Restoration of unitarity in anisotropic quantum cosmology

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Quantum Cosmology

Gravity $\rightarrow$ no universally accepted quantum theory

Cosmology is an arena where quantum principles are applied to a gravitational system.

Motivation

When the universe was small, smaller than Planck length, classical gravity would not work.

A quantum picture is required.

Hope: Might resolve the singularity.

The action

The relevant action for gravity:

$$\mathcal{A} = \int_M d^4x\sqrt{-g}\,R + 2\int_{\partial M} d^3x\sqrt{h}\,h_{ab}\,K^{ab} + \int_M d^4x\sqrt{-g}\,L_m$$

h_{ab}: induced metric over spatial hypersurface

∂M: boundary of the four dimensional manifold M

K^{ab}: extrinsic curvature.

L_m: Lagrangian for the matter

(Units: $c = 16\pi G = \hbar = 1$).

Quantization scheme

Choose the metric in a given form using symmetry

(Minisuperspace)

Einstein-Hilbert action is written in terms of the metric

The metric for the space-section, h_{ij} and the matter degrees of freedom are the relevant variables.

Conjugate momenta are defined

The Hamiltonian is formed.

Usual canonical quantization:
Replace the momenta by the corresponding operators, e.g.,
If Π_{ij} is the momentum conjugate to the dynamical variable h_{ij}, then

$$[h_{ij}, \Pi^{ij}] = \imath$$

Hamiltonian constraint $\rightarrow$ H $= 0$

Wheeler-DeWitt equation

$$H\Psi = 0.$$

Problems with quantum cosmology

There are many:

Problem of interpretation

Problem of the identification of a time parameter

And many others.

Anisotropic models are believed to be nonunitary !

This is the one we shall deal with.

This leads to a non-conservation of probability.

The observed universe is isotropic.

But, this isotropy is not a theoretical requirement .

So this nonunitary leads to an inconsistency in the quantization scheme.

Intersting to note: In the absence of a properly oriented scalar time parameter, this nonconservation of probability is often obscure!

Choice of a time parameter

The cosmic time t is a coordinate.

Problem: To pick up a properly oriented time

The evolution of a fluid in the model comes to the rescue.

The action with a fluid

If the matter sector contains a perfect fluid.

The action becomes:

$$\mathcal{A} = \int_M d^4x\sqrt{-g}\,R + 2\int_{\partial M} d^3x\sqrt{h}\,h_{ab}\,K^{ab} + \int_M d^4x\sqrt{-g}\;P$$

P is the pressure due to the perfect fluid.

Schutz Formalism

Take a break, consider the fluid

The velocity vector: $u_\nu = \frac{1}{h}(\epsilon_{,\nu} + \theta S_{,\nu})$.[1]

h, S, ϵ and θ: the velocity potentials having their own evolution equations.

(The potentials connected with vorticity are dropped.)

u^μ is normalized as $u^\nu u_\nu = 1$.

h and $S \rightarrow$ specific enthalpy and specific entropy respectively.

Only two are actually used: ϵ and S.
ϵ and h are related by $u^\mu \epsilon_{,\mu} = -h$,
θ can be settled using the normalization.

[1] B.F. Schutz, PRD **4**, 3559 (1971)
V.G. Lapchinskii and V.A. Rubakov, Theor. Math. Phys. **33**, 1076 (1977)

Schutz Formalism

The fluid density ρ, the pressure P, specific enthalpy (h) and specific entropy(S) are connected by standard thermodynamical relations.

We shall use a barotropic fluid, given by the equation of state

$$P = \rho,$$

$$\alpha \to \text{a constant.}$$

Problem with anisotropic models

Widely believed → Anisotropic models are nonunitary [2]!!

An explicit example → Bianchi I model[3]

$$ds^2 = n^2 dt^2 - \left[e^{(\beta_0+\beta_+ + \sqrt{3}\beta_-)} dx^2 + e^{(\beta_0+\beta_+ - \sqrt{3}\beta_-)} dy^2 + e^{(\beta_0 - 2\beta_+)} dz^2 \right],$$

The W-D equation

$$\left(\frac{\partial^2}{\partial \beta_0^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} \right) \phi = 24 E \phi e^{3(1-\alpha)\beta_0},$$

indeed yields a non-unitary evolution!

[2] N. Pinto-Neto and J.C. Fabris, CQG **30**, 143001 (2013)

[3] F.G. Alvarenga, A.B. Batista, J.C. Fabris, N.A. Lemos and S.V. B. Goncales, GRG **35**, 1639 (2003).

Problem with anisotropic models

Reason ?

Hyperbolicity ?

Perhaps NO.

Improper Operator ordering?

Perhaps YES !

Problem Resolved?

For Bianchi V, it was shown that with an operator ordering, probability conservation holds good for large "time"[4]

Now it is quite established → Unitarity can in fact be restored

Examples: Bianchi I[5], Bianchi V and IX[6] and Kantowski-Sachs[7]

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[4] B. Majumder and NB, GRG **45**, 1 (2013)
[5] S. Pal and NB, PRD **90**, 104001 (2014)
[6] S. Pal and NB, PRD **91** 044042 (2015)
[7] S. Pal and NB, CQG, **32**, 205005 (2015)

Bianchi III cosmology

The Bianchi type III model is given by the metric

$$ds^2 = n^2 dt^2 - e^{2\sqrt{3}\beta_+} dr^2 - e^{-2\sqrt{3}(\beta_+ + \beta_-)} \left[d\theta^2 + \sinh^2(\theta)\, d\phi^2\right].$$

Lapse function n, β_+ and β_- $\rightarrow$ functions of time t.

Bianchi III metric in this form $\rightarrow$ similar to the Kantowski-Sachs cosmology.

(The hyperbolic coefficient of $d\phi^2$ is replaced by a sinusoidal function.)

The Hamiltonian

Given the action, the Hamiltonian for the gravity sector can be written as

$$H_g = \frac{n}{24} e^{\sqrt{3}(\beta_+ + 2\beta_-)} \left[-p_{\beta_-}^2 + p_{\beta_+}^2 + 48 e^{-2\sqrt{3}\beta_-} \right].$$

p_i's are the momenta, canonically conjugate to β_i's.

The Hamiltonian

We effect the canonical transformations,

$$T = -p_S \exp(-S) p_\epsilon^{-\alpha-1},$$
$$p_T = p_\epsilon^{\alpha+1} \exp(S),$$
$$\epsilon' = \epsilon + (\alpha+1) \frac{p_S}{p_\epsilon}.$$
$$p'_\epsilon = p_\epsilon.$$

The Hamiltonian for the fluid sector

$$H_f = n e^{\alpha\sqrt{3}(\beta_+ + 2\beta_-)} p_T.$$

The net (super) Hamiltonian is given by $H = H_g + H_f$.

A variation with respect to n yields the Hamiltonian constraint,

$$e^{\sqrt{3}(1-\alpha)(\beta_+ + 2\beta_-)} \left\{ -p_{\beta_-}^2 + p_{\beta_+}^2 + 48 e^{-2\sqrt{3}\beta_-} \right\} + 24 p_T = 0.$$

Back to Quantization scheme

Promote the dynamical variables to operators.

$p_i \mapsto -\imath\hbar\partial_{\beta_i}$ for $i = 0, +, -,$

and $p_T \mapsto -\imath\hbar\partial_T$.

This mapping is equivalent to postulating the fundamental commutation relations

$$[\beta_i, p_j] = \imath\hbar\delta_{ij}\mathbb{I}.$$

Hamiltonian constraint $\rightarrow$ $H = 0$.

Wheeler-DeWitt equation

$$H\psi = 0,$$

Some points to note

The Poisson brackets $\{\epsilon', p'_\epsilon\} = 1$ and $\{T, p_T\} = 1$ are satisfied

Other Poisson brackets $\rightarrow 0$. $\rightarrow$ Ensures the canonical structure.

$\frac{dT}{dt} > 0$,

T has the same sign as the cosmic time !

And, is a monotonically increasing function.

Quantization continued

General perfect fluid: $\alpha \neq 1$
We propose following operator ordering:

$$\Big[- e^{\frac{\sqrt{3}}{2}(1-\alpha)(\beta_+ + 4\beta_-)} \frac{\partial}{\partial \beta_+} e^{\frac{\sqrt{3}}{2}(1-\alpha)\beta_+} \frac{\partial}{\partial \beta_+}$$

$$+ e^{\sqrt{3}(1-\alpha)(\beta_+ + \beta_-)} \frac{\partial}{\partial \beta_-} e^{\sqrt{3}(1-\alpha)\beta_-} \frac{\partial}{\partial \beta_+}$$

$$+ 48 e^{-2\sqrt{3}\beta_-} e^{\sqrt{3}(1-\alpha)(\beta_+ + 2\beta_-)} \Big] \Psi$$

$$= 24\imath \frac{\partial \Psi}{\partial T}.$$

Quantization continued

We effect a transformation of variables as

$\chi_+ \equiv e^{-\frac{\sqrt{3}}{2}(1-\alpha)\beta_+}$ & $\chi_- \equiv e^{-\sqrt{3}(1-\alpha)\beta_-}$,

and use separability ansatz $\Psi = \phi(\chi_+, \chi_-)e^{-\imath ET}$:

$$H_g\phi = -\frac{1}{\chi_-^2}\frac{\partial^2\phi}{\partial\chi_+^2} + \frac{1}{\chi_+^2}\frac{\partial^2\phi}{\partial\chi_-^2} + 48\chi_-^{\frac{2\alpha}{1-\alpha}}\chi_+^{-2}\phi = 24E\phi.$$

Unitarity restored

Now it is easy to see that one can use Neumann's theorem which states that

"A symmetric operator $\hat{A}$ defined on domain $\mathcal{D}$ has equal deficiency index, if there exists a norm preserving anti-unitary conjugation map $C : \mathcal{D} \rightarrow \mathcal{D}$ such that $[\hat{A}, C] = 0$, which, in turn, shows that $\hat{A}$ admits self-adjoint extension".

H_g satisfies the conditions !!

$C \rightarrow$ the map which takes ϕ to ϕ^*.
Hamiltonian admits self-adjoint extension i.e a unitary evolution.

The same analysis holds for the Kantowski-Sachs model as well.

Rationale behind the operator ordering

The kinetic term $\frac{\partial^2 \phi}{\partial \chi_{\pm}^2}$ multiplied with $\chi_{\mp}^2$.

Hence, the condition for H_g being symmetric is same as the condition for a standard Laplacian to be symmetric

We have following condition,

$$\left[\phi \frac{\partial \phi^*}{\partial \chi_{\pm}} - \phi^* \frac{\partial \phi}{\partial \chi_{\pm}}\right]_0^{\infty} = 0.$$

Ordering plays a role in making H_g a symmetric operator.
Once it is guaranteed to be a symmetric operator, the self-adjoint extension is obvious
(following Neumann's theorem.)

The particular operator ordering is a sufficient condition for making H_g symmetric.

A specific example

Stiff fluid ($P = \rho$)

$$\left\{ \frac{\partial^2}{\partial \beta_-^2} - \frac{\partial^2}{\partial \beta_+^2} + 48 e^{-2\sqrt{3}\beta_-} \right\} \Psi = 24\imath \frac{\partial \Psi}{\partial T},$$

Separation of variables, $\Psi = \phi(\beta_-)\psi(\beta_+)e^{-\imath ET}$,

$$\left\{ \frac{\partial^2}{\partial \beta_-^2} + 3k_+^2 + 48 e^{-2\sqrt{3}\beta_-} \right\} \phi = 24E\phi,$$

$$\left[\frac{\partial^2}{\partial \beta_+^2} + 3k_+^2 \right] \psi = 0.$$

A specific example

With $||\psi|| \equiv \int_{-\infty}^{\infty} d\beta_+ \psi\psi^*$,
The solution is unitary;
The norm for the β_+ sector is time independent and finite (by explicit construction of wavepacket).
The equation for the β_- sector can be recast in the standard self-adjoint form (using the variable $\chi \equiv e^{-\sqrt{3}\beta_-}$),

$$\frac{d}{d\chi}\left(\chi\frac{d\phi}{d\chi}\right) + \left(16\chi - \frac{8E - k_+^2}{\chi}\right)\phi = 0,$$

with inner product given by $\langle\phi_1|\phi_2\rangle \equiv \int_0^{\infty} d\chi\ \chi\ \phi_1^*(\chi)\phi_2(\chi)$.
This Hamiltonian for β_- sector is self-adjoint as well, ensuring a unitary time evolution.

Any hidden price?

Does anisotropy remain intact?

YES!

One can check the classical anistropy:

$$\sigma^2 = \frac{1}{12}\Big[\big(\frac{\dot{g}_{11}}{g_{11}} - \frac{\dot{g}_{22}}{g_{22}}\big)^2 + \big(\frac{\dot{g}_{22}}{g_{22}} - \frac{\dot{g}_{33}}{g_{33}}\big)^2 + \big(\frac{\dot{g}_{33}}{g_{33}} - \frac{\dot{g}_{11}}{g_{11}}\big)^2\Big]$$

is indeed a nonzero object[8]

[8] S. Pal, CQG, **33**, 045007 (2016)

Hidden Price

Scale invariance is lost !!

Any other symmetry?

May be yes.....

However, Noether symmetry appears to be respected.

Summary

- The threat of nonconservation of probablity is not real!
- Anisotropic models with constant spatial curvature (Bianchi I, V, IX) as well as varying spatial curvature (Bianchi III, KS), on proper operator ordering, show unitary evolution.
- In fact, thanks to Neumann's theorem, as all the Bianchi models and KS, possess a symmetric Hamiltonian, a self-adjoint extension is always possible.
- The unitarity is restoterd, not at the cost of anisotropy.