicality of the post-measurement states.

- Non-positivity and/or singularity of the P-distributions of the post-measurement states indicate impossibility of quantum-to-classical transition in the concerned case.

Our approach (with Wigner distribution)

- We also applied measurement of the reference coarse-grained observable O_Δ on certain types of two-mode states and looked for negativity of the Wigner distributions of the post-measurement states (non-selective case) – to certify non-classicality of the post-measurement states.

- Singularities and signatures of phase-space distribution functions in the selective cases are more severe.

- Initial states are chosen with different degrees of presence of 'non-classicality' at the single-mode level.

Case 1: On quantum-to-classical transition for $|\psi_\alpha\rangle = (1/\sqrt{2}) \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle)$

State formulation

- Choose the ONB $\{|o_n\rangle : n = 0, \pm 1, \pm 2, \ldots\}$ as: $|o_n\rangle = |2n\rangle$ for $n = 0, 1, 2, \ldots$ and $|o_{-n}\rangle = |2n-1\rangle$ for $n = 1, 2, \ldots$.

- **Coherent state $|\alpha\rangle$:** For any $\alpha \in \mathbb{C}$, $|\alpha\rangle \equiv e^{-|\alpha|^2/2} \times [\sum_{n=0}^{+\infty} (\alpha^{2n}/\sqrt{(2n)!}) \times |o_n\rangle + \sum_{n=1}^{+\infty} (\alpha^{(2n-1)}/\sqrt{(2n-1)!}) \times |o_{-n}\rangle]$.

- **Even and odd coherent states:** $|\alpha_e\rangle = N_e(|\alpha\rangle + |-\alpha\rangle)$, $|\alpha_o\rangle = N_o(|\alpha\rangle - |-\alpha\rangle)$ with $N_{e/o} = 1/\sqrt{2 \pm 2e^{-2|\alpha|^2}}$.

- We will study quantum-to-classical transition of $|\psi_\alpha\rangle = (1/\sqrt{2}) \times (|\alpha_e\rangle \otimes |\alpha_o\rangle + |\alpha_o\rangle \otimes |\alpha_e\rangle)$.

Coarse-grained observables for BCHSH inequality

- **Choice of dichotomic observables:** $O^k(\theta_j) \equiv V(\theta_j)^\dagger O^k_+ V(\theta_j) - V(\theta_j)^\dagger O^k_- V(\theta_j)$ with $j \in \{a, c\}$ for Alice and $j \in \{b, d\}$ for Bob.
- **Action of unitary** $V(\theta_j)$: $V(\theta_j)|\alpha_e\rangle = \cos\theta_j|\alpha_e\rangle + \sin\theta_j|\alpha_o\rangle$, $V(\theta_j)|\alpha_o\rangle = \sin\theta_j|\alpha_e\rangle - \cos\theta_j|\alpha_o\rangle$. For simplicity, we take: $U(\theta) = V(\theta)$.
- **Choice of coarse-grained observables:** $O_\delta(\theta_j) \equiv \sum_k P_\delta(k) O^k(\theta_j)$, $O_\Delta(\theta_j; \theta_0) \equiv \int_0^{2\pi} P_\Delta(\theta_j - \theta_0) U(\theta_j)^\dagger O^0(\theta_0) U(\theta_j) d\theta_j$.
- **Expectation values:** $E_{ab}(\delta) \equiv \langle\psi_\alpha|O_\delta(\theta_a) \otimes O_\delta(\theta_b)|\psi_\alpha\rangle$, etc.
- **Bell quantity:** $B_\delta(\theta_a, \theta_b, \theta_c, \theta_d) \equiv E_{ab}(\delta) + E_{cb}(\delta) + E_{ad}(\delta) - E_{cd}(\delta)$.

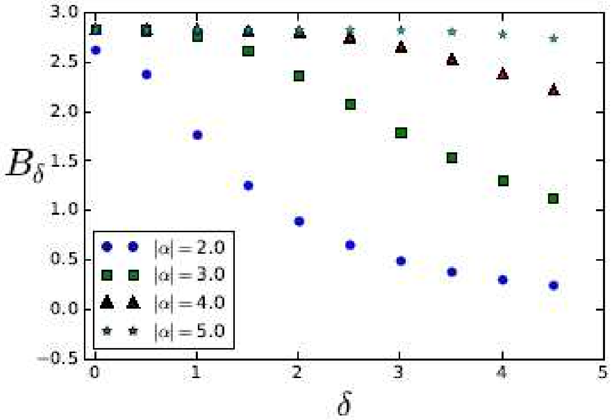
The maximum values: B_δ and $B_\Delta(\theta_0)$

- $B_\delta = \left\{\max_{\theta_a,\theta_b,\theta_c,\theta_d} \mathcal{F}(\theta_a,\theta_b,\theta_c,\theta_d)\right\} \times \left\{-1 + A + B - \frac{1}{4}(A+B)^2\right\} + \frac{1}{2}(A-B)^2$,

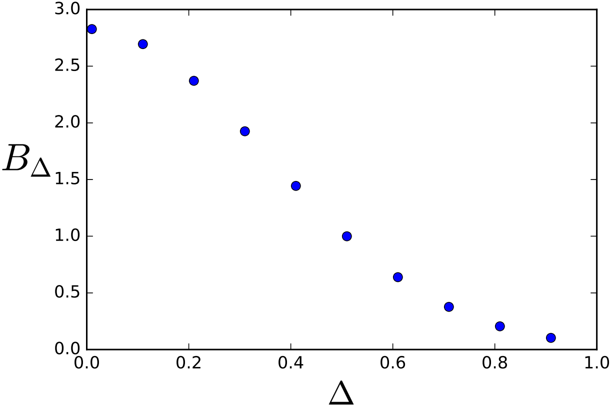
- with $A \equiv 2C_e^2 \sum_{k=0}^{\infty}\sum_{n=0}^{k} P_\delta(k)\frac{(|\alpha|^2)^{2n}}{(2n)!}$, $B \equiv 2C_o^2 \sum_{k=1}^{\infty}\sum_{n=0}^{k-1} P_\delta(k)\frac{(|\alpha|^2)^{2n+1}}{(2n+1)!}$, $C_e^2 = (\cosh|\alpha|^2)^{-1}$, and $C_o = (\sinh|\alpha|^2)^{-1}$.

- $\mathcal{F}(\theta_a,\theta_b,\theta_c,\theta_d) = \cos(2\theta_a + 2\theta_b) + \cos(2\theta_c + 2\theta_b) + \cos(2\theta_a + 2\theta_d) - \cos(2\theta_c + 2\theta_d)$ with max. being $2\sqrt{2}$.

- $B_\Delta(\theta_0) = B_\Delta = 2\sqrt{2}e^{-4\Delta^2}$.



3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
0
1
2
3
4
5
B_δ
δ
$|\alpha| = 2.0$
$|\alpha| = 3.0$
$|\alpha| = 4.0$
$|\alpha| = 5.0$


3.0
2.5
2.0
1.5
1.0
0.5
0.0
B_Δ
0.0
0.2
0.4
0.6
0.8
1.0
Δ

Post-measurement state for reference coarse-graining

- Given the measurement direction θ_a, Alice performs measurement of the dichotomic observable $O^0(\theta_1)$ (chosen with apriori probability $P_\Delta(\theta_1 - \theta_a)$) on her system.
- Given the measurement direction θ_b, Bob performs measurement of the dichotomic observable $O^0(\theta_2)$ (chosen with apriori probability $P_\Delta(\theta_2 - \theta_b)$) on his system.
- **Post-measurement state:** (w.r.t. $\{|\alpha_e\alpha_e\rangle, |\alpha_e\alpha_o\rangle, \ldots\}$)

$$\rho_{ref} = \begin{pmatrix} a & b & c & d \\ b & \frac{1}{2} - a & d & -c \\ c & d & \frac{1}{2} - a & -b \\ d & -c & -b & a \end{pmatrix}$$

Post-measurement state for reference coarse-graining (continued)

- $a = \frac{1}{16}\left(3 - e^{-8\Delta^2}\left\{\cos(4\theta_a) + \cos\left(4\theta_b\right)\right\} - e^{-16\Delta^2}\cos(4\theta_a + 4\theta_b)\right)$
- $b = \frac{1}{16}\left(e^{-8\Delta^2}\left\{\sin(4\theta_a) - \sin\left(4\theta_b\right)\right\} - e^{-16\Delta^2}\sin(4\theta_a + 4\theta_b)\right)$
- $c = \frac{1}{16}\left(e^{-8\Delta^2}\left\{-\sin(4\theta_a) + \sin\left(4\theta_b\right)\right\} - e^{-16\Delta^2}\sin(4\theta_a + 4\theta_b)\right)$
- $d = \frac{1}{16}\left(1 - e^{-8\Delta^2}\left\{\cos(4\theta_a) + \cos\left(4\theta_b\right)\right\} + e^{-16\Delta^2}\cos(4\theta_a + 4\theta_b)\right)$

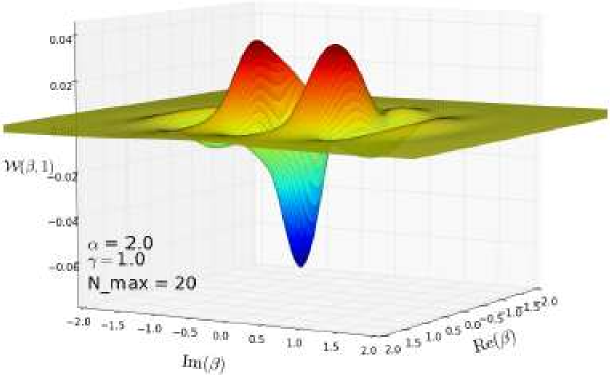
P-distribution of ρ_{ref}:

- $\mathcal{P}_{ref}(\beta, \gamma) = \sum_{i,j,k,l,\in\{e,o\}} \rho_{i,j,k,l} P_{ij}(\beta) P_{kl}(\gamma)$
- $\rho_{i,j,k,l} = \mathrm{Tr}\left(\rho_{ref} |i\rangle\langle j| \otimes |k\rangle\langle l|\right) \qquad i,j,k,l \in \{e,o\}$
- $P_{ee}(\beta) = N_e^2 \left\{1 + e^{-2|\alpha|^2} \hat{A}(\alpha)\right\} \left[\delta^{(2)}(\alpha - \beta) + \delta^{(2)}(\alpha + \beta)\right]$ is the P-distribution of $|\alpha_e\rangle\langle\alpha_e|$, etc.
- $\hat{A}(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (2\alpha)^n \left(\frac{\partial}{\partial\alpha}\right)^n$
- The P-distribution is highly singular function – no classical analogue.

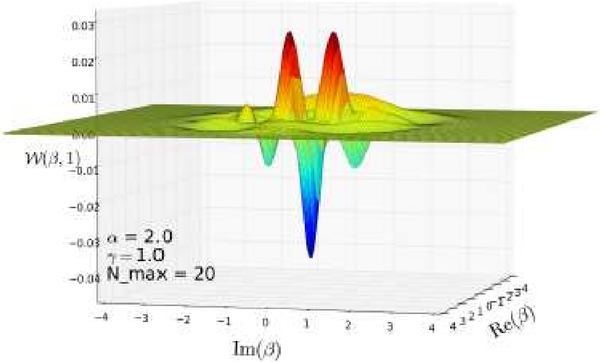
Post-measurement state for resolution coarse-graining

- Given the measurement direction θ_a, Alice performs measurement of the dichotomic observable $O^k(\theta_a)$ (chosen with apriori probability $P_\delta(k)$) on her system.
- Given the measurement direction θ_b, Bob performs measurement of the dichotomic observable $O^m(\theta_b)$ (chosen with apriori probability $P_\delta(m)$) on his system.
- **Post-measurement state:** Difficult to get a closed form analytically.
- We fix our basis by choosing $(\theta_a, \theta_b) = (\frac{\pi}{4}, \frac{3\pi}{4})$.
- **Negativity of Wigner distribution:** $\mathcal{N}_\rho = \frac{1}{2} \int_{\beta,\gamma \in \mathbb{C}} (|\mathcal{W}(\beta,\gamma)| - \mathcal{W}(\beta,\gamma))\, d^2\beta d^2\gamma$

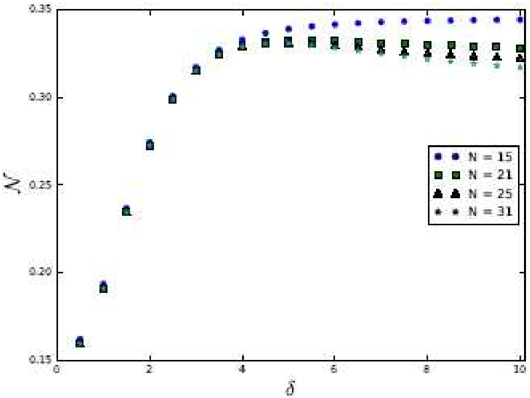
$$|\psi> \propto |\alpha_e \alpha_o> + |\alpha_o \alpha_e>$$



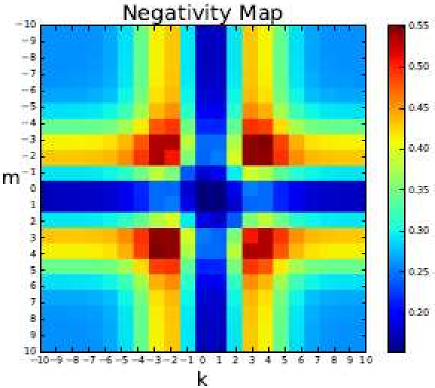
Wigner function $\mathcal{W}_{\psi_\alpha}(\beta, \gamma)$ of $|\psi_\alpha\rangle$ with γ constant, N_max is the maximum number of photons in each mode (color online).



Wigner function $\mathcal{W}(\beta,\gamma)$ of the post-measurement state of $|\psi_\alpha\rangle$ after sharp measurement with $O^4(\frac{\pi}{4}) \otimes O^4(\frac{3\pi}{4})$, N_max is the maximum number of photons in each mode

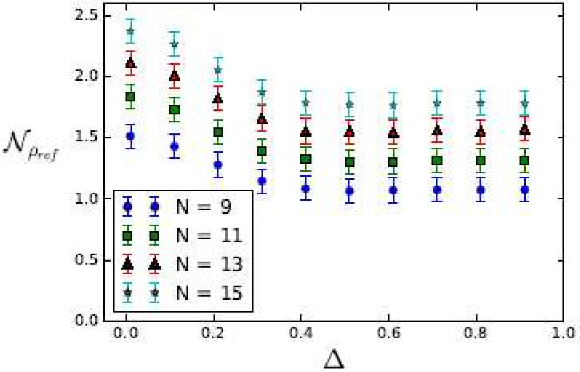


Wigner function negativity of the post-measurement state of $|\psi_\alpha\rangle$ vs. the resolution coarse-graining parameter after measurement with $O_\delta(\frac{\pi}{4}, \frac{3\pi}{4})$ for different truncations N, with $\alpha = 2$. Neg. increases with δ.

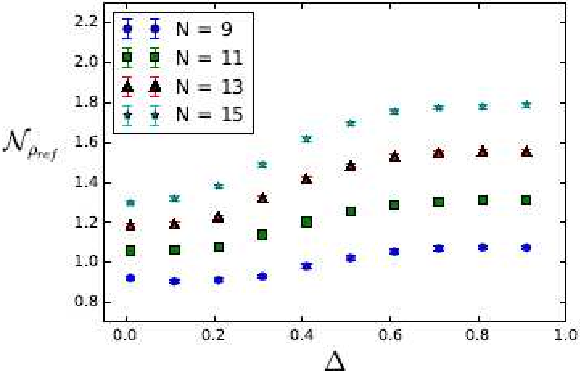


Wigner function negativity of the post-measurement state of $|\psi_\alpha\rangle$ after sharp measurement with $O^k(\frac{\pi}{4}) \otimes O^m(\frac{3\pi}{4})$ for different values of k and m. N_{ψ_α} has less -ve value for $k=m=0$ compared to other values of k and m.

Case 2: On quantum-to-classical transition for $|\psi_N\rangle = (1/\sqrt{2}) \times (|0\rangle \otimes |N\rangle + |N\rangle \otimes |0\rangle)$



Wigner function negativity of the post-measurement state of $|\psi_N\rangle$ after the reference coarse-grained measurement with $O_\Delta(\frac{13\pi}{100}, \frac{21\pi}{50})$: $N_{\rho_{ref}}$ increases with the photon no. N.

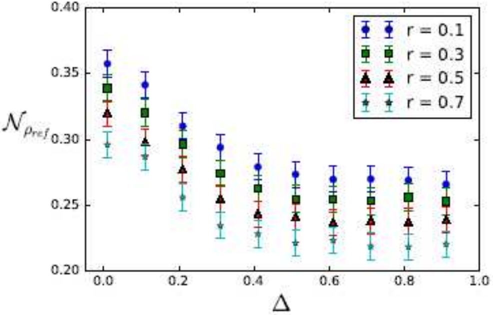


Wigner function negativity of the post-measurement state of $|\psi_N\rangle$ after the reference coarse-grained measurement $O_\Delta(0,0)$. $\mathcal{N}_{\rho_{ref}}$ increases with Δ.

Case 3: On quantum-to-classical transition for $|\psi_r\rangle = (1/\sqrt{2}) \times (|\Psi^r_+\rangle \otimes |\Psi^r_-\rangle + |\Psi^r_-\rangle \otimes |\Psi^r_+\rangle)$

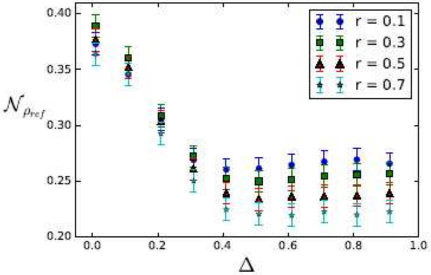
Introduction of single-mode non-classicality via squeezing

- $|\psi_r\rangle = \frac{1}{\sqrt{2}}\left(|\Psi^r_+\rangle|\Psi^r_-\rangle + |\Psi^r_-\rangle|\Psi^r_+\rangle\right)$,
- $|\Psi^r_+\rangle$ and $|\Psi^r_-\rangle$ are two-photon-added and one-photon-added squeezed vacuum states, respectively.
- They are defined as: $|\Psi^r_+\rangle \equiv \frac{1}{\cosh^2 r\sqrt{2+\tanh^2 r}}(\hat{a}^\dagger)^2 S(r)|0\rangle$, $|\Psi^r_-\rangle \equiv \frac{1}{\cosh r}\hat{a}^\dagger S(r)|0\rangle$.
- $S(r) = \exp\left(\frac{r}{2}(\hat{a}^2 - (\hat{a}^\dagger)^2)\right)$ is the squeezing operator with real squeezing parameter r.
- $B_\Delta = 2\sqrt{2}e^{-4\Delta^2}$.



Wigner function negativity of the post-measurement state of $|\psi_r\rangle$ after the reference coarse-grained measurement $O_\Delta(\frac{53\pi}{100}, \frac{2\pi}{3})$.

$W_{\rho_{ref}}$ decreases with the non-classicality parameter r.



Wigner function negativity of the post-measurement state of $|\psi_r\rangle$ after the reference coarse-grained measurement $O_\Delta(\frac{\pi}{4}, \frac{3\pi}{4})$. For fixed Δ, $\mathcal{N}_{\rho_{ref}}$ does not always behave monotonically with respect to r.

Discussion

Conclusion

- Even though the violation of any local realistic inequality does indicate some ‘non-classical’ behaviour of the bipartite state at hand, the non-violation does not guarantee any ‘classicality’ unless we can provide a local realistic model of the state.

- As the notion of non-classicality is a well-established feature in quantum optics, we looked at the P-distribution as well as the negativity of the Wigner function of the post-measurement state in the case of non-selective measurement involving both reference as well as resolution coarse-graining, by starting from some suitably chosen two-mode entangled states.

Conclusion (continued)

- Contrary to general indication of quantum-to-classical transition via non-violation of the Bell-CHSH inequality, we found the presence of non-classicality in the post-measurement states (of certain classes of states) irrespective of whether we choose reference or resolution coarse-graining.

- We hope that our study will help better understand the notion of quantum-to-classical transition.

Open questions

- To compare the three cases presented in this paper in a consistent way, it will be pertinent to study the behavior of non-classicality of the post-measurement state with respect to average photon number in the initial state $\langle\psi_i|(N_1 \otimes \mathbb{I} + \mathbb{I} \otimes N_2)|\psi_i\rangle$, where, N_1 and N_2 are single mode photon number operators, for a fixed value of the coarse-graining parameter.

- We would like to determine the effect of rotations more general than what has been considered here for reference coarse-graining to study quantum-to-classical transition.

Open questions (continued)

- The present study can, in principle, be extended to multipartite systems.

- We are also interested in relating our analysis with other approaches to quantum-to-classical transitions such as the one put forth in Gisin and his collaborators [*Phys. Rev. Lett.* **113**, 090403 (2014)].

Perspectives on non-commutative phase-space extension

- Bell-type inequalities have been extended to CV systems using Wigner functions on (commutative) phase-space.
- Extension of Wigner function formalism has been made by Bastos et al. [JMP **49**, 072101 (2008); CMP **299**, 709 (2010)] in the case of non-commutative phase-space.
- Unfortunately, in the case of CV BCHSH inequality (*i.e.*, with dichotomic observations), better non-loaclity is **not** revealed with non-commutative Wigner functions – so far as squeezed states are concerned [Bastos et al., PRD **93**, 104055 (2016)].
- But there is still a lot of room to see the effect of phase-space non-commutativity for other types of Bell-type inequalities, particularly with infinite-valued measurements [*e.g.*, Ketterer et al., PRA **91**, 012106 (2015)].

Perspectives on non-commutative phase-space extension (continued)

- To study quantum-to-classical transition in non-commutative phase-space setup – in the line of Quantum Optics – it may be necessary to develop the non-commutative version of Glauber-Sudarshan P-function, like in the case of Wigner function, mentioned above.

- It is also useful to study the effect of phase-space non-commutativity on system dynamics in the context of quantum-to-classical transition.

THANK YOU!