The Sorkin-Johnston State: Coupling to Gravity and Casimir Energy

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Current Developments in Quantum Field Theory and Gravity

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Table of contents

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Motivation

- Causal set approach to quantum gravity
- Entanglement entropy in QFT
- Cosmology

Scalar field in a curved spacetime background

- (M, g): Globally hyperbolic spacetime.

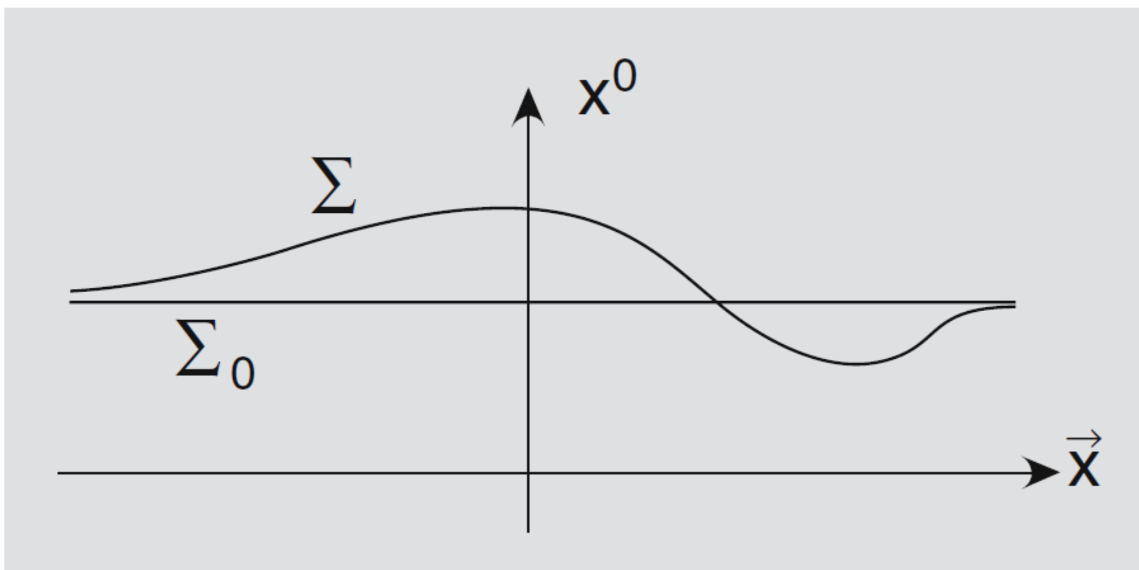
- Klein-Gordon equation: $(\nabla_a \nabla^a + m^2)\varphi = 0$

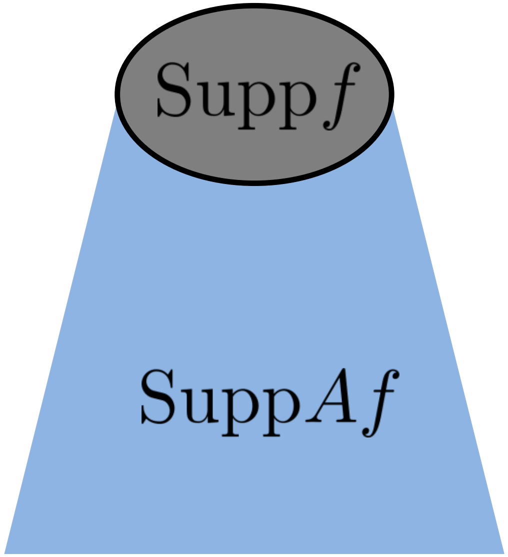
- $\mathcal{S} = \{\varphi \in C^\infty(M, \mathbb{R}) : \ (\nabla_a \nabla^a + m^2)\varphi = 0, \ \varphi|_{\Sigma_0} \in C_0^\infty(\Sigma_0)\}$

- Breakdown of Stone-von Neumann theorem.

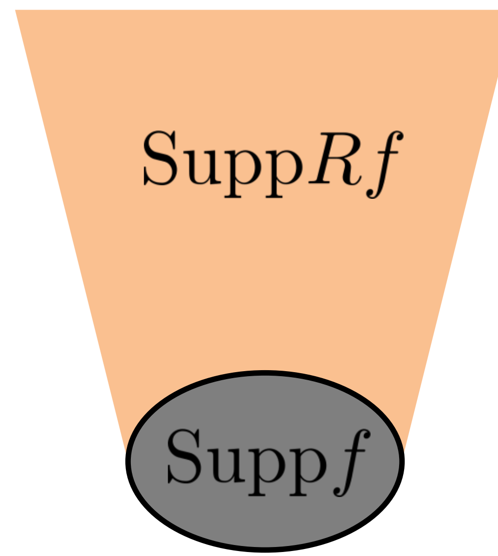
Symplectic structure

$$\sigma(\varphi_1, \varphi_2) := \int_{\Sigma_t} (\varphi_1 \nabla_\mu \varphi_2 - \varphi_2 \nabla_\mu \varphi_1) n_\mu \sqrt{-h}\, d^3x$$





$$(\Box + m^2)Af = f$$



$$(\Box + m^2)Rf = f$$

$$Ef := Af - Rf$$

$$\Delta(x, y) = G_A(x, y) - G_R(x, y)$$

Properties of the map $E : C_0^\infty(M) \to \mathcal{S}$

(i) It is surjective: $\forall \varphi \in \mathcal{S} \quad \exists f_\varphi \ \text{ t.q. } \ \varphi = E f_\varphi$

(ii) "Huge" kernel: $C_0^\infty(M)/\mathrm{Ker}E \cong \mathcal{S}$

Quantization

- CCR: $$[\Phi(f), \Phi(g)] = i\sigma(Ef, Eg)$$
- Representation:
  - Complexification: $\mathcal{S} \to \mathcal{S}_{\mathbb{C}}$
  - Sesquilinear form: $(\varphi_1, \varphi_2)_{KG} := i\sigma(\bar{\varphi}_1, \varphi_2)$
  - Choice of $\mathcal{H} \leq \mathcal{S}_{\mathbb{C}}$ such that:
    - $S_{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$
    - $(\mathcal{H}, (\ ,\ )_{KG})$ becomes a Hilbert space
  - Representation of CCR algebra on $\mathcal{F}_S(\mathcal{H})$

Sorkin-Johnston vacuum

- Properties of the integral kernel *E:*
  - Antisymmetric
  - "Self-adjoint": $\overline{i\Delta(y,x)} = i\Delta(x,y)$
- Consequence:

  iE is (formally) self-adjoint on $L^2(M)$
- **<u>Definition.</u>**

$$\boxed{\omega_{SJ}(\Phi(f)\Phi(g)) := \langle \bar{f}, (iE)^+ g\rangle_{L^2(M)}}$$

Concretely:

$$iEf(x) = \int_M i\Delta(x,y)f(y)dV_y$$

$$\int_M i\Delta(x,y)T_k(y)dV_y = \lambda_k T_k(x)$$

$$\begin{aligned} W(x,y) \quad &:= \quad \langle SJ|\hat{\phi}(x)\hat{\phi}(y)|SJ\rangle \\ &= \quad \sum_{k=1}^{\infty} \lambda_k T_k^+(x) T_k^+(y)^* \quad (\lambda_k > 0) \end{aligned}$$

Properties

- Conmutador: $i\Delta(x,y) = W(x,y) - W^*(x,y)$
- Positividad: $\int_{\mathcal{M}} dV_x \int_{\mathcal{M}} dV_y f^*(x) W(x,y) f(y) \geq 0$
- Soportes ortogonales: $\int_{\mathcal{M}} dV_y W(x,y) W^*(y,z) = 0$

• The first two conditions have to be satisfied by the 2-point function of *any* state.

• The third condition singles out the SJ state.

• It can be interpreted as the requirement that the Wightman function be the ``positive frequency part” of the Pauli-Jordan function, regarded as an integral operator.

A simple example:
SJ state for HO on the interval [-T, T]

"Field equation":

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

Pauli-Jordan function:

$$i\Delta(t, t') = \frac{1}{2\omega}\left(e^{-i\omega(t-t')} - e^{i\omega(t-t')}\right)$$

Eigenvalue problem:

$$\int_{-T}^{T} i\Delta(t,t')\Psi(t')dt' = \lambda\Psi(t)$$

Ansatz:

$$\Psi(t) = Ae^{-i\omega t} + Be^{i\omega t}$$

Eigenvalue equation:

$$\begin{pmatrix} 2T & \frac{\sin(2\omega T)}{\omega} \\ -\frac{\sin(2\omega T)}{\omega} & -2T \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 2\omega\lambda \begin{pmatrix} A \\ B \end{pmatrix}$$

Solution:

$$\lambda_\pm = \pm\frac{T}{\omega}\sqrt{1 - \frac{\sin^2(2\omega T)}{4\omega^2 T^2}},$$

$$(A, B) = (2\omega(\omega\lambda_\pm + T), -\sin(2\omega T)).$$

Normalization (with respect to $L^2([-T,T])$ norm!):

$$\|\Psi\|^2 = 2T(A^2 + B^2) + 2AB\frac{\sin(2\omega T)}{\omega}$$

2-point function:

$$\langle q(t)q(t')\rangle_{SJ} = \lambda_+ \frac{\Psi_+(t)\Psi_+^*(t')}{\|\Psi_+\|^2}$$

$$= \lambda_+ \frac{(Ae^{-i\omega t} + Be^{i\omega t})(Ae^{i\omega t'} + Be^{-i\omega t'})}{2T(A^2 + B^2) + 2AB\frac{\sin(2\omega T)}{\omega}}$$

$$= \frac{\left(\frac{\lambda_+}{T}\right)\left(\frac{A}{T}e^{-i\omega t} + \frac{B}{T}e^{i\omega t}\right)\left(\frac{A}{T}e^{i\omega t'} + \frac{B}{T}e^{-i\omega t'}\right)}{2\left(\left(\frac{A}{T}\right)^2 + \left(\frac{B}{T}\right)^2\right) + 2\frac{A}{T}\frac{B}{T}\frac{\sin(2\omega T)}{\omega T}}$$

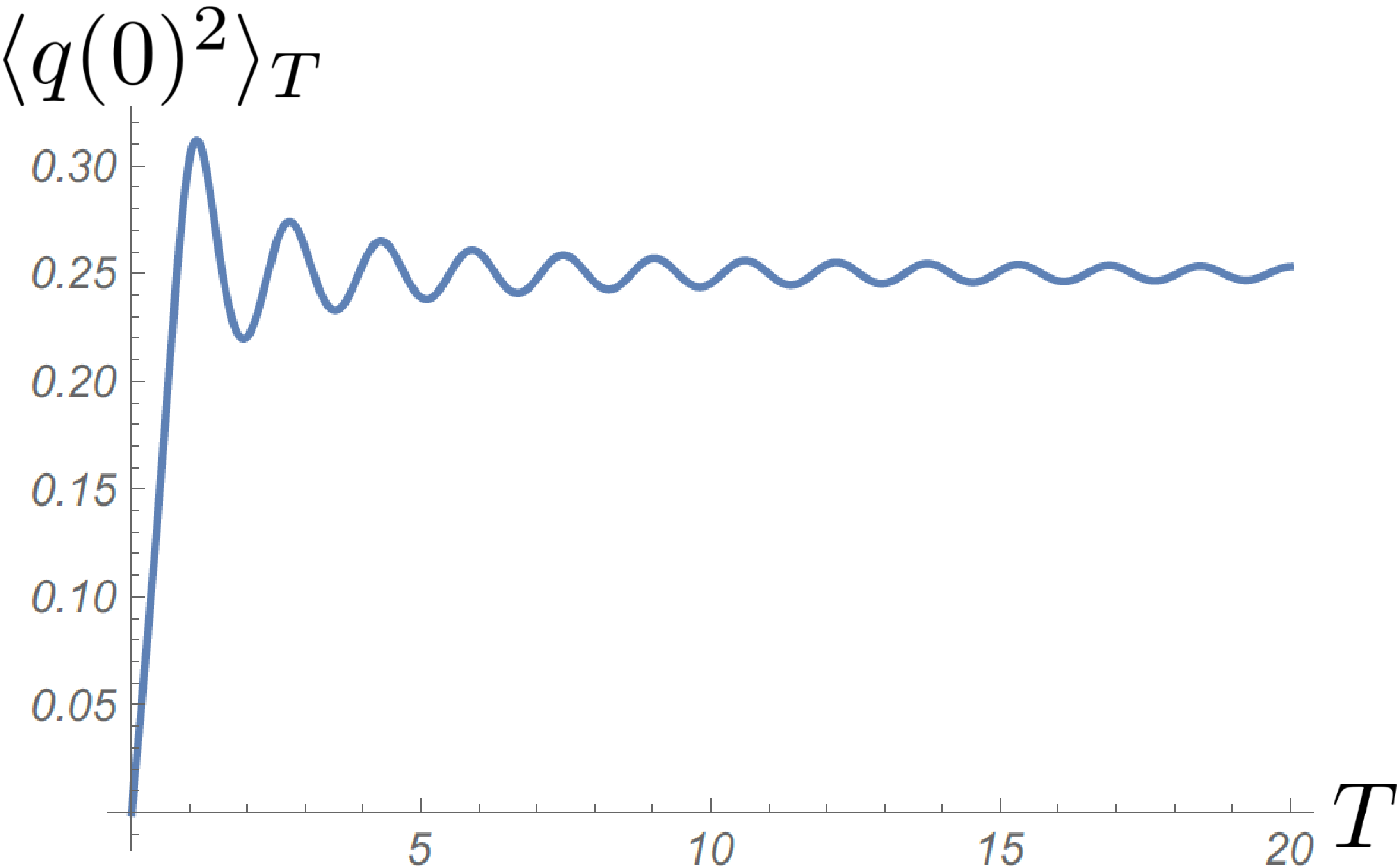
$$\frac{\lambda_+}{T} = \frac{1}{\omega}\sqrt{1 - \frac{\sin^2(2\omega T)}{4\omega^2 T^2}} \xrightarrow[(T \to \infty)]{} \frac{1}{\omega},$$

$$\frac{A}{T} = 2\omega\left(1 + \frac{\lambda_+}{T}\omega\right) \xrightarrow[(T \to \infty)]{} 4\omega\,,$$

$$\frac{B}{T} = -\frac{\sin(2\omega T)}{T} \xrightarrow[(T \to \infty)]{} 0$$

Limiting case:

$$\lim_{T\to\infty} \langle q(t)q(t')\rangle_{SJ} = \frac{e^{-i\omega(t-t')}}{2\omega}$$



$\langle q(0)^2 \rangle_T$
0.30
0.25
0.20
0.15
0.10
0.05
5
10
15
20
T

SJ state in a causal diamond (1+1)

$$u = \frac{1}{\sqrt{2}}(t+x), \quad u = \frac{1}{\sqrt{2}}(t-x)$$

$$\mathcal{M} : \{-\ell \leq u \leq \ell, -\ell \leq v \leq \ell\}$$

$$ds^2 = -2du\,dv, \qquad \partial_u \partial_v \phi(u,v) = 0.$$

Eigenfunctions with positive eigenvalues:

$$\begin{aligned} f_k(u,v) &:= e^{-iku} - e^{-ikv}, \quad k = \frac{n\pi}{\ell}, n = 1, 2, \ldots \\ g_k(u,v) &:= e^{-iku} + e^{-ikv} - 2\cos(k\ell), \\ k_n \in \mathcal{K} &= \{k \in \mathbb{R} | \tan(k\ell) = 2k\ell, k > 0\} \end{aligned}$$

Wightman function

$$W_{SJ}(u,v;u'v') = \frac{1}{4\pi}\Big\{-\log\left[1-e^{-\frac{i\pi(u-u')}{2\ell}}\right] - \log\left[1-e^{-\frac{i\pi(v-v')}{2\ell}}\right] + \log\left[1+e^{-\frac{i\pi(u-v')}{2\ell}}\right] + \log\left[1+e^{-\frac{i\pi(v-u')}{2\ell}}\right]\Big\} + \epsilon(u,v;u',v').$$

"correction" term

Coupling to gravity

- We are interested in the dynamics of a mssles field in two dimensions.
- Stress-energy tensor renormalization. We find a significative contribution from the “correction” term (ϵ).
- Backreaction.

Stress-energy tensor

$$T_{ab}^{\mathrm{ren}}(x) = T_{ab}(x) - T_{ab}^{0}(x)$$

$$T_{ab}^{0} = \langle T_{ab}(x) \rangle_{\Omega} = \lim_{x' \to x} \mathcal{D}_{ab}(x, x') G^{(1)}(x, x');$$

$$\mathcal{D}_{ab}(x, x') = \tfrac{1}{2} \left[\nabla_a \nabla'_b + \nabla'_a \nabla_b \right]$$

$$\frac{\partial}{\partial \ell} T_{ab}^{0}(x) = \lim_{x' \to x} \frac{\partial}{\partial \ell} \mathcal{D}_{ab}(x, x') G^{(1)}(x, x')$$

$$T_{ab}^{0} = T_{ab}^{\mathrm{box}} + T_{ab}^{\epsilon}$$

$$\langle T_{ab}(t,x)\rangle = -\frac{(1-\sigma)\pi}{96\ell^2}(\eta_{ab} + 2u_a u_b)$$
$$-\left(\frac{\pi}{32\ell^2\cos^2\left(\frac{\pi x}{2\sqrt{2}\ell}\right)} + \frac{x^2}{4\pi\ell^4}\log\tan^2\left(\frac{\pi x}{2\sqrt{2}\ell}\right)\right)\eta_{ab}.$$

Expectation value of T (w.r.t SJ state) diverges, for finite ℓ, at the positions $x = \pm\sqrt{2}\ell$

→ Consider coupling to gravity

Coupling to gravity

- Introduce a metric according to the following ansatz:

$$ds^2 = \exp(2\varphi)(-dt^2 + dx^2)$$

- Impose $\nabla^a T_{ab} = 0$
- Result:

$$\exp(2\varphi) = \frac{\pi}{32\ell^2 \cos^2\left(\frac{\pi x}{2\sqrt{2}\ell}\right)} + \frac{x^2}{4\pi\ell^4} \log\tan^2\left(\frac{\pi x}{2\sqrt{2}\ell}\right)$$

- Curvature: $R = -2e^{-2\varphi}\partial_x^2\varphi$

- Asymptotic behavior:

$$R = -\frac{8\pi}{\ell^2} + \frac{12\pi}{\ell^4}(x \pm \sqrt{2}\ell)^2 + \ldots$$

- Trace anomaly: $\langle T^a{}_a \rangle = \frac{c}{24\pi} R$

Casimir effect (on a cylinder)

$$\mathcal{W}(x,x') := \langle 0|\varphi(x)\ \ \varphi(y)|0\rangle = \frac{1}{2\pi}\int \frac{dp}{2E_p} e^{ip(x-x')},$$

$$\mathcal{W}_L(x,x') := \langle 0_L|\varphi_L(x)\ \ \varphi_L(y)|0_L\rangle = \frac{1}{L}\sum_{n\in\mathbb{Z}} \frac{e^{ik_n(x-x')}}{2E_n},$$

$$G^{(1)}(x,y) := \langle 0|\{\varphi(x),\varphi(y)\}|0\rangle = \frac{1}{\pi}K_0(\mu|x-y|),$$

$$G^{(1)}_L(x,y) := \langle 0_L|\{\varphi_L(x),\varphi_L(y)\}|0_L\rangle = \frac{1}{L}\sum_{n\in\mathbb{Z}} \frac{\cos(k_n(x-y))}{\sqrt{\mu^2+k_n^2}}.$$

$$\langle 0_L| :\varphi_L(x)^2: |0_L\rangle \;:=\; \lim_{x'\to x} \left(\mathcal{W}_L(x,x') - \mathcal{W}(x,x')\right)$$

$$\varphi(x)\varphi(y) = \frac{1}{2}\left(\varphi(x)\varphi(y) + \varphi(y)\varphi(x)\right) + \frac{1}{2}\left(\varphi(x)\varphi(y) - \varphi(y)\varphi(x)\right)$$

$$[\varphi(x), \varphi(y)] = i\Delta(x,y)$$

$$\langle 0_L| :\varphi_L(x)^2: |0_L\rangle \;=\; \lim_{x'\to x} \left(\frac{1}{2}G_L^{(1)}(x,x') - \frac{1}{2}G^{(1)}(x,x')\right).$$

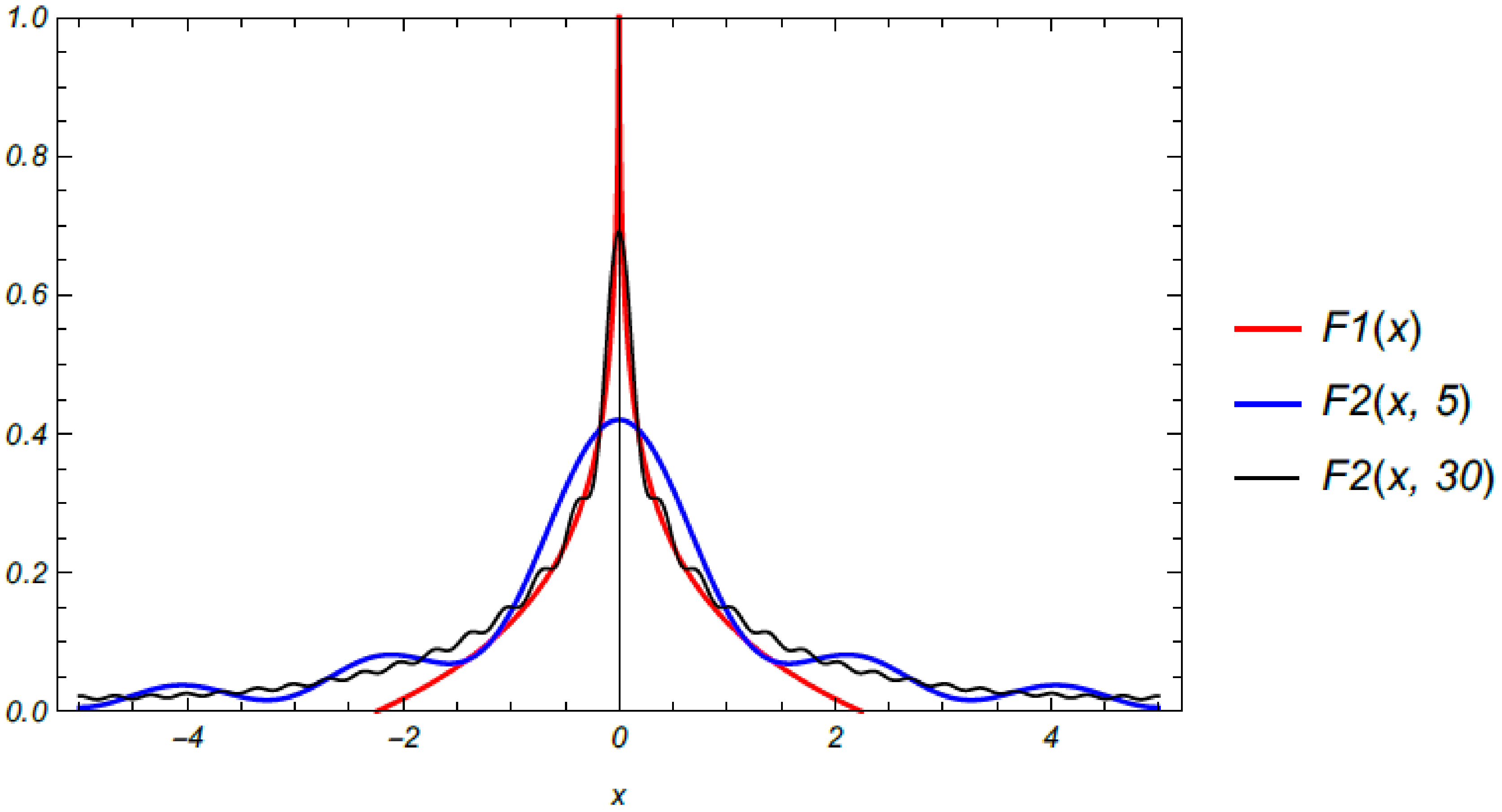
$$G^{(1)}(x^1, y^1) := \langle 0 | \{\varphi(x), \varphi(y)\} | 0 \rangle \Big|_{x^0 = y^0} = \frac{1}{\pi} K_0(\mu |x^1 - y^1|),$$

$$G_L^{(1)}(x^1, y^1) := \langle 0_L | \{\varphi_L(x), \varphi_L(y)\} | 0_L \rangle \Big|_{x^0 = y^0} = \frac{1}{L} \sum_{n \in \mathbb{Z}} \frac{\cos(k_n(x^1 - y^1))}{\sqrt{\mu^2 + k_n^2}}$$

$$K_0(x) = -\ln x - \gamma_E + \ln 2 + o(x^2).$$

$$F_1(x) := \frac{1}{2\pi} \ln(2/\mu) - \frac{\gamma_E}{2\pi} + \frac{1}{2\pi} \ln \frac{1}{|x|},$$

$$F_2(x, N) := \frac{1}{L} \sum_{n=1}^{N} \frac{\cos k_n x}{\sqrt{\mu^2 + k_n^2}} + \frac{1}{2\mu L}.$$

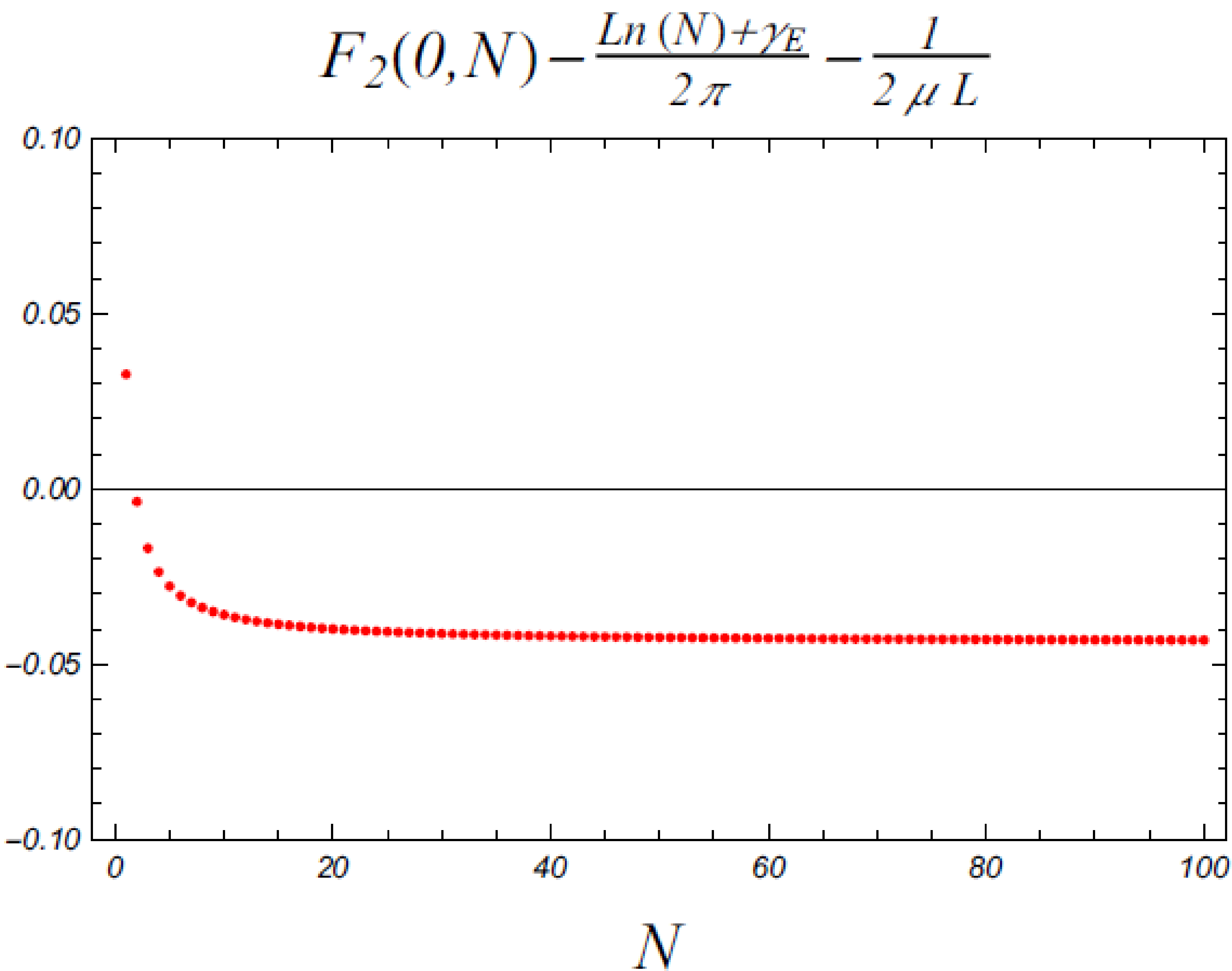


It is clear that both F1 and F2 have the same UV behavior. This just reflects the fact that the corresponding states are Hadamard. To visualize this, we plot both functions near x = 0.

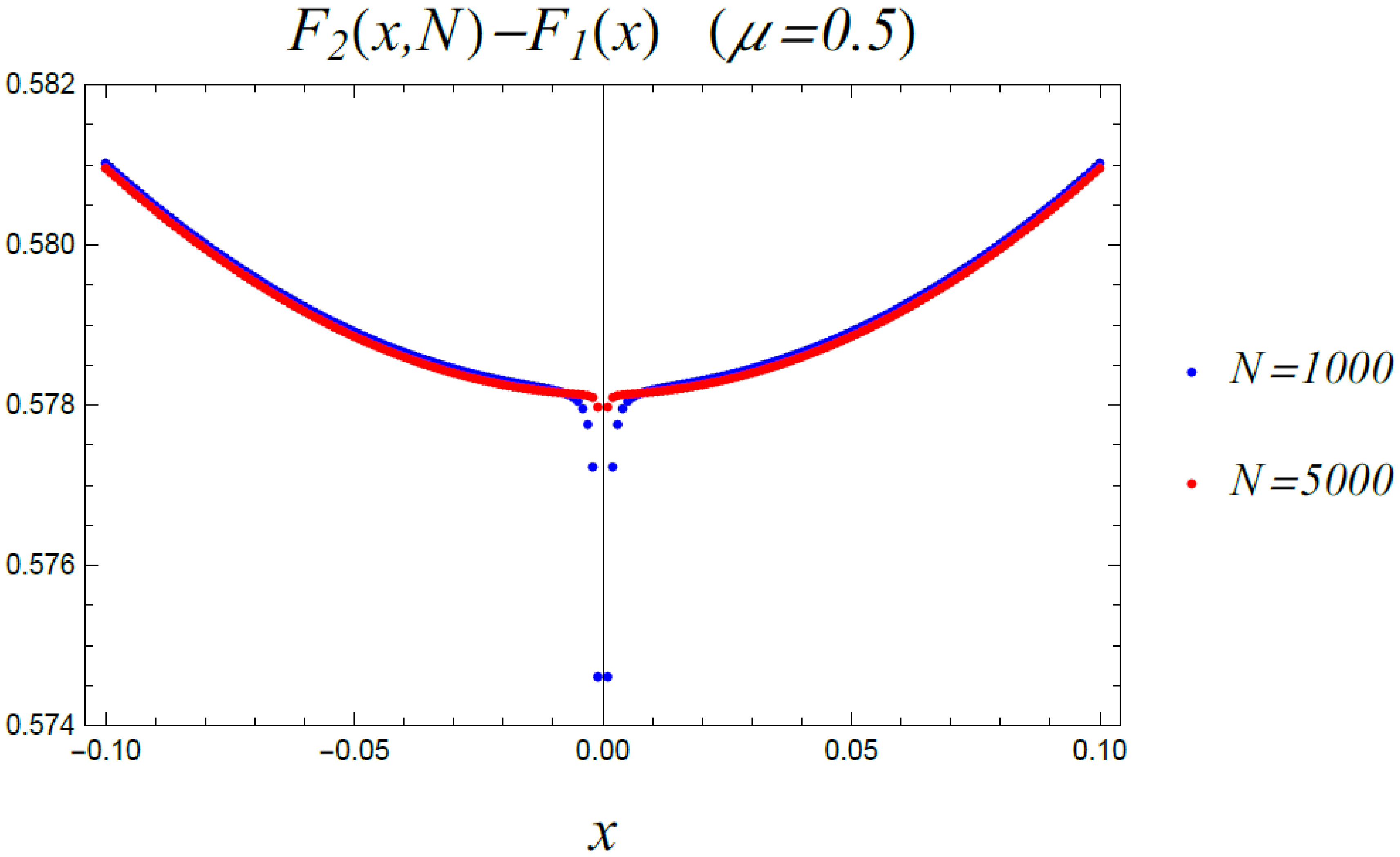
- We want to obtain a finite value for

$$\langle 0_L| :\varphi_L(0)^2: |0_L\rangle = \lim_{x\to 0}\lim_{N\to\infty}\left(F_2(x,N) - F_1(x)\right).$$

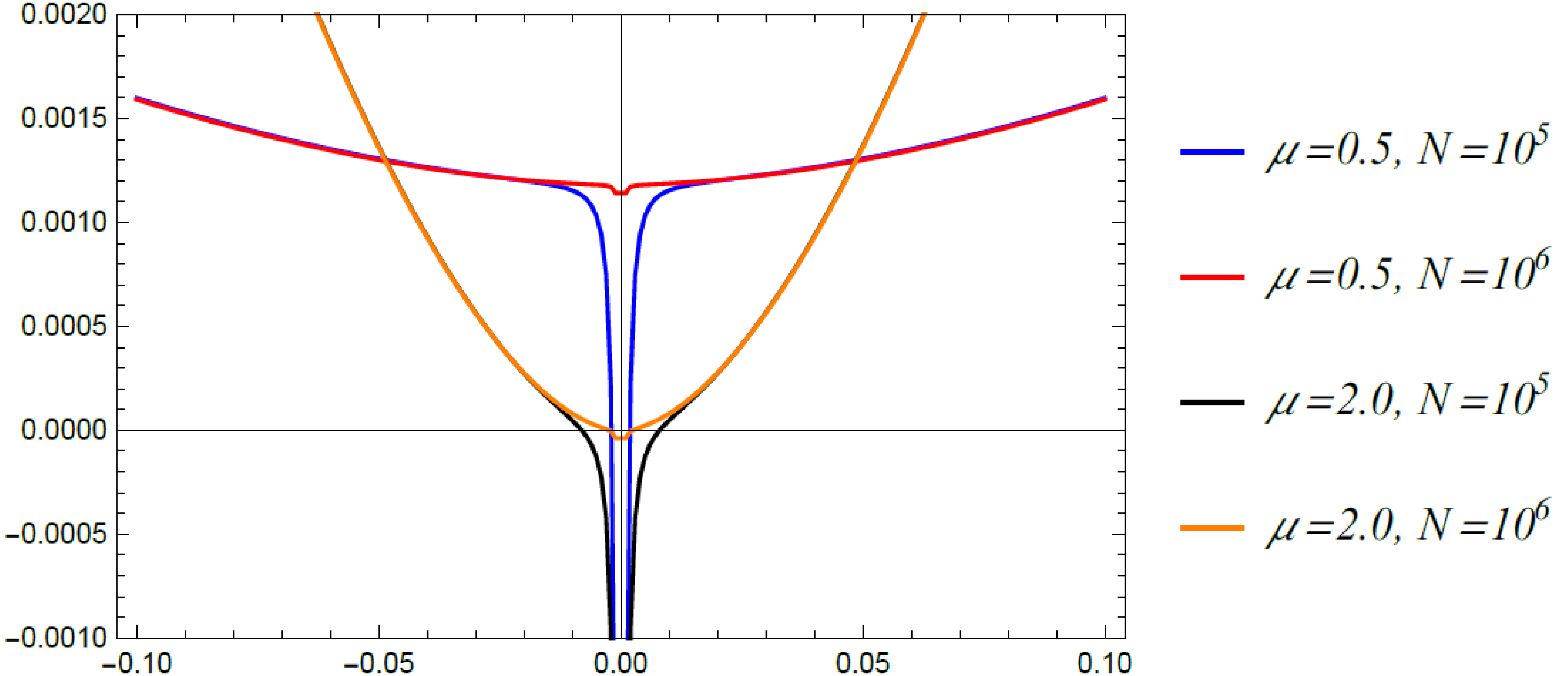
- But we cannot do this directly, as $\lim_{x\to 0} F_1(x) = \infty$.
- We know $\mathcal{W}_L(x,y) - \mathcal{W}(x,y)$ is *smooth.*
- Compute $\lim_{N\to\infty} F_2(x,N) - F_1(x)$ at *fixed* x.
- Then we can repeat the same computation for a sequence $\{x_k\}_k$ such that x_k approaches zero as k grows.

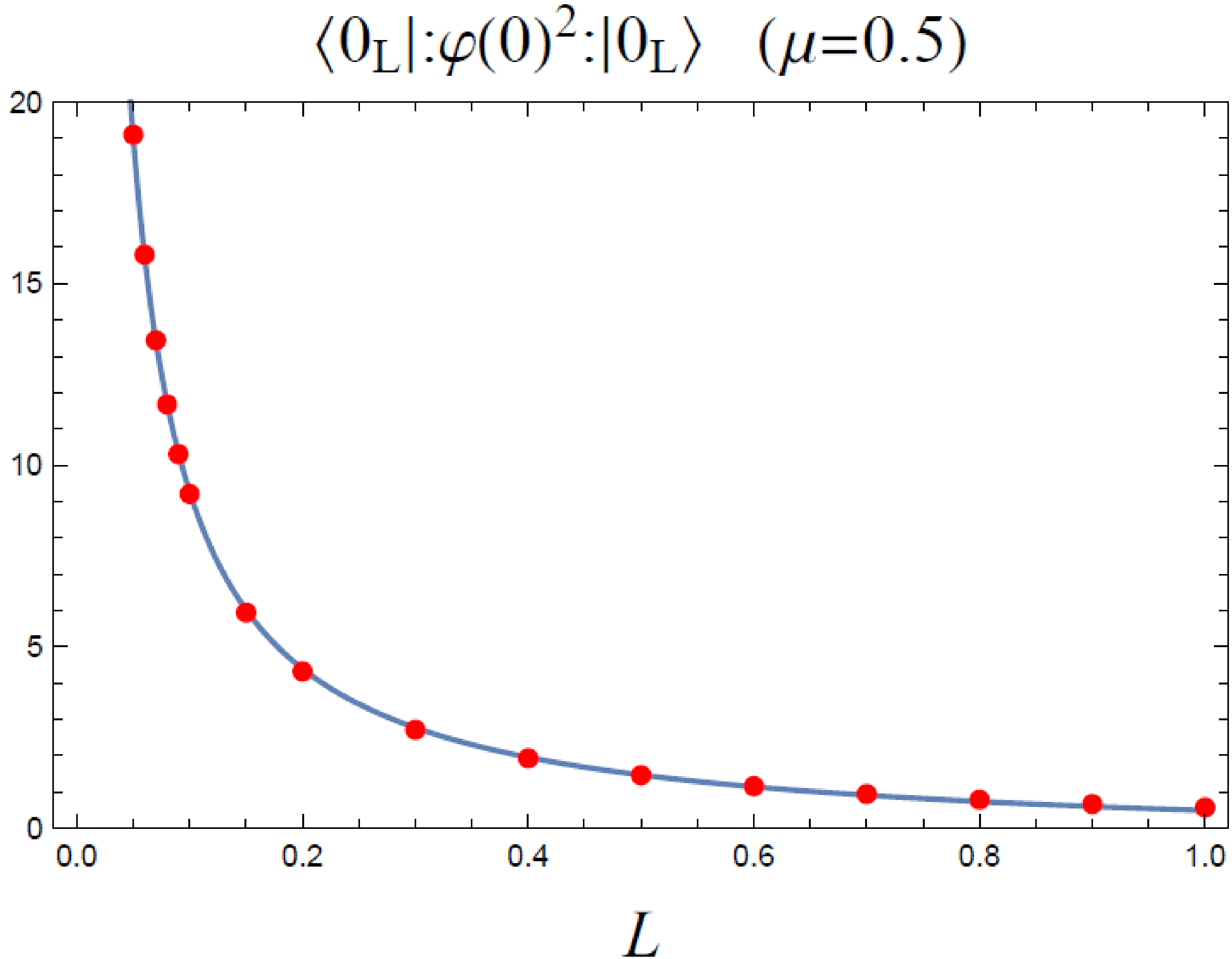


$F_2(0,N) - \frac{Ln(N)+\gamma_E}{2\pi} - \frac{1}{2\mu L}$
0.10
0.05
0.00
−0.05
−0.10
0
20
40
60
80
100
N



$F_2(x,N)-F_1(x)$ ($\mu=0.5$)
0.582
0.580
0.578
0.576
0.574
−0.10
−0.05
0.00
0.05
0.10
x
$N=1000$
$N=5000$


0.0020
0.0015
0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.10
−0.05
0.00
0.05
0.10
$\mu = 0.5,\ N = 10^5$
$\mu = 0.5,\ N = 10^6$
$\mu = 2.0,\ N = 10^5$
$\mu = 2.0,\ N = 10^6$



$\langle 0_L | :\varphi(0)^2: | 0_L \rangle \quad (\mu=0.5)$
20
15
10
5
0
0.0
0.2
0.4
0.6
0.8
1.0
L

- Using the previous numerical analysis as a "benchmark", we can now turn to the computation of correlation functions using the SJ vacuum.
- But first we check our method against the familiar result

Casimir energy for the cylinder

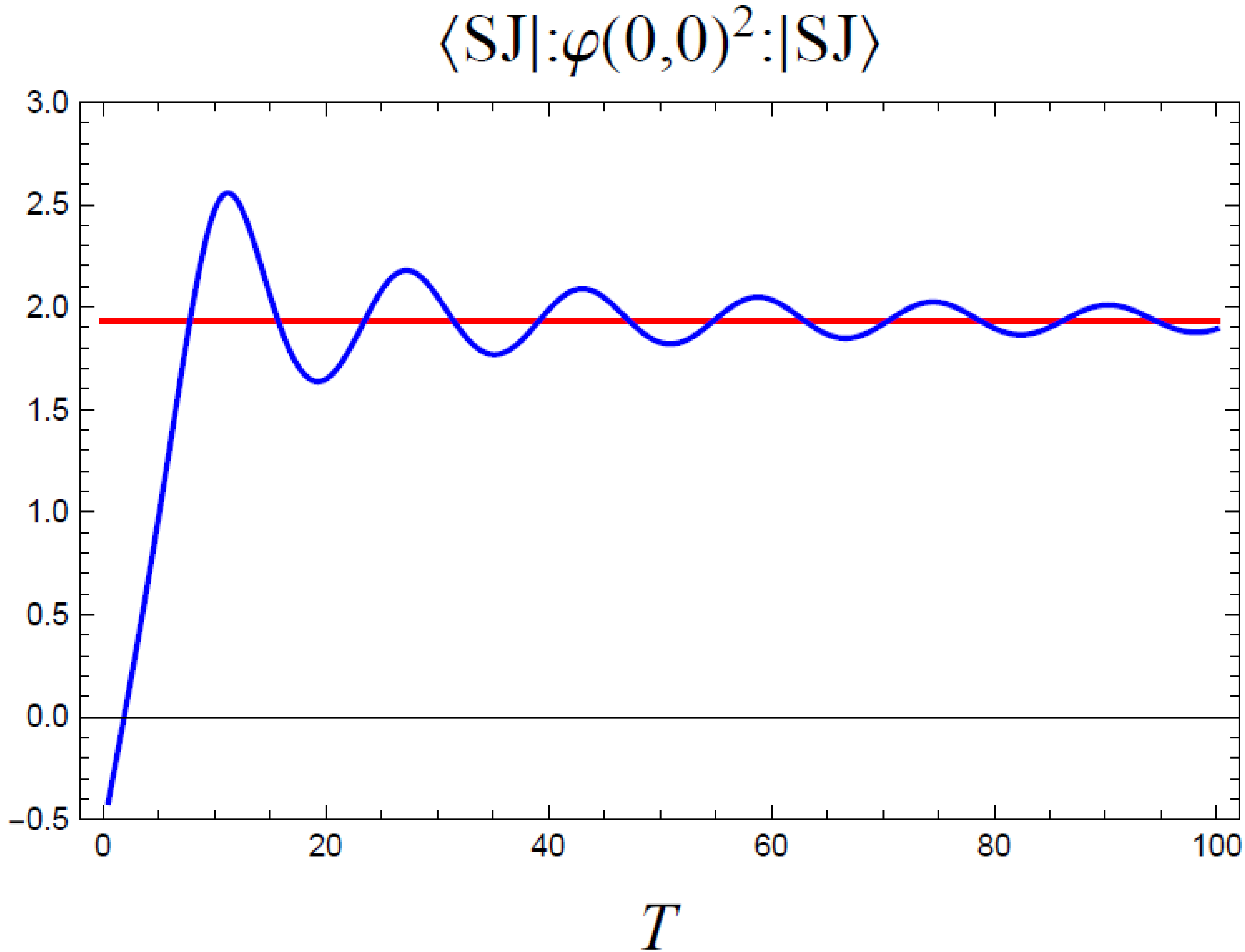
$$\langle : \mathcal{H}(x) : \rangle = \frac{1}{2} \text{``}\langle : \partial_0\varphi(x)\partial_0\varphi(x) + \partial_1\varphi(x)\partial_1\varphi(x) : \rangle\text{''}$$
$$:= \frac{1}{2}\lim_{y\to x}\left[\left(\partial_{x^0}\partial_{y^0} + \partial_{x^1}\partial_{y^1}\right)\left(\mathcal{W}_L(x,y) - \mathcal{W}(x,y)\right)\right].$$

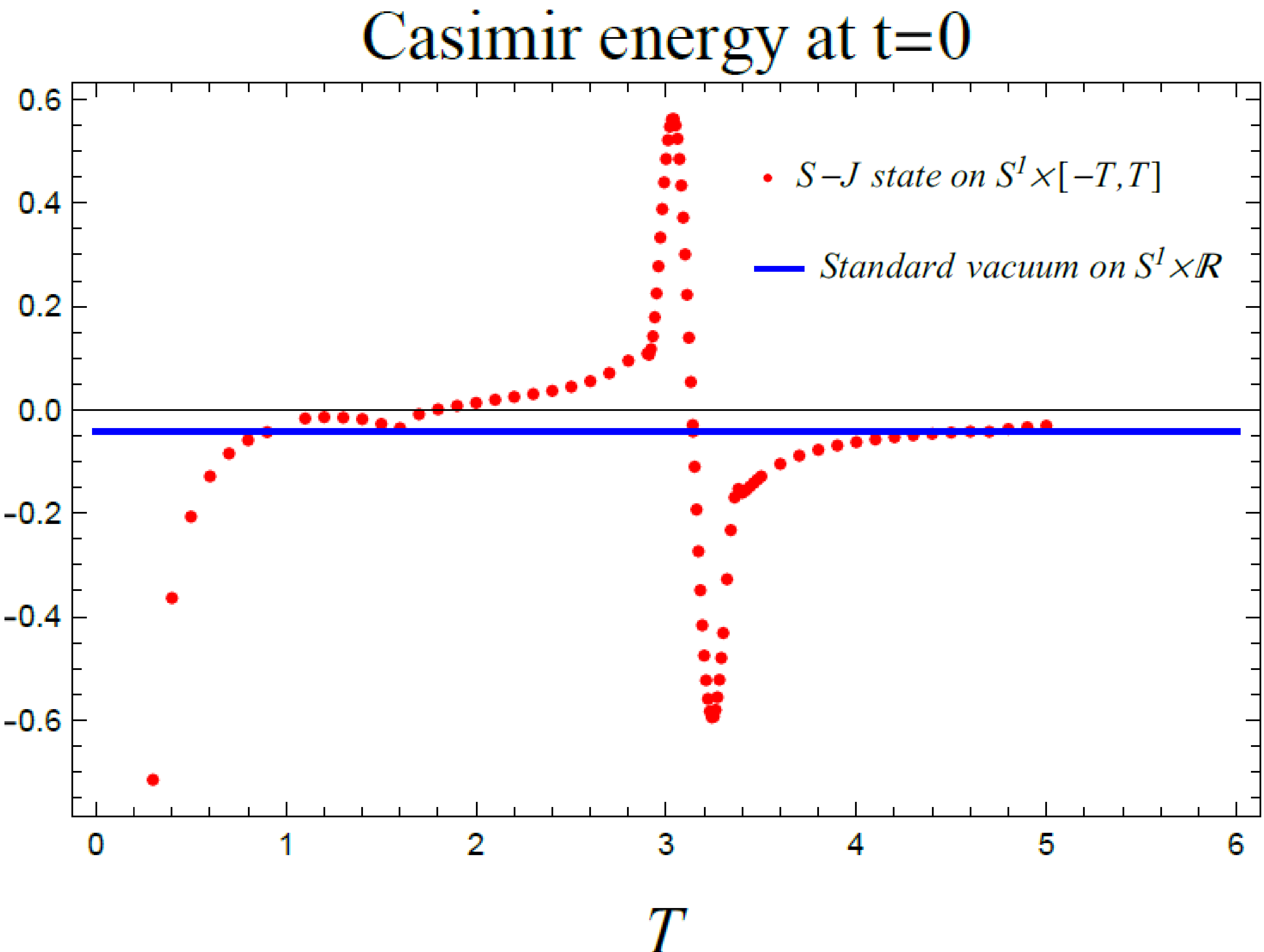
Using the identity $\frac{\partial}{\partial x^\nu}\mathcal{W}_{(L)}(x,y) = -\frac{\partial}{\partial y^\nu}\mathcal{W}_{(L)}(x,y)$, we obtain:

$$\langle : \mathcal{H}(0) : \rangle = -\frac{1}{2}\lim_{x\to 0}\left(\partial_{x^0}^2 + \partial_{x^1}^2\right)\left(\mathcal{W}_L(x,0) - \mathcal{W}(x,0)\right)$$
$$= -\frac{1}{2}\lim_{x\to 0}\left(\partial_{x^0}^2 + \partial_{x^1}^2\right)\left(-\frac{1}{4\pi}\ln\left(\left[1 - e^{-\frac{i}{R}(x^0 - x^1)}\right]\left[1 - e^{-\frac{i}{R}(x^0 - x^1)}\right]\right)\right.$$
$$\left. + \frac{1}{4\pi}\ln\left(-(x^0)^2 + (x^1)^2\right)\right).$$

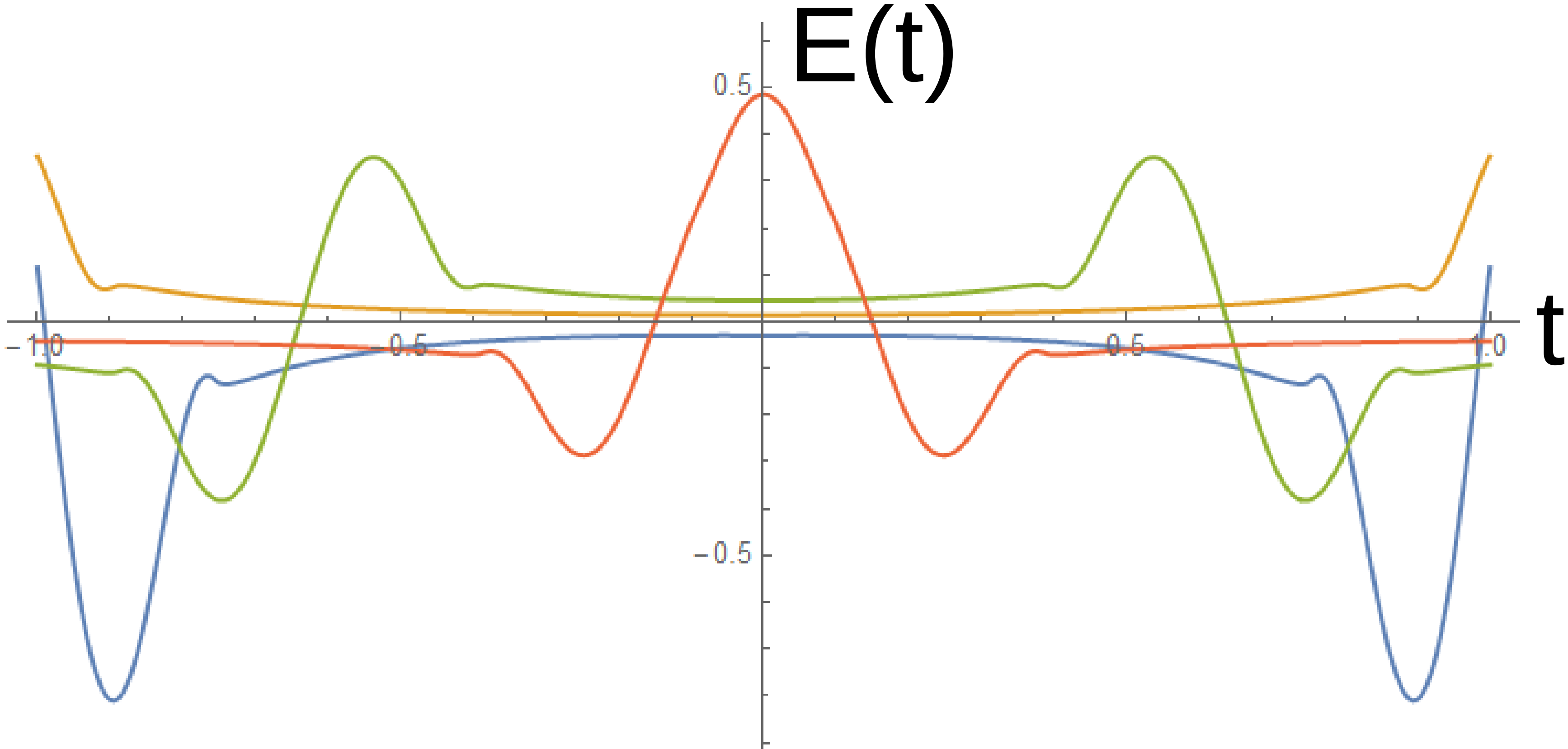
$$
\begin{aligned}
\langle : \mathcal{H}(0) : \rangle &= -\frac{2}{4\pi} \lim_{u\to 0} \left(-\partial_u^2 \ln \left(1 - e^{-\frac{i}{R}u} \right) + \partial_u^2 \ln u \right) \\
&= -\frac{1}{2\pi} \lim_{u\to 0} \left(-\frac{i}{R} \partial_u \left(\frac{e^{-\frac{i}{R}u}}{1 - e^{-\frac{i}{R}u}} \right) + \partial_u \frac{1}{u} \right) \\
&= -\frac{1}{2\pi} \lim_{u\to 0} \left[\left(\frac{i}{R} \right)^2 \frac{e^{\frac{i}{R}u}}{\left(e^{\frac{i}{R}u} - 1 \right)^2} - \frac{1}{u^2} \right] \\
&= -\frac{1}{2\pi} \lim_{u\to 0} \left(\frac{1}{R^2} \frac{1}{2(1 - \cos(u/R))} - \frac{1}{u^2} \right) \\
&= -\frac{1}{24\pi R^2}.
\end{aligned}
$$

We turn now to the numerical computation, using the SJ state..

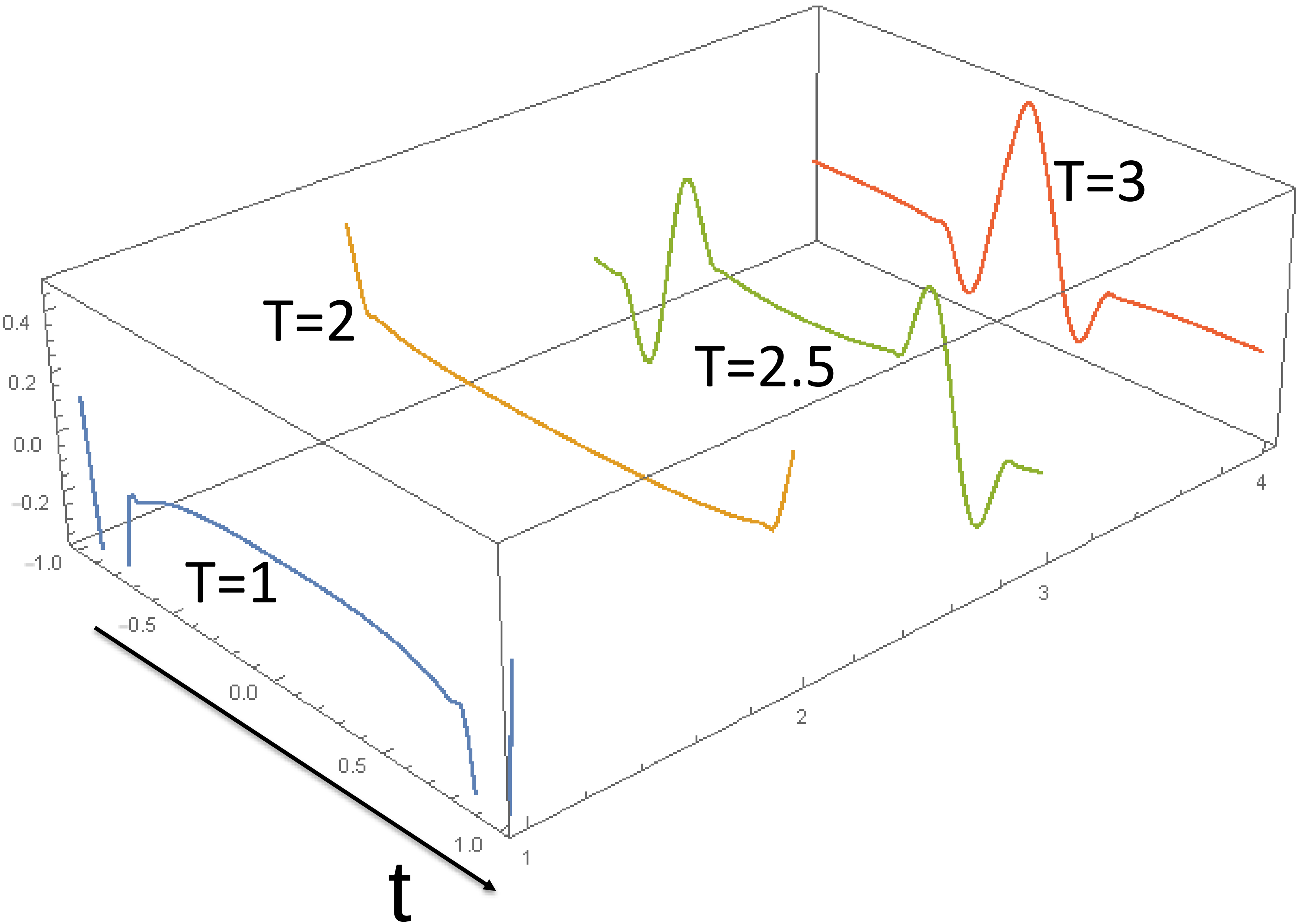

⟨SJ|:$\varphi(0,0)^2$:|SJ⟩
3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
0
20
40
60
80
100
T



Casimir energy at t=0
S−J state on $S^1\times[-T,T]$
Standard vacuum on $S^1\times\mathbb{R}$
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
1
2
3
4
5
6
T



- Casimir energy as a function of time, for different values of T.
- T=1 (blue), T=2 (orange), T=2.5 (green)
  T=3 (red)

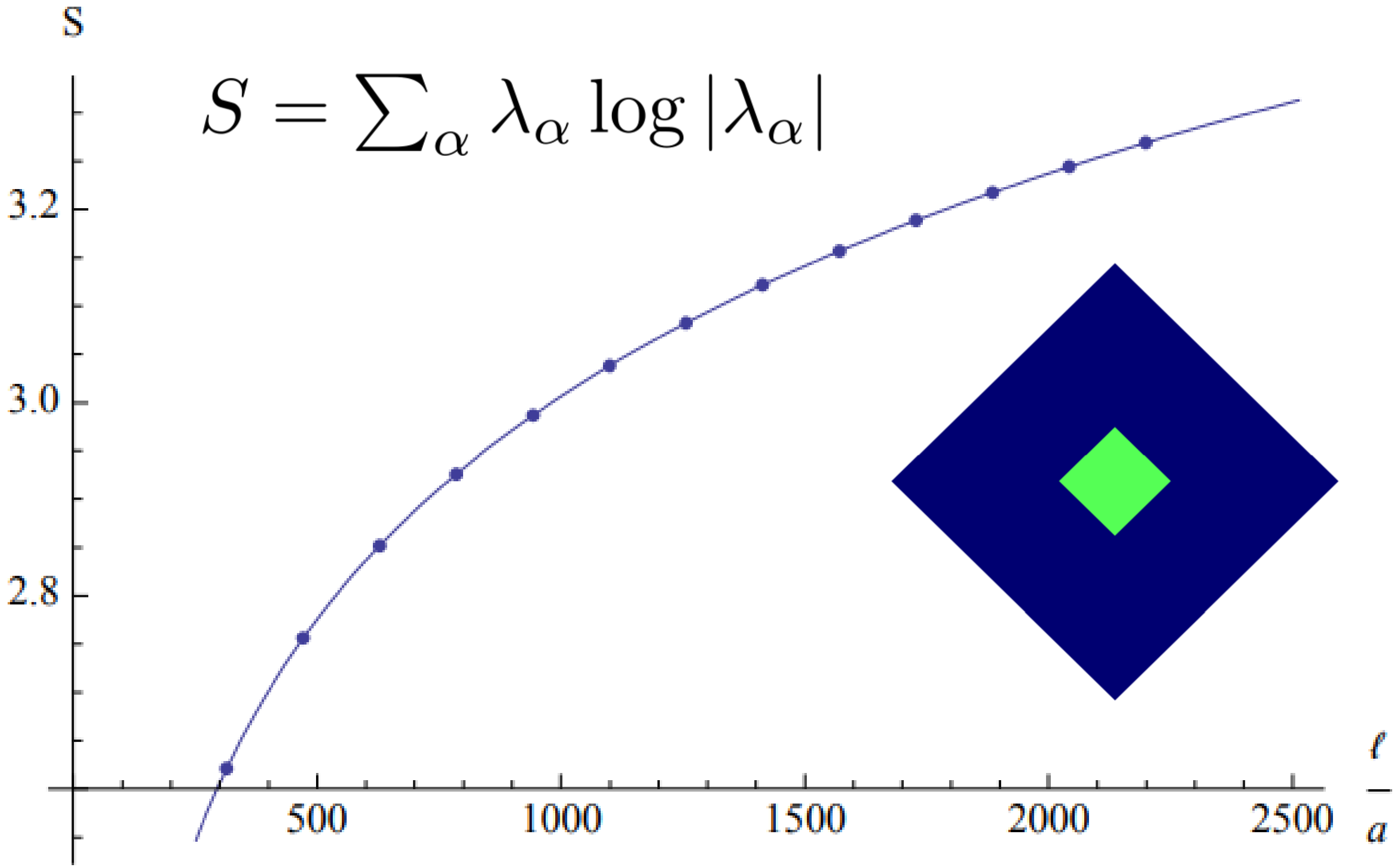

T=1
T=2
T=2.5
T=3
t
0.4
0.2
0.0
-0.2
-1.0
-0.5
0.0
0.5
1.0
1
2
3
4

Entanglement entropy

(Sorkin 2012; Saravani, Sorkin & Yazdi 2013)

$$W v_\alpha = i\lambda_\alpha E v_\alpha \quad (E v_\alpha \neq 0)$$

$$S = \sum_\alpha \lambda_\alpha \log|\lambda_\alpha|$$



Thanks for your attention!