Constraints on the Generalized Uncertainty Principle and Rainbow Gravity functions from black-hole thermodynamics

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December 7, 2018

Outline

INTRODUCTION

Minimum measurable length and it's consequences

- The idea of existence of an observer independent minimal length scale arises naturally in Quantum Gravity theories in the form of effective minimal uncertainty in position.
- This minimal length expected to be close or equal to the Planck length occurs in String theory, black hole physics, doubly special relativity, Lorentzian dynamical triangulations, non-commutative geometry, loop quantum gravity, to name a few.
- One of the manifestations of the inclusion of a minimal length is the modification of Heisenberg Uncertainty Principle to the so-called Generalized Uncertainty Principle(GUP). (*Fabio Scardigli, Phys.Lett.B452:39-44,1999*)
- Alternatively an observer independent invariant length scale indicates the possibility of generalization of the Einstein special relativity (SR) which is called Doubly Special Relativity(DSR).
- In curved space-time it is possible to make generalization of DSR leading to double general relativity, where the geometry is represented by a one parameter family of energy-dependent metrics forming a rainbow of metrics or gravity's rainbow.

Motivations

- We have studied the following thermodynamic properties of different black holes by use of several GUP representations and Rainbow Gravity functions, available in the literature:
  - ▷ The mass-temperature relation
  - ▷ Heat capacity expression
  - ▷ The critical masses(below which the thermodynamic quantities become ill-defined)
  - ▷ The remnant masses (at which the radiation process stops)
  - ▷ The Entropy
- By using the expression of Entropy the well known area theorem has been derived.
- The existence of a logarithmic correction to the entropy of a black hole is a universal predication coming from all approaches which analyze leading-order corrections to the entropy of a black hole.
- The leading-order corrections to the entropy of a black hole has to have the form of a logarithmic correction.

Constraints on the GUP from black hole thermodynamics

Metric in d-dimension and the simplest GUP

- The metric for a Schwarzschild black hole with mass M in d-dimensions can be written as

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-3}}\right) dt^2 + \frac{1}{\left(1 - \frac{\mu}{r^{d-3}}\right)} dr^2 + r^2 d\Omega_{d-2}^2 \tag{1}$$

where

$$\mu = \frac{16\pi G_d M}{(d-2)\Omega_{d-2}}; \quad G_d = \frac{1}{M_p^{d-2}}; \quad \Omega_{d-2} = \frac{2\pi^{\left(\frac{d-1}{2}\right)}}{\Gamma(\frac{d-1}{2})} . \tag{2}$$

- The simplest GUP is given by (*R.J. Adler, P. Chen, D.I. Santiago, Gen. Rel. Grav. 33 (2001) 2101.*)

$$\delta x \delta p \geq \frac{\hbar}{2} \left\{ 1 + \frac{\beta^2 l_p^2}{\hbar^2} (\delta p)^2 \right\} \tag{3}$$

where l_p is the Planck length and β is a dimensionless constant.

- We use natural units $c = 1 = \hbar$ and $k_B = 1$.

Momentum and Position uncertainty for Schwarzschild black Hole in d-dimension

- Near the horizon the position uncertainty of an emitted particle will be order of the Schwarzschild radius:

$$\delta x = \varepsilon r_h \tag{4}$$

  where ε is a calibration factor.

- Uncertainty in the momentum for the particle can be written as

$$\delta p = T. \tag{5}$$

- The temperature of the emitted particle will be identical to the temperature of the black hole.

Mass-Temperature Relation, Heat Capacity and Entropy for Schwarzschild black Hole in d-dimension

- Thus the mass-temperature relation of the black hole would be

$$M = a' \left[\frac{1}{T} + \frac{\beta^2}{M_p^2} T \right]^{d-3} \tag{6}$$

where

$$a' = \left(\frac{d-3}{4\pi} \right)^{(d-3)} \left[\frac{(d-2)\Omega_{d-2} M_p^{d-2}}{16\pi} \right]; \quad l_p = M_p{}^{-1}. \tag{7}$$

- The heat capacity of the black hole reads

$$C = \frac{dM}{dT} = a'(d-3) \left[\frac{1}{T} + \frac{\beta^2}{M_p^2} T \right]^{d-4} \left[-\frac{1}{T^2} + \frac{\beta^2}{M_p^2} \right]. \tag{8}$$

- The black hole entropy from the first law of black hole thermodynamics can be determined as

$$S = \int c^2 \frac{dM}{T} = \int C \frac{dT}{T} . \tag{9}$$

Entropy of the five dimensional Schwarzschild BH

- Considering the simplest GUP the entropy then can be calculated as follows

$$S = 2a'_{(d=5)}\left(\frac{1}{3T^3} + \frac{\beta^4}{M_p^4}T\right). \tag{10}$$

- This expression of entropy is not an acceptable form for the entropy of a black hole as the leading order corrections have to be logarithmic in nature.
- Now we consider a more general form of GUP, namely linear GUP (*Das S., Vagenas E. C. and Ali A. F., Phys. Lett. B, 690 (2010) 407*)

$$\delta x \delta p \geq \frac{\hbar}{2}\left\{1 - \frac{\alpha l_p}{\hbar}\delta p + \frac{\beta^2 l_p^2}{\hbar^2}(\delta p)^2\right\}. \tag{11}$$

- The expression for the entropy of the black hole thus can be obtained as

$$\begin{aligned} S &= \frac{A}{4{l_p}^3} + \frac{3\alpha}{(128\pi)^{\frac{1}{3}}}\left(\frac{A}{4{l_p}^3}\right)^{\frac{2}{3}} - \frac{3\beta^2}{(256\pi^2)^{\frac{1}{3}}}\left(\frac{A}{4{l_p}^3}\right)^{\frac{1}{3}} \\ &\quad - \frac{3\alpha\beta^2}{16\pi}\ln\left[\frac{\alpha}{M_p} + \frac{(16\pi)^{\frac{1}{3}}}{M_p}\left(\frac{A}{4{l_p}^3}\right)^{\frac{1}{3}}\right] \end{aligned} \tag{12}$$

Entropy of the six dimensional Schwarzschild BH

- The corrections to the entropy from the simplest GUP reads

$$\begin{aligned} S &= \frac{A}{4{l_p}^4} - \beta^2 \frac{3\sqrt{3}}{4\sqrt{2}\pi}\left(\frac{A}{4{l_p}^4}\right)^{\frac{1}{2}} - \frac{\beta^4}{M_p^4} \\ & -\frac{\beta^4}{M_p^4}\ln\left[\left(\frac{128\pi^2}{M_p}\right)^{\frac{1}{4}} l_p \left(\frac{A}{4{l_p}^4}\right)^{\frac{1}{4}}\right]. \end{aligned} \tag{13}$$

- The corrections to the entropy from the linear GUP reads

$$\begin{aligned} S = \frac{A}{4l_p^4} + \frac{\alpha}{\sqrt{\pi}}\left(\frac{2}{3}\right)^{\frac{1}{4}}\left(\frac{A}{4{l_p}^4}\right)^{\frac{3}{4}} - \frac{(\alpha^2-\beta^2)}{\pi}\frac{3\sqrt{3}}{4\sqrt{2}}\left(\frac{A}{4{l_p}^4}\right)^{\frac{1}{2}} \\ -\frac{27\alpha^2\beta^2}{16\pi^2} + \frac{27\alpha^2\beta^2}{32\pi^2}\ln\left[\frac{\alpha}{M_p} + \frac{4\pi}{3M_p}\left(\frac{3}{2}\right)^{\frac{1}{4}}\left(\frac{A}{4{l_p}^4}\right)^{\frac{1}{4}}\right]. \end{aligned} \tag{14}$$

Entropy of the seven dimensional Schwarzschild BH

- Entropy of a black hole from the simplest GUP can be obtained as

$$S = \frac{A}{4l_p^5} - \frac{5\beta^2}{(4\pi)^{\frac{4}{5}}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{3}{5}} + 5\beta^2 \frac{M_p^3}{6\pi^2} \ . \tag{15}$$

- Corrections to the entropy from the linear GUP reads

$$\begin{aligned} S \ &= \ \frac{A}{4l_p^5} - 5\alpha \frac{(2)^{\frac{3}{5}}}{(\pi)^{\frac{2}{5}}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{4}{5}} + \frac{5\beta^2}{3(2\pi)^{\frac{4}{5}}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{3}{5}} \\ &+ \frac{45\alpha\beta^2}{(2)^{\frac{11}{5}}(\pi)^{\frac{6}{5}}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{2}{5}} + \frac{5\beta^4}{(2)^{\frac{8}{5}}(\pi)^{\frac{3}{5}}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{1}{5}} \\ &- \frac{19\alpha\beta^4}{8\pi^2} + \frac{3\alpha\beta^4}{4\pi^2} \ln\left[\frac{\alpha}{M_p} + (2\pi)^{\frac{2}{5}} \left(\frac{A}{4{l_p}^5}\right)^{\frac{1}{4}}\right] . \end{aligned} \tag{16}$$

Entropy for modified GUP in five dimensions

- A modified GUP containing a cubic and quadratic powers of the momentum uncertainty can be proposed as

$$\delta x \delta p \geq \frac{\hbar}{2}\left\{1+\frac{\beta^2 l_p^2}{\hbar^2}(\delta p)^2+\frac{\gamma^3 l_p^3}{\hbar^3}(\delta p)^3\right\} \quad (17)$$

  where γ is a suitable parameter in the theory.
- The entropy corresponding to this GUP can now be written as

$$S = \frac{A}{4{l_p}^3} - \frac{3\beta^2}{(16\pi)^{\frac{2}{3}}}\left(\frac{A}{4{l_p}^3}\right)^{\frac{1}{3}} + \frac{33\beta^2\gamma^3}{2(16\pi)^{\frac{5}{3}}}\left(\frac{A}{4{l_p}^3}\right)^{-\frac{2}{3}} - \frac{3\gamma^3}{16\pi} - \frac{2\gamma^3}{16\pi}\ln(4\pi) - \frac{2\gamma^3}{16\pi}\ln A \,. \quad (18)$$

- Coefficient of the logarithmic term depends upon the coefficient of the cubic power of the momentum uncertainty in this new GUP.

Constraints on rainbow gravity functions

Basics of rainbow gravity

- Rainbow gravity generalizes the modified dispersion relations in DSR to curved spacetime

$$E^2 f^2 (E/E_p) - p^2 g^2 (E/E_p) = m^2 c^4 \tag{19}$$

where E_p is the Planck Energy and the functions $f(E/E_p)$ and $g(E/E_p)$ are called rainbow functions.

- These functions are responsible for the modification of the energy-momentum relation in the ultraviolet limit.
- In the infrared limit, they reproduce standard dispersion relation

$$\lim_{E/E_p \to 0} f(E/E_p) = 1 \; ; \lim_{E/E_p \to 0} g(E/E_p) = 1 \; . \tag{20}$$

- In the limit $E/E_p \to 0$, usual general relativity is recovered.

- The field equations of Einstein are also modified as

$$G_{\mu\nu}(E/E_p) = 8\pi G(E/E_p)\, T_{\mu\nu}(E/E_p) \tag{21}$$

where the energy dependent Newton's universal gravitational constant $G(E/E_p)$ becomes the conventional Newton's universal gravitational constant $G = G(0)$ in the limit $E/E_p \to 0$.

- As a consequence, corresponding black hole metrics also get redefined.
- We have considered specific rainbow gravity functions which are motivated from Loop quantum gravity

$$f(E/E_p) = 1 \; ; \; g(E/E_p) = \sqrt{1 - \eta \left(\frac{E}{E_p}\right)^n} \tag{22}$$

where η is the rainbow parameter.

The Schwarzschild metric, Entropy

- The metric is being considered

$$ds^2 = -\frac{1}{f^2(E/E_p)}\left(1-\frac{2MG}{r}\right)dt^2 + \frac{1}{g^2(E/E_p)}\left(1-\frac{2MG}{r}\right)^{-1}dr^2 + \frac{r^2}{g^2(E/E_p)}d\Omega^2. \quad (23)$$

- The expression for the entropy can be recast in the form

$$S = \frac{A}{4} + \frac{\pi^{\frac{n}{2}}\eta}{(2-n)}\frac{1}{\left(\frac{A}{4}\right)^{\left(\frac{n}{2}-1\right)}} + \frac{3\pi^n\eta^2}{8(1-n)}\frac{1}{\left(\frac{A}{4}\right)^{(n-1)}} + \mathcal{O}(\eta^3) \quad (24)$$

where we have set $l_p = 1$.

- It is evident that the integration is not valid for $n = 1, 2$. That means there are no logarithmic corrections to the semi-classical result for the values of $n \geq 3$.

Entropy for $n = 1$ and $n = 2$

- Taking into account the universality of the logarithmic corrections then the values of n gets restricted to $n = 1, 2$.
- For $n = 1$, the entropy expression upto $\mathcal{O}(\eta^2)$ in terms of horizon area yields

$$S = \frac{A}{4} + \eta\sqrt{\pi}\sqrt{\frac{A}{4}} + \frac{3\pi\eta^2}{8}\ln\left(\frac{A}{4}\right) + \frac{3\pi\eta^2}{8}\ln(4\pi). \tag{25}$$

- For $n = 2$, the entropy expression upto $\mathcal{O}(\eta^2)$ becomes

$$S = \frac{A}{4} + \frac{\eta\pi}{2}\ln\left(\frac{A}{4}\right) + \frac{\eta\pi}{2}\ln(4\pi) - \frac{3\pi^2\eta^2}{8}\frac{1}{\left(\frac{A}{4}\right)} . \tag{26}$$

The RN metric and the Entropy

- The rainbow gravity inspired RN black hole metric reads

$$ds^2 = -\frac{1}{f(E/E_p)^2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{g(E/E_p)^2}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + \frac{r^2}{g(E/E_p)^2} d\Omega^2 \ . \quad (27)$$

- The expression for the entropy for a small argument η yields

$$S = S_{BH} + \frac{\pi^{\frac{n}{2}}\eta}{(2-n)}\frac{1}{S_{BH}{}^{(\frac{n}{2}-1)}} + \frac{3\pi^n\eta^2}{8(1-n)}\frac{1}{S_{BH}{}^{(n-1)}} + \mathcal{O}(\eta^3) \ . \quad (28)$$

- $S_{BH} = \pi r_0^2$ is the semi-classical Bekenstein-Hawking entropy for the rainbow gravity inspired RN black hole.
- Once again no logarithmic corrections are present in the expression for the entropy.

Entropy for $n = 1$ and $n = 2$

- For $n = 1$, the entropy expression upto $\mathscr{O}(\eta^2)$ in terms of horizon area yields

$$S = \frac{A}{4} + \eta\sqrt{\pi}\sqrt{\frac{A}{4}} + \frac{3\pi\eta^2}{8}\ln\left(\frac{A}{4}\right). \tag{29}$$

- For $n = 2$, the entropy expression upto $\mathscr{O}(\eta^2)$ becomes

$$S = \frac{A}{4} + \frac{\eta\pi}{2}\ln\left(\frac{A}{4}\right) - \frac{3\pi^2\eta^2}{8}\frac{1}{\left(\frac{A}{4}\right)}\ . \tag{30}$$

Conclusion

Conclusion

- Modifications of the thermodynamic properties of Schwarzschild and Reissner-Nordström black holes taking into account the effects of generalized uncertainty principle as well as rainbow gravity functions have been investigated.
- Findings of Reissner-Nordström black holes reduced to Schwarzschild black hole in the $Q \to 0$ limit.
- The leading-order corrections to the entropy of any thermodynamic system are logarithmic in nature to constraint the form of the GUP.
- In higher dimensional space time the presence of logarithmic term in the entropy expression does not refer to arbitrary choice of GUP.
- Rather the order(s) of the momentum uncertainty in GUP depends upon the dimension of the black hole.
- In all even dimensions the simplest form of the GUP might be enough to produce the correct form for the corrections to the entropy of a black hole.

Conclusion (cont.)

- In odd dimensions an odd power of momentum uncertainty in the GUP might be needed to produce the correct form for the corrections to the entropy of a black hole.
- The computation of the entropy does not contain the universal logarithmic corrections for all values of the parameter n appearing in the rainbow gravity functions.
- Only for the values of $n = 1, 2$ appearing in the rainbow gravity functions that the logarithmic corrections to the semi-classical area law for the entropy are found to exist.

This talk is based on the work I have done in collaboration with Dr. Sunandan Gangopadhyay.

References

- S Gangopadhyay, A Dutta, Mir Faizal, "Constraints on the Generalized Uncertainty Principle from Black Hole Thermodynamics"; 'EPL, 112 (2015) 20006'.
- S Gangopadhyay, A Dutta, "Constraints on rainbow gravity functions from black hole thermodynamics" ; 'EPL, 115 (2016) 50005'.

THANK YOU...