A consistent search for the Hawking effect for extremal Kerr black holes

Subhajit Barman

Department of Physical Sciences
Indian Institute of Science Education and Research Kolkata

IISER KOLKATA

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The Kerr spacetime

- The Kerr line-element in Boyer-Lindquist coordinates

$$ds^2 = -\left(1 - \frac{rr_s}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma \sin^2\theta}{\rho^2} d\phi^2 - \frac{2arr_s \sin^2\theta}{\rho^2} dt d\phi \,,$$

  where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$, $\Delta = r^2 + a^2 - rr_s$.

- $\Delta = 0$ gives coordinate singularities at $r_h = \frac{r_s}{2} + \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}$ and $r_c = \frac{r_s}{2} - \sqrt{\left(\frac{r_s}{2}\right)^2 - a^2}$.

- The surface gravity at the event horizon $\varkappa_h = \sqrt{r_s^2 - 4a^2}/2r_s r_h$ and the Hawking temperature $T_H = \varkappa_h/(2\pi k_B)$.

- In extremal limit $a \to r_s/2$ surface gravity $\varkappa_h \to 0$ and $T_H \to 0$.

- The bogoliubov coefficients do not satisfy the required normalization condition for extremal black holes[1].

[1] F. G. Alvarenga, A. B. Batista, J. C. Fabris, and G. T. Marques, Phys. Lett. A320, 83 (2003), arXiv:gr- qc/0306030

Reduction of massless free scalar field action

- Action for a minimally coupled massless free scalar field $S_\Phi = \int d^4x \left[-\frac{1}{2}\sqrt{-g} g^{\mu\nu} \nabla_\mu \Phi(x) \nabla_\nu \Phi(x) \right]$.
- Decompose first $\Phi(t,r,\theta,\phi) = \sum_{lm} e^{im\phi} \Phi_{lm}(t,r,\theta)$ and $\tilde{\Phi}_{lm}(t,r,\theta) \simeq \mathscr{S}_{lm}(\theta)\varphi_{lm}(t,r_\star)/\sqrt{r^2+a^2}$.
- Action in asymptotic regions far from the horizon becomes, $S_\Phi = \sum_{lm} S_{lm}$, with $S_{lm} \simeq \int dt dr_\star \left[\frac{1}{2}\partial_t \varphi^*_{lm} \partial_t \varphi_{lm} - \frac{1}{2}\partial_{r_\star}\varphi^*_{lm}\partial_{r_\star}\varphi_{lm} \right]$.
- $dr_\star = \frac{r^2+a^2}{\Delta} dr$ and solution to redefined reduced field equation is $\varphi_{lm}(t,r_\star) \sim \frac{1}{\sqrt{2\pi\tilde{\omega}}} e^{-i\tilde{\omega}(t \pm r_\star)}$.
- Near-null coordinates for observer $\mathbb{O}^-$ and $\mathbb{O}^+$ $\tau_\mp = t \mp (1-\epsilon) r_\star$; $\xi_\mp = -t \mp (1+\epsilon) r_\star$.

Hamiltonian formulation[2]

- For two observers line-elements $ds^2 = \frac{\epsilon}{2}\left[-d\tau_\pm^2 + d\xi_\pm^2 + \frac{2}{\epsilon}d\tau_\pm d\xi_\pm\right]$.
- Scalar field Hamiltonians $H_\varphi^\pm = \int d\xi_\pm \frac{1}{\epsilon}\left[\left\{\frac{\Pi^2}{2} + \frac{1}{2}(\partial_{\xi_\pm}\varphi)^2\right\} + \Pi\,\partial_{\xi_\pm}\varphi\right]$.

  momentum $\Pi(\tau_\pm, \xi_\pm) = \epsilon(\partial_{\tau_\pm}\varphi) - (\partial_{\xi_\pm}\varphi)$ and poisson bracket relation $\{\varphi(\tau_\pm, \xi_\pm), \Pi(\tau_\pm, \xi'_\pm)\} = \delta(\xi_\pm - \xi'_\pm)$.
- Fourier transform $\varphi = \frac{1}{\sqrt{V_\pm}}\sum_k \tilde{\phi}_k e^{ik\xi_\pm}$; $\Pi = \frac{1}{\sqrt{V_\pm}}\sum_k \sqrt{q}\,\tilde{\pi}_k e^{ik\xi_\pm}$.
- Fourier field modes satisfy $\{\tilde{\phi}_k, \tilde{\pi}_{-k'}\} = \delta_{k,k'}$ and Hamiltonians $H_\varphi^\pm = \sum_k \frac{1}{\epsilon}\left[\left(\frac{1}{2}\tilde{\pi}_k\tilde{\pi}_{-k} + \frac{1}{2}|k|^2\tilde{\phi}_k\tilde{\phi}_{-k}\right) - \frac{ik}{2}\left(\tilde{\pi}_k\tilde{\phi}_{-k} - \tilde{\pi}_{-k}\tilde{\phi}_k\right)\right] = \sum_k \frac{1}{\epsilon}(\mathcal{H}_k^\pm + \mathcal{D}_k^\pm)$.
- Using relations $\varphi(\tau_-, \xi_-) = \varphi(\tau_+, \xi_+)$, $\Pi(\tau_+, \xi_+) = (\partial\xi_-/\partial\xi_+)\Pi(\tau_-, \xi_-)$ we express $\tilde{\phi}_\kappa = \sum_k \tilde{\phi}_k F_0(k, -\kappa)$, $\tilde{\pi}_\kappa = \sum_k \tilde{\pi}_k F_1(k, -\kappa)$.

[2] S. Barman, G. M. Hossain, and C. Singha, Phys. Rev. D97, 025016 (2018), arXiv:1707.03614

Number density of Hawking quanta

- Coefficient functions satisfy $F_1(\pm|k|,\kappa) = \mp\frac{\kappa}{|k|}\, F_0(\pm|k|,\kappa)$ using their general form $F_n(k,\kappa) = \frac{1}{\sqrt{V_- V_+}} \int d\xi_+ \left(\frac{\partial\xi_-}{\partial\xi_+}\right)^n\, e^{ik\xi_- + i\kappa\xi_+}$.
- A relation $\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) \equiv \sum_{k>0} \frac{\kappa}{k}\left[|F_0(-k,\kappa)|^2 - |F_0(k,\kappa)|^2\right] = 1$ must satisfy from simultaneous satisfaction of Poisson brackets $\{\tilde{\phi}^-_k, \tilde{\pi}^-_{-k'}\} = \delta_{k,k'}$ and $\{\tilde{\phi}^+_\kappa, \tilde{\pi}^+_{-\kappa'}\} = \delta_{\kappa,\kappa'}$.
- With a suitable redefinition of the field modes $\mathcal{H}^\pm_k = \frac{1}{2}\pi_k^2 + \frac{1}{2}k^2\phi_k^2$, $\{\phi_k^2, \pi_{k'}^2\} = \delta_{k,k'}$ and the diffeomorphism generators vanish.
- In Fock quantization $|0_-\rangle = \prod_k \otimes |0_k\rangle$ and $\langle\hat{\mathcal{H}}^-_k\rangle = \frac{1}{2}|k|$.
- Then the number density of Hawking quanta can be expressed as $N_{\tilde{\omega}} = N_\kappa \equiv \frac{\langle_- 0|\hat{\mathcal{H}}^+_\kappa|0_-\rangle}{\kappa} - \frac{1}{2} = \mathbb{S}_+(\kappa)$.

Coefficient functions for extremal Kerr black holes

- For extremal Kerr black hole tortoise coordinate $r_\star = r + r_s \ln\left(\frac{2r - r_s}{r_s}\right) - \frac{r_s^2}{2r - r_s}$ is used to get relation $\xi_+ = \xi_- + 2r_s \ln\left(\frac{\xi_-}{\sqrt{2}r_s}\right) - \frac{2r_s^2}{\xi_-}$ which approximates to $\xi_+ \approx \xi_- - \frac{2r_s^2}{\xi_-}$.
- Introducing regulator δ the coefficient functions become
$$F_0^\delta(\pm k, \kappa) = \int \frac{d\xi_-}{\sqrt{V_- V_+}} \left(1 + \frac{2r_s^2}{\xi_-^2}\right) e^{-(\delta+i)\frac{2r_s^2\kappa}{\xi_-} - [\delta|\kappa \pm k| - i(\kappa \pm k)]\,\xi_-} .$$
- We define quantities $b_\pm = \sqrt{2}r_s\,[\delta|\kappa \pm k| - i\,(\kappa \pm k)]$, $b_0 = \sqrt{2}r_s\kappa(\delta + i)$, $\xi = (\xi_-/\sqrt{2}r_s)$, $m_\star \equiv (\kappa V_-/2\pi) = (|b_0|V_-/2\pi\sqrt{2}r_s)$, $\gamma \equiv (V_-/V_+)$ and get
$$F_0^\delta(\pm k, \kappa) = \frac{\sqrt{2}\,r_s}{\sqrt{V_- V_+}} \int_{\xi^L}^{\xi^R} d\xi \left(1 + \frac{1}{\xi^2}\right) e^{-b_\pm \xi - b_0/\xi} .$$
- We evaluate the coefficient $F_0^\delta(-\kappa, \kappa)$ separately and obtain
$$|F_0^\delta(-\kappa, \kappa)|^2 = \gamma\,[1 + \mathcal{O}\{\ln(m_\star)/m_\star\}] .$$

Consistency condition

- When $b_- \neq 0$ we have $F_0^\delta(\pm k, \kappa) = \frac{\sqrt{2}\, r_s}{\sqrt{V_- V_+}} \left(\frac{b_0 + b_\pm}{b_0\, b_\pm}\right) [z_\pm K_1(z_\pm)]$.
- *lhs* of consistency $\mathbb{S}_-^\delta(\kappa) - \mathbb{S}_+^\delta(\kappa) = |F_0^\delta(-\kappa, \kappa)|^2 + \frac{\gamma\zeta(2)}{2\pi^2} + \frac{\gamma\, \mathcal{S}(1,\infty)}{4\pi^2 m_\star}$, where
  $\mathcal{S}(s_0, s_1) = \sum_{s=s_0}^{s_1} \left[\frac{2m_\star}{s^2}\left\{|\tilde{z}K_1(\tilde{z})|^2 - 1\right\} + \frac{(s-m_\star)}{s^2}\left\{|\tilde{z}K_1(\tilde{z})|^2 - |\tilde{z}K_1(|\tilde{z}|)|^2\right\}\right]$
  and $\tilde{z} \equiv \tilde{z}(s) = \sqrt{4|b_0|^2 s/m_\star}\,(\delta + i)$.
- $\lim_{z\to 0} |zK_1(z)| = 1$, $\lim_{z\to\infty} |zK_1(z)| = 0$ enables one to choose λ_1, λ_2 such that $|\tilde{z}(\lambda_1 m_\star)| \ll 1$, $|\tilde{z}(\lambda_2 m_\star)| \gg 1$ & $|\tilde{z}K_1(\tilde{z})|^2 \leq d_1$, $|\tilde{z}K_1(|\tilde{z}|)|^2 \leq d_2$.
- We may express $\mathcal{S}(1,\infty) = \mathcal{S}(\lambda_1 m_\star, \lambda_2 m_\star) + \mathcal{S}(\lambda_2 m_\star + 1, \infty)$ where
  $\mathcal{S}(\lambda_1 m_\star, \lambda_2 m_\star) \leq (d_1 + d_2)\left[\ln\left(\frac{\lambda_2}{\lambda_1}\right) + \frac{\lambda_2 - \lambda_1}{\lambda_2\lambda_1}\right]$, $\mathcal{S}(\lambda_2 m_\star + 1, \infty) = \frac{\pi}{2\delta} e^{-4|b_0|\delta\sqrt{\lambda_2}}\,[1 + \mathcal{O}(\delta)]$.
- We observe that $\delta \to 0$ leads to a violation of the consistency condition.

Consistency condition and Number density

- In the limit $m_\star \to \infty$ we get $\mathbb{S}_-(\kappa) - \mathbb{S}_+(\kappa) = \gamma \left[1 + \frac{1}{2\pi^2}\,\zeta(2)\right] = 1$.
- Now $\zeta(2) = \frac{1}{6}\pi^2$, then we must have $\gamma = (12/13)$ which says volume regulators should be varied together as $\left(\frac{\xi_-^L}{\sqrt{2}r_s}\right) = 12\left(\frac{\sqrt{2}r_s}{\xi_-^R}\right)$.
- One can express $\mathbb{S}_+^\delta(\kappa) = \frac{\gamma}{4\pi^2 m_\star}\left[\mathcal{S}_+(m_\star, \lambda_2 m_\star) + \mathcal{S}_+(\lambda_2 m_\star + 1, \infty)\right]$ by defining $\mathcal{S}_+(s_0, s_1) = \sum_{s=s_0}^{s_1} \frac{(s-m_\star)}{s^2}\,|\tilde{z}K_1(|\tilde{z}|)|^2$ where $\mathcal{S}_+(m_\star, \lambda_2 m_\star) \le d_2\left[\ln(\lambda_2) - 1 + \frac{1}{\lambda_2}\right]$, $\mathcal{S}_+(\lambda_2 m_\star + 1, \infty) = \frac{\pi}{2} e^{-4|b_0|\sqrt{\lambda_2}}\left[1 + \mathcal{O}\left(\frac{1}{\lambda_2}\right)\right]$.
- In the limit $m_\star \to \infty$ we have $\mathbb{S}_+^\delta(\kappa) \le 0$ and by definition $\mathbb{S}_+^\delta(\kappa) \ge 0$. Then we have $\lim_{m_\star \to \infty} \mathbb{S}_+^\delta(\kappa) = 0$.
- The number density of Hawking quanta for extremal Kerr black holes[3] become $N_{\tilde{\omega}} = N_{\omega - m\Omega_h} = \langle \hat{N}_\kappa^+ \rangle = 0$.

[3] S. Barman and G. M. Hossain (2018), arXiv:1809.09430

THANK YOU FOR LISTENING!