Number of microstates of Hawking Radiation and, Branching Ratio of a Black Hole

Pratik Ghosal, Rajarshi Ray

Bose Institute,CAPSS
Saltlake,Sector-V
Kolkata 700091

December 4, 2018

Hawking Radiation

• Stephen Hawking first studied the effects of Quantum Fields in Schwarzschild spacetime, and discovered that black hole (of mass M say) emits thermal radiations[1] with a temperature $T_{BH} = \frac{1}{8\pi M}$.

[1] S.W. Hawking, Comm. Math. Phys. 43(1975) 199

Hawking Radiation

• Stephen Hawking first studied the effects of Quantum Fields in Schwarzschild spacetime, and discovered that black hole (of mass M say) emits thermal radiations[1] with a temperature $T_{BH} = \frac{1}{8\pi M}$.
• The origin of these radiations were accounted for by considering vacuum fluctuations near the event horizon and tunneling of one of the virtual particles through the horizon, while letting the other particle to escape as radiation.

[1] S.W. Hawking, Comm. Math. Phys. 43(1975) 199

Hawking Radiation

• Stephen Hawking first studied the effects of Quantum Fields in Schwarzschild spacetime, and discovered that black hole (of mass M say) emits thermal radiations[1] with a temperature $T_{BH} = \frac{1}{8\pi M}$.
• The origin of these radiations were accounted for by considering vacuum fluctuations near the event horizon and tunneling of one of the virtual particles through the horizon, while letting the other particle to escape as radiation.
• However, the original calculations by Hawking and later by others, based on Bogoliubov transformation, has no direct connection with tunneling phenomenon.

[1] S.W. Hawking, Comm. Math. Phys. 43(1975) 199

Hawking Radiation as Tunneling

- The heuristic picture of Hawking radiation as Tunneling was first made clear by Wilczek and Parikh[2].

[2] M.K. Parikh, F. Wilczek, Phys. Rev. Lett. 85(2000) 5042

[3] R. Banerjee, B.R. Majhi, Phys. Lett. B 675(2009) 243-245

Hawking Radiation as Tunneling

• The heuristic picture of Hawking radiation as Tunneling was first made clear by Wilczek and Parikh[2].

• Wilczek and Parikh considered dynamical geometry due to **self-gravitation of the emitted particle**, enforced energy conservation, and obtained "**non thermal**" corrections to the radiation spectrum.

[2] M.K. Parikh, F. Wilczek, Phys. Rev. Lett. 85(2000) 5042

[3] R. Banerjee, B.R. Majhi, Phys. Lett. B 675(2009) 243-245

Hawking Radiation as Tunneling

• The heuristic picture of Hawking radiation as Tunneling was first made clear by Wilczek and Parikh[2].

• Wilczek and Parikh considered dynamical geometry due to **self-gravitation of the emitted particle**, enforced energy conservation, and obtained "**non thermal**" corrections to the radiation spectrum.

• Strictly thermal spectrum of Hawking radiation with appropriate Hawking temperature is also obtained in the tunneling fromalism[3].

[2] M.K. Parikh, F. Wilczek, Phys. Rev. Lett. 85(2000) 5042

[3] R. Banerjee, B.R. Majhi, Phys. Lett. B 675(2009) 243-245

Black hole "information loss" paradox

• Strictly thermal radiation spectrum, as obtained by Hawking, possessed a disturbing problem. It implies that information is lost in the black hole evaporation process[4].

• In other words, Hawking radiation process apprears to be non-unitary, in contrast to Quantum mechanics.

[4] S.W. Hawking, *"Breakdown of Predictability in Gravitational Collapse"*, Phys. Rev. D 14(1976) 2460

[5] W.H Zurek, Phys. Rev. Lett. 49(1982) 1683-1686

Black hole "information loss" paradox

• Strictly thermal radiation spectrum, as obtained by Hawking, possessed a disturbing problem. It implies that information is lost in the black hole evaporation process[4].

• In other words, Hawking radiation process apprears to be non-unitary, in contrast to Quantum mechanics.

• As a result of non-unitary evolution, the entropy of the final system of radiations (after complete evaporation of black hole) is greater than initial Black hole entropy[5].

$$S_{rad} = \frac{4}{3} S_{BH} \tag{1}$$

[4] S.W. Hawking, *"Breakdown of Predictability in Gravitational Collapse"*, Phys. Rev. D 14(1976) 2460

[5] W.H Zurek, Phys. Rev. Lett. 49(1982) 1683-1686

Non-thermal Hawking radiation "preserves" unitarity

- As Parikh[6] pointed out, the non-thermal radiation spectrum is consistent with unitary evolution.

[6] M.K. Parikh, "Energy Conservation and Hawking Radiation", arXiv:hep-th/0402166

Non-thermal Hawking radiation "preserves" unitarity

- As Parikh[6] pointed out, the non-thermal radiation spectrum is consistent with unitary evolution.
- The tunneling rate for non-thermal radiation is given by,

$$\Gamma \sim exp(-8\pi E(M - \frac{E}{2})) \quad (2)$$
$$= exp(\Delta S_{BH}) \quad (3)$$

[6] M.K. Parikh, "Energy Conservation and Hawking Radiation", arXiv:hep-th/0402166

Non-thermal Hawking radiation "preserves" unitarity

• As Parikh[6] pointed out, the non-thermal radiation spectrum is consistent with unitary evolution.

• The tunneling rate for non-thermal radiation is given by,

$$\Gamma \sim exp(-8\pi E(M - \frac{E}{2})) \quad (2)$$

$$= exp(\Delta S_{BH}) \quad (3)$$

• In Quantum Mechanics, the rate of a unitary process is $\Gamma(i \rightarrow f) = |M_{fi}|^2.(phase\ space\ factor)$
where, M_{fi} is the amplitudefor the process

[6] M.K. Parikh, "Energy Conservation and Hawking Radiation", arXiv:hep-th/0402166

Non-thermal Hawking radiation "preserves" unitarity

- The phase space factor is obtained by summing over all possible final states (exponent of the final entropy) and averaging over all possible initial states (exponent of the initial entropy).

[7] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Non-thermal Hawking radiation "preserves" unitarity

- The phase space factor is obtained by summing over all possible final states (exponent of the final entropy) and averaging over all possible initial states (exponent of the initial entropy).
- Therefore, the rate is

$$\Gamma \sim \frac{e^{S_{final}}}{e^{S_{initial}}} = exp(\Delta S_{BH})$$

[7]Zhang et. al., Phys. Lett. B 675(2009), 98-101

Non-thermal Hawking radiation "preserves" unitarity

- The phase space factor is obtained by summing over all possible final states (exponent of the final entropy) and averaging over all possible initial states (exponent of the initial entropy).
- Therefore, the rate is

$$\Gamma \sim \frac{e^{S_{final}}}{e^{S_{initial}}} = exp(\Delta S_{BH})$$

- These non-thermal radiations carries hidden correlations shared between them and the entropy of the final system of radiations (after complete evaporation of black hole) is euqual to the initial Black hole entropy[7].

[7] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Counting "number of microstates" of the complete Hawking radiation

- In the paper titled "*Hidden Messenger Revealed in Hawking Radiation: a Resolution to the Paradox of Black Hole Information Loss*"[8], Zhang et. al. considered the following scenario.

[8] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Counting "number of microstates" of the complete Hawking radiation

- In the paper titled "*Hidden Messenger Revealed in Hawking Radiation: a Resolution to the Paradox of Black Hole Information Loss*"[8], Zhang et. al. considered the following scenario.
- A black hole of mass M is completely evaporated by emission of 'n' particles (we consider $n = 2$ for the following example).

[8] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Counting "number of microstates" of the complete Hawking radiation

- In the paper titled "*Hidden Messenger Revealed in Hawking Radiation: a Resolution to the Paradox of Black Hole Information Loss*"[8], Zhang et. al. considered the following scenario.
- A black hole of mass M is completely evaporated by emission of 'n' particles (we consider $n = 2$ for the following example).
- The first particle emitted carries an energy E_1 with probability

$$\Gamma(E_1) = e^{-8\pi E_1 (M - \frac{E_1}{2})}$$

[8] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Counting "number of microstates" of the complete Hawking radiation

- After that a second particle with energy E_2 is emitted from a BH of mass $M - E_1$, with probabilty

$$\Gamma(E_2|E_1) = e^{-8\pi E_2(M - E_1 - \frac{E_2}{2})}$$

and completely evaporates the BH $(E_1 + E_2 = M)$.

Counting "number of microstates" of the complete Hawking radiation

- After that a second particle with energy E_2 is emitted from a BH of mass $M - E_1$, with probabilty

$$\Gamma(E_2|E_1) = e^{-8\pi E_2(M - E_1 - \frac{E_2}{2})}$$

and completely evaporates the BH ($E_1 + E_2 = M$).

- Total probability for the occurence of these event is

$$\begin{aligned} P(E_1, E_2) &= \Gamma(E_1)\Gamma(E_2|E_1) \\ &= e^{-4\pi M^2} \end{aligned} \quad (4)$$

Counting "number of microstates" of the complete Hawking radiation

- After that a second particle with energy E_2 is emitted from a BH of mass $M - E_1$, with probabilty

$$\Gamma(E_2|E_1) = e^{-8\pi E_2(M - E_1 - \frac{E_2}{2})}$$

and completely evaporates the BH $(E_1 + E_2 = M)$.

- Total probability for the occurence of these event is

$$\begin{aligned} P(E_1, E_2) &= \Gamma(E_1)\Gamma(E_2|E_1) \\ &= e^{-4\pi M^2} \end{aligned} \qquad (4)$$

- Now, if a particle of energy E_2 is emitted first, then followed by a particle of energy E_1, then total probabilty for the event would be

$$P(E_2, E_1) = \Gamma(E_2)\Gamma(E_1|E_2) = e^{-4\pi M^2}$$

Counting "number of microstates" of the complete Hawking radiation

- This is true in general for any number of particles, irresepective of their sequence of emission.

i.e,

$$\begin{aligned} P(E_1, ..., E_n) &= \Gamma(E_1)\Gamma(E_2|E_1)\Gamma(E_3|E_1, E_2)...\Gamma(E_n|E_1, ...E_{n-1}) \\ &= e^{-4\pi M^2} \end{aligned} \quad (5)$$

Counting "number of microstates" of the complete Hawking radiation

• This is true in general for any number of particles, irresepective of their sequence of emission.
i.e,

$$P(E_1,...,E_n) = \Gamma(E_1)\Gamma(E_2|E_1)\Gamma(E_3|E_1,E_2)...\Gamma(E_n|E_1,...E_{n-1}) = e^{-4\pi M^2} \quad (5)$$

• Thus we have a system of n particles, with total energy $\Sigma_{i=1}^{n} E_i = M$. This system can exist in several microstates, as denoted by the sequence of emission.
$(E_1, E_2, E_3, ..., E_n)$
$(E_2, E_1, E_3, ..., E_n)$
$(E_7, E_n, E_3, ..., E_2)$ etc.

Counting "number of microstates" of the complete Hawking radiation

- All these microstates are equally probable with probability $P = e^{-4\pi M^2}$. Since, all microstates are equally probable, number of microstates is given by,

$$\Omega = \frac{1}{P} = e^{4\pi M^2} \tag{6}$$

and the entropy of the system is given by,

$$S_{rad} = ln\Omega = 4\pi M^2 = S_{BH} \tag{7}$$

But..

• According to the calculations by Zhang et. al., number of microstates, when the black hole is exhausted by
1 particle
or, n particles
or, n' particles is

$$\Omega = e^{4\pi M^2}$$

• However, in the case of single particle decay, number of microstates which is given by permutations of sequence of emission, can be nothing but 1!

But..

- Now, we ask the question, what would be the number of microstates when the number of particles exhausting the black hole is unknown? i.e, We want to consider all possible cases where, number of particles may vary from 1 to infinty.

But..

- Now, we ask the question, what would be the number of microstates when the number of particles exhausting the black hole is unknown? i.e, We want to consider all possible cases where, number of particles may vary from 1 to infinty.
- Then, total number of microstates would be the sum of microstates for all possible number of particles (as, they are mutually exclusive microstates).

$$\begin{aligned}\Omega_{total} &= \Sigma_{n=1}^{\infty}\Omega_n \\ &= e^{4\pi M^2} + e^{4\pi M^2} + \ldots \\ &= \infty\end{aligned}$$

How to correct these fallacies?

• We note that each microstates for 1 particle decay, n particle decay, or n' particle decay occurs with the same probability.

$$P(E_1 = M) = e^{-4\pi M^2}$$
$$P(E_1, ..., E_n) = \Gamma(E_1)...\Gamma(E_n|E_1..E_{n-1}) = e^{-4\pi M^2}$$
$$P(E_1, ..., E_{n'}) = \Gamma(E_1)...\Gamma(E_{n'}|E_1..E_{n'-1}) = e^{-4\pi M^2}$$

How to correct these fallacies?

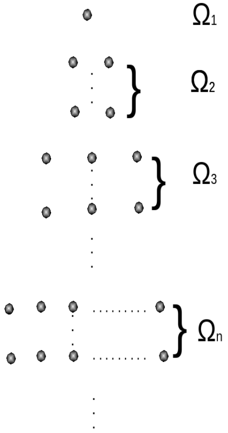
• We note that each microstates for 1 particle decay, n particle decay, or n' particle decay occurs with the same probability.

$$P(E_1 = M) = e^{-4\pi M^2}$$
$$P(E_1, ..., E_n) = \Gamma(E_1)...\Gamma(E_n|E_1..E_{n-1}) = e^{-4\pi M^2}$$
$$P(E_1, ..., E_{n'}) = \Gamma(E_1)...\Gamma(E_{n'}|E_1..E_{n'-1}) = e^{-4\pi M^2}$$

• We consider the number of microstates for different number of particles (Ω_n for n particles) to be different. Such that,

$$\Omega_{total} = \Sigma_{n=1}^{\infty}\Omega_n \neq \infty$$

Calculating number of microstates correctly



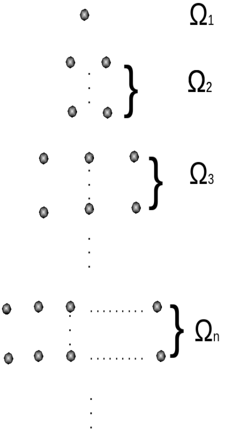
- Therfore, the total number of microstates is given by,

$$\Omega_{total} = \frac{1}{P} = e^{4\pi M^2}$$

- Now we have,

$$\begin{aligned} \Sigma_{n=1}^{\infty}\Omega_n &= e^{4\pi M^2} \quad (8) \\ &= \Sigma_{n=1}^{\infty}\frac{(4\pi M^2)^{n-1}}{(n-1)!} \end{aligned}$$

Calculating number of microstates correctly



- We assume: $\Omega_n = \frac{(4\pi M^2)^{n-1}}{(n-1)!}$

- Therfore, the total number of microstates is given by,

$$\Omega_{total} = \frac{1}{P} = e^{4\pi M^2}$$

- Now we have,

$$\begin{aligned} \Sigma_{n=1}^{\infty} \Omega_n &= e^{4\pi M^2} \qquad (8) \\ &= \Sigma_{n=1}^{\infty} \frac{(4\pi M^2)^{n-1}}{(n-1)!} \end{aligned}$$

How to calculate

• So, we have a system of n particles with total energy M. Microcanonical ensemble.

How to calculate

- So, we have a system of n particles with total energy M. Microcanonical ensemble.
- But we have no volume defined for the system.
- We cannot use the phase-space description.

Defining microstates

• An emitted particle carries with it not only energy, but also entropy from the black hole. The entropy carried by a single particle is defined as[9]

$$S = -ln\Gamma$$

where, Γ is the emission probability of the particle.

[9] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Defining microstates

• An emitted particle carries with it not only energy, but also entropy from the black hole. The entropy carried by a single particle is defined as[9]

$$S = -ln\Gamma$$

where, Γ is the emission probability of the particle.

• Case I: A particle with energy E_1 is emitted first, then a second particle with energy E_2 is emitted, completely evaporating the BH. Then,

$$S_1 = -ln\Gamma(E_1) = 8\pi E_1(M - E_1/2)$$

$$S_2 = -ln\Gamma(E_2|E_1) = 8\pi E_2(M - E_1 - E_2/2) = 4\pi(M - E_1)^2$$

[9] Zhang et. al., Phys. Lett. B 675(2009), 98-101

Defining microstates

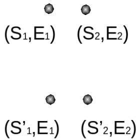
- Case II: A particle with energy E_2 is emitted first, then a second particle with energy E_1 is emitted, completely evaporating the BH. Then,

$$S_1' = -ln\Gamma(E_2) = 8\pi E_2(M - E_2/2) = 4\pi(M^2 - E_1^2)$$
$$S_2' = -ln\Gamma(E_1|E_2) = 8\pi E_1(M - E_2 - E_1/2) = 4\pi E_1^2$$

Defining microstates

- Case II: A particle with energy E_2 is emitted first, then a second particle with energy E_1 is emitted, completely evaporating the BH. Then,

$$S_1' = -ln\Gamma(E_2) = 8\pi E_2(M - E_2/2) = 4\pi(M^2 - E_1^2)$$
$$S_2' = -ln\Gamma(E_1|E_2) = 8\pi E_1(M - E_2 - E_1/2) = 4\pi E_1^2$$



- So, we can label the microstates based on the entropy contents of the constuting particles.

Defining microstates

- Since the total entropy of the system is S_{BH}, our *definition for microstates* are: different combinations of single particle entropies such that the total entropy is fixed.

For a system of n particles,

$$\Sigma_{i=1}^{n} S_i = S_{BH} \tag{9}$$

Where, S_i is the entropy of the i-th particle.

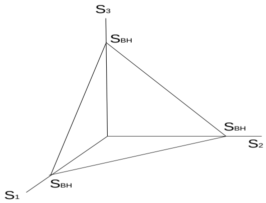
Defining microstates

• Since the total entropy of the system is S_{BH}, our *definition for microstates* are: different combinations of single particle entropies such that the total entropy is fixed.
For a system of n particles,

$$\Sigma_{i=1}^{n} S_i = S_{BH} \tag{9}$$

Where, S_i is the entropy of the i-th particle.

The above equation is the equation of an $(n-1)$ dimensional plane $\mathcal{S}$ in an n dimensional "entropy space", and all the microstates are on this plane.



Counting microstates

- If we allow the total entropy of the system to vary between S_{BH} and $S_{BH} + \Delta$, then we get a region ($\mathcal{R}$) bounded by planes, $\Sigma_{i=1}^{n} S_i = S_{BH}$ and $\Sigma_{i=1}^{n} S_i = S_{BH} + \Delta$, where Δ is the amount of fluctuation.

Counting microstates

• If we allow the total entropy of the system to vary between S_{BH} and $S_{BH} + \Delta$, then we get a region ($\mathcal{R}$) bounded by planes, $\Sigma_{i=1}^{n} S_i = S_{BH}$ and $\Sigma_{i=1}^{n} S_i = S_{BH} + \Delta$, where Δ is the amount of fluctuation.

• Now,

no. of microstates = Surface area of the plane $\mathcal{S}$ = Volume of the region $\mathcal{R}$ / Δ

Area of $\mathcal{S}$ is

$$\mathcal{A}_{n-1} = V_n / \Delta$$

Counting microstates

- The volume of the region $\mathcal{R}$ is given by

$$V_n = \frac{S_{BH}^{(n-1)}}{(n-1)!}\Delta$$

Counting microstates

- The volume of the region $\mathcal{R}$ is given by

$$V_n = \frac{S_{BH}^{(n-1)}}{(n-1)!}\Delta$$

- Therefore, number of microstates for n particle decay is:

$$\Omega_n = \frac{S_{BH}^{(n-1)}}{(n-1)!} \tag{10}$$

Counting microstates

- The volume of the region $\mathcal{R}$ is given by

$$V_n = \frac{S_{BH}^{(n-1)}}{(n-1)!}\Delta$$

- Therefore, number of microstates for n particle decay is:

$$\Omega_n = \frac{S_{BH}^{(n-1)}}{(n-1)!} \tag{10}$$

- Using this expression we get back the previous expression for total number of microstates:

$$\Omega_{total} = \Sigma_{n=1}^{\infty} \frac{S_{BH}^{(n-1)}}{(n-1)!} = e^{S_{BH}}$$

Branching Ratio

- We see each of the Ω_n microstates of a particular n-particle decay mode occurs with probability $P = e^{-S_{BH}}$.

Branching Ratio

- We see each of the Ω_n microstates of a particular n-particle decay mode occurs with probability $P = e^{-S_{BH}}$.
- Therefore, total probability that the black hole decays into n particles is given by

$$p_n = P\Omega_n = \frac{S_{BH}^{(n-1)}}{(n-1)!} e^{-S_{BH}} \qquad (11)$$

This gives the branching ratio of the black hole after complete evaporation.

Summary

We have taken a revisit to the problem of *hidden correlations between Hawking radiations*, and found some fallacies regarding the calcuation of microstates. We have tried to fix those fallacies, and have found an expression for the branching ratios of a BH on complete evaporation.


Thank you