Boundary Terms in Gravitational Action

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Outline

1. Action principle in general relativity.
2. Constructing a well-posed action for general relativity as well as in Lovelock theories in relation to null surfaces.

Main References

- **SC** and Parattu, arXiv:1806.08823.
- **SC**, Parattu and Padmanabhan, arXiv:1703.00624.
- **SC**, arXiv:1607.05986.
- Parattu, **SC** and Padmanabhan, arXiv:1602.07546.
- Parattu, **SC**, Majhi and Padmanabhan, arXiv:1501.08765.

The action for General Relativity

- The covariant action for general relativity is the Ricci scalar. In four dimension it yields,

GR Action

$$16\pi G\mathcal{A} = \int_{\mathcal{V}} d^4x \, \sqrt{-g}R(g, \partial g, \partial^2 g)$$

- The Ricci scalar is constructed from the Riemann tensor, which contains second derivatives of the metric.

Variation of the Action

- Vary the gravitational action.
- We will consider the difference of the action for two metric configuration, g_{ab} and a varied one, $g_{ab} + \delta g_{ab}$.
- This leads to,

variation

$$16\pi G\delta\mathcal{A} = \int_{\mathcal{V}} d^4x\sqrt{-g}\left(R_{ab} - \frac{1}{2}Rg_{ab}\right)\delta g^{ab}$$
$$+ \epsilon\int_{\partial\mathcal{V}} d^3y\sqrt{\mid h\mid}n^a h^{bc}\left(\partial_b\delta g_{ac} - \partial_a\delta g_{bc}\right)$$
$$+ \text{Boundary Terms with } \delta g^{ab}$$

A possibility to make the action well-posed

SC, Parattu and Padmanabhan, arXiv:1703.00624

Charap and Nelson, J. Phys. A 16, 1661 (1983)

- There are many alternatives to make the Einstein-Hilbert action well-posed. Among them the boundary term proposed by Gibbons and Hawking is the famous one.

Basic Idea

$$\delta\mathcal{A} = \int d^4x \; (\text{Equation of Motion Term})\, \delta\, (\text{Dynamical Variable})$$
$$+ \int d^3x \; (\text{Conjugate Momentum})\, \delta\, (\text{Variables to be fixed})$$
$$+ \int d^3x \; \delta\, (\text{Boundary Term}) + \int d^3x\, (\text{Total Divergence Term})$$

Is it Possible?

Padmanabhan, MPLA 29, 1450037 (2014)

Gibbons and Hawking, Phys. Rev. D 15, 2752 (1977)

York, Phys. Rev. Lett 28, 1082 (1972)

- The variation of the Einstein-Hilbert action involves,

Variation on Surface

$$\mathcal{B} = \mathrm{Tot.\ Derv.} - \delta\left(2\nabla_a n^a\right) + \left(\nabla_i n_j - \epsilon n_i a_j\right)\delta g^{ij}$$

- Using the properties of the extrinsic curvature

Finally

$$\sqrt{h}\mathcal{B} = \mathrm{Tot.\ Derv.} - \delta\left(2K\sqrt{h}\right) + (K h_{ij} - K_{ij})\delta h^{ij}\sqrt{h}$$

- $2K\sqrt{h}$ term to be added to the gravitational action.

Surface Variation on a null surface

Parattu, SC, Majhi and Padmanabhan, arXiv:1501.08765

- Starting from the surface variation we can evaluate it for a null surface as well. This will lead to,

Surface Variation

$$\mathcal{B} = \text{Tot. Derv.} - 2\delta\left[\sqrt{q}\left(\Theta+\kappa\right)\right] + \sqrt{q}\left[\Theta_{ab} - \left(\Theta+\kappa\right)q_{ab}\right]\delta q^{ab} + 2\sqrt{q}\left[\left(\Theta+\kappa\right)k_c + \ell^b\nabla_c k_b\right]\delta\ell^c$$

- Thus for null surfaces we need to add a term $2\sqrt{q}\left(\Theta+\kappa\right)$ to the Ricci scalar.
- When this modified action is varied we will get two surface contributions, one from δq^{ab} and another from $\delta\ell^c$.

Applications

- Can we derive the structure of the boundary term in the form language? What happens when the boundaries are not smooth? These have been answered in:
    1. I. Jubb, J. Samuel, R. Sorkin and S. Surya, CQG 34, 065006 (2017).
    2. L. Lehner, R.C. Myers, E. Poisson, R.D. Sorkin, PRD 94, 084046 (2016) and others.
- The proposal "Complexity of a boundary state in CFT = Gravitational Action in the bulk" crucially requires the boundary term for null surfaces. These results appear in:
    1. S. Chapman, H. Marrochio and R.C. Myers, JHEP, 1701, 062 (2017)
    2. J. Maltz and L. Susskind, Phys. Rev. Lett. 118, 101602 (2017) and others.
- These results have also been used in various other contexts, e.g., de Sitter spacetime, degrees of freedom, AdS Soliton etc.

Variation of the Gauss-Bonnet Lagrangian

SC, Parattu and Padmanabhan, arXiv:1703.00624

Myers, Phys. Rev. D 36, 392 (1987)

Variation

$$\begin{aligned}\text{Surface Variation} &= \text{Total Divergence} \\ &+ (\text{Conjugate Momentum})\,\delta\,(\text{Variables to be fixed}) \\ &+ \delta\Big\{8\ {}^{(D-1)}R^a_bK^b_a - 4\ {}^{(D-1)}RK \\ &+ \epsilon\left(\frac{4}{3}K^3 + \frac{8}{3}K^a_bK^b_cK^c_a - 4KK^a_bK^b_a\right)\Big\}\end{aligned}$$

- A similar calculation yields the corresponding boundary term for Lanczos-Lovelock gravity as well.

Null Boundary Terms in Gauss-Bonnet Gravity

SC and Parattu, arXiv:1806.08823

Boundary Term Gauss-Bonnet

$$\begin{aligned}\sqrt{q}\ \mathcal{B}_{\rm GB} =& 4\sqrt{q}\Big[\kappa\left({}^{(D-2)}R - 2\Theta^p_q\Psi^q_p + 2\Theta\Psi\right) + 2\Psi^p_q\mathcal{L}_\ell\Theta^q_p \\ & - 2\Psi\frac{d\Theta}{d\lambda} + 2\Theta^a_c\Theta^c_b\Psi^b_a - 2\Psi\Theta^a_b\Theta^b_a \\ & - 4\left(D^m\Theta - D^a\Theta^m_a\right)\left(k^q\nabla_m\ell_q\right) \\ & - 2\left(k^q\nabla_m\ell_q\right)\left(k^p\nabla_a\ell_p\right)\left(\Theta q^{ma} - \Theta^{ma}\right)\Big]\end{aligned}$$

Summary

1. Elaborated on the associated complications with gravitational action principle.
2. Possible resolutions to make the variational principle well posed for null surfaces.
3. The correct action associated with spacelike/timelike and null surfaces in the context of Lanczos-Lovelock gravity have been put forward.

Thank You