Constructive Holography

Current developments in Quantum Field Theory and Gravity, S.N. Bose National Centre for Basic Sciences,

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Gauge theory / gravity duality

The gauge theory / gravity duality, captured in the equation

$$\mathrm{CFT}_d \qquad = \qquad \mathrm{AdS}_{d+1} \tag{1}$$

is a remarkable but still very mysterious duality.

Most famous example is the duality between $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$ and IIB string theory on asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime with N units of 5-form flux.

A much easier example is provided by the duality between the free $O(N)$ vector model and higher spin gravity. This is the example we study.

Higher spin theories in AdS

In any dimension Vasiliev wrote a set of consistent, fully non-linear, gauge invariant equations of motion for higher spin gravity. They admit a vacuum solution which is AdS spacetime.

In the simplest bosonic 4d theory, the linearized spectrum around the AdS vacuum is an infinite tower of massless higher spin fields with spin $s = 2, 4, 6, \ldots$ plus a scalar with mass $m^2 = -2/l_{AdS}^2$.

Full quantum theory is not yet known. We don't know of any action that reproduces Vasiliev's equations.

Vasiliev equations

The Vasiliev equations in 4d are

$$d\mathcal{A} + \mathcal{A} * \mathcal{A} = B * \kappa \, dz^{\alpha} dz_{\alpha} + B * \bar{\kappa} d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \tag{2}$$

$$dB + \mathcal{A} * B - B * \pi(\mathcal{A}) = 0 \tag{3}$$

Twelve dimensional theory with 4 spinor variables $y^{A}, y^{\dot{A}}$ use to book keep the spin degrees of freedom, 4 spinor variables $z^{A}, z^{\dot{A}}$ and four spacetime coordinates x^{μ}.

Linearized HS Spectrum

Linearizing the Vasiliev equations around AdS vacuum solution, one finds the usual equations for a scalar (with mass $m^2 = -2$), linearized graviton and free massless Higher Spin fields (Fronsdal).

$$(\nabla^2 - \kappa^2)\varphi_{\mu_1\cdots\mu_s} = 0 \qquad \kappa^2 = (s-2)(s+d-3) - 2 \tag{4}$$

$$\nabla^\mu \varphi_{\mu\mu_2\cdots\mu_s} = 0 \qquad g^{\mu\nu}\varphi_{\mu\nu\nu_3\cdots\mu_s} = 0 \tag{5}$$

At higher orders in perturbation theory, one reads off cubic, quartic, etc. couplings. Could in principle reconstruct a Lagrangian in terms of the physical Higher Spin fields but its very hard in practice.

Free equations are ordinary wave equations. Interactions involve higher derivatives of arbitrarily high order so the Vasiliev theory is a higher derivative theory of gravity.

Higher Spins and AdS/CFT

The AdS/CFT correspondence predicts that such theories exist.

Vasiliev higher spin theories have the correct spectrum to be dual to simple CFTs with matter transforming in the vector rep of the gauge group.

Consider a free theory of N free real scalar fields

$$S = \int d^3x \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i \tag{6}$$

Has $O(N)$ global symmetry under which the scalar transforms as a vector. Vasiliev theory is dual to the singlet sector of this $O(N)$ vector model.

This is different from familiar examples of AdS/CFT, where the CFT side is usually a gauge theory with matrix (adjoint) fields.

Single Trace CFT Primaries

The free theory has an infinite tower of conserved Higher Spin currents which are primary operators of the form

$$j^{\mu_1\mu_2\cdots\mu_s} \sim \sum_k c_k \ \partial_{\mu_1}\cdots\partial_{\mu_k}\phi^i \ \partial_{\mu_{k+1}}\cdots\partial_{\mu_s}\phi^i \tag{7}$$

$$\partial_\mu j^{\mu\mu_2\cdots\mu_s} = 0 \tag{8}$$

$$\Delta(j^{\mu_1\cdots\mu_s}) = s+1 \tag{9}$$

In the singlet sector, these currents and the scalar operator $\phi^i\phi^i$ with $\Delta = 1$ are all the "single trace" primaries of the CFT.

Higher Spin Holography

The standard AdS/CFT dictionary identifies single trace primary operators in the CFT with single particle states in AdS.

The conserved currents are dual to gauge fields (massless Higher Spin fields).

The scalar operator is dual to a bulk scalar with $m^2 = \Delta(\Delta - d)$.

This precisely matches the spectrum of Vasiliev's minimal bosonic theory in AdS_4.

Interacting Higher Spin Holography

The dual (gravitational) higher spin fields must be interacting in order to reproduce the non-vanishing correlation functions of Higher Spin currents in the CFT.

The large N limit corresponds, as usual, to weak interactions in the bulk.

So AdS/CFT implies that consistent theories of interacting massless higher spins in AdS indeed have to exist, as they should provide AdS duals to free vector like CFTs (in the singlet sector).

Higher Spins and AdS/CFT

The conjecture that the (singlet sector of) the $O(N)$ vector model is dual to the bosonic Vasiliev theory was made by Klebanov and Polyakov (2002) (earlier closely related work: Sundborg; Witten; Sezgin, Sundell).

One can also consider the Wilson-Fischer fixed point reached by a relevant double trace deformation of the free theory.

The IR fixed point (the "critical vector model") is an interacting CFT. Single trace spectrum is a scalar operator of dimension $\Delta = 2 + O(N^{-1})$ and approximately conserved Higher Spin currents of dimension $\Delta = s + 1 + O(N^{-1})$.

Tests of the duality

The 4d Higher Spin/3d-vector model dualities have passed nontrivial tests at the level of correlation functions (Giombi, Yin; Maldacena-Zhiboedov; Didenko-Skvortsov; Boulanger, Kessel, Skvortsov, Taronna;...)

One loop tests of higher spin/vector model dualities can be obtained by comparing sphere partition functions. (Giombi, Klebanov; Jevicki, Yoon;...)

Taken together this is very strong support in favor of the duality.

Constructive Holography

Mechanism by which gravitational physics is manifested from a strongly coupled gauge theory is elusive.

Original CFT has field theory coupling as loop expansion parameter. Gravitational description has $1/N$ as loop counting parameter. $\Rightarrow$ highly non-trivial hint into structure of holographic reorganization of CFT.

Collective field theory (Jevicki, Sakita) achieves this: formulates theory in gauge invariant variables. Resulting field theory explicitly has $1/N$ as loop expansion parameter. Reorganization of dynamics is highly non-trivial: non-linear collective dynamics are induced by Jacobian of change of variables.

Bi-local Holography

Free vector model in original variables

$$\langle \cdots \rangle = \int [d\phi^i] \; e^{iS} \; \cdots \tag{10}$$

Collective variable is the $O(N)$ singlet $\sigma(x, y)$

$$\sigma(x, y) = \frac{1}{N} \sum_{i=1}^{N} \phi^i(x) \phi^i(y) \tag{11}$$

After changing variables

$$\langle \cdots \rangle = \int [d\sigma] \; J[\sigma] e^{iS[\sigma]} \; \cdots \tag{12}$$

$\sigma(x, y)$ is a function of $2d$ variables: AdS_{d+1}, S_{d-1}

(Das, Jevicki, hep-th/0304093)

Jacobian from path integral

Collective variable is the $O(N)$ singlet $\sigma(x, y)$

$$S = \int d^3x \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \qquad \sigma(x, y) = \frac{1}{N}\sum_{i=1}^{N}\phi^i(x)\phi^i(y) \tag{13}$$

Schwinger-Dyson equations of original theory

$$0 = \int [d\phi] \frac{\partial}{\partial\phi^i(x_1)}\left(\phi^i(x_2)F[\sigma]e^{iS}\right) \tag{14}$$

must match the Schwinger-Dyson equations after changing variables

$$0 = \int [d\sigma] \int d^dx \frac{\partial}{\partial\sigma(x, x_1)}\left(\sigma(x, x_2)J[\sigma]F[\sigma]e^{iS[\sigma]}\right) \tag{15}$$

Jacobian from Path Integral

The Jacobian obeys

$$\int d^d x \sigma(x, x_1) \frac{\delta \log J}{\delta \sigma(x, x_2)} = (N - L^d \delta^d(0)) \delta^{(d)}(x_1 - x_2) \tag{16}$$

which is solved by

$$\log J = (N - L^d \delta^d(0)) \mathrm{Tr} \log \sigma \tag{17}$$

Using Jacobian (i) one can reproduce large N perturbative expansion for vector models and (ii) reproduces non-perturbative physics (e.g. gap equations for NJL/Gross-Neveu models)

(dMK, Rordigues hep-th/9605079)

Jacobian from Hamiltonian

Use equal time bilocals and rewrite Hamiltonian

$$\sigma(t,\vec{x}_1,\vec{x}_2)=\frac{1}{N}\sum_i \phi^i(t,\vec{x}_1)\phi^i(t,\vec{x}_2) \tag{18}$$

Kinetic term is

$$\sum_i \frac{1}{i}\frac{\partial}{\partial\phi^i(t,\vec{x})}\frac{1}{i}\frac{\partial}{\partial\phi^i(t,\vec{x})} \tag{19}$$

Use chain rule

$$\begin{aligned}\frac{\partial}{\partial\phi^i(t,\vec{x})} &= \int d^{d-1}x_1\int d^{d-1}x_2\frac{\partial\sigma(t,\vec{x}_1,\vec{x}_2)}{\partial\phi^i(t,\vec{x})}\frac{\partial}{\partial\sigma(t,\vec{x}_1,\vec{x}_2)}\\ &= \int d^dx_1\phi^i(x_1)\frac{\partial}{\partial\sigma(t,\vec{x}_1,\vec{x})}+\int d^dx_2\phi^i(x_2)\frac{\partial}{\partial\sigma(t,\vec{x},\vec{x}_2)}\end{aligned}$$

Jacobian from Hamiltonian

The resulting Hamiltonian is not (manifestly) hermittian. This is because of the non-trivial measure.

Determine the Jacobian by requiring

$$H_{\text{eff}} = J^{\frac{1}{2}} H J^{\frac{-1}{2}} \tag{20}$$

obeys

$$H_{\text{eff}} - H_{\text{eff}}^{\dagger} = 0 \tag{21}$$

which in the end gives

$$\log J = (N - L^{d-1}\delta^{d-1}(0))\mathrm{Tr}\log\sigma \tag{22}$$

(Jevicki, Sakita)

Bilocal Holography

$$S_{\rm col}[\sigma] \;=\; S[\sigma] \;-\; \frac{N}{2}\,{\rm Tr}\log\sigma\,, \qquad (23)$$

$$\sigma(x_1,x_2) \;=\; \frac{1}{N}\sum_{i=1}^{N}\phi_i(x_1)\phi_i(x_2)\,. \qquad (24)$$

$$S[\sigma] \;=\; \frac{N}{2}\int d^dx\Big[-\Big(\nabla_x^2\sigma(x,x')\Big)_{x'=x}\Big] \qquad (25)$$

Saddle-point solution σ_0 represents the two-point function

$$\sigma_0(x_1,x_2) \;=\; \frac{1}{N}\sum_{i=1}^{N}\big\langle\phi_i(x_1)\phi_i(x_2)\big\rangle\,. \qquad (26)$$

Expanding around the saddle-point

$$\sigma(x_1,x_2) \;=\; \sigma_0(x_1,x_2) \;+\; |x_{12}|^2\sqrt{\frac{2}{N}}\eta(x_1,x_2)\,, \qquad (27)$$

Bilocal Holography

$$S_{(2)} = \frac{1}{2} \int \prod_{k=1}^{4} d^d x_k \; \eta(x_1, x_2) \, \widehat{\mathcal{L}}_{\rm bi} \, \eta(x_3, x_4) \, . \tag{28}$$

The bi-local Laplacian is given by

$$\widehat{\mathcal{L}}_{\rm bi} = |x_{12}|^2 |x_{34}|^2 \left[\Phi_0^{-1}(x_1, x_3) \Phi_0^{-1}(x_2, x_4) \right] . \tag{29}$$

The bilocal Laplacian can be expressed in terms of two Casimir operators of the conformal symmetry (with generators L_{AB} $(A, B = -1, 0, 1, \cdots, d)$)

$$C_2 \equiv \frac{1}{2} L_{AB} L^{AB} \qquad C_4 \equiv \frac{1}{2} \widetilde{C}_4 - \frac{1}{2} C_2^2 \qquad \widetilde{C}_4 = \frac{1}{2} L_A{}^B L_B{}^C L_C{}^D L_D{}^A$$

Bilocal Holography

Eigenvalues for a primary with conformal dimension h and spin s are

$$\begin{aligned} C_2(\Delta, s) &= -\Delta(\Delta - d) - s(s + d - 2)\,, \\ \widetilde{C}_4(\Delta, s) &= \Delta^2(\Delta - d)^2 + \tfrac{1}{2}d(d-1)\Delta(\Delta - d) \\ &\quad + s^2(s + d - 2)^2 + \tfrac{1}{2}(d-1)(d-4)s(s + d - 2)\,. \end{aligned} \tag{30}$$

Bi-local Laplacian (for free theory) is related to Casimirs as (dMK, Jevicki, Rodrigues, Yoon)

$$\Box_{\mathrm{bi}\,12} \equiv C_4 + \frac{1}{4}C_2^2 + \frac{d^2 - 3d + 4}{4}C_2 = \frac{1}{4}|x_{12}|^4\, \nabla_{x_1}^2 \nabla_{x_2}^2\,, \tag{31}$$

Bilocal Holography

$$\Box_{\text{bi}12}\,\psi_{c,s}(x_1,x_2) \;=\; \lambda_{c,s}\,\psi_{c,s}(x_1,x_2)\,. \tag{32}$$

where s is spin and $\Delta \equiv \frac{d}{2} + c$.

$$\psi_\varepsilon(x_1,c_0;x_2,c_0;x_3,c,s) \sim \frac{(Z_{12,3}\cdot\varepsilon)^s}{|x_{12}|^{2\Delta_0-\Delta}|x_{23}|^{\Delta}|x_{31}|^{\Delta}} \tag{33}$$

where ϵ is a null polarization vector and

$$Z^\mu_{12,3} \equiv \frac{|x_{13}||x_{23}|}{|x_{12}|}\left(\frac{x^\mu_{13}}{|x_{13}|^2} - \frac{x^\mu_{23}}{|x_{23}|^2}\right) \tag{34}$$

$\Delta_0 \equiv \frac{d}{2} + c_0 \equiv \frac{d-2}{2}$ is the conformal dimension of the d-dimensional free scalar field.

$$\lambda_{c,s} \;=\; \frac{1}{4}\left[\left(\Delta - \frac{d}{2}\right)^2 - \left(s + \frac{d}{2} - 2\right)^2\right]\left[\left(\Delta - \frac{d}{2}\right)^2 - \left(s + \frac{d}{2}\right)^2\right]. \tag{35}$$

Map

Bi-local reconstruction of AdS fields is a map between η and bulk higher spin fields H

$$\begin{aligned} \eta(x_1^i, x_2^i) &= \int_{\rm AdS} \mathcal{M}(x_1^i, x_2^i | x^i, z, S)\, H(x^i, z, S)\,, \\ H(x^i, z, S) &= \int_{\rm Bi-local} \mathcal{M}^{-1}(x^i, z, S | x_1^i, x_2^i)\, \eta(x_1^i, x_2^i)\,, \end{aligned} \tag{36}$$

z is radial coordinate of AdS and x^i are the remaining coordinates.

Basis of map is group theory and the action of the conformal group. On bi-local $\sigma(x_1, x_2)$ we have kinematic action of two copies of $SO(1, d+1)$

$$L_{AB} = L^1_{AB} + L^2_{AB} \tag{37}$$

and on the higher spin fields there is a nontrivial realization (spinning particle in AdS) of $SO(1, d+1)$

$$\mathcal{M} = \sum_{\Delta, s, p^\mu} \psi_{\rm AdS}({\rm AdS}_{d+1}, S_{d-1} | \Delta, s, p_\mu) \psi_{\rm bi}(x_1, x_2 | \Delta, s, p_\mu) \tag{38}$$

Momentum Map for $d = 2$: AdS

$$C_2 = 2(L^2 + \overline{L}^2), \qquad C_4 = -(L^2 - \overline{L}^2)^2$$

with $(p_\pm = -\frac{1}{2}(p^0 \mp p^1))$

$$L^2 = -\frac{1}{4}\left(z\sqrt{4p_+p_- - p_z^2} + p_\theta\right)^2, \qquad \overline{L}^2 = -\frac{1}{4}\left(z\sqrt{4p_+p_- - p_z^2} - p_\theta\right)^2$$

$$\Box^2_{\mathrm{AdS}_3\times S^1} = C_4 + \frac{1}{4}C_2^2 = 4L^2\overline{L}^2$$

Changing $z \to i\frac{\partial}{\partial p_z}$ and $p_\theta \to -i\frac{\partial}{\partial\theta}$, we have

$$\Box_{\mathrm{AdS}_3\times S^1} = \frac{1}{2}\left[\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\theta^2}\right] \qquad \phi \equiv -\arcsin\left(\frac{p_z}{\sqrt{4p_+p_-}}\right)$$

Eigenfunctions are

$$\psi_{\Delta,s} = e^{\pm(i\Delta\phi + is\theta)}$$

Lead to the eigenvalues $(-\Delta^2 + s^2)/2$ as expected.

Momentum Map for $d = 2$: Bilocal

$$L^2 = \left(\frac{\partial}{\partial k_{1+}} - \frac{\partial}{\partial k_{2+}}\right)^2 k_1^+ k_2^+ \,, \qquad \bar{L}^2 = \left(\frac{\partial}{\partial k_{1-}} - \frac{\partial}{\partial k_{2-}}\right)^2 k_{1-} k_{2-} \,, \tag{39}$$

where $k_{i\pm} \equiv -\frac{1}{2}(k_i^0 \mp k_i^1)$. The Laplacian $\Box_{\rm bi} = 4L^2\bar{L}^2$ becomes

$$\Box_{\rm bi} = 2\left(\frac{\partial}{\partial k_{1+}} - \frac{\partial}{\partial k_{2+}}\right)\left(\frac{\partial}{\partial k_{1-}} - \frac{\partial}{\partial k_{2-}}\right)\sqrt{k_{1+}k_{2+}k_{1-}k_{2-}} \,, \tag{40}$$

Fourier transformation of spinning three-point functions gives the bi-local wave functions

$$\psi(\vec{k}_1, \vec{k}_2) = \delta(\vec{k}_1 + \vec{k}_2 - \vec{p}) F(k_{\rm dif+}) F(k_{\rm dif-}) \,, \tag{41}$$

where $k_{\rm dif\pm} \equiv k_{1\pm} - k_{2\pm}$ and

$$F(k_{\rm dif}^+) F(k_{\rm dif}^-) = \frac{e^{i\frac{s}{2}\left(\arcsin\frac{k_{\rm dif+}}{p_+} - \arcsin\frac{k_{\rm dif-}}{p_-}\right)}\, e^{i\frac{\Delta}{2}\left(\arcsin\frac{k_{\rm dif+}}{p_+} + \arcsin\frac{k_{\rm dif-}}{p_-}\right)}}{\sqrt{k_{1+}k_{2+}k_{1-}k_{2-}}} \,. \tag{42}$$

Momentum Map for $d = 2$: Comparison

Comparing with the AdS and bi-local wave functions we conclude

$$\theta = \frac{1}{2}\arcsin\frac{k_{\text{dif}+}}{p_+} - \frac{1}{2}\arcsin\frac{k_{\text{dif}-}}{p_-}\,, \tag{43}$$

$$\phi = \frac{1}{2}\arcsin\frac{k_{\text{dif}+}}{p_+} + \frac{1}{2}\arcsin\frac{k_{\text{dif}-}}{p_-}\,. \tag{44}$$

In summary, we have the Lorentzian version of the momentum map from $(k_{1\pm}; k_{2\pm})$ to $(p_\pm, p^z, \theta)$

$$p_\pm = k_{1\pm} + k_{2\pm}\,, \tag{45}$$

$$p^z = -2\sqrt{k_{1+}}\sqrt{k_{1-}} + 2\sqrt{k_{2+}}\sqrt{k_{2-}}\,, \tag{46}$$

$$\theta = -\arctan\frac{\sqrt{k_{2+}}}{\sqrt{k_{1+}}} + \arctan\frac{\sqrt{k_{2-}}}{\sqrt{k_{1-}}}\,, \tag{47}$$

and its inverse map is given by

$$k_{1\pm} = \frac{p_\pm}{2}\Big[1+\sin(\phi\pm\theta)\Big] \quad , \qquad k_{2\pm} = \frac{p_\pm}{2}\Big[1-\sin(\phi\pm\theta)\Big]\,. \tag{48}$$

Tests of the map

The map passes a number of non-trivial tests:

1. Taking $z \to 0$ in the expression

$$H(x^i, z, S) = \int_{\mathrm{Bi-local}} \mathcal{M}^{-1}(x^i, z, S | x_1^i, x_2^i)\, \eta(x_1^i, x_2^i)\,, \tag{49}$$

   we recover the CFT expression for the higher spin primaries.
2. Equations governing small fluctuations are the Fronsdal equations.
3. 3 point correlators come out correctly. Loop expansion in collective field theory is the $1/N$ expansion of the original vector model.

Future Directions

One would like to demonstrate that the collective field theory is a theory of gravity. There are no gauge invariances left: the collective description is a fully gauge fixed description.

Does collective field theory reproduce the fully gauge fixed Vasiliev equations? (Hard to demonstrate.)

Investigate: locality of higher spin theory, cubic couplings, quartic couplings, critical vector model, ...

Gravity dual to SYK?

Future Directions

Collective field theory for one matrix model:

$$L = \frac{1}{2}\mathrm{Tr}\dot{M}^2 - \mathrm{Tr}V(M) \qquad (50)$$

Collective variables:

$$\phi_k(t) = \mathrm{Tr}(e^{ikM}) \qquad \phi(x,t) = \int_{-\infty}^{\infty} dk\, e^{ikx}\phi_k(t) \qquad (51)$$

The collective field theory is a *LOCAL* field theory in 1+1 dimensions. One can recover the gauge fixed string field theory of the non-critical $c = 1$ string theory. (Jevicki, Sakita; Das, Jevicki)

Future Directions

For multi-matrices we can't easily construct a collective field theory. The space of gauge invariants is enormous and its difficult to formulate the change of variables.

For one matrix models the space of gauge invariants is the space of eigenvalues. This simplicity is what makes progress possible.

What is a useful set of variables for the multi-matrix problem? It is on this very first step that we get stuck.

Thanks for your attention!