INVESTIGATIONS IN THREE DIMENSIONAL FIELD THEORIES

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (SCIENCE)
OF
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PRADIP MUKHERJEE

DEPARTMENT OF PHYSICS
A. B. N. SEAL COLLEGE, COOCH BEHAR

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**INVESTIGATIONS IN THREE DIMENSIONAL FIELD THEORIES**" submitted by **Sri Pradip Mukherjee** who got his name registered on 27.12.97 for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Dr. Rabin Banerjee** and that neither this thesis nor any part of it has been submitted for any degree / diploma or any other academic award any where before.

13.4.99.

DR. RABIN BANERJEE (Sole Supervisor)
Reader and Acting Administrative Officer
S. N. Bose National Centre for Basic Sciences
Block - JD, Sector III, Salt Lake
Calcutta 700 091

DR. RABIN BANERJEE
Reader in Physics and
Acting Administrative Officer
S N Bose National Centre for
Basic Sciences, Salt Lake
Calcutta-700091

In the fond memory of Late Smt. Krishna Basu

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Pradip Mukherjee

A. B. N. Seal College, Cooch Behar

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Chapter 1

OVERVIEW

1.1 Introduction

Classical and quantum field theories may be consistently formulated in arbitrary d dimensional space. Low dimensional field theories, for which $d \leq 3$, form an important class of such theories. Present challenges of theoretical physics come from the strongly interacting systems which are not amenable to solutions by ordinary perturbation techniques. The low dimensional theories serve as 'laboratories' where various nonperturbative methods have been developed and tested with comparatively lesser efforts. Also such low dimensional theories correspond to concrete situations in various phenomenology. In this dissertation we focus attention on (2+1)- dimensional field theoretic models which are being actively investigated in the recent times. Of course the scope of such studies is vast. Our research concentrates on certain aspects of three - dimensional field theories that are of current interest. Specifically, we concentrate on the Chern - Simons (C - S) coupled theories [1, 2]. The C - S interaction has numerous interesting properties [3], which make it useful in many branches of physics and mathematics. Below we mention a few of those properties which are related to our area of research. Subsequently an overview of the thesis is presented in the remaining sections.

The abelian C - S term in (2+1) dimensions is given by,

$$\mathcal{L}_{CS} = \frac{k}{4}\epsilon_{\mu\nu\lambda}A^{\mu}F^{\nu\lambda}, \tag{1.1}$$

where A^{μ} is the C - S gauge field and $F^{\mu\nu}$ is the corresponding field tensor,

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{1.2}$$

Here $\epsilon^{\mu\nu\lambda}$ is the totally antisymmetric symbol with $\epsilon^{012} = 1$. We work in a Minkowskian space with the diagonal metric tensor,

$$g_{\mu\nu} = \mathrm{diag}(1, -1, -1). \tag{1.3}$$

A nonabelian generalisation of (1.1) is described by the Lagrangian density,

$$\mathcal{L}_{CS}{}^{N} = \frac{k}{4}\epsilon_{\mu\nu\lambda}\mathrm{tr}(A^{\mu}F^{\nu\lambda} + \frac{2}{3}A^{\mu}A^{\nu}A^{\lambda}). \tag{1.4}$$

A^{μ} and $F^{\mu\nu}$ are now matrices,

$$A_{\mu} = A_{\mu a}T^{a} \tag{1.5}$$

and

$$F_{\mu\nu} = F_{\mu\nu a}T^{a} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}], \tag{1.6}$$

where T^{a} are the generators of the gauge group $\mathcal{G}$ in which A takes its values. T^{a} satisfy the Lie algebra of the nonabelian group $\mathcal{G}$,

$$[T^{a}, T^{b}] = f^{abc}T^{c}, \tag{1.7}$$

f^{abc} being the structure constants. The abelian C - S Lagrangian (1.1) is a special case of (1.4); the trilinear term in (1.4) vanishes due to the antisymmetry of the ϵ - symbol when the gauge group is abelian.

The C - S Lagrangian (1.1) may be reduced to the following form by neglecting total boundary terms [1],

$$\mathcal{L}_{CS} = kA_2\dot{A}_1 + kA_0F_{12}. \tag{1.8}$$

Equation (1.8) shows that the fields A_1 and A_2 are canonically conjugate to each other. The field A_0 is nothing but a Lagrange multiplier which enforces the Gauss law constraint. This situation is very different from the conventional Maxwell (Yang - Mills) gauge field where A_i may be regarded as coordinate fields, canonically conjugate to the electric field $E_i = F_{0i}$. The C - S gauge field thus has no dynamics of its own - it is a nonpropagating field whose dynamics comes from the fields to which it is coupled. A remarkable fallout of this is that the C - S gauge field can be coupled to both Poincare and Galileo symmetric theories, a unique feature in the class of gauge theories.

Let us now consider the coupling of the C - S gauge field with external source. A representative example is the nonlinear O(3) sigma model which admits a conserved current j^μ. The C - S term may be minimally coupled to the theory by extending the Lagrangian as

$$\mathcal{L}' = \mathcal{L} + j_\mu A^\mu + \mathcal{L}_{CS}, \tag{1.9}$$

where $\mathcal{L}_{CS}$ is given by (1.1). In case of coupling with complex scalars, the extension (1.9) is formally the same but the current j_μ is now improved so that it is covariantly conserved ($D_\mu j^\mu = 0$). The equation of motion satisfied by A_μ is obtained by the

[1] We are considering the abelian C - S term for simplicity, but all these remarks apply to the nonabelian generalisation (1.4).

usual method to be

$$\frac{k}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = j^{\mu}. \tag{1.10}$$

Using equation (1.10) A_μ can be expressed in terms of the variables of the matter sector $\mathcal{L}$ which shows in another way that the C - S field is devoid of any independent dynamics. However, the global topological effects associated with the C - S interaction leads to a spectrum of interesting phenomena with the inclusion of source j^μ. These include the generation of topologically massive gauge mode [2], transmutation of statistics [4, 5], production of novel solitonic excitations [6, 7, 8, 9] etc.

The action corresponding to the pure C - S term is generally covariant irrespective of any metric [10]. This leads to the development of the C - S theories as topological field theories [11]. The metric independence of the C - S action has crucial effect on the form of the symmetry generators of the theory (1.9). These symmetry generators are constructed from the symmetric, gauge invariant energy momentum (EM) tensor obtained by varying the action with respect to a background metric,

$$\Theta^{\mu\nu} = \frac{\delta \mathcal{A}}{\delta g_{\mu\nu}}. \tag{1.11}$$

and finally setting the metric flat. Due to the metric independence, the pure C - S part of the action does not contribute to Θ. Nevertheless, it contributes to the EM tensor obtained from Noether's definition. A clean method of studying the symmetry of a gauge theory is provided by the gauge independent approach [12, 13] where the constraints are only weakly zero [14]. Thus the EM tensor is required to be augmented by some suitable linear combinations of the constraints of the theory to generate appropriate symmetry generators [15]. Using this freedom it is possible to understand the difference between (1.11) and Noether's definition, and to formulate a consistent definition of intrinsic spin [16]. This connection is explicitly discussed in section 2 of this chapter where our studies of the spin of the C - S vortices are outlined.

The Euler - Lagrange (E - L) equations following from the Lagrangian (1.9) are nonlinear in general. These nonlinear field equations admit finite energy solutions which correspond to extended structures moving without dissipation [17]. Solitons, instantons etc. are such solutions which have become paradigms of modern field theory [18]. Their quantization crucially depends on the solutions of the classical E - L equations. Such solutions are however very difficult to find. In this context, a class of theories enjoy a very prominent place where a set of first order equations may be found which satisfies the second order E - L equations. The first order equations so obtained are observed to exhibit some kind of self - duality. In this sense the corresponding theories may be called self - dual theories. Henceforth in the following the term 'self - dual' will be used to refer to the cases where the equations of motion may be factorised in first order forms [3]. From (1.10) we find that in C - S theories the equations of motion satisfied by the gauge - field is already in the self - dual form. This may be contrasted with the Maxwell (Yang - Mills) case where the gauge field satisfies second order equations of motion. The C - S coupled theories are thus prone to provide a large variety of self - dual theories [3]. Since the soliton sector of the C - S theories provide candidate excitations for the realization of anyons, the self - dual C - S models are extremely important in the context of anyonic quantum field theory with applications to such planar models as quantum Hall effect, anyonic superconductivity etc. Connections of the self - dual theories with related extended supersymmetric structures have also been revealed. We have worked on a new type of self - duality in connection with the partially gauged O(3) nonlinear sigma models , an overview of which is presented in section 3 of this chapter.

The C - S theories (1.9) being gauge theories, are necessarily constrained. Using the reduction (1.8) it is easy to read off the Gauss law constraint which generates gauge transformations at a certain instant of time in the Hamiltonian approach [14, 19]. A suitable gauge fixing condition is to be invoked to eliminate the redundancy of the theory arising out of the gauge symmetry. This amounts to reduction of the phase space using the chosen gauge. One way of the reduction of phase space is to solve the Gauss constraint along with the gauge condition, thereby eliminating the C - S gauge field in terms of the matter variables. This is the symplectic method [20] of phase space reduction. Alternatively, one may follow Dirac's method [14] of constrained Hamiltonian analysis. It is instructive to compare the different reduced phase space calculations. We have studied the aspects of symmetry of a nonrelativistic matter theory coupled with the nonabelian C - S term (1.4), using alternative methods of phase space reduction. Outline of this work is contained in section 4 below.

1.2 Analysis of spin in (2+1)-dimensional field theories coupled with the Chern - Simons term

Perhaps the most interesting and fascinating aspect of (2+1)- dimensional models is the possibility of existence of particles with arbitrary spin and statistics, interpolating continuously between bosons and fermions. This may be understood from the following group- theoretic arguments [10]. The rotation group in two dimensions is the abelian group U(1). The group elements satisfy no non - trivial algebra and therefore study of infinitesmal rotations does not fix the possible representations as in the three dimensional analogue. However, a finite rotation by 2π must bring all the physical quantities to their initial setting. Since the squared modulus of the wavefunction is physically observable only, the wave function may gain an extra phase $e^{i\theta}$ on rotation by 2π where θ is any real number. The corresponding angular momentum eigenvalues (J) are then given by

$$J = \hbar(\frac{\theta}{2\pi} + n), \tag{1.12}$$

where n is an integer. We find from (1.12) that any real spin is allowed. The number θ is defined modulo 2π and $\theta = 0$ (π) correspond to bosons (fermions). In analogy with bosons and fermions, particles carrying arbitrary real spin are called anyons. Traditionally the anyons are said to have fractional spin though any real spin value is actually allowed [21]. By a generalised spin-statistics connection [22] the anyons obey fractional (arbitrary) statistics. Fractional spin and statistics is a comparatively new idea in physics [23] which has been made popular by the pioneering works of Wilczek [24]. Since then the idea has been vigorously pursued in the literature [10, 21, 25]. Indeed, this is not only due to the mathematical curiosity involved. Anyons have been found useful in understanding fractional quantum Hall effect and they are believed to be responsible for high- T_c superconductivity [10, 26, 27].

It is natural to expect that the full significance of fractional spin and the spin-statistics connection may be understood in the context of relativistic field theories. From the quantum mechanics of anyons it is known that fractional spin and statistics is induced when the configuration space is multiply connected. So in field theories we look for field configurations that fall in disjoint classes. The nonlinear field equations do admit soliton solutions that are classified into topologically distinct classes in various (2+1)-dimensional field theories [17]. Such theories, thus, appear to be candidates for providing anyon - like excitations.

A well - known example of a (2+1)-dimensional field theoretical model supporting topologically stable soliton solutions is the **non- linear** O(3) sigma model [28]. The Lagrangian of the model is given by,

$$\mathcal{L} = \frac{1}{2f}\partial_\mu n^a \partial^\mu n^a, \tag{1.13}$$

where $n^a(x)$ are a triplet of fields satisfying,

$$n^a n^a = 1 \tag{1.14}$$

and f is a parameter having dimension of length.

From equation (1.14) we find that the components n^a define a vector in the internal space, always confined on the unit sphere with centre at the origin. Static solutions of the problem are mappings from the physical space to the internal space. For the finiteness of energy of the soliton configurations , we require,

$$n^a = n_0^a \tag{1.15}$$

at spatial infinity, where n_0^a is constant. The physical space is thus one point compactified to S^2. The static field configurations are then maps $n : S^2 \to S^2$ which fall into distinct homotopy classes [17, 29]

$$\Pi_2(S_2) = Z. \tag{1.16}$$

Time - dependent solutions are obtained from static soliton solutions by letting the location of the solitons depend on time. The corresponding current,

$$j^\mu(x) = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda}\epsilon_{abc} n^a \partial_\nu n^b \partial_\lambda n^c, \tag{1.17}$$

is identically conserved irrespective of the equations of motion. We then get a conserved charge,

$$Q = \frac{1}{8\pi}\int d^2x \epsilon_{abc}\epsilon^{ij}\partial_i n^b \partial_j n^c n^a \tag{1.18}$$

which gives the winding number of the mapping (1.16). Since Q characterizes the homotopy classes it is called the topological charge. Accordingly, j^μ is the topological current. The soliton solutions of the model (1.13) with Q $=$ 1 are called baby skyrmions.

The time dependent solutions of (1.13) are mappings from S^3 to S^2. Here closed curves on S^3 are mapped onto points of S^2 and there exists an invariant called the Hopf invariant, which classifies the space of field histories in distinct homotopy classes,

$$\Pi_3(S_2) = Z. \tag{1.19}$$

Analytically the Hopf invariant is given by the Hopf interaction,

$$H = -\frac{1}{2\pi}\int d^3x A_\alpha j^\alpha, \tag{1.20}$$

with A_α defined implicitly through

$$j^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{1.21}$$

The nonlinear O(3) sigma model (1.13) with the addition of the Hopf interaction (1.20) becomes,

$$\mathcal{L} \to \mathcal{L} + \theta H, \tag{1.22}$$

where θ is an arbitrary real parameter. The resulting theory is non - local because A_μ defined via (1.21) is essentially a non - local function of n^a. The soliton solutions of the extended theory are endowed with fractional spin $\frac{\theta}{2\pi}$ [4]. Alternatively, adding the following terms

$$\Delta\mathcal{L} = -\frac{\theta}{\pi}(j^\mu A_\mu - \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda) \tag{1.23}$$

with the Lagrangian (1.13) does the trick [4, 30]. The kinetic term for the gauge field is easily recognised to be the celebrated Chern- Simons term (1.1). The topological solitons of the (2+1)- dimensional O(3) sigma model are thus endowed with fractional spin when coupled with the C - S gauge field. The model has been treated in the Hamiltonian approach a la Dirac [14] where the structure of the canonical (Noether) angular momentum has been shown to be shifted due to the inclusion of the Hopf interaction [31]. A gauge- independent analysis [12, 13] shows conclusively that the fractional spin induced is a physical effect and not an artifact of the gauge [16].

Subsequently, it has been found that the C - S coupling is instrumental in inducing fractional spin in a number of field theoretic models [32]. The study of the spin of the topological solitons of the C - S theories is therefore very important in the context of anyon physics. There are various approaches of computing the spin of the Chern - Simons vortices but the results from different methods do not always agree [33]. In the original method proposed by Wilczek and Zee [4] it is shown that the wave function of a Q=1 soliton in the model becomes a multivalued function for adiabatic space rotation of the soliton. If the wave- function is n-ply valued for the space rotation, then the value of the spin for the soliton is 1/n. In another method the field angular momentum for a static soliton configuration is calculated and any nonzero expression thus obtained is identified with the spin of the soliton. The charged vortex, which is a topological soliton in the (2+1)-dimensional Chern-Simons-Higgs model, was shown to have fractional spin basing on this consideration [8, 9, 34]. However, it happens that the spin value obtained in the static limit of the field angular momentum does not always give consistent results [33]. A remarkable example is the nonlinear O(3) sigma model coupled with the Hopf or Chern - Simons term. Since the Hopf or C - S action is generally covariant without reference to the metric, the EM tensor constructed according to (1.11) does not get any contribution from these terms.The resulting expression is the same as that found for the usual sigma model (1.13),

$$\Theta^s_{\mu\nu} = \frac{2}{f^2}(2\partial_\mu n^a \partial_\nu n^a - g_{\mu\nu}\partial_\alpha n^a \partial_\alpha n^a). \tag{1.24}$$

This EM tensor is symmetric, gauge invariant and appears in the Dirac - Schwinger quantum conditions [35]. We will henceforth refer to this as the Schwinger EM tensor. The angular momentum integral is obtained from the Schwinger EM tensor as

$$J = \int d^2\mathbf{x}\epsilon_{ij}x_i\Theta^s_{0j}. \tag{1.25}$$

Since the (0,j) component of $\Theta^s_{\mu\nu}$ explicitly involves a time derivative of n^a, J vanishes in the static configuration. The definition of [8, 9, 34] then predicts zero spin of the baby skyrmions. However, we have just observed that it follows from quite general topological arguments that they carry fractional spin [4]. The connection of the baby skyrmions with the quasiparticles found in the quantum Hall state has been established, where the anomalous spin of the excitations play a crucial role [36]. So the vanishing spin of the baby skyrmions predicted by the definition of [8, 9, 34] is clearly a contradiction. The static limit of the angular momentum cannot thus be identified with fractional spin consistently. The reason for this may be ascribed to the fact that unlike the Noether angular momentum, the Schwinger angular momentum does not naturally split in orbital and spin parts [37]. The method followed in [8, 9, 34] involves integration over detailed field configuration for which one needs to resort to some suitable ansatz. The analysis depends again on how one accounts for the contribution from the singularities of the configuration which is why the results obtained by the same method do not always agree [8, 34].

A new field theoretic method of computation of the spin of the C - S vortices was advanced in [16] in connection with the nonlinear O(3) sigma model coupled with the C - S term. The method is based on constraints, the role of which proves crucial in generating space time symmetry transformations of the fields. The analysis was carried out in a gauge independent setting [12, 13] and the Schwinger EM tensor was required to be improved by suitable linear combinations of the constraints of the theory [15] to generate proper transformations of the fields. The EM tensor obtained from the canonical (Noether) prescription was then constructed. The angular momentum obtained from the (improved) Schwinger EM tensor was found to differ from the angular momentum obtained from the Noether EM tensor (J_c) by a total boundary term K,

$$K = -\theta \int d^2x \partial_i (x^i A_j A^j - x^j A^i A_j). \tag{1.26}$$

The boundary term does not offend local transformations of the fields under rotation and so J, like J_c, generates appropriate transformation of the fields under rotation For nonsingular field configurations the difference term vanishes but for the singular vortex structures, it gives nonvanishing contribution. The value of the difference of J and J_c in a rotationally symmetric configuration is found to be independent of the origin of coordinates. It is, therefore, interpreted as the spin of the vortices Equation (1.26) serves as the spin formula. Note that for the calculation of spin from the definition (1.26) only the asymptotic configuration of the gauge field is necessary. This form is dictated by the requirements of rotational symmetry and the Gauss constraint of the theory to be

$$A^i(x) = \frac{Q}{2\pi} \epsilon_{ij} \frac{x^j}{|x|^2}, \tag{1.27}$$

where Q is the topological charge (1.18). Using equation (1.26) we get

$$K = \frac{Q^2}{2\pi} \theta. \tag{1.28}$$

For $Q = 1$ this result agrees with the earlier computations [4, 31]. The definition of [16], unlike [8, 9, 34] yields results in this example which are compatible with general topological arguments [4] and is in conformity with the experimental observations [36].

We have seen that the definition of [16] gives a consistent method of finding the spin of the baby skyrmions, which are the $Q = 1$ excitations of the nonlinear O(3) sigma model coupled with the C - S term. The question comes whether this scheme is applicable to the C - S theories in general. This question has been addressed in our works on the C - S vortices [38, 39]. We found that the definition of [16] can be used to develop a general method of calculation of the spin of the C - S vortices, both relativistic and nonrelativistic.

Among the relativistic C - S theories the C - S - H model is a prominent one. The theory may be reduced to the self - dual form. Remarkably, the model provides both topological and nontopological solitons [34]. Using the definition of [16] we find a spin formula identical with (1.26) and the value of the spin for the topological solitons comes out to be

$$K = \frac{\pi k n^2}{e^2}, \tag{1.29}$$

where e is the coupling constant, k the strength of the C - S interaction and n is the topological number. The similarity of the expression (1.29) with equation (1.28) is easily recognised. The signature of the spin [40] is the same as that of the elementary excitations of the model [41]. In the earlier computation an opposite sign was found where the static limit of the angular momentum was identified as the spin. As a result the spin - statistics connection was not possible to establish from the usual Aharonov - Bohm [42] phase and a new interaction had to be invoked [43] to account for the extra phase.

We have investigated the spin of the vortices of a number of relativistic C - S models. These include abelian and nonabelian [7] C - S - H models and a generalisation of the C - S - H model[44]. In all the cases the spin formula were found to be identical in form with (1.26) and the resulting value of the spin were found to be given by (1.29) irrespective of the details of the matter sector. The topological origin of the spin of the C - S vortices was thus clearly revealed.

The method of [16] is not directly applicable to the nonrelativistic models. This is because the Galileo symmetric models cannot be made generally covariant and the powerful method of constructing the Schwinger EM tensor is not available. However, a gauge invariant EM tensor can still be constructed by using the equations of motion [2][45]. The spin of the nonrelativistic C - S vortices was defined as,

$$K = J - J^N, \tag{1.30}$$

replacing the Schwinger angular momentum in the earlier definition by J, the gauge invariant EM tensor obtained from the equations of motion [39]. A unified method was thus developed to analyze the spin of the vortices of the relativistic and nonrelativistic C - S theories. There are some calculations of the spin of the nonrelativistic C - S vortices in the literature [46, 47] but a general methodology was lacking. We have applied our definition to the nonrelativistic model of a Schrodinger field coupled with the C - S term [48]. The analysis of spin was found to be in complete parallel with that of relativistic models.

[2]This time the EM tensor is not symmetric as a consequence of the specific nature of the Galilean space - time.

1.3 Exploration of a new self - duality in connection with the O(3) nonlinear sigma model in (2+1)-dimensions

Investigations of anyonic excitations in relativistic field theories bring us in contact with the study of solitons in (2+1)-dimensional field theories coupled with the Chern - Simons (C - S) term. Classically, the soliton configurtions are to be obtained by solving the E - L equations. These second order nonlinear equations are, however, very difficult to solve. In this context the self - dual theories enjoy a unique place because here we obtain a set of first order equations that automatically satisfy the E - L equations. The C - S theories are all the more suitable for building self - dual theories because here the E - L equation satisfied by the gauge field is already in the first order, self - dual form. In this dissertation we discuss a new type of self - duality in the nonlinear O(3) sigma model [49, 50].

In the usual (2+1)-dimensional nonlinear O(3) sigma model (1.13) discussed earlier in this chapter, the self - dual point is achieved by minimising the energy functional corresponding to the static field configuration in a given topological sector characterised by a fixed value of the topological charge (1.18). The trick of minimisation is due to Bogomol'nyi [51] which has become a standard for the self - dual theories in general. The Bogomol'nyi conditions lead us to the first order self - dual equations, which for the model (1.13) are exactly integrable. The self - dual configurations are topological lumps which are characterized by the the second homotopy of the field as a mapping from the physical space to the internal space, equation (1.16). The solutions are obtained in terms of rational functions and enjoy scale invariance. Due to this scale invariance the excitations may change shape arbitrarily without loss of energy. Numerical simulations amply confirm this [52]. This leads to a difficulty of particle interpretation of these excitations on quantization.

Various methods have been proposed in the literature to break the scale invariance of the self dual solitons of the nonlinear sigma model [53, 54]. Of them, a particularly interesting method is to partially gauge the global O(3) symmetry of the model. By gauging a part of the global symmetry of (1.13) with the Maxwell [54] or the C - S interaction [55], one can have another self - duality. In order to satisfy the Bogomol'nyi bounds, one requires to include a suitable self - interaction in the extended model. Such models will be referred to as the gauged nonlinear O(3) sigma models.

Both the models of [54] and [55] share a common feature - their soliton sectors suffer from the problem of infinite degeneracy. In case of Maxwell coupling [54] we get neutral solutions with quantized energy in a given topological sector but having *arbitrary* magnetic flux. For the theory with the C - S coupling the solutions become charge - flux composites with both charge - flux and angular momentum *arbitrary* for fixed topological charge [55]. We consider this degeneracy physically undesirable, specifically in the context of our experience with the C - S vortices [38, 39]. The degeneracy of solutions actually frustrates the motivation for gauging because the scale invariance appears in a mutated form as a result of the arbitrariness of the magnetic flux [54]. We naturally enquire whether it is possible to construct gauged O(3) nonlinear sigma models which will have the welcome features of self - dual solutions with broken scale - invariance *without* the degeneracy. We have demonstrated this possibility first for the C - S coupled theory [49] and later on, for the Maxwell coupling [50].

Indeed, the degeneracy of the solutions is not an essential outcome of the problem. We have shown that it is actually connected with the structure of the vaccum. If the self interaction potential is so chosen to allow symmetry breaking vaccua the degeneracy of the solitons is lifted [49, 50]. The corresponding models differ significantly from the earlier gauged sigma models [54, 55] due to the novel topology introduced in the process. The homotopy sectors of the solutions of [54, 55] are classified by (1.16) just as the usual sigma model. In contrast, the solutions of our models [49, 50] fall into the homotopy

$$\Pi_1(S_1) = Z, \tag{1.31}$$

which proves to be responsible for the lifting of the degeneracy. These new features in our models lead to new type of self - dual soliton configurations in connection with the gauged O(3) nonlinear sigma model in (2 + 1) dimensions.

We have performed detailed investigations of the new gauged sigma models with symmetry breaking vaccua. Topological classification of the soliton solutions were studied along with the general properties of the solutions like charge, flux, energy and spin. Saturation of the self dual limits was explicitly demonstrated. The corresponding self - dual equations have solutions with broaken scale invariance as desired. We have also studied the integrability of the solutions and discussed numerical methods of solving the same taking the model with the Maxwell coupling as example.

1.4 Analysis of space - time symmetries in a nonabelian Chern - Simons theory

The C - S theories are necessarily constrained systems on account of the gauge symmetry. The issue of symmetry of the gauge theories is a subtle one because the imposition of the gauge condition forces the explicit transformation relations of the fields to deviate from the canonical structure. Mention may be made of such a well known theory as the Maxwell electrodynamics [15]. The example of the C - S theories are special because the C - S gauge field is nondynamical and can be eliminated from the phase space by solving the Gauss constraint along with the gauge fixing condition [45]. Alternatively, the phase space reduction can be carried out [56] by Dirac's method of constrained Hamiltonian analysis [14]. While these approaches give the same result for the abelian models, the nonabelian C - S theories offer a few surprises. It was observed [57] that the (classical) Poincare covariance gets violated in a theory where the nonabelian C - S term is coupled with fermions. The calculations were done in the axial gauge which enabled the elimination of the gauge degrees of freedom in terms of the matter variables. However, the (classical) Poincare invariance was shown to be preserved [58] by following Dirac's method which retains all degrees of freedom. The issue of symmetry is even less transparent in the quantum level, due to the ordering ambiguities [48, 59]. The C - S term may be coupled with both Poincare and Galileo symmetric models. Purely Galilean invariant models are useful to study problems which are difficult to study within the full formalism of special relativity. We have, therefore, analysed in details, the classical and quantal Galilean algebra for a model involving the coupling of nonabelian C - S three form with nonrelativistic matter [60]. A gauge independent formulation [12, 13] of the canonical constrained structure was carried out which proved immensely helpful in the verification of the

Galilean algebra avoiding the problems related with gauge fixing. In the classical level the gauge fixed computations were done in both symplectic [20] and Dirac [14] approaches. By demanding the equivalence of different approaches some restrictions on the Green function were established. These conditions were again proved to be instrumental in closing the Galilean algebra in the quantum case in the gauge fixed approach. It was NOT necessary to assume vanishing self interaction as in the earlier approaches,since this condition was validated by two alternative arguments based on algebraic consistency.

1.5 Organisation of the thesis

We are now in a position to describe the organisation of the thesis. In the next chapter the analysis of the spin of the Chern - Simons vortices is presented. Chapter 3 contains our works on the partially gauged sigma models. Study of the Galilean symmetry of a nonabelian Chern - Simons matter system constitute the subject of chapter 4. Finally in chapter 5 some observations regarding the possible elaborations and new directions of research following from our studies has been discussed.The thesis is based on the results of the following works [3]:

[1] R. Banerjee and P. Mukherjee, Spin of the Chern - Simons Vortices [38].

[2] R. Banerjee and P. Mukherjee, Some comments on the Spin of the Chern - Simons Vortices [39].

[3] P. Mukherjee, On the question of degeneracy of topological solitons in a gauged O(3) sigma model [49].

[3]Numbers in the parenthesis give bibliographical references

[4] P. Mukherjee, Magnetic vortices in a gauged O(3) sigma model [50].

[5] R. Banerjee and P. Mukherjee, Galilean symmetry in a nonabelian Chern - Simons matter system [60].

Chapter 2

ANALYSIS OF SPIN OF THE CHERN - SIMONS VORTICES

2.1 Introduction

This chapter contains a detailed discussion of our works [38, 39] on the spin of the Chern - Simons (C - S) vortices. The different techniques of obtaining the spin of the solitonic excitations of the C - S theories were reviewed in the previous chapter. It was argued that the definition of [16] appears to be a consistent field theoretic method of computing the spin of the C - S vortices. We adopt this definition as the basis for our analysis. After a brief discussion of the method we demonstrate in detail, its application to the Chern - Simons - Higgs (C - S - H) model [8, 9 34] [1] Spin of both the topological and the nontopological solitons of the model is discussed

[1] Related analysis of the constraint structure of the model and its classical Poincare covariance are discussed in the appendices

and the effect of inclusion of the Maxwell term is examined. In the next section the analysis is extended to a generalisation of the C - S - H model [44]. Subsequently, we resort to the nonabelian C - S - H model [7] to demonstrate how our analysis fits into the nonabelian matter system coupled with the C - S term.

The method of [16] is suitable for relativistic field theories. But the topological C - S interaction can be coupled to both Poincare and Galileo symmetric models [45, 56, 48, 61]. The latter possibility is very useful in view of the applications of the C - S theories in condensed matter physics, which is essentially nonrelativistic. A nonrelativistic generalisation of the method of [16] was therefore developed [39] This has been discussed in the last section of the present chapter.

2.2 The method

To put our method in the proper perspective we first recall the usual method of defining the spin of the C - S vortices. Here the total angular momentum in the static soliton configuration is defined to be the spin of the vortices [8, 9, 34]. This angular momentum integral is constructed from the symmetric energy momentum (EM) tensor obtained by varying the action with respect to a background metric Since this EM tensor is relevant in formulating the Dirac - Schwinger conditions [35] it will henceforth be referred as the Schwinger EM tensor. Correspondingly, the angular momentum following from this EM tensor usually goes by the name of Schwinger. It is both symmetric and gauge invariant , and also occurs naturally in the context of the general theory of relativity. For these properties it is also interpreted as the physical angular momentum. Contrary to the Noether angular momentum, however, the Schwinger angular momentum does not have a natural splitting into an orbital and a spin part [37]. Thus it is not transparent how the value of the angular momentum in the static limit may be identified with the intrinsic spin of the vortices.

Now an alternative field-theoretic definition of the spin of the C - S vortices was given in [16] which has been applied successfully in the recent past [62], and will also be used in the present thesis. Here one abstracts the canonical part from the physical angular momentum. The canonical part is obtained by using the conventional Noether definition. Both the canonical as well as the physical angular momentum are obtained from the improved versions of the corresponding EM tensors to properly account for the constraints of the theory. Now the Noether angular momentum contains the orbital part plus the contribution coming from the spin degrees of freedom as appropriate for generating local transformation of the fields under Lorentz transformations [63]. Its difference from the physical angular momentum is shown to be a total boundary containing the C - S gauge field only, the value of which depends on the asymptotic limit of the C - S field. This difference term does not affect local transformation of the fields under rotation. It vanishes for nonsingular configurations. However, for the C - S vortex configurations we get a nonzero contribution. This contribution is found to be independent of the origin of the coordinate system. Consequently we interpret it to be the intrinsic spin of the C - S vortices. The connection of the anomalous spin with the topological C - S interaction is thus clearly revealed.

2.3 Spin of the solitons of the Chern - Simons - Higgs model

Let us first consider the Chern - Simons - Higgs (C - S -H) Lagrangian considered in [8, 9, 34],

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) + \frac{k}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - V(|\phi|), \tag{2.1}$$

where the covariant derivative is defined by,

$$D_\mu = \partial_\mu + ieA_\mu \tag{2.2}$$

and the potential $V(|\phi|)$ has a symmetry breaking minimum at $|\phi| = v$ containing only renormalisable interactions [34]. In the assymetric phase of the theory there exists topologically stable vortex solutions whereas in the symmetric phase there is no topological invariant. However, there exists nontopological vortices in this phase whose stability is ensured by the equations of motion. Nontopological solitons of nonzero vorticity are called nontopological vortices. We thus have topological and nontopological vortices in the model.

By neglecting total boundary terms the Lagrangian (2.1) can be written explicitly in the symplectic form,

$$\mathcal{L} = \pi\dot{\phi} + \pi^*\dot{\phi}^* + kA_2\dot{A}_1 - (\pi\pi^* - D_i\phi^* D_i\phi) + A_0[k\epsilon^{ij}\partial_i A_j + ie(\phi\pi - \phi^*\pi^*)], \tag{2.3}$$

where π (π^*) is the momentum conjugate to ϕ (ϕ^*),

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^* - ieA_0\phi^*, \tag{2.4}$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi} + ieA_0\phi. \tag{2.5}$$

From equation (2.3) we can immediately read off the basic equal time brackets

$$\{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \{\phi^*(\mathbf{x}), \pi^*(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \tag{2.6}$$

$$\{A_i(\mathbf{x}), A_j(\mathbf{y})\} = \frac{1}{k}\epsilon_{ij}\delta(\mathbf{x} - \mathbf{y}) \tag{2.7}$$

according to the symplectic arguments [2][20]. The field A_0 is observed to be a Lagrange multiplier which enforces the Gauss constraint,

$$G = k\epsilon^{ij}\partial_i A_j + ie(\phi\pi - \phi^*\pi^*) \approx 0. \tag{2.8}$$

The energy momentum tensor following from Schwinger's [35] definition is given by,

$$\Theta_{\mu\nu} = \frac{\delta\mathcal{A}}{\delta g^{\mu\nu}}, \tag{2.9}$$

where $\mathcal{A}$ is the action of the model (2.1). By a straightforward calculation we get,

$$\Theta_{\mu\nu} = (D_\mu\phi)(D_\nu\phi)^* + (D_\mu\phi)^*(D_\nu\phi) - g_{\mu\nu}[(D_\sigma\phi)^*(D^\sigma\phi) - V(|\phi|)], \tag{2.10}$$

where the Chern-Simons term, being covariant without reference to the metric, does not contribute. In the presence of the constraint (2.8) a more general expression for $\Theta_{\mu\nu}$ follows. This is called the total energy momentum tensor [15],

$$\Theta^T_{\mu\nu} = \Theta_{\mu\nu} + \wedge_{\mu\nu}G \tag{2.11}$$

where $\wedge_{\mu\nu}$ are multipliers that can be fixed by requiring that the fields transform normally under the various space - time generators [12, 15]. Note that since G is the generator of time - independent gauge transformations, the gauge invariance of $\Theta^T_{\mu\nu}$ is preserved on the constraint surface, i.e.,

$$\{\Theta^T_{\mu\nu}, G\} \approx 0. \tag{2.12}$$

[2] An alternative derivation of the basic brackets based on Dirac's method of Constrained Hamiltonian analysis is presented in appendix 1

Since we are concerned with the angular momentum operator defined by,

$$J = \int d^2\mathbf{x}\epsilon^{ij} x_i \Theta^T_{0j}, \tag{2.13}$$

let us concentrate on the $(0j)$ component of (2.11). It is easy to verify, using the algebra (2.6), (2.7) and the relations (2.10), (2.11), that the correct transformations under spatial translations,

$$\{\chi, \int \Theta^T_{0i}\} = \partial_i \chi, \tag{2.14}$$

where χ generically denotes the basic fields $(\phi, \phi^*, \pi, \pi^*, A_i)$, are obtained for the choice,

$$\wedge_{0i} = -A_i. \tag{2.15}$$

One can check that with $\wedge_{0\mu} = -A_\mu$ the correct transformation properties follow under all the space time generators (translations, rotations, boosts) defined from (2.11)[3]. Hence the desired structure of J simplifies to,

$$J = \int d^2\mathbf{x}\epsilon^{ij} x_i [\pi \partial_j \phi + \pi^* \partial_j \phi^* - k\epsilon^{lm} A_j \partial_l A_m]. \tag{2.16}$$

We now focus on the canonical angular momentum which requires Noether's definition of the energy momentum tensor,

$$\begin{aligned} \Theta^N_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} \partial_\nu \phi^* + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\sigma)} \partial_\nu A^\sigma - g_{\mu\nu}\mathcal{L} \\ &= (D_\mu \phi)^* \partial_\nu \phi + (D_\mu \phi) \partial_\nu \phi^* + \frac{k}{2} \epsilon_{\rho\mu\sigma} A^\rho \partial_\nu A^\sigma - g_{\mu\nu}\mathcal{L}. \end{aligned} \tag{2.17}$$

Note that, in contrast to the case of Schwinger's definition (2.10), the Chern-Simons term does contribute to $\Theta^N_{\mu\nu}$. Once again it is possible to define, in analogy with (2.11), a total (canonical) energy momentum tensor by addition to (2.17) a term proportional to the Gauss constraint. This time, however, such a term is absent

[3]The corresponding calculations are shown in appendix 2

because the correct transformations are already generated by (2.17). Hence the canonical (Noether) angular momentum can be directly obtained from (2.17),

$$J_{ij}^N = \int d^2\mathbf{x}(x_i\Theta_{0j}^N - x_j\Theta_{0i}^N + \frac{\partial\mathcal{L}}{\partial(\partial^0 A^k)}(g_i^k g_j^l - g_j^k g_i^l)A_l), \tag{2.18}$$

which can be simplified in terms of the single component of J_{ij}^N as,

$$\begin{aligned} J^N &= \int d^2\mathbf{x}\epsilon^{ij}(x_i\Theta_{0j}^N + \frac{k}{2}\epsilon_{il}A^l A_j) \\ &= \int d^2\mathbf{x}[\epsilon^{ij}x_i(\pi\partial_j\phi + \pi^*\partial_j\phi^* - \frac{k}{2}\epsilon_{lm}A^l\partial_j A^m) + \frac{k}{2}A^j A_j], \end{aligned} \tag{2.19}$$

where we have used (2.17) for simplifications.

Let us next compute the difference between the two angular momentum operators (2.16) and (2.19),

$$K = J - J^N = -\frac{k}{2}\int d^2\mathbf{x}\partial^i[x_i A_j A^j - A_i x_j A^j], \tag{2.20}$$

which can be further expressed in a compact form,

$$K = -\int d^2\mathbf{x}\partial^i[\pi_i\epsilon^{jk}x_j A_k], \tag{2.21}$$

where

$$\pi_i = \frac{\partial\mathcal{L}}{\partial\dot{A}^i} = \frac{k}{2}\epsilon_{ij}A^j, \tag{2.22}$$

is the momentum conjugate to A^i. The above relation is effectively a constraint which has been accounted by the symplectic bracket (2.7)[4]. It is now observed that the difference K between the two expressions of angular momentum is a boundary term. For nonsingular configurations this vanishes. For singular configurations however, K need not vanish. This is precisely what happens for the Chern-Simons vortices.

[4]See appendix 1

For topological vortices of the model (2.1) the matter field ϕ bears a representation of the broken U(1) symmetry,

$$\phi \approx v e^{in\theta}, \tag{2.23}$$

where n is the topological charge. The requirement of finite energy of the configuration dictates that asymptotically,

$$eA_i = n\epsilon_{ij}\frac{x^j}{|x|^2}. \tag{2.24}$$

The above form is rotationally symmetric and obeys Gauss law. As a consequence the magnetic flux is quantized,

$$\Phi = \int d^2\mathbf{x}\epsilon_{ij}\partial_i A_j = \frac{2\pi n}{e}. \tag{2.25}$$

After a straightforward calculation using (2.20) and (2.24), we obtain

$$K = \frac{\pi k n^2}{e^2}. \tag{2.26}$$

Note that our analysis leading to equation (2.26) does, at no point presupposes any specific form of the self - interaction $V(|\phi|)$ except for the general requirements that it has symmetry breaking minima at $|\phi| = v$ and contains only renormalizable interactions. Particularly, it was not assumed that the self - dual point is saturated. Thus the expression for spin, equation (2.26), holds for the topological solitons of the model in general.

When the potential is chosen so as to satisfy the self - dual limits, a class of vortex solutions to the self - dual equations are obtained whose stability is ensured by the equations of motion and not by any topological criterion [34]. These solutions correspond to the symmetric minimum of the self interaction and are termed nontopological vortices. For these the magnetic flux Φ is arbitrary. The asymptotic form of the gauge field is now expressed as ,

$$A_i = \frac{\Phi}{2\pi}\epsilon_{ij}\frac{x^j}{|x|^2} \tag{2.27}$$

and the spin computed from (2.20) is

$$K = \frac{k\Phi^2}{4\pi}. \tag{2.28}$$

Equations (2.26) and (2.28) give the spin of the topological and nontopological vortices of the C - S - H model respectively. Notice that the sign of the spin is +ve in both the cases which is again the same as that of the elementary excitations of the model [41]. In earlier analysis [34] there was a difference of sign which was explained by the introduction of a new interaction [43]. This is not necessary in the present discussion.

The spin of the vortices of the self - dual C - S - H model, expressed by equations (2.26) and (2.28), is proportional to the strength k of the C - S interaction which is completely arbitrary. The value of K modulo integer is the 'fractional' part of the spin of the vortices. Thus the method of [16] justifies the claim that the charged vortices of the C - S - H model are candidate excitations for anyons, just as it did for the C - S coupled nonlinear O(3) sigma model. At this point it may be recalled that the method of [8, 9, 34] predicted fractional spin of the C - S - H vortices but the application of the same method to the C - S coupled nonlinear O(3) sigma model yielded incorrect result[33].

2.4 Effect of the Maxwell term

It should be pointed out that (2.21) is the master equation which yields the fractional spin of Chern-Simons vortices in any theory. This is because the matter sector is completely absent in (2.21). Furthermore K is just a boundary term which reveals the topological origin of fractional spin. In this sense this establishes a correspondence with the approach of [4, 33] where the argument for determining spin depends on the topology associated with the relevant homotopy class. An immediate fall-out of (2.21) is that the result for the spin should be unaffected by including the Maxwell term in the original Lagrangian. This is because at large distances the low derivative Chern-Simons term dominates over the higher derivative Maxwell term. Consequently asymptotic effects (as, for instance, given by (2.21)) are insensitive to the Maxwell term. This can also be checked explicitly. Addition of the Maxwell piece modifies the momenta (2.22) to,

$$\Pi_i = \frac{k}{2}\epsilon_{ij}A^j - \frac{1}{e^2}(\partial_0 A_i - \partial_i A_0) \tag{2.29}$$

The piece involving A_0 will not contribute to the boundary term (2.20). This is because A_0 is a Lagrange multiplier which can always be chosen so that the boundary term vanishes. The asymptotic configuration (2.24) does not have any explicit time dependence so that the piece $\partial_0 A_i$ also vanishes. Consequently the original result (2.26) for K is reproduced.

2.5 Application to the generalised Chern - Simons - HIggs model

Let us next consider the generalised Chern-Simons-Higgs (GCSH) model introduced in [44]. The Lagrangian is defined by

$$\mathcal{L} = 2\sqrt{2}\epsilon^{\mu\nu\rho}[\eta^4 A_\rho - 2i(\eta^2 - \frac{1}{2}|\phi|^2)\phi(D_\rho\phi)^*]F_{\mu\nu} + 4(\eta^2 - |\phi|^2)^2|D_\mu\phi|^2 - V, \quad (2.30)$$

where $V(|\phi|)$ has symmetry breaking minima at $|\phi| \rightarrow \eta$.

The GCSH model was constructed by noting the analogy between the C - S - H model with the abelian Higgs model [64] on the one hand and the construction of the generalised abelian Higgs model [65] from the abelian Higgs model on the other. Like the C - S - H model, the GCSH model also allow self dual vortices [44]. One expects these vortices to carry fractional spin like the C - S - H vortices. However, this spin has not been computed earlier.

To compute the spin of the GCSH vortices we use the master equation (2.21). The asymptotic expressions for A_i and π_i are required. Since in the assymetric phase (which is relevant for the existence of topological invariants) ϕ does not vanish at ∞, A is required to go to the asymptotic limit (2.24) so that $D_i\phi$ vanishes asymptotically. The later is required for finite energy of the configurations. Now from the Lagrangian (2.30) we get

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = 4\sqrt{2}\epsilon^{ij}\eta^4 A_j - 8\sqrt{2}i\epsilon^{ij}(\eta^2 - \frac{1}{2}|\phi|^2)(\phi(D_j\phi)^*). \quad (2.31)$$

Since $D_i\phi$ vanishes assymptotically,

$$\pi^i \longrightarrow 4\sqrt{2}\epsilon^{ij}\eta^4 A_j \quad (2.32)$$

when $|x| \to \infty$. Using (2.24) and (2.32) in our master equation (2.21), the desired expression for the fractional spin can be derived,

$$K = 8\sqrt{2}\pi\eta^4 n^2 \tag{2.33}$$

2.6 Nonabelian Chern - Simons - Higgs model

Let us finally extend our analysis to non-abelian vortex configurations. As a typical example we consider the nonabelian generalisation of the C - S - H model. The Higgs field ϕ now forms a representation of the nonabelian group $\mathcal{G}$. The generators T^a of the group are assumed to satisfy the Lie algebra

$$[T^a, T^b] = f_c^{ab} T^c \tag{2.34}$$

The Lagrangian of the nonabelian C - S - H model is given by

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \frac{k}{2}\epsilon^{\alpha\beta\gamma}[A_{\alpha a}\partial_\beta A_\gamma^a + \frac{1}{3}f^{abc}A_{\alpha a}A_{\beta b}A_{\gamma c}] - V(|\phi|). \tag{2.35}$$

The gauge field A_μ now assumes values in the group space

$$A_\mu = A_{\mu a} T^a \tag{2.36}$$

and D_μ is the covariant derivative defined by

$$D_\mu = \partial_\mu + A_\mu. \tag{2.37}$$

The second term in the expression of $\mathcal{L}$ is the nonabelian C - S term, equation (1.4) Vortex solutions are associated with spontaneously broken gauge symmetries via the Higgs Fields. The form of the potential $V(|\phi|)$ is thus required to be so chosen in order to have maximal symmetry breaking of the gauge group $\mathcal{G}$. As usual, particular expression for V is not required in our calculations.

Proceeding now, as in the abelian models, one can show that the fractional spin is given by the non-abelian analogue of (2.21),

$$K = -\int d^2\mathbf{x}\partial^i[\pi_{ia}\epsilon^{jk}x_j A_k^a], \tag{2.38}$$

where 'a' denotes the group index. The asymptotic form of the nonabelian vortex configuration [58] is structurally identical to (2.24),

$$A_i^a = -\epsilon_{ij}\frac{x^j}{|x|^2}\frac{Q^a}{2\pi k}, \tag{2.39}$$

where,

$$Q^a = \int d^2\mathbf{x} J_0^a, \tag{2.40}$$

with J_μ^a being the nonabelian current,

$$J_\mu^a = \phi^* T^a (D_\mu \phi) - (D_\mu \phi)^* T^a \phi, \tag{2.41}$$

which enters the Euler-Lagrange equation as,

$$k\epsilon^{\mu\beta\sigma}(\partial_\beta A_\sigma^a + \frac{1}{2}f^{abc}A_\beta^b A_\sigma^c) = J^{\mu a} \tag{2.42}$$

The normalisation in (2.39) is determined [58] from the fact that it should obey the charge-flux identity following from the time-component of (2.42). Inserting (2.39) in (2.38) yields the expression,

$$K = -\frac{\pi k Q^a Q_a}{(2\pi k)^2}. \tag{2.43}$$

Till now the analysis is completely general because the gauge group has not yet been specified. If we consider this group to be SU(2), which is the simplest example admitting nonabelian vortices, it is possible to express $Q^a Q_a$ in (2.43) in terms of the vortex number n. Requiring the finiteness of energy demands that $(D_i\phi)_m$ should vanish asymptotically, in analogy with the abelian case. Using the ansatz (2.23) [5] and (2.39) this implies,

$$[(q_a T^a + in)v]_m = 0; q_a = -\frac{Q_a}{2\pi k} \tag{2.44}$$

For nontrivial solutions of v this yields the condition,

$$\det |(q_a T^a + in)| = 0. \tag{2.45}$$

For SU(2), the T^a are just the Pauli matrices. Using their standard representation, one obtains from (2.45),

$$q_a q^a = -n^2. \tag{2.46}$$

Substituting in (2.43) yields,

$$K = \pi k n^2, \tag{2.47}$$

which is structurally identical to the abelian result (2.26). There is a difference of e^2 because the coupling constant was absorbed in the definition of the gauge potential (2.39). For the particular case of $n = 1$, where detailed vortex configurations have been worked out [7], the result agrees with (2.47) in magnitude. A difference of sign occurs. In [7] the spin is defined as the static limit of the Schwinger angular momentum like [8, 9, 34]. The difference of sign observed is therefore well recognised as a typical outcome from the experience of the calculations performed above.

[5]Note that v is now a column vector

2.7 Extension to the non-relativistic models

Now we will apply the same general method to the non-relativistic models. Consider the Lagrangian

$$\mathcal{L} = i\phi^* D_t\phi - \frac{1}{2m}(D_k\phi)^*(D_k\phi) + \frac{k}{2}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \tag{2.48}$$

where ϕ is a bosonic Schrodinger field. The model (2.48) is invariant under the Galilean transformations and not under the transformations of the Poincare group. Note that the Galilean transformations take time and space on an unequal footing. So space-time metric is not defined. In writing (2.48) we adopt a spatial Eucledian metric, covariant and contravariant components are thus not to be distinguished.

The action of model (2.48) cannot be made generally covariant. The powerful method of constructing a gauge invariant energy momentum(EM) tensor as formulated by Schwinger, is thus not available. Nonetheless, it is possible to construct a gauge invariant EM tensor by appealing to the equations of motion [45]. Our program is then clear. We will find a gauge-invariant momentum density from the matter current obtained by using the equations of motion. These equations will then be exploited to show the conservation of the corresponding momentum. We work in the gauge-independent formalism in contrast with the gauge fixed approach of [45]. A suitable linear combination of the Gauss constraint is to be added with the gauge invariant momentum operator, in order to generate correct transformation of the fields under spatial translation. A gauge invariant angular momentum is then constructed using this momentum density. The canonical angular momentum obtained by Noether's prescription is now subtracted from it. The spin of the vortices is, as usual, defined by

$$K = J - J^N, \tag{2.49}$$

which is exactly similar to equation (2.20) with the exception that J is now the gauge invariant angular momentum constructed by using the equations of motion

From the Lagrangian (2.48) we write the Euler-Lagrange equation corresponding to A_μ,

$$k\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda = j_\mu, \tag{2.50}$$

where j_μ is given by,

$$\begin{aligned} j_0 &= \phi^*\phi, & (2.51) \\ j_i &= \frac{1}{2im}[\phi^*(D_i\phi) - \phi(D_i\phi)^*]. & (2.52) \end{aligned}$$

Observe that (2.50) leads to a continuity equation

$$\partial_0 j_0 + \partial_i j_i = 0. \tag{2.53}$$

Hence j_0 and j_i can be interpreted as the matter density and current density respectively.

From the E - L equation corresponding to A_0 we get the Gauss constraint of the theory

$$G = \phi^*\phi - k\epsilon_{ij}\partial_i A_j \approx 0. \tag{2.54}$$

Now we come to the construction of the gauge invariant momentum operator. The (0-i) -th component of the EM tensor T_{0i} (i.e. the momentum density) is obtained from the matter current

$$T_{0i} = \frac{i}{2}[\phi^*(D_i\phi) - \phi(D_i\phi)^*]. \tag{2.55}$$

It is straightforward to verify, using the equations of motion, that T_{0i} indeed satisfies the appropriate continuity equation,

$$\partial_0 T_{0i} + \partial_k T_{ki} = 0, \tag{2.56}$$

where T_{ki} is the stress - tensor [66],

$$T_{ki} = \frac{1}{2m}[(D_k\phi)^*(D_i\phi) + (D_k\phi)(D_i\phi)^* - \partial_i(\phi^* D_k\phi + \phi(D_k\phi)^*)] \tag{2.57}$$

Using the expression(2.55) of T_{0i} we construct a gauge invariant momentum operator

$$P_i = \int d^2\mathbf{x} T_{0i}. \tag{2.58}$$

Exploiting (2.56) and neglecting the boundary term we find that P_i is indeed conserved,

$$\frac{dP_i}{dt} = 0. \tag{2.59}$$

The boundary term vanishes due to the condition that the covariant derivative $D_i\phi$ is zero on the boundary which is required to keep the energy finite. For proper transformation of the fields under spatial translation we require to supplement T_{0i} by the Gauss constraint,

$$T_{0i}^T = T_{0i} + A_i G, \tag{2.60}$$

and the corresponding momentum operator

$$P_i = \int d^2\mathbf{x}[\frac{i}{2}(\phi^* D_i\phi - \phi(D_i\phi)^*) + A_i G] \tag{2.61}$$

turns out to be an appropriate generator of spatial translation. Note that equation (2.59) is still valid on the constraint surface.

We now come to the construction of J, the gauge invariant angular momentum, from the momentum density (2.61),

$$J = \int d^2\mathbf{x}\epsilon_{ij}x_i[\frac{i}{2}(\phi^* D_j\phi - \phi(D_j\phi)^*) + A_j G] \tag{2.62}$$

The canonical angular momentum J^N is obtained from Noether's theorem as [48],

$$J^N = \int d^2\mathbf{x}[\epsilon_{ij}x_i(\frac{i}{2}(\phi^*\partial_j\phi - \phi(\partial_j\phi)^*) - \frac{k}{2}\epsilon_{mn}A_m\partial_j A_n) + \frac{k}{2}A_j A_j] \tag{2.63}$$

Substituting (2.62) and (2.63) in (2.49) we obtain,

$$K = -\frac{k}{2}\int d^2\mathbf{x}\partial_i[x_iA^2 - x_jA_jA_i]. \tag{2.64}$$

Observe that the master formula (2.64) for the calculation of spin is identical with equation (2.20). The asymptotic form of A_i following from general considerations already elaborated leads to the same structure as in (2.24). Inserting this in (2.64) exactly reproduces (2.26) as the spin of the vortices.

We note in passing that self - dual soliton solutions can be obtained by including a quartic self - interaction in (2.48) [45], which are interpreted as the nonrelativistic limit of the nontopological vortices of the relativistic Chern - Simons - Higgs model considered previously. The spin of these solitons can be calculated by (2.64) using the asymptotic form (2.27). The result comes out to be identical with (2.28). This is expected because the existence of the fractional spin is connected to the Chern - Simons piece which is a topological term. The spin of the model (2.48) with quartic self - interaction and an external magnetic field was calculated earlier [46, 47]. Their method was in spirit akin to that of [8, 9, 34] but they had to subtract the background contribution to get the spin. The result of [46, 47] scales as our result with the vortex number and the same comments apply to this comparision as made earlier in connection with the C - S - H model.

2.8 Conclusion

We recall from Chapter 1 that the usual method of defining the spin as the static limit of the physical angular momentum yields contradictory results when applied to compute the spin of the solitons of the Chern - Simons (C - S) coupled O(3) nonlinear sigma model [33]. In this connection we have observed that a consistent result is obtained when we apply our general formalism for computing the spin in the C - S theories [16], which exploits the constraints of the theory. In this method the physical angular momentum is first constructed from the gauge invariant symmetric energy momentum tensor obtained by varying the action with respect to a background metric and then keeping the metric flat. Since this EM tensor is used in formulating the Dirac - Schwinger conditions for the covariance of a theory [35] it is named as the Schwinger EM tensor. The angular momentum following from this EM tensor is called Schwinger angular momentum. The canonical part is then abstracted from the physical angular momentum. This is done by subtracting the angular momentum obtained from the canonical (Noether) EM tensor, from the Schwinger angular momentum. (Incidentally, both the EM tensors are improved by including appropriate linear combination of the constraints so as to get proper transformation of the fields) The difference was found to be nonzero for singular configurations. In particular for C - S vortices this difference was shown to be independent of the origin of the coordinate system. Consequently we interpret it as the intrinsic spin of the vortices. The formula for the spin comes out to be model independent and contrary to other approaches where detailed field configurations are necessary, only the asymptotic form of the gauge field is required for its evaluation.

The spin of the topological and nontopological vortices of the C - S - H model was reviewed by the general formalism mentioned above. The spin of both types of vortices of the model comes out with the same sign. We also find that the sign of the spin of the topological vortices is the same as that of the elementary excitations of the model [41]. This is a satisfactory result because the spin-statistics connection is then respected with the usual Aharonov - Bohm phase. Notably, in [34] an opposite sign was found so that a new interaction was required to account properly for this phase [43].

A remarkable feature of the method is its general applicability. We have considered the effect of adding the Maxwell term to the C - S - H model and found that the spin is unchanged. This is expected because in the asymptotic limit the lower derivative C - S term dominates over the Maxwell piece. We have also applied our method to a generalisation of the C - S - H model [44] and its non-abelian counterpart [7]. In the latter case an explicit expression was derived for the SU(2) group. Our result, valid for any winding number n agrees in magnitude with the special case ($n = 1$) discussed earlier [7].

Our formalism is directly applicable to the relativistic theories but the Chern - simons interaction enjoys the rare distinction of being suitable to be coupled to both Poincare and Galileo symmetric models [45, 56, 48, 61]. We were thus motivated to extend our formalism to the nonrelativistic theories. Moreover, a systematic discussion of the spin in such theories is nonexistent. The main problem here is that Galileo symmetric theories cannot be made generaly covariant and the powerful method of constructing the gauge - invariant Schwinger EM tensor is not available. However, a gauge - invariant EM tensor can still be constructed by using the equations of motion [45]. We extended our formalism to the nonrelativistic models by using this gauge - invariant EM tensor. The resulting spin formula was identical with that obtained for the relativistic theories. The complete parallel of the method and the emergent spin formula is a pointer to the topological origin of the C - S term, responsible for the induction of the fractional spin, either in relativistic or nonrelativistic models.

2.9 Appendix 1 : Constraint structure of the C - S - H model

We present here the detailed analysis of the constraint structure of the theory (2.1). The independent fields in the model are ϕ,ϕ^* and A^μ. The corresponding momenta π,π^* and π^μ are obtained from the definition of the canonical momenta as,

$$\begin{aligned} \pi &= \dot{\phi}^* - ieA^0\phi^*, & (2.65)\\ \pi^* &= \dot{\phi} + ieA^0\phi, & (2.66)\\ \pi^0 &= 0, & (2.67)\\ \pi^i &= \frac{k}{2}\epsilon^{ij}A_j. & (2.68) \end{aligned}$$

From these expressions we can immediately identify the primary constraints of the theory

$$\begin{aligned} \pi^0 &\approx 0, & (2.69)\\ P^i = \pi^i - \frac{k}{2}\epsilon^{ij}A_j &\approx 0. & (2.70) \end{aligned}$$

The constraint (2.69) has vanishing Poisson bracket (PB) with the others. It is therefore first class [14]. However, the constraints (2.70) are second class. The constraints should be preserved in time so that their PBs with the Hamiltonian are at least weakly zero. These conditions may generate secondary constraints.

The Hamiltonian density according to the canonical definition is obtained as,

$$\mathcal{H}_c = \pi\pi^* - ieA^0(\phi\pi - \phi^*\pi^*) - |\partial_i\phi|^2 - ieA^i(\phi\partial^i\phi^* - \phi^*\partial^i\phi) - e^2A_iA^i|\phi|^2 - \frac{k}{2}\epsilon^{ij}(A_0\partial_iA_j + A_i\partial_jA_0) + V(|\phi|) \tag{2.71}$$

The primary Hamiltonian is obtained by adding linear combinations of the primary constraints with (2.71) as

$$H = \int d^2\mathbf{x}(\mathcal{H}_c + u_0\pi_0 + u_i P_i), \tag{2.72}$$

where u_0 and u_i are arbitrary functions. Conserving the constraint (2.69) with this Hamiltonian we get,

$$\{\pi_0(t,\mathbf{x}), H(t)\} = 0, \tag{2.73}$$

where $\{\ \ \}$ denote equal time PBs. The condition (2.73) gives a secondary constraint,

$$G = ie(\phi\pi - \phi^*\pi^*) + k\epsilon^{ij}\partial_i A_j \approx 0. \tag{2.74}$$

Carrying on the iterative process we get no more secondary constraint. Thus equations (2.69),(2.70) and (2.74) give the complete constraint structure of the theory.

From (2.70) and (2.74) we find that P and G have nonvanishing PB. Thus there are apparently three second class constraints. We can, however, construct the following linear combination of the constraints P and G,

$$F = \partial_i P^i + G \approx 0, \tag{2.75}$$

which has vanishing PBs with both (2.69) and (2.70). The complete constraint structure of the theory thus consists of two first class constraints (2.69) and (2.75) and two second class constraints given by (2.70).

The second class constraints (2.70) point to a redundancy of the configuration space variables which can be eliminated by replacing the PBs by the Dirac (star) brackets (DB) defined by,

$$\{\chi(\mathbf{x}),\eta(\mathbf{y})\}^* = \{\chi(\mathbf{x}),\eta(\mathbf{y})\} - \int d^2\mathbf{z}d^2\mathbf{z}'\{\chi(\mathbf{x}, P_i(\mathbf{z})\}P_{ij}^{-1}(\mathbf{z},\mathbf{z}')\{P_j(\mathbf{z}'),\eta(\mathbf{y})\}, \tag{2.76}$$

where χ,η are generic fields and P_{ij} is the matrix of the Poisson brackets of the constraints,

$$P_{ij}(\mathbf{x},\mathbf{y}) = \{P_i(\mathbf{x}), P_j(\mathbf{y})\}. \tag{2.77}$$

Our basic fields are ϕ,ϕ^* and A^μ. Since ϕ,ϕ^* and A^0 have vaishing PBs with P_i, their DBs are the same as the corresponding PBs. The nontrivial DBs are,

$$\{A_i(\mathbf{x}), A_j(\mathbf{y})\}^* = \frac{1}{k}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y}), \tag{2.78}$$

$$\{A_i(\mathbf{x}), \pi_j(\mathbf{y})\}^* = \frac{1}{2}g_{ij}\delta(\mathbf{x}-\mathbf{y}), \tag{2.79}$$

$$\{\pi_i(\mathbf{x}), \pi_j(\mathbf{y})\}^* = \frac{k}{4}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y}). \tag{2.80}$$

The second class constraints (2.70) are now strongly implemented. The equations (2.79) and (2.80) then follow from (2.78) and (2.70). Thus (2.78) is the only nontrivial independent DB. This is the same bracket as (2.7) obtained from the symplectic arguments. Again, since P_i are strongly equal to zero, the constraints F given by (2.75) coincide with G. It is easy to check that G is the generator of gauge transformations at fixed time with respect to the DBs of the theory,

$$\int d^2\mathbf{x}\alpha(\mathbf{x})\{G(\mathbf{x}), \phi(\mathbf{y})\}^* = -ie\alpha(\mathbf{y})\phi(\mathbf{y}), \tag{2.81}$$

$$\int d^2\mathbf{x}\alpha(\mathbf{x})\{G(\mathbf{x}), A_k(\mathbf{y})\}^* = \partial_k\alpha(\mathbf{y}). \tag{2.82}$$

The constraint (2.74) is therefore identified with the Gauss constraint of the theory.

2.10 Appendix 2 : Classical Poincare covariance of the C - S - H model

We discuss the classical Poincare covariance of the Chern - Simons - Higgs model (2.1) using alternative definitions of the energy momentum (EM) tensor. These calculations provide detailed derivation of some of the results quoted in this chapter and provide justification for some comments made therein.

We first consider the Schwinger EM tensor given by equation (2.10). The corresponding total EM tensor is

$$\Theta^T_{\mu\nu} = \Theta_{\mu\nu} + \Lambda_{\mu\nu} G, \tag{2.83}$$

where $\Lambda_{\mu\nu}$ are some (yet undetermined) functions of the space - time coordinates. The corresponding total Hamiltonian density is given by,

$$\begin{aligned} \Theta^T_{00} = \Theta_{00} + \Lambda_{00} G &= \pi^* \pi - \partial_i \phi \partial^i \phi^* - ie A_i (\phi \partial^i \phi^* - \phi^* \partial^i \phi) && (2.84) \\ &- e^2 A_i A^i \phi \phi^* + V(|\phi|) + \Lambda_{00} G && (2.85) \end{aligned}$$

Now we find,

$$\begin{aligned} \{\phi(\mathbf{x}), \int d^2\mathbf{y} \Theta^T_{00}(\mathbf{y})\} &= \pi^*(\mathbf{x}) + \Lambda_{00} \cdot ie\phi(\mathbf{x}) && (2.86) \\ &= \dot{\phi} + ie(A_0 + \Lambda_{00})\phi(\mathbf{x}), && (2.87) \end{aligned}$$

where we have used the basic equal time brackets (2.6) and (2.7).

Equation (2.87) shows that if we chose

$$\Lambda_{00} = -A_0, \tag{2.88}$$

then the expected time development of $\phi(\mathbf{x})$ is obtained:

$$\{\phi(\mathbf{x}), H^T\} = \dot{\phi}. \tag{2.89}$$

With the choice (2.88) we can straightforwardly verify that

$$\begin{aligned} \{\phi^*(\mathbf{x}), H^T\} &= \dot{\phi}, && (2.90) \\ \{A_i(\mathbf{x}), H^T\} &= \dot{A}_i. && (2.91) \end{aligned}$$

From (2.83) we construct the candidate generator of spatial translation

$$P_i = \int d^2\mathbf{x}\Theta^T_{0i} \tag{2.92}$$

$$= \int d^2\mathbf{x}[\pi^*(\partial_i\phi^* - ieA_i\phi^*) + \pi(\partial_i\phi + ieA_i\phi) + \wedge_{0i}G]. \tag{2.93}$$

Now

$$\{\phi(\mathbf{x}), P_i\} = \partial_i\phi(\mathbf{x}) + ieA_i\phi(\mathbf{x}) + ie\wedge_{0i}\phi(\mathbf{x}) \tag{2.94}$$

and we find that for proper transformation of $\phi(\mathbf{x})$ under spatial translation, we require

$$\wedge_{0i} = -A_i. \tag{2.95}$$

It is indeed pleasant to observe that the choice(2.95) also gives proper transformations of the fields ϕ^* and A_i,

$$\{\phi^*(\mathbf{x}), P_i\} = \partial_i\phi^*(\mathbf{x}), \tag{2.96}$$

$$\{A_k(\mathbf{x}), P_i\} = \partial_i A_k(\mathbf{x}) - \int d^2\mathbf{y}\partial^y_k A_i(\mathbf{y})\delta(\mathbf{x}-\mathbf{y}) \tag{2.97}$$

$$= \partial_i A_k(\mathbf{x}), \tag{2.98}$$

where in the last equation the boundary term vanishes due to the delta function term.

From the above analysis we find that the choice

$$\wedge_{0\mu} = -A_\mu \tag{2.99}$$

gives appropriate transformation of the fields under space - time translations.

Now let us come to the cases of rotations and boosts. The generators $J_{\mu\nu}$ are defined as

$$J_{\mu\nu} = \int d^2\mathbf{x}\mathcal{J}^T_{0\mu\nu} \tag{2.100}$$

where,

$$\mathcal{J}^T_{\delta\mu\nu} = x_\mu \Theta^T_{\delta\nu} - x_\nu \Theta^T_{\delta\mu}. \tag{2.101}$$

Using (2.83) for $\Theta^T_{\mu\nu}$ and the basic equal time brackets it can be readily shown that

$$\begin{aligned} \{\phi(\mathbf{x}), J_{0i}\} &= x_0\partial_i\phi(\mathbf{x}) - x_i\partial_0\phi(\mathbf{x}), & (2.102)\\ \{\phi^*(\mathbf{x}), J_{0i}\} &= x_0\partial_i\phi^*(\mathbf{x}) - x_i\partial_0\phi^*(\mathbf{x}) & (2.103)\end{aligned}$$

and

$$\begin{aligned} \{\phi(\mathbf{x}), J_{ij}\} &= x_i\partial_j\phi(\mathbf{x}) - x_j\partial_i\phi(\mathbf{x}), & (2.104)\\ \{\phi^*(\mathbf{x}), J_{ij}\} &= x_i\partial_j\phi^*(\mathbf{x}) - x_j\partial_i\phi^*(\mathbf{x}). & (2.105)\end{aligned}$$

Also a straighforward calculation yields

$$\begin{aligned} \{A_k(\mathbf{x}), J_{0i}\} &= x_0\partial_i A_k(\mathbf{x}) - x_i\partial_0 A_k(\mathbf{x}) + g_{ik}A_0(\mathbf{x}), & (2.106)\\ \{A_k(\mathbf{x}), J_{ij}\} &= x_i\partial_j A_k(\mathbf{x}) - x_j\partial_i A_k(\mathbf{x}) - g_{ik}A_j(\mathbf{x}) + g_{jk}A_i(\mathbf{x}), & (2.107)\end{aligned}$$

where we have dropped boundary terms containing delta function terms as in (2.97)

Remember that the fields ϕ, ϕ^* are Lorentz scalars whereas A_i is Lorentz vector. Inspection of the above equation then reveals that all the fields transform appropriately under rotations and Lorentz boosts. Since all the fields satisfy normal transformation properties under space - time translations and rotations, the classical Poincare algebra is trivially preserved. We also note that the matrix J_{ij} for rotation generators has only one independent component [6] which is identified with the angular momentum operator J, equation (2.16).

[6]This is due to the trivial nature of the rotation group in two spatial dimensions, see section 2 of chapter 1.

The analysis proceeds in exactly similar fashion with the EM tensor (2.17) obtained from Noether's prescription. We extend it as in (2.83). But now we find that the choice

$$\wedge_{\mu\nu} = 0 \tag{2.108}$$

is appropriate. In other words, the Noether EM tensor generates proper transformations of the fields under space - time translation without any improvement. The generators of rotations and boosts are now given by [67],

$$J_{\mu\nu} = \int d^2\mathbf{x} \mathcal{J}^T_{0\mu\nu} \tag{2.109}$$

where,

$$\mathcal{J}^T_{\delta\mu\nu} = x_\mu \Theta^T_{\delta\nu} - x_\nu \Theta^T_{\delta\mu} + \frac{\partial \mathcal{L}}{\partial(\partial^\delta A^\rho)} \Sigma^{\rho\sigma}_{\mu\nu} A_\sigma, \tag{2.110}$$

with

$$\Sigma^{\rho\sigma}_{\mu\nu} = g^\rho_\mu g^\sigma_\nu - g^\rho_{\nu} g^\sigma_\mu \tag{2.111}$$

The difference of the expression (2.110) with the corresponding expression (2.100) is to be noticed. The canonical definition splits naturally in the orbital and spin parts unlike the Schwinger definition. Using equation (2.17) we can verify that the fields transform appropriately under rotations and boosts.

The demonstration of the Poincare covariance using the alternative definitions of the EM tensors clearly reveals the crucial role played by the Gauss constraint. The angular momentum operators obtained from these EM tensors are different by a boundary term which *does not* affect the local transformation of fields under Poincare group.

Chapter 3

GAUGED SIGMA MODELS: A NEW SELF DUALITY

3.1 Introduction

From the general problem of determination of the spin of the vortex solutions of the Chern - Simons (C - S) theories, we now proceed to the particular problem of self - duality in the context of the gauged nonlinear O(3) sigma models. Specifically, in the present chapter we will elaborate on the findings of a new type of self - duality [49, 50] in such models.

The low dimensional sigma models have been studied extensively over a long period of time [68]. In (2+1) dimensions the model provides topologically stable soliton solutions [28] which are exactly integrable in the Bogomol'nyi limit [51]. These solutions find wide application in condensed matter physics [26]. A characteristic feature of the solutions is that they are expressed in terms of rational functions and thereby enjoy scale invariance. This scale invariance presents a problem of particle interpretation on quantisation because the size of the solitons can change arbitrarily without costing any energy. Numerical simulations of the interaction of the solitons reveal the problem clearly [52].

Several methods of breaking the scale invariance of the solutions have been discussed in the literature [53, 54, 69]. Of them a particularly interesting method is to gauge the U(1) subgroup of the full symmetry of the usual sigma model [54]. The gauge field dynamics was chosen to be governed by the Maxwell term and a particular form of self interaction was included to saturate the Bogomol'nyi bounds. In this connection it is interesting to observe that the self - dual point is not unique and different choices of the form of the potential is possible. This observation is crucial in relation to our studies as will be evident soon.

The soliton solutions of the partially gauged O(3) sigma model considered in [54] are electrically neutral but endowed with magnetic flux. This mimicks the topological solitons of the (2+1) dimensional Higgs model [64]. However unlike the later, the magnetic flux is not quantised by the topological charge and is, instead, arbitrary. The solutions thus do not qualify as vortices, being infinitely degenerate in each topological sector. The magnetic flux takes the role of the size parameter here. Since solutions with arbitrary magnetic flux are degenerate in energy, the scaling degeneracy of the pure O(3) sigma model persists in the present model in a mutated form [54].

An extension of the study in the partially gauged O(3) sigma model in (2+1) dimensions was performed in [55]. In this work the gauge field dynamics was assumed to be dictated by the Chern - Simons (C - S) term instead of the Maxwell piece. The model was shown to admit soliton solutions with broken scale invariance. These are now charge - flux composites carrying fractional spin. The topological solitons of the model are again infinitely degenerate. Though the energy is quantised, both the charge - flux and angular momentum are arbitrary.

We thus find that the studies of the gauged O(3) sigma models are plagued with the problem of degeneracy [54, 55]. As we have already mentioned, due to this degeneracy these models are not able to remove the scaling degeneracy of the pure O(3) sigma models completely - the degeneracy only mutates. This degeneracy is thus physically undesirable. However, this is not mandatory. We have demonstrated that the degeneracy of the topological solitons of both [54, 55] is related to the fact that their vaccum structures fail to break the gauged U(1) symmetry. In this connection it is to be remembered that the form of the self - interaction to be assumed in this type of models to saturate the Bogomol'nyi bounds is not unique. We can chose this interaction in such a way which has symmetry breaking minima.

The spontaneous breaking of the U(1) symmetry corresponds to the introduction of novel topology in the corresponding models. Remarkably, the solutions of the resulting models do not show any form of degeneracy. In the case of the model with the Maxwell coupling, we get neutral magnetic vortices with the magnetic flux quantised by the topological number [50]. For the C - S coupling, the solutions are charge - flux composites as expected, with fractional spin. In contrast with [55], all these parameters have fixed values in a particular topological sector [49]. These self - dual solutions give, therefore, a new type of excitations which do not suffer from the problem of degeneracy. They have the physically desirable feature of breaking the scale invariance of the usual sigma model. The new self - duality exhibited by the models will be analysed in the following. To place the discussions in the proper perspectives let us begin with a short review of the soliton structure of the usual sigma model in (2+1) dimensions and earlier attempts of gauging the model.

3.2 The O(3) nonlinear sigma model

The Lagrangian of the model is given by (1.13) which can be written by suppressing the dimensionful parameter f for convenience as,

$$\mathcal{L} = \frac{1}{2}\partial_\mu n_a \partial^\mu n_a. \tag{3.1}$$

This is subject to the constraint

$$n_a n_a = \mathbf{n} \cdot \mathbf{n} = 1, \tag{3.2}$$

where n_a (a = 1,2,3) are a triplet of scalar fields constituting a vector in the internal space the tip of which lies on a sphere of unit radius with centre at the origin. We use Greek letters to denote the Lorentz indices and Latin letters from the beginning of the alphabet to denote the internal space components. The set $\mathbf{i}^a$ (a = 1,2,3) is an orthogonal basis in the later. Vectors in the internal space will be expressed by boldface letters with the 'dot' and 'cross' product symbols between them meaning the corresponding operations in that space. Letters from the middle of the latin alphabet will denote the physical space components. Unless specified otherwise, repeated indices will always mean a summation over them. We will work in a Minkowski space with the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1)$.

The energy functional obtained from eqn. (3.1) in the static configuration is given by

$$E = \frac{1}{2}\int d^2\mathbf{x}\partial_i n_a \partial_i n_a \tag{3.3}$$

which is of course constrained by eqn. (3.2). From eqn. (3.3) we find that the vaccum structure is

$$n_a = n_a^{(0)}, \tag{3.4}$$

where $n_a^{(0)}$ is an arbirary constant vector. The direction of $\mathbf{n}(0)$ is arbitrary. This indicates a spontaneous breaking of the O(3) symmetry of the model. The finite energy solutions of the model demand the boundary condition

$$\lim\ n_a = n_a^{(0)}. \tag{3.5}$$

Note that as we approach infinity from any direction, $\mathbf{n}$ must tend to the same limit, otherwise the angular part of the gradient will contribute in eqn. (3.3) even at infinity making the energy infinite. This has very important topological significance. The physical infinity is one point compactified leading to the compactification of R_2 to S_2^{Phy}. The static configurations are mappings from the physical space to the internal sphere S_2^{int}. Hence these are labelled according to the nontrivial homotopy

$$\Pi_2(S_2) = Z. \tag{3.6}$$

Different homotopy sectors are classified according to the conserved topological charge Q,

$$Q = \frac{1}{8\pi} \int d^2\mathbf{x} \epsilon_{ij} \mathbf{n} \cdot (\partial_i \mathbf{n} \times \partial_j \mathbf{n}), \tag{3.7}$$

which is equal to the winding number of the mapping.

Now let us consider a fixed direction in the internal space, given by the constant unit vector $\mathbf{i}_3 = (0,0,1)$ for example. The model (3.1) and the constraint (3.2) are invariant under rotations about $\mathbf{i}_3$. These rotations form a SO(2) (U(1)) subgroup of the full rotational symmetry of the model (3.1). When this subgroup is gauged the corresponding models become partially gauged nonlinear O(3) sigma models. In

(2 + 1) dimensions the dynamics of the gauge field A may be dictated alternatively by the Maxwell or the C - S term. Different classes of gauged nonlinear O(3) sigma models are thereby obtained. In [54] the gauge field dynamics was assumed to be dictated by the Maxwell term. A covariant derivative

$$D_\mu \mathbf{n} = \partial_\mu \mathbf{n} + A_\mu \mathbf{i}_3 \times \mathbf{n} \tag{3.8}$$

was defined. The resulting Lagrangian of the model is

$$\mathcal{L} = \frac{1}{2} D_\mu \mathbf{n} \cdot D^\mu \mathbf{n} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + U(\mathbf{n}), \tag{3.9}$$

where

$$U(\mathbf{n}) = -\frac{1}{2}(1 - \mathbf{i}_3 \cdot \mathbf{n})^2 \tag{3.10}$$

was the assumed form of the self - interaction and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We immediately observe that the vaccum structure corresponds to

$$n_3 = 1, n_1 = n_2 = 0, \tag{3.11}$$

which is invariant under rotations about $\mathbf{i}_3$. The form of the chosen potential does not lead to the spontaneous breaking of the gauged U(1) symmetry. The finite energy configurations thus correspond to the mapping of the physical infinity to the north pole of the internal sphere and classified by the same homotopy equation (3.6) as the usual sigma model. The same comments are also due to the work [55] where the gauge field dynamics was chosen to be dictated by the Chern - Simons kinetic term. The model Lagrangian here is

$$\mathcal{L} = \frac{1}{2} D_\mu \mathbf{n} \cdot D^\mu \mathbf{n} + \frac{k}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} - \frac{1}{2k^2}(1 + \mathbf{i}_3 \cdot \mathbf{n})(1 - \mathbf{i}_3 \cdot \mathbf{n})^3. \tag{3.12}$$

The third term is the chosen form of self-interaction. We find that in both the gauged O(3) sigma models the topological sectors are classified by the second homotopy equation (3.6) and the gauged symmetry is not spontaneously broken. The topological solitons of the models are infinitely degenerate in a particular sector, a feature not desirable from the point of view of particle interpretation. Our studies show that the self - interactions can be alternatively chosen so as to allow symmetry breaking minima and the resulting models provide self-dual soliton solutions free of the problem of degeneracy.

3.3 Maxwell coupling

Our model with the Maxwell coupling is given by the Lagrangian (3.9) but with the potential $U(\mathbf{n})$ given by

$$U(\mathbf{n}) = -\frac{1}{2}(\mathbf{i_3}.\mathbf{n})^2. \tag{3.13}$$

Explicitly, our model Lagrangian is

$$\mathcal{L} = \frac{1}{2}D_\mu \mathbf{n} \cdot D^\mu \mathbf{n} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\mathbf{i_3} \cdot \mathbf{n})^2. \tag{3.14}$$

Note that the minima of the potential correspond to,

$$n_3 = 0 \text{ and } n_1^2 + n_2^2 = 1 \tag{3.15}$$

Clearly classical vaccua correspond to configurations where the vector $\mathbf{n}$ is spatially uniform and points in an arbitrary direction perpendicular to $\mathbf{i_3}$. The gauged U(1) symmetry is spontaneously broken. The physical situation is thus very different from the model considered in [54]. Its consequences will now be investigated.

3.3.1 Equations of motion

The classical equations of motion are given by the Euler - Lagrange (E - L) equations of the model (3.14) subject to the constraint (3.2). We use the Lagrange multiplier technique and consider the augmented form of the Lagrangian

$$\mathcal{L}' = \mathcal{L} + \lambda|\mathbf{n}|^2, \tag{3.16}$$

where $\lambda(x)$ is the Lagrange multiplier. Extremizing the action corresponding to the Lagrangian (3.16) we derive

$$D_\nu(D^\nu \mathbf{n}) = 2\lambda \mathbf{n} - (\mathbf{i}_3 \cdot \mathbf{n})\mathbf{i}_3. \tag{3.17}$$

Using eqn. (3.2) we can eliminate λ from eqn. (3.17) to get,

$$D_\nu(D^\nu \mathbf{n}) = [D_\nu(D^\nu \mathbf{n}) \cdot \mathbf{n}]\mathbf{n} - \mathbf{i}_3 n_3 + n_3^2 \mathbf{n}, \tag{3.18}$$

which is the equation of motion for $\mathbf{n}$. Similarly the equation of motion for A_μ is obtained as,

$$\partial_\nu F^{\nu\mu} = j^\mu, \tag{3.19}$$

where j_μ is the conserved U(1) current given by

$$j^\mu = -\mathbf{i}_3 \cdot \mathbf{J}^\mu, \tag{3.20}$$

with

$$\mathbf{J}^\mu = \mathbf{n} \times D^\mu \mathbf{n}. \tag{3.21}$$

Using the equations of motion for $\mathbf{n}$, eqn.(3.18), we can show that $\mathbf{J}_\mu$ satisfies

$$D_\mu \mathbf{J}^\mu = (\mathbf{i}_3 \times \mathbf{n})n_3. \tag{3.22}$$

The structure of the equations (3.20) and (3.21) shows explicitly that j_μ is gauge invariant, as it should be. Putting $\mu = 0$ we get,

$$j^0 = (n_2\dot{n}_1 - n_1\dot{n}_2) - A^0(1 - n_3^2). \tag{3.23}$$

So, for static configurations,

$$j^0 = -A^0(1 - n_3^2). \tag{3.24}$$

Now, putting $\mu = 0$ in eqn. (3.19) we find, for static configurations,

$$\nabla^2 A^0 = -A^0(1 - n_3^2). \tag{3.25}$$

From the last equation it is evident that we can chose

$$A^0 = 0, \tag{3.26}$$

which along with eqn. (3.24) gives,

$$j^0 = 0. \tag{3.27}$$

The U(1) charge of the static configurations then vanish which implies that the excitations of the model are electrically neutral.

3.3.2 Homotopy classification

A symmetric gauge invariant energy - momentum tensor for the model (3.14) can be constructed by the standard method of varying the action with respect to a background metric,

$$\Theta_{\mu\nu} = \frac{\delta \mathcal{A}}{\delta g^{\mu\nu}}. \tag{3.28}$$

As is well-known Θ_{00} gives the energy density. Starting from eqn. (3.14), it is straightforward to obtain the energy-functional as,

$$E = \frac{1}{2}\int d^2x[D_0\mathbf{n}\cdot D^0\mathbf{n} - D_i\mathbf{n}\cdot D^i\mathbf{n} + n_3^2 - 2(F_0^iF_{0i} - \frac{1}{4}F^2)], \tag{3.29}$$

Under static conditions and the solution (3.26) the energy-functional becomes,

$$E = \frac{1}{2}\int d^2x[(D_i\mathbf{n})\cdot(D_i\mathbf{n}) + F_{12}^2 + n_3^2] \tag{3.30}$$

which is of course subject to the constraint (3.2). Equation (3.30) shows that the boundary conditions for finite energy configurations require $\mathbf{n}$ to go to one of the symmetry breaking minima given by eqn. (3.15), as spatial infinity is approached. A particular static field configuration then maps the infinite circle of the physical space to the equatorial circle of the internal sphere. These solutions are classified according to the homotopy,

$$\Pi_1(S_1) = Z, \tag{3.31}$$

which is different from the usual O(3) sigma model (3.6) and for that matter, from the gauged model of [54]. The choice (3.13) in eqn. (3.14) induces the new topology in the model which is a direct consequence of the spontaneous breaking of the gauged symmetry.

The influence of the new topology (3.31) on the physics of the model is crucial. Defining

$$\psi = n_1 + in_2, \tag{3.32}$$

we observe that ψ at the physical infinity bears a representation of the gauged U(1) symmetry,

$$\psi \approx e^{in\theta}, \tag{3.33}$$

where n is the number of times the equatorial circle on the internal sphere is wrapped. Clearly n is the topological number labelling the homotopy sectors (3.31). Now using eqn. (3.32), we can prove the identity,

$$D_i\mathbf{n}\cdot D_i\mathbf{n} = |(\partial_i + iA_i)\psi|^2 + (\partial_i n_3)^2. \tag{3.34}$$

From eqn. (3.33) and eqn. (3.34) we observe that for finite energy configurations we require

$$\mathbf{A} = \mathbf{e}_\theta \frac{n}{r}, \tag{3.35}$$

on the boundary.

The asymptotic form (3.35) allows us to compute the magnetic flux

$$\Phi = \int B d^2x = \int_{boundary} A_\theta r d\theta = 2\pi n. \tag{3.36}$$

The above equation shows that the magnetic flux is quantized in each topological sector. Thus in contrast with [54], the topologically stable soliton solutions of the model (3.14) have quantized magnetic flux. Note that the quantization of the magnetic flux is ensured by the novel topology (3.31), the origin of which traces back to the choice of the self-interaction potential (3.13) having symmetry breaking minima.

It will be an interesting exercise to compute the spin of the excitations using the method of the previous Chapter. A straightforward calculation shows that the spin K is given by equation (2.21). The momentum π_i is now given by

$$\pi_i = -F_{0i} \tag{3.37}$$

which vanishes due to equations (3.26) and (3.35). The excitations are spinless which is expected because the electrodynamic interaction does not give rise to 'fractional' spin like the C - S interaction. To explore further implication of the new topology we look for a gauge invariant conserved topological current. We propose a trial candidate

$$K_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}[\mathbf{n}\cdot D^\nu \mathbf{n} \times D^\lambda \mathbf{n} - F^{\nu\lambda} n_3], \tag{3.38}$$

which is motivated by the form of the topological current of the usual sigma model. It can be shown easily that K_μ is indeed conserved,

$$\partial_\mu K^\mu = 0. \tag{3.39}$$

The corresponding conserved charge is

$$T = \int d^2x K_0. \tag{3.40}$$

Using eqn. (3.38) and eqn. (3.40) we obtain the expression of the topological charge,

$$T = \int d^2x[\frac{1}{8\pi}\epsilon_{ij}\mathbf{n} \cdot (\partial^i \mathbf{n} \times \partial^j \mathbf{n})] + \frac{1}{4\pi}\int_{boundary} n_3 A_\theta r d\theta, \tag{3.41}$$

where r,θ are polar coordinates in the physical space and $A_\theta = \mathbf{e}_\theta \cdot \mathbf{A}$. The second term vanishes due to eqn. (3.15) and the first term is the topological charge of the usual O(3) sigma model. The latter is known to give the number of times the physical space is wrapped over the internal space. Now if the equatorial circle is traversed once, the physical space is mapped on a hemisphere of the internal sphere. We thus expect that

$$T = \frac{n}{2}. \tag{3.42}$$

Evidently half integral values of T are allowed. This is a again a new feature of our model.

3.3.3 Self-dual solutions

So far we were discussing the general nature of the finite energy solutions of the model. Our next pursuit is to show that in the present model self-duality conditions can be obtained by satisfying Bogomol'nyi limit. The following identity

$$D_i\mathbf{n} \cdot D_i\mathbf{n} = \frac{1}{2}(D_i\mathbf{n} \pm \epsilon_{ij}\mathbf{n} \times D_j\mathbf{n})^2 \pm \epsilon_{ij}\mathbf{n} \cdot D_i\mathbf{n} \times D_j\mathbf{n}, \tag{3.43}$$

may be used to rearrange the energy functional as

$$E = \frac{1}{2}\int d^2x[\frac{1}{2}(D_i\mathbf{n} \pm \epsilon_{ij}\mathbf{n} \times D_j\mathbf{n})^2 + (F^{12} \pm n_3)^2] \pm 4\pi T, \tag{3.44}$$

where T is the topological charge given by eqn. (3.41). In a particular topological sector labelled by fixed value of T, the following equations

$$D_i\mathbf{n} \pm \epsilon_{ij}\mathbf{n} \times D_j\mathbf{n} = 0, \tag{3.45}$$

$$F_{12} \pm n_3 = 0, \tag{3.46}$$

minimize the energy functional (3.44). In equations (3.45) and (3.46) the $\pm$ sign corresponds to positive and negetive values of the topological charge.

If we define the dual of $D_i\mathbf{n}$ as

$$\overline{D_i\mathbf{n}} = \mp\epsilon_{ij}\mathbf{n} \times D_j\mathbf{n}, \tag{3.47}$$

then the equations (3.45) are self - dual in that sense. The duality operation (3.47) must be consistent i.e.

$$\overline{\overline{D_i\mathbf{n}}} = D_i\mathbf{n} \tag{3.48}$$

should be preserved. This can be checked easily

$$\begin{aligned} \overline{\overline{D_i\mathbf{n}}} &= \mp\epsilon_{ij}\mathbf{n} \times \overline{D_j\mathbf{n}} \\ &= \mp\epsilon_{ij}\mathbf{n} \times (\mp\epsilon_{jl}\mathbf{n} \times D_l\mathbf{n}) \\ &= -\delta_{il}\mathbf{n} \times (\mathbf{n} \times D_l\mathbf{n}) \\ &= -[(\mathbf{n} \cdot D_i\mathbf{n})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})D_i\mathbf{n}]. \end{aligned} \tag{3.49}$$

Using the constraint (3.2) it can be shown that

$$\mathbf{n} \cdot D_i\mathbf{n} = 0 \tag{3.50}$$

Using (3.2) and (3.50) we get from (3.49) the equation (3.48). Due to the self - duality of the equations (3.45) corresponding to the duality operation (3.47) the set of equations (3.45) and (3.46) are referred to as self - dual equations [3]. This is the reason why the corresponding theories go by the name self - dual. In the context of our works the self - dual equations (3.45) and (3.46) have the significance that they minimize the energy functional in the static limit. This means that these equations extremize the action in this limit and hence solution to these equations form a subset of solutions of the second order Euler - Lagrange equations (3.18) and (3.19). It is useful to verify this explicitly.

Starting from eqn. (3.45), we get after some algebra,

$$D_i(D_i\mathbf{n}) = -\epsilon_{ij}D_i\mathbf{n} \times D_j\mathbf{n} + (n_3\mathbf{i_3} - n_3^2\mathbf{n}). \tag{3.51}$$

Using eqn. (3.45) for $D_i\mathbf{n}$ in the r.h.s. of eqn. (3.51) again, we obtain

$$D_i(D_i\mathbf{n}) = [\mathbf{n} \cdot D_k(D_k\mathbf{n})] \cdot \mathbf{n} + (n_3\mathbf{i_3} - n_3^2\mathbf{n}) \tag{3.52}$$

where the constraint (3.2) and the condition (3.50) has been used and F_{12} is substituted by equation (3.46). Equation (3.52) is the E - L equation for $\mathbf{n}$, equation (3.18), under the static limit and eqn. (3.2).Similarly, we can verify that equation (3.19) is satisfied.

3.3.4 Integrability of the self-dual equations

The first order equations (3.45) and (3.46) can be cast in an equivalent second order equation by projecting the target space S^2 steriographically onto C U ∞ [17]. A point on S^2 denoted by n_1,n_2,n_3 subject to the constraint (3.2) corresponds to a point (ω_1, ω_2) on the complex plane so that,

$$\omega_1 = \frac{n_1}{1+n_3}, \omega_2 = \frac{n_2}{1+n_3} \tag{3.53}$$

The plane on which the projection is made is parallel to the n_1, n_2plane and contains the north pole. We can now define the complex variable ω by

$$\omega = \omega_1 + i\omega_2 \tag{3.54}$$

The self-dual equations (3.45) and (3.46) can now be exploited to show that ω satisfies

$$\begin{aligned} D_1\omega &= \mp i D_2\omega && (3.55)\\ F_{12} &= \mp\frac{(1-|\omega|^2)}{(1+|\omega|^2)} && (3.56)\end{aligned}$$

where D_j now stands for $\partial_j + iA_j$. Equations (3.55) and (3.56) imply the following second order equation for $\phi = \ln\omega$,

$$\nabla^2(\frac{\phi+\phi^*}{2}) = \tan\frac{\phi+\phi^*}{2} \tag{3.57}$$

This nonlinear Laplace equation falls outside the small class of such equations which are exactly integrable [70]. It then implies that the self-dual equations (3.45) and (3.46) are not exactly integrable. To obtain details of the solutions of these equations some numerical method must be adopted. There is an important class of solutions which are physically useful and easily amenable to numerical solutions. These are the rotationally symmetric solutions which we are going to study next.

3.3.5 Rotationally symmetric solutions

We consider the well-known rotationally symmetric ansatz for the solutions of eqn. (3.45) and eqn. (3.46), subject to the constraint (3.2) and the boundary conditions (3.15) and (3.35) [54, 71],

$$n_1(r,\theta) = \sin g(r)\cos n\theta,$$

$$n_2(r,\theta) = \sin g(r)\sin n\theta,$$

$$n_3(r,\theta) = \cos g(r),$$

$$\mathbf{A}(r,\theta) = -\mathbf{e}_\theta \frac{na(r)}{r}. \tag{3.58}$$

Equation (3.15) demands the boundary condition

$$g(r) \to \pm\frac{\pi}{2} \text{ as } r \to \infty \tag{3.59}$$

and equation (3.35) requires that

$$a(r) \to -1 \text{ as } r \to \infty \tag{3.60}$$

The boundary conditions at the origin follow from the condition that the fields be nonsingular there. From the last equation of (3.58) we observe that the condition

$$a(0) = 0 \tag{3.61}$$

is required. The energy functional (3.30) involves the gradient of the fields n^a which must be nonsingular at the origin. Imposition of the condition demands that

$$n \sin g(0) = 0. \tag{3.62}$$

Since n is the degree of vorticity which does not vanish for nontrivial solutions, we require $g(0) = 0$ or $\pm\pi$. These considerations suggest the following boundary conditions

$$g(r) \to 0 \text{ or } \pm\pi \tag{3.63}$$

$$a(r) \to 0 \tag{3.64}$$

as $r \to 0$. Substituting the ansatz (3.58) in (3.45) and (3.46), we get the following first order equations for g(r) and a(r),

$$g'(r) = \pm\frac{n(a+1)}{r}\sin g, \tag{3.65}$$

$$a'(r) = \mp\frac{r}{n}\cos g, \tag{3.66}$$

where the upper sign holds for +ve T and the lower sign corresponds to -ve T. If we substitute the ansatz (3.58) in the expression (3.41) of T we get

$$T = -\frac{n}{2}[\cos g(\infty) - \cos g(0)]. \tag{3.67}$$

Using the allowed boundary conditions (3.59) and (3.63) for g(r) in (3.67) we find that $T = \pm\frac{n}{2}$. This result is in conformity with our general considerations[1]. Equation (3.67) also allow us to identify the appropriate boundary condition at the origin for +ve (-ve) T. If g(0) = 0, T is +ve and if g(0) = $\pm\pi$, T is -ve. Henceforth, for convenience, we will assume +ve T. The equations to be solved are then,

$$\begin{aligned} g'(r) &= \frac{n(a+1)}{r}\sin g, && (3.68)\\ a'(r) &= -\frac{r}{n}\cos g, && (3.69) \end{aligned}$$

with the boundary conditions

$$g(0) = 0, a(0) = 0, g(\infty) = \frac{\pi}{2}, a(\infty) = -1. \tag{3.70}$$

3.3.6 Numerical Solution of the self - dual equations

The equations (3.68) and (3.69) have a regular singular point at $r = 0$. Due to this singularity the solution cannot be started from $r = 0$ and boundary conditions are to be imposed at a finite but very small value of r [54]. We take $r = 10^{-6}$ to start. A power series solution of eqn. (3.68) and eqn.(3.69) is sought,

$$\begin{aligned} a(r) &= a^{(1)}(r) + a^{(2)}(r) + ..., && (3.71)\\ g(r) &= g^{(1)}(r) + g^{(2)}(r) + ..., && (3.72) \end{aligned}$$

[1]See equation (3.38).

in the neighberhood of $r = 0$. The ascending terms of the series are progessively of higher order in r. Substituting eqn.s (3.71) and (3.72) in eqn.s (3.68) and (3.69) and equating terms of a particular order we get

$$g(r) = A_n r^n + O(r^{n+2}), \tag{3.73}$$

$$a(r) = -\frac{r^2}{2n} + O(r^{2n+2}), \tag{3.74}$$

where A_n is arbitrary, so far as the behaviour near the origin is considered. The value of A_n determines the conditions at infinity. If the value is too large the conditions at infinity are overshooted, whereas, if the value is too small g(r) vanishes asymptotically after reaching a maximum. There is a critical value of $A_n = A_n^{crit}$ which enables the solution started from $r = 10^{-6}$ to reach the asymptotic conditions. The situation is comparable with similar findings elsewhere [34]. Equations (3.73) and (3.74) allow us to impose the boundary condition at the origin at a point near $r = 0$. This is necessary because $r = 0$ is a regular singular point of the differential equations (3.68) and (3.69) and as such, cannot be handled directly by the numerical algorithm. In figures 1 and 2 we show the solutions for $g(r)$ and $a(r)$ for $n = 1$. The solutions for other n values can be similarly obtained and the critical value of A_n is found to decrease with increasing n.

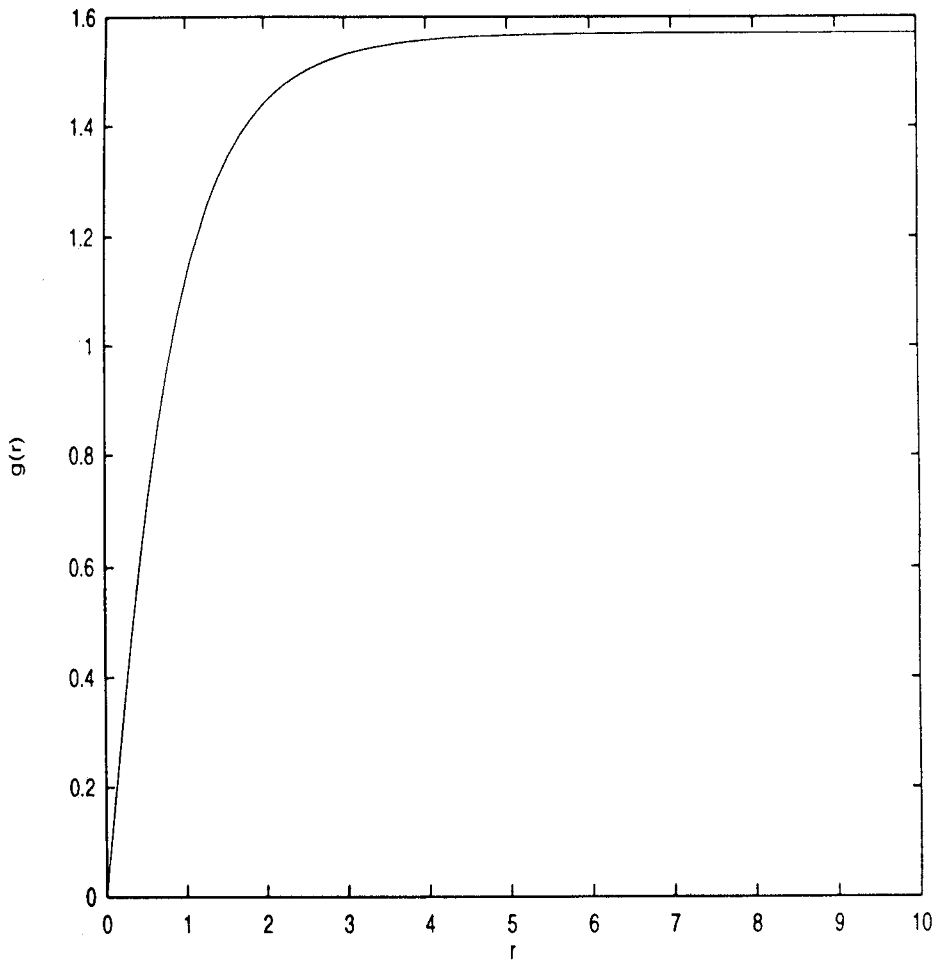

Figure 3.1: The solution for g(r) when n =1

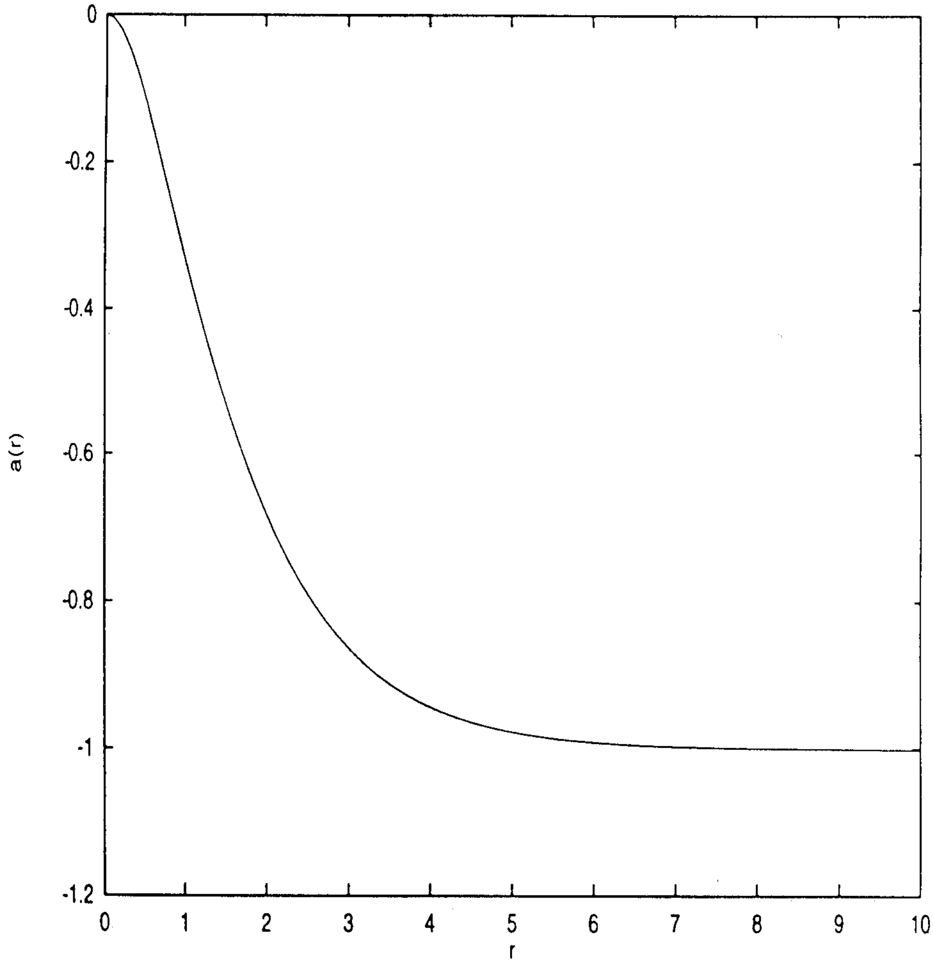

Figure 3.2: The solution for a(r) when n =1

3.4 Chern - Simons gauge coupling

We have mentioned in the introduction that in (2+1) dimensions the gauge field dynamics may be governed by the C - S term instead of the Maxwell piece. Field theories with C - S coupling have been actively pursued for a long period of time . The field is rich with interesting and exotic consequences. In particular the C - S term is found to induce fractional spin and statistics in the soliton sector of various models as we have elaborately discussed in the previous chapter. Earlier attempt of gauging the O(3) sigma model by the C - S coupling suffers the problem of degeneracy [55]. The solutions have arbitrary magnetic flux and angular momentum in a particular topological class, which is certainly unappetising in view of the experience with the C - S vortices. Moreover, the arbitrariness of the magnetic flux leads to excitations of arbitrary size which are degenerate in energy, a fact which frustrates the very motivation of gauging. The remeady is by now clear. It lies in the choice of the self interaction, the form of which must be so as to break the gauged symmetry. Simultaneously, the requirement of self - duality is to be satisfied. We chose the potential satisfying this twin requirements. The model Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} D_\mu \mathbf{n} \cdot D^\mu \mathbf{n} + \frac{k}{4} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + U(\mathbf{n}), \tag{3.75}$$

subject to the constraint (3.2). The last term

$$U(\mathbf{n}) = -\frac{1}{2k^2} n_3^2 (1 - n_3^2) \tag{3.76}$$

is the assumed self interaction. Note that the minima of the potential arise when either,

$$n_1 = 0, \;\; n_2 = 0 \;\; and \;\; n_3 = \pm 1, \tag{3.77}$$

or,

$$n_3 = 0 \;\; and \;\; n_1^2 + n_2^2 = 1. \tag{3.78}$$

Equation(3.77) corresponds to a vaccum structure where the U(1) symmetry is not spontaneously broken whereas in eqn. (3.78) spontaneous symmetry breaking takes place. For obvious reasons we will refer to eqn. (3.77) as the symmetric and eqn. (3.78) as the symmetry breaking minima. This scenario is different from the model with the Maxwell coupling, eqn. (3.14), where only symmetry breaking minima were obtained. We will find that the symmetric vaccua corresponds to soliton solutions whose stability is not guranteed by any topological criterion. Such solitons are termed as the nontopological solitons. The symmetry breaking minima again lead to topologically stable solutions. Thus, under these limits we get both topological and nontopological soliton solutions. Note that this soliton structure is different from that of [55] but comparable with the soliton solutions of the Chern-Simons-Higgs model [34].

3.4.1 Equations of motion

The Euler - Lagrange equations of the system (3.75) are derived subject to the constraint (3.2) by the Lagrange multiplier technique just as in section 3,

$$\begin{aligned} D_\nu(D^\nu \mathbf{n}) &= [D_\nu(D^\nu \mathbf{n}) \cdot \mathbf{n}]\mathbf{n} - \frac{1}{k^2}\mathbf{i}_3 n_3(1 - 2n_3^2) \\ &+ \frac{1}{k^2} n_3^2(1 - 2n_3^2)\mathbf{n}, \qquad (3.79) \\ \frac{k}{2}\epsilon^{\mu\nu\lambda} F_{\nu\lambda} &= j^\mu. \qquad (3.80) \end{aligned}$$

The conserved current j^μ is again expressed by

$$j^\mu = -\mathbf{i}_3 \cdot \mathbf{J}^\mu, \qquad (3.81)$$

with

$$\mathbf{J}^\mu = \mathbf{n} \times D^\mu \mathbf{n}. \qquad (3.82)$$

These relations are identical with eqn. (3.20) and eqn. (3.21). Note that eqn. (3.80) is already of first order i.e. in the self-dual form. This is a special feature of the low derivative C - S interaction [3]. Using eqn. (3.80) we will now derive an important characteristic of the model. From the $\mu = 0$ component of (3.80) we get,

$$k\epsilon_{ij}\partial^i A^j = j^0. \tag{3.83}$$

But

$$B = -\epsilon_{ij}\partial_i A_j = \mathrm{curl}\mathbf{A} \tag{3.84}$$

is the magnetic field. Using eqn. (3.84) and integrating eqn. (3.83) over the entire space we get,

$$\Phi = -\frac{Q}{k}, \tag{3.85}$$

where Q is the charge and Φ is the magnetic flux. The solitons are charge-flux composites in contrast with the model of the previous section. This is again another universal feature of the C - S vortices.

3.4.2 Topological classification of the solutions.

To analyse the topological properties of the finite energy solutions we again go to the static limit. The energy functional can be derived from Schwinger's energy - momentum tensor [35],

$$E = \frac{1}{2}\int d^2\mathbf{x}[(D_i\mathbf{n})\cdot(D_i\mathbf{n}) + A_0^2(1-n_3^2) + \frac{1}{k^2}n_3^2(1-n_3^2)]. \tag{3.86}$$

Now from (3.81) and (3.82) we get

$$j^0 = -A^0(1-n_3^2). \tag{3.87}$$

Expressing j^0 with the help of (3.83) and (3.84) we get from (3.87)

$$A^0 = \frac{kB}{1 - n_3^2}. \tag{3.88}$$

Equation (3.88) forces B to vanish whenever n_3 becomes equal to 1. We will use this fact below. Meanwhile, using (3.88) in eqn. (3.86), A^0 can be eliminated from the expression of the energy functional. The expression now becomes

$$E = \frac{1}{2} \int d^2x[(D_i\mathbf{n}) \cdot (D_i\mathbf{n}) + \frac{k^2 B^2}{1 - n_3^2} + \frac{1}{k^2} n_3^2 (1 - n_3^2)]. \tag{3.89}$$

From (3.89) we note that for finite energy configurations the potential $\mathbf{n}$ must asymptotically go to its symmetric minimum (3.77) or to the symmetry breaking minimum (3.78). Let us consider these two cases separately.

case1:symmetric minimum

In this case the physical infinity is one point compactified to the north or south pole of the internal sphere. The situation is analogous to the model considered in [55]. The finite energy solutions are classified according to the homotopy

$$\Pi_2(S_2) = Z, \tag{3.90}$$

just as the pure sigma model. The gauge field $\mathbf{A}$ can go to can go to any asymptotic limit. Since $n_3 \to 1$ at infinity the magnetic field vanishes asymptotically due to (3.88). The magnetic flux therefore remains finite. However both the magnetic flux and and angular momentum assume arbitrary values. We thus see that these configurations are not quite interesting from the point of view of breaking the scale invariance of the solutions.

case2: symmetry breaking minimum

The situation changes dramatically when we consider the symmetry breaking minimum (3.78). Now the infinite circle on the physical space is mapped on the equatorial circle of the internal sphere. The degree of this mapping , n, is the topological number labelling different sectors of the homotopy group

$$\Pi_1(S_1) = Z. \tag{3.91}$$

This state of affairs is analogous to the problem considered in the previous section, see the discussions under equation (3.31). As a result of this the gauge field ceases to remain arbitrary. The asymptotic limit of the gauge field is given by

$$\mathbf{A} = \mathbf{e}_\theta \frac{n}{r}, \tag{3.92}$$

which is the same equation as (3.35) and derived in the same fashion. In fact the asymptotic form is sufficient to compute the charge, magnetic flux and spin of the model. The magnetic flux is

$$\Phi = \int B d^2x = \int_{boundary} A_\theta r d\theta = 2\pi n. \tag{3.93}$$

The charge Q is then automatically fixed by (3,85)

$$Q = -2\pi k n. \tag{3.94}$$

The spin of the excitations is determined by the method of Chapter2. Explicitly this spin K is given by

$$K = -\frac{k}{2}\int_{boundary} \partial^i[x_i A^2 - A_i x_j A^j] d^2x = \pi k n^2. \tag{3.95}$$

Equations (3.93) to (3.95) shows that the flux, charge and spin of the excitations are quantised in each topological sector. The degeneracy is totally lifted which enables us to interpret the excitations as extended particles in a consistent manner. Equation (3.38) still gives a relevant topological current and (3.40) is the expression for the corresponding charge. We find that for the configurations corresponding to the symmetry breaking minima the topological charge is given by

$$T = \frac{n}{2}. \tag{3.96}$$

This expression actually gives the number of times the internal sphere is wrapped. For n=1, $T = \frac{1}{2}$ which means that one hemisphere is covered when the equatorial circle is traversed once.

Looking back at equation (3.41) we find that for the symmetric minimum case T is not quantised. This is due to the presence of the second term in (3.41),

$$\frac{1}{4\pi}\int A_\theta . r d\theta, \tag{3.97}$$

which has arbitrary value as A_θ is arbitrary on the infinite circle. In this sense the finite energy solutions corresponding to the symmetric vaccum will be called "nontopological". This can be put in an alternative way. The magnetic flux Φ remain arbtrary for these solitons. Due to the charge flux relation Φ is however conserved. Thus Φ can be used to label the "nontopological" solitons. These results may be compared with the nontopological solitons of the Chern - Simons -Higgs model [34].

3.4.3 Self - dual equations

We then turn to show that the model satisfies Bogomol'nyi conditions. The energy - functional (3.89) can be rearranged as

$$E = \frac{1}{2}\int d^2x[\frac{1}{2}(D_i\mathbf{n} \pm \epsilon_{ij}\mathbf{n} \times D_j\mathbf{n})^2 + \frac{k^2}{1-n_3^2}(F_{12} \pm \frac{1}{k^2}n_3(1-n_3^2))^2] \pm 4\pi T, \tag{3.98}$$

where use has been made of the identity (3.43). We immediately get the self - dual equations

$$D_i\mathbf{n} \pm \epsilon_{ij}\mathbf{n} \times D_j\mathbf{n} = 0, \tag{3.99}$$

$$F_{12} \pm \frac{1}{k^2}n_3(1-n_3^2) = 0, \tag{3.100}$$

which minimises the energy functional in a particular topological sector, the upper sign corresponds to +ve and the lower sign corresponds to -ve value of the topological charge.

We will now show the consistency of (3.99) and (3.100) using the well-known Ansatz [54, 71],

$$\begin{aligned} n_1(r,\theta) &= \sin g(r) \cos n\theta, \\ n_2(r,\theta) &= \sin g(r) \sin n\theta, \\ n_3(r,\theta) &= \cos g(r), \\ \mathbf{A}(r,\theta) &= -\mathbf{e}_\theta \frac{na(r)}{r} \end{aligned} \tag{3.101}$$

From (3.78) we observe that we require the boundary condition

$$g(r) \to \pm\frac{\pi}{2} \text{ as } r \to \infty \tag{3.102}$$

and equation (3.92) dictates that

$$a(r) \to -1 \text{ as } r \to \infty. \tag{3.103}$$

Remember that equation (3.92) was obtained so as the solutions have finite energy. Again, for the fields to be well defined at the origin we require

$$g(r) \to 0 \text{ or } \pi \text{ and } .a(r) \to 0 \; as \; r \to 0 \tag{3.104}$$

Substituting the ansatz (3.101) into (3.99) and (3.100), we find that

$$g'(r) = \pm\frac{n(a+1)}{r} \sin g, \tag{3.105}$$

$$a'(r) = \mp\frac{r}{nk^2} \sin^2 g \cos g, \tag{3.106}$$

where the upper sign holds for +ve T and the lower sign corresponds to -ve T.Equations (3.105) and (3.106) are not exactly integrable.They may be solved numerically subject to the appropriate boundary conditions to get the exact profiles.

Using the ansatz (3.101) we can explicitly compute the topological charge T by performing the integration in (3.41).The result is

$$T = -\frac{n}{2}[\cos g(\infty) - \cos g(0)] \tag{3.107}$$

So we find that according to (3.102) and (3.104) T $=\pm\frac{n}{2}$ which is in agreement with our observation (3.42). Note that g(0) $=$ 0 corresponds to +ve T and g(0) $= \pi$ corresponds to -ve T. If we take +ve T we find g(r) bounded between 0 and $\frac{\pi}{2}$ is consistent with (3.102),(3.104) and (3.105).Again a(r) bounded between 0 and -1 is consistent with (3.92),(3.88) and (3.106).Thus for +ve topological charge the ansatz (3.101) with the following boundary conditions

$$\begin{aligned} g(0) &= 0, \;\; a(0) = 0, \\ g(\infty) &= \frac{\pi}{2}, \;\; a(\infty) = -1, \end{aligned} \tag{3.108}$$

are consistent with the Bogomol'nyi conditions. Similarly the consistency may be verified for -ve T. Equations (3.105) and (3.106) can be solved by the same numerical algorithm discussed in the previous section.

3.5 Conclusion

The nonlinear O(3) sigma model in (2+1) dimensions support self - dual soliton solutions [28]. The self - dual equations are exactly integrable and the solutions are expressed in terms of rational functions. The solutions are thus scale - invariant which poses a problem in the particle identification on quantisation [52]. An interesting method of breaking the scale - invariance is to partially gauge the model. A particular form of self - interaction is to be included in order to saturate the Bogomol'nyi bounds [51]. The form of this self - interaction is very important. In [54] the gauge field dynamics is chosen to be dictated by the Maxwell term. An alternative possibility is the Chern - Simons coupling [55]. Both in [54, 55] the solitons are found to be infinitely degenerate in each topological sector. We have demonstrated that this degeneracy is not an essential feature of the problem. The degeneracy of the solutions was shown to be lifted by suitably choosing the self - interaction so that the gauged symmetry is spontaneously broken. A new type of the partially gauged nonlinear sigma models was thereby discovered where a novel topology is introduced due to the symmetry breaking. This has been demonstrated for both Maxwell and Chern - Simons couplings [49, 50].

A detailed discussion of our model with the Maxwell coupling has been presented. The Euler - Lagrange equations were derived by the Lagrange multiplier technique. The excitations of the model were shown to be electrically neutral using these equations of motion. The homotopy classification of the finite energy solutions exhibited the novel topology introduced in our model. The asymptotic limit of the gauge field obtained from the requirement of finite energy immediately provided the quantization of the magnetic flux in a given topological sector instead of being degenerate as in [54]. The correspondence of the lifting of the degeneracy with the new topology was thus clearly revealed. A gauge invariant toplogical current was then constructed. The toplogical charge following from this current was found to assume both half integral and integral values which is again a new feature of our model. The expression of the topological charge was used to minimize the static energy functional in a particular topological sector *a la* Bogomol'nyi [51]. A set of coupled first order equations involving the field functions were obtained. Since these equations correspond to the minimization of the energy functional in the static limit, they are expected to satisfy the E - L equations of motion. This has been demonstrated explicitly. The self - duality of the set of the first order equations were explored. Integrability of the set was investigated by casting the equations in an equivalent second order form. The resulting nonlinear Laplace equation was found to fall outside the small class of those equations that are exactly integrable. Numerical solutions of the self - dual equations were discussed by adopting the well - known rotationally symmetric ansatz [54, 71]. The field profiles for $T = \frac{1}{2}$ were explicitly demonstrated.

The analysis of the model where the partial gauging was implemented by the C - S coupling mimicked the previous case of Maxwell coupling. The excitations are no longer electrically neutral. These are now charge - flux composites which is a characteristic feature of the C - S theories. From a homotopy classification of the solutions, the charge - flux and the spin of the excitations were derived where we have employed the method discussed in chapter 2. The novel topology of the solutions was again proved to be instrumental in lifting the degeneracy of the solutions observed in a similar model [55]. The self - dual equations were derived and the consistency of the equations and the boundary conditions were discussed by adopting a roatationally symmetric ansatz.

Chapter 4

SYMMETRY IN A NONABELIAN CHERN SIMONS SYSTEM

4.1 Introduction

Theories with the Chern -Simons (C - S) gauge field coupling constitute the focal theme of our studies in three dimensional field theories. In chapter 2 we have discussed a general aspect of the C - S vortices, namely, the fractional spin. Later, in chapter 3 our attention was shifted to the aspect of self duality. In all these calculations we have mostly worked in a gauge independent setting [12, 13]. Of course in the final stage of calculation of the spin or in assuming a rotationally symmetric ansatz for the self dual configurations, a definite gauge fixing was tacitly assumed. But we still have not found suitable occassions to discuss the subtleties of gauge fixing including its correspondence with the gauge independent analysis Again our analysis so far remained to be classical , to be more accurate, almost classical. Thus we did not bother about the ordering of different fields and such issues which occur in a full fledged quantum calculation. As we have indicated in the overview, both these issues of gauge fixing and ordering problems connected with quantization appear in our studies of the space time symmetries in connection with a nonrelativistic matter theory coupled with the nonabelian C - S term. These studies will be reported in the present chapter.

Systems of point particles carrying non-abelian charge interacting with a non-abelian gauge potential have been considered over the last two decades [72]. Similar models in 2+1 dimensions, where the kinetic term of the gauge field is given by the Chern-Simons three form instead of the usual Yang-Mills piece, have been actively investigated in recent years [57, 58, 73, 74]. In this context it is interesting to note that it is possible to construct models which are Galilean invariant [45, 56, 59, 48, 75, 76, 77] rather than Poincare invariant. This is because the Chern Simons term does not have an elementary photon associated with it so that the Bargmann super - selection rule can be accomodated. Purely Galilean - invariant models are useful to study problems which are difficult when analysed within the full formalism of special relativity.

An important issue in the context of theories involving non-abelian Chern-Simons term is the study of relevant space-time symmetries associated with either Galilean or Poincare transformations. For instance it was claimed [57] that (classical) Poincare covariance gets violated in a theory where the non-abelian Chern-Simons term is coupled to fermions. The calculations were done in the axial gauge which enabled the elimination of the gauge degrees of freedom in favour of the matter variables. Alternatively, it has been shown [58] that by formulating the model in Dirac's [14] constrained approach which retains all degrees of freedom, the (classical) Poincare covariance is preserved. It is thus clear that the issue of symmetries is rather subtle and requires a thorough and systematic investigation. Indeed since Chern-Simons matter systems are constrained systems, it is possible to discuss different formalisms depending on how one accounts for the constraints. For instance it was shown [48] that while the quantum Galilean algebra was preserved following a gauge independent approach [12, 13, 15, 16], there was a violation in the Coulomb gauge. Similarly, an unconventional ordering of operators, different from the usual normal ordering, was suggested in [59] to recover the quantum Galilean algebra in the axial gauge. Both these analyses were performed for abelian models. The situation is even less transparent for nonabelian models. Indeed, as far as we are aware, the systematic study of the galilean algebra of such a system remains to be performed. Incidentally, these models are interesting in their own right because of the nonabelian anyonic states supported by them as shown in recent works [74, 78].

The main obstacle that hinders an analysis of nonabelian models is the existence of nonlinear constraints. The usual Coulomb gauge fails to "solve" the Gauss constraint and recourse is taken to axial or holomorphic gauges [74, 78]. In formulating a quantised version, however, the holomorphic gauge poses problems in defining a hermitian Hamiltonian. The axial gauge, therefore, remains a natural choice [74], but care must be exercised in the handling of boundary terms which are a usual consequence of this gauge.

The object of this chapter is to analyse in details both the classical and quantal Galilean algebra of a model involving the coupling of nonabelian Chern Simons three form with nonrelativistic matter. In oreder to avoid the complexities of gauge fixing in nonabelian theories, a gauge independent formulation [12, 13, 16] for the canonical constrained structure has been presented in section 2. The closure of the classical Galilean algebra on the constraint surface is then demonstrated. It is also shown that the various galilean generators are gauge invariant on this surface. Consequently it is expected, although not mandatory, that the classical Galilean algebra should be preserved for gauge fixed computations.This aspect has been considered in sections 3 and 4, where different possibilities have been discussed for doing the reduced space computations in the axial gauge. In the symplectic approach [74, 20] the gauge degrees of freedom are eliminated in favour of the matter degrees of freedom by solving the Gauss constraint. The space time generators are then defined in the axial gauge. The classical Galilean algebra is now verified by using the fundamental brackets in the matter sector. Surprisingly, it is found that the complete galilean algebra does not close in general. Specifically, the bracket of the angular momentum with the Hamiltonian does not vanish; rather it is proportional to a boundary term. This term vanishes provided some additional restrictions on the Green functions are imposed. The reduced space formulation is briefly summarised following Dirac's [14] constrained formalism with a view to compare with the symplectic analysis. It is shown that the algebra in the pure gauge sector differs by a boundary term in the two approaches which is exactly identical to the boundary term in the bracket of the angular momentum and the Hamiltonian found in the symplectic approach. In other words the same conditions on the Green functions simultaneously preserve the equivalence between the Dirac and the symplectic approaches, as well as the classical galilean symmetry of the model. Section 5 provides a quantum analysis, both in the gauge independent and gauge fixed (Dirac) approaches. A conventional normal ordering is invoked and the Galilean generators are defined to preserve hermiticity. The closure of the quantum Galilean algebra in the gauge independent formalism is straightforward, exactly as happened for the classical analysis. This is basically due to the simple algebra in the mixed sector which considerably irons out the ordering problems. The reduced space computation, on the contrary, is more involved. Now the mixed sector algebra is nontrivial leading to complicated ordering isues. An immediate fallout of this is that the equations of motion for either gauge or matter fields acquire quantum corrctions which are of $O(\hbar^2)$. In the closure of the galilean algebra, the bracket of the angular momentum with the Hamiltonian does not vanish but contains terms proportional to $O(\hbar^2)$, thereby revealing their quantum nature. Some of these are boundary terms which are similar to the ones already encountered in the classical analysis, which may therefore be eliminated by adopting identical restrictions on the Green functions. There also occurs a term that drops out provided the self interaction vanishes. That the latter is possible is shown by two alternative arguments based on algebraic consistency . Section 6 contains the concluding remarks.

4.2 Gauge independent formulation of the model

In this section we study the clasical Galilean symmetry of the model without any explicit gauge fixing. The model comprises the Schrodinger Lagrangian minimally coupled with the non abelian Chern-Simons term [74],

$$\mathcal{L} = i\psi^\dagger D_0\psi - \frac{1}{2}(D_i\psi)^\dagger(D_i\psi) - k\epsilon_{\alpha\beta\gamma}tr(A^\alpha\partial^\beta A^\gamma + \frac{2}{3}A^\alpha A^\beta A^\gamma), \qquad (4.1)$$

where the covariant derivative is defined by,

$$D_\mu = \partial_\mu + A_\mu \qquad (4.2)$$

and $A_\mu = A_{\mu a}T^a$ with the antihermitian matrices T^a normalised as,

$$tr(T^aT^b) = -\frac{1}{2}g^{ab}, \tag{4.3}$$

where g^{ab} is the metric [74] in group space. The Schrodinger field ψ is an N-component column vector in a certain representation of T^a. The Lagrangian (4.1) can be written in a canonical form by working out the traces,

$$\mathcal{L} = i\psi^+\dot{\psi} - \frac{k}{2}\epsilon_{ij}A_i^a\dot{A}_j^a - \frac{1}{2}(\partial_i\psi^+ - \psi^+A_i)(\partial_i\psi + A_i\psi) + A_{0a}G^a, \tag{4.4}$$

where

$$G^a = i\psi^+T^a\psi + \frac{k}{2}\epsilon_{ij}(2\partial_iA_j^a + f^{abc}A_i^bA_j^c). \tag{4.5}$$

Since $\mathcal{L}$ has now been expressed in the desired canonical form, it is simple to read-off the relevant brackets[1] using the symplectic approach [74, 20]. The nontrivial brackets are

$$\{\psi_n(\mathbf{x}), \psi_m^*(\mathbf{x}')\} = -i\delta_{nm}\delta(\mathbf{x}-\mathbf{x}'), \tag{4.6}$$

$$\{A_i^a(\mathbf{x}), A_j^b(\mathbf{x}'\} = \frac{1}{k}\epsilon_{ij}g^{ab}\delta(\mathbf{x}-\mathbf{x}'). \tag{4.7}$$

It is clear from (4.4) that A_0^a is a Lagrange multiplier which enforces the constraint,

$$G^a \approx 0. \tag{4.8}$$

This constraint is just the analogue of the usual Gauss constraint in electrodynamics, being the generator of the time independent non-abelian gauge transformations. Using the brackets (4.6-4.7), it is straightforward to verify this property,

$$\int d^2\mathbf{x}\alpha_a(\mathbf{x})\{\psi(\mathbf{x}'), G^a(\mathbf{x})\} = \alpha(\mathbf{x}')\psi(\mathbf{x}') \tag{4.9}$$

$$\int d^2\mathbf{x}\alpha_b(\mathbf{x})\{A_i^a(\mathbf{x}'), G^b(\mathbf{x})\} = -\partial_i\alpha^a(\mathbf{x}') + f^{abc}\alpha_b(\mathbf{x}')A_{ic}(\mathbf{x}') \tag{4.10}$$

[1] All brackets are referred to at equal times

According to Dirac's [14] classification, therefore, $G^a(x)$ is a first class constraint. Indeed it is easy to obtain the involutive algebra,

$$\{G^a(\mathbf{x}), G^b(\mathbf{x}')\} = f_c^{ab} G^c(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') \tag{4.11}$$

The equations of motion obtained from (4.1) are found to be,

$$iD_0\psi = -\frac{1}{2}D_iD_i\psi \tag{4.12}$$

$$\frac{k}{2}\epsilon_{\alpha\beta\gamma}F^{\beta\gamma} = J_\alpha \tag{4.13}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}A^b_\mu A^c_\nu \tag{4.14}$$

is the field tensor and

$$J_0 = T_a J^a_0 = T_a(-i\psi^+ T^a\psi), \tag{4.15}$$

$$J_i = T_a J^a_i = -\frac{1}{2}T_a[\psi^+ T^a D_i\psi - (D_i\psi)^+ T^a\psi], \tag{4.16}$$

are the non-abelian charge density and spatial current density respectively. As usual, the time-component of (4.13) yields the Gauss constraint (4.8).

Going over to the Hamiltonian formalism we observe that the momenta canonically conjugate to the Lagrange multiplier A_0 is a constraint,

$$\pi^a_0 \approx 0 \tag{4.17}$$

This, together with G^a, form the complete set of constraints. The relations,

$$\{\pi^a_0(\mathbf{x}), G^b(\mathbf{x}')\} = \{\pi^a_0(\mathbf{x}), \pi^b_0(\mathbf{x}')\} = 0 \tag{4.18}$$

along with (4.11) constitute the full involutive algebra among the constraints. The canonical Hamiltonian is immediately written on inspecting (4.4),

$$H_c = \int d^2\mathbf{x}(\frac{1}{2}(D_i\psi)^+(D_i\psi) - A^a_0 G_a) \tag{4.19}$$

Using (4.6-4.7) it is easy to verify that H_c correctly generates the equations of motion,

$$\partial_0 \psi_n = \{\psi_n, H_c\} \tag{4.20}$$

$$\partial_0 A_i^a = \{A_i^a, H_c\} \tag{4.21}$$

Let us next discuss the symmetries under various space-time transformations. Consider an infinitesimal transformation,

$$x_\mu \to x'_\mu = x_\mu + \delta x_\mu \tag{4.22}$$

$$\phi_n(x) \to \phi'_n(x') = \phi_n(x) + \delta\phi_n(x) \tag{4.23}$$

with,

$$\delta x_\mu = \Lambda_{\mu\nu} \delta\omega^\nu \tag{4.24}$$

$$\delta\phi_n = \Phi_{n\nu} \delta\omega^\nu \tag{4.25}$$

where $\phi_n(x)$ generically denotes the fields in the Lagrangian and ν can be a single or double index. Then the invariance of any Lagrangian under the above transformations leads to a conserved current [67],

$$J_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} \Phi_{n\nu} - \theta_{\mu\sigma} \Lambda_\nu^\sigma \tag{4.26}$$

where,

$$\theta_{\mu\sigma} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_n)} \partial_\sigma \phi_n - \mathcal{L} g_{\mu\sigma} \tag{4.27}$$

is the canonical (Noether) EM tensor.

With this general input it is straightforward to obtain the various Galileo generators of the present model. For example, under space translations,

$$x_i \to x'_i = x_i - \delta\omega_i, \tag{4.28}$$

$$x_0 \to x'_0 = x_0, \tag{4.29}$$

the fields do not transform ($\Phi_{n\nu} = 0$) and the relevant generator is given by

$$\begin{aligned} P_i &= \int d^2\mathbf{x}\theta_{0i}(\mathbf{x}) \\ &= \int d^2\mathbf{x}(i\psi^+\partial_i\psi - \frac{k}{2}\epsilon_{kj}A^a_k\partial_i A_{ja}). \end{aligned} \tag{4.30}$$

Once again using (4.6-4.7) the normal transformation properties for the fields may be checked,

$$\{A_k(\mathbf{x}), P_i\} = \partial_i A_k(\mathbf{x}), \tag{4.31}$$

$$\{\psi(\mathbf{x}), P_i\} = \partial_i\psi(\mathbf{x}). \tag{4.32}$$

Similarly, under infinitesimal spatial rotations with angle θ,

$$t' = t, \tag{4.33}$$

$$x'_i = x_i + \theta\epsilon_{ij}x_j, \tag{4.34}$$

the fields transform as,

$$A'_0(x') = A_0(x), \psi'(x') = \psi(x), \tag{4.35}$$

$$A'_i(x') = A_i(x) + \theta\epsilon_{ij}A_j(x). \tag{4.36}$$

Comparing with the general transformation laws (4.22 - 4.25) we find,

$$\delta x_i = \theta\epsilon_{ij}x_j, \wedge_{ijk} = \delta_{ij}x_k, A^a_{ijk} = \delta_{ij}A^a_k. \tag{4.37}$$

The rotation generator after an antisymmetrisation now follows from (4.26),

$$\begin{aligned} J_{ij} &= \int d^2\mathbf{x}J_{0[ij]} \\ &= \int d^2\mathbf{x}(x_{[i}\theta_{0j]} + \frac{k}{2}\epsilon_{[im}A_{ma}A^a_{j]}). \end{aligned} \tag{4.38}$$

Since there is only one component, we may express this as,

$$J = \int d^2\mathbf{x}(\epsilon_{ij}x_i\theta_{0j} + \frac{k}{2}A_{ja}A_j^a). \tag{4.39}$$

The basic fields obey covariant transformation laws,

$$\{\psi(\mathbf{x}), J\} = \epsilon_{ij}x_i\partial_j\psi(\mathbf{x}), \tag{4.40}$$

$$\{A_i^a(\mathbf{x}), J\} = \epsilon_{jk}x_j\partial_k A_i^a(\mathbf{x}) + \epsilon_{ij}A_j^a(\mathbf{x}). \tag{4.41}$$

Finally, we come to the Galileo boosts,

$$x_i \to x_i' = x_i - \vartheta_i t. \tag{4.42}$$

The fields transform as,

$$\psi'(x', t') = \psi(x, t) - i\vartheta_i x_i \psi(x, t), \tag{4.43}$$

$$A_i'(x', t') = A_i(x, t), \tag{4.44}$$

$$A_0'(x', t') = A_0(x, t) + \vartheta_i A_i(x, t). \tag{4.45}$$

It can be verified that the action corresponding to (4.1) is invariant under these transformations. Comparing (4.42 - 4.45) with (4.22 - 4.25) yields the correspondence,

$$\wedge_{ij} = -t\delta_{ij}, \Phi_{nj} = -i\psi_n x_j. \tag{4.46}$$

so that the boost generator may be obtained from (4.26) as,

$$\begin{aligned} K_i &= \int d^2\mathbf{x} J_{0i} \\ &= \int (\frac{\partial L}{\partial \dot{\psi}_n}\Phi_{ni} - \theta_{0j}\wedge_{ji})d^2\mathbf{x} \\ &= tP_i + \int d^2\mathbf{x} x_i \psi^+ \psi. \end{aligned} \tag{4.47}$$

This definition of the boost agrees with [45, 48] but differs from [59, 75] where the intermediate sign is minus instead of plus as occurs in (4.47). This difference in signatures was also noticed and clarified in an earlier work [48]. It should however be mentioned that the boost generator has been derived here from first principles. Under these boosts the basic fields have the usual transformation properties,

$$\{\psi(\mathbf{x}), K_i\} = t\partial_i\psi(\mathbf{x}) - ix_i\psi(\mathbf{x}), \tag{4.48}$$

$$\{A_j(\mathbf{x}), K_i\} = t\partial_i A_j(\mathbf{x}). \tag{4.49}$$

We have thus shown that the basic fields transform covariantly under all the (Galilean) space-time generators. Consequently it is expected that the complete Galilean algebra ought to be satisfied. Indeed an explicit computation reveals that,

$$\{P_i, P_j\} = \{P_i, H_c\} = \{K_i, K_j\} = 0, \tag{4.50}$$

$$\{P_i, J\} = \epsilon_{ij} P_j, \tag{4.51}$$

$$\{K_i, J\} = \epsilon_{ij} K_j, \tag{4.52}$$

$$\{P_i, K_j\} = \delta_{ij}\int d^2\mathbf{x}\psi^+\psi = \delta_{ij} M, \tag{4.53}$$

$$\{H_c, K_i\} = P_i + \int d^2\mathbf{x} A_i^a G_a \approx P_i, \tag{4.54}$$

$$\{J, H_c\} = \epsilon_{ij}\int d^2\mathbf{x} x_i A_0^a \partial_j G_a \approx 0. \tag{4.55}$$

The last two brackets reduce to the conventional result on the constraint surface. Thus, on this surface classical Galilean covariance of the model has been demonstrated. An identical conclusion also holds in the abelian model [48].

The last part of this section is devoted to show that the generators entering in the above (Galilean) algebra are all gauge invariant. In that case these generators can be regarded as physical entities. Using the basic brackets (4.6 - 4.7), the algebra of the Gauss constraint (4.5) with the various generators may be explicitly calculated to yield,

$$\{P_i, G^a(\mathbf{x})\} = -\partial_i G^a(\mathbf{x}) \approx 0, \tag{4.56}$$

$$\{H_c, G^a(\mathbf{x})\} = 0, \tag{4.57}$$

$$\{J, G^a(\mathbf{x})\} = -\epsilon_{ij} x_i \partial_j G^a(\mathbf{x}) \approx 0, \tag{4.58}$$

$$\{K_i, G^a(\mathbf{x})\} = -t\partial_i G^a(\mathbf{x}) \approx 0. \tag{4.59}$$

Thus all the generators are found to be gauge invariant on the constraint surface defined by (4.8). This completes the gauge independent formulation of the model. The independent canonical pairs are (A_1, A_2) and (ψ, ψ^+). Classical Galilean algebra is satisfied. Furthermore gauge invariance of the relevant generators implies that this algebra should also be preserved in a gauge fixed analysis. However, this gauge invariance is valid in a weaker sense, because it assumes that $G_a \approx 0$ implies $\partial_i G_a \approx 0$. To understand this implication in a simpler setting, let us consider the corresponding abelian case. The Gauss operator G is the generator of time - independent gauge transformations. Since

$$\{J, G(\mathbf{x})\} = -\epsilon_{ij} x_i \partial_j G(\mathbf{x}) \tag{4.60}$$

We find that under a gauge transformation

$$A_i \quad \rightarrow \quad A_i - \partial_i \Lambda \tag{4.61}$$

the change in J is given by

$$\begin{aligned}
\Delta J &= \int d^2\mathbf{x} \Lambda(\mathbf{x}) \{J, G(\mathbf{x})\} \\
&= -\int d^2\mathbf{x} \Lambda(\mathbf{x}) \epsilon_{ij} x_i \partial_j G \\
&= -\int d^2\mathbf{x} \partial_j (\Lambda(\mathbf{x}) \epsilon_{ij} x_i G) + \int d^2\mathbf{x} \epsilon_{ij} x_i \partial_j \Lambda(\mathbf{x}) G
\end{aligned} \tag{4.62}$$

where we have exploited the antisymmetry of the ϵ - symbol in the last line. Clearly, the second term of the r.h.s of the above equation vanishes on the constraint surface. The first term, however, vanishes only when we rule out singular vortex structures. In presence of the latter, only

$$J_p = J + K \tag{4.63}$$

is strictly gauge invariant, where $K$is that part of the physical angular momentum which gives the anomalous spin of the vortices (see the discussions of article 2.7). The Noetherian expressions used in equations (4.56) - (4.59) are gauge invariant, modulo singular vortex configurations. This also explains the interpretation of the weaker form of gauge invariance ($\partial_i G_a \approx 0$). It is thus interesting and instructive to explicitly perform the gauge fixed computations. This will provide fresh insights into the model.

4.3 Gauge fixed formulation : The symplectic approach

The basic idea of this formulation is to work in a reduced space by eliminating the gauge freedom. There are different ways to achieve this target. In the symplectic approach [20] one explicitly solves the Gauss constraint (4.8) by imposing an additional (gauge) condition, thereby eliminating the gauge degrees of freedom in favour of the matter variables. The brackets involving the gauge fields are now computed from the sole knowledge of the fundamental algebra in the matter sector. This was also the course adopted in [74]. It is important to point out, however, that whatever gauge condition is chosen, the resulting solution is nonlocal since it involves the inversion of a derivative. In this sense the concept of a local action breaks down. An alternative approach which does not require the explicit solution of the Gauss constraint is to follow Dirac's [14] constrained analysis. In this case the Poisson brackets get replaced by the corresponding Dirac brackets. We shall show that the brackets in the pure gauge sector, computed by these two methods, differ by a boundary term involving the Green function. This term vanishes on imposing certain conditions on the Green function. Interestingly, these conditions are again required to prove the closure of the Galilean algebra. We now proceed with the reduced space analysis in the symplectic approach and the Dirac analysis will be discussed subsequently.

A particularly effective gauge choice is the axial gauge,

$$A_1^a = 0, \tag{4.64}$$

since it linearises the Gauss constraint,

$$G^a = k\partial_1 A_2^a - J_0^a = 0, \tag{4.65}$$

so that the other component of the gauge field is given by,

$$A_2^a = \frac{1}{k}\int d^2\mathbf{x}' G(\mathbf{x}-\mathbf{x}')J_0^a(\mathbf{x}'), \tag{4.66}$$

where $G(\mathbf{x}-\mathbf{x}')$ is the Green function,

$$\partial_1 G(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}'). \tag{4.67}$$

The algebra of the gauge sector is now completely governed by the basic bracket (4.6) in the matter sector. Using (4.64) and (4.66) it follows,

$$\{A_1^a(\mathbf{x}), A_1^b(\mathbf{x}')\} = \{A_1^a(\mathbf{x}), A_2^b(\mathbf{x}')\} = 0, \tag{4.68}$$

$$\{A_2^a(\mathbf{x}), A_2^b(\mathbf{x}')\} = -\frac{1}{k^2}\int d^2\mathbf{y} G(\mathbf{x}-\mathbf{y})G(\mathbf{x}'-\mathbf{y})f^{abc}J_{0c}(\mathbf{y}). \tag{4.69}$$

Likewise it is easy to obtain the algebra of the mixed sector,

$$\{A_1^a(\mathbf{x}), \psi_n(\mathbf{x}')\} = 0, \tag{4.70}$$

$$\{A_2^a(\mathbf{x}), \psi_n(\mathbf{x}')\} = \frac{1}{k}(T^a\psi(\mathbf{x}'))_n G(\mathbf{x} - \mathbf{x}'). \tag{4.71}$$

The algebra involving A_0^a is inconsequential since it is a Lagrange multiplier and not a dynamical variable. Note that there is an important subtlety in the solution (4.66). It does not represent a unique solution for A_2^a. There is an arbirtrariness because if $A_2^a(x)$ is a solution then $A_2'^a(x) = A_2^a(x) + f^a(x_0, x_2)$ is also a solution. On the other hand there is a residual gauge freedom that survives the axial gauge (4.64) [58],

$$A_\mu^a(x) \to A_\mu'^a(x) = A_\mu^a(x) + \partial_\mu \alpha^a(x_0, x_2) + f^{abc} A_{\mu b}(x) \alpha_c(x_0, x_2). \tag{4.72}$$

In the abelian theory it is possible, by choosing $f^a = \partial_2 \alpha^a$, to account for the residual gauge freedom and regard (4.66) as a unique solution for the gauge field. For the nonabelian theory at hand, however, the presence of the extra piece in (4.72) complicates matters. Indeed if we take,

$$f^a(x_0, x_2) = \partial_2 \alpha^a(x_0, x_2) + f^{abc} A_{2b}(x) \alpha_c(x_0, x_2), \tag{4.73}$$

we find that while the l.h.s. depends on (x_0, x_2) only, the r.h.s. depends on all x, so that it becomes impossible to find a solution for f^a. The arbitrariness in (4.66), therefore, persists just as the residual gauge freedom due to (4.72) remains.

The implementation of a specific gauge is known to modify the manifest covariant transformation of the basic fields [15]. For instance in the radiation gauge the boost law is found to be altered [48]. In the axial gauge, on the other hand, manifest rotational symmetry is violated. This implies that the transformation (4.41) under rotations will be modified to preserve the gauge condition (4.64). Since manifest covariance is spoilt it becomes imperative to verify the Galilean symmetry by working out the algebra (4.50 - 4.55) involving the gauge invariant generators. A detailed analysis shows that apart from one exception the complete Galilean algebra (4.50 - 4.55) is reproduced. The only nontrivial bracket is given by,

$$\begin{aligned}\{J, H_c\} &= \frac{1}{k}\int d^2\mathbf{x}d^2\mathbf{y}d^2\mathbf{z}\{G(\mathbf{x}-\mathbf{y})G(\mathbf{y}-\mathbf{z}) - G(\mathbf{x}-\mathbf{z})G(\mathbf{y}-\mathbf{z})\} \\ &\quad f_{abc}A_2^a(\mathbf{x})J_2^b(\mathbf{y})J_0^c(\mathbf{z}),\end{aligned} \tag{4.74}$$

where the current J_2^b and charge density J_0^c are defined in (4.15-4.16), and A_2^a is given in (4.66). It is possible to simplify the r.h.s. of (4.74) by replacing J_0^c using (4.65),

$$\begin{aligned}\{J, H_c\} &= \int d^2\mathbf{x}d^2\mathbf{y}d^2\mathbf{z}\{G(\mathbf{x}-\mathbf{y})G(\mathbf{y}-\mathbf{z}) - G(\mathbf{x}-\mathbf{z})G(\mathbf{y}-\mathbf{z})\} \\ &\quad f_{abc}A_2^a(\mathbf{x})J_2^b(\mathbf{y})\partial_1 A_2^c(\mathbf{z}).\end{aligned} \tag{4.75}$$

Using (4.67), one can further simplify to obtain,

$$\{J, H_c\} = -\int d^2\mathbf{x}d^2\mathbf{y}d^2\mathbf{z}\partial_1^z\{G(\mathbf{x}-\mathbf{z})G(\mathbf{y}-\mathbf{z})A_2^c(\mathbf{z})\}f_{abc}A_2^a(\mathbf{x})J_2^b(\mathbf{y}). \tag{4.76}$$

As pointed out in the previous section the algebra of the gauge invariant generators must be independent of the choice of gauge. Since the complete Galilean algebra was demonstrated earlier, it implies that $\{J, H_c\}$ must vanish in the axial gauge. The r.h.s. of (4.76) shows that this is not true in general. A simple way to establish compatibility is to demand that the boundary term vanishes, i.e.,

$$\int d^2\mathbf{z}\partial_1^z\{G(\mathbf{x}-\mathbf{z})G(\mathbf{y}-\mathbf{z})A_2^c(\mathbf{z})\} = 0. \tag{4.77}$$

The above relation gives a restriction on the connection $G(\mathbf{x}-\mathbf{y})$. Note that this connection appears squared which must be regularised [74, 79] to make it meaningful.

The regularisation must be such that the above condition (4.77) is satisfied. In that case the complete Galilean algebra is reproduced. It is useful to compare this analysis with Dirac's gauge fixed approach which is given below.

4.4 Gauge fixed formulation : Dirac's approach

In contrast to the symplectic approach the Hamiltonian analysis of Dirac [14] distinguishes between first class and second class constraints. The gauge freedom generated by the first class constraint $G^a \approx 0$ is eliminated by initially choosing a gauge $\chi^b \approx 0$ so that,

$$\det ||\{G^a, \chi^b\}|| \neq 0 \tag{4.78}$$

Then the complete set of constraints $G^a \approx 0, \chi^b \approx 0$ becomes second class which can be strongly implemented by working with Dirac (star) brackets,

$$\{\phi^a(\mathbf{x}), \phi^b(\mathbf{y})\}^* = \{\phi^a(\mathbf{x}), \phi^b(\mathbf{y})\} - \int d^2z d^2z' \{\phi^a(\mathbf{x}), \Omega^c(\mathbf{z})\} \Omega^{-1}_{cd}(\mathbf{z}, \mathbf{z}') \{\Omega^d(\mathbf{z}'), \phi^b(\mathbf{y})\}, \tag{4.79}$$

where Ω^{-1}_{cd} is the inverse of the matrix defined by the Poisson brackets $\{\Omega_c, \Omega_d\}$ of the complete set of constraints $\Omega_c = G_c, \chi_c \approx 0$. The ordinary brackets in (4.79) merely refer to the fundamental brackets (4.6 - 4.7).

It is worthwhile to highlight some of the fundamental distinctions between the implementation of constraints in the symplectic [20, 74] and Dirac [14] approaches. Contrary to the symplectic case, all the degrees of freedom (either gauge or matter) are retained in the Dirac analysis. There is no need for an explicit solution of the Gauss constraint (4.65) leading to the non-local structure (4.66). This also avoids the inherent arbitrariness in the solution (4.66,4.67).

The next step is to compute the Dirac brackets among the basic fields in the axial gauge. The matrix of the Poisson brackets of the constraints is given by,

$$\Omega_{ij}^{ab}(\mathbf{x},\mathbf{y}) = ||\{\Omega_i^a(\mathbf{x}), \Omega_j^b(\mathbf{y})\}|| = -\begin{pmatrix} 0 & \partial_1^x \\ \partial_1^x & 0 \end{pmatrix} \delta(\mathbf{x}-\mathbf{y}) g^{ab}, \tag{4.80}$$

where $\Omega_1^a = A_1^a \approx 0$ and $\Omega_2^a = G^a \approx 0$. The corresponding inverse matrix is found to be,

$$(\Omega_{ij}^{ab}(x,y))^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} G(\mathbf{x}-\mathbf{y}) g^{ab}, \tag{4.81}$$

where the connection has been defined in (4.67). From the basic brackets (4.6, 4.7) and using the definition of Dirac brackets in (4.79), we find the gauge fixed algebra,

$$\{\psi_n(\mathbf{x}), A_2^a(\mathbf{y})\}^* = \frac{1}{k} G(\mathbf{x}-\mathbf{y}) [T^a \psi(\mathbf{x})]_n, \tag{4.82}$$

$$\{A_2^a(\mathbf{x}), A_2^b(\mathbf{y})\}^* = \frac{1}{k} f^{abc} G(\mathbf{x}-\mathbf{y})(A_{2c}(\mathbf{x}) - A_{2c}(\mathbf{y})) \tag{4.83}$$

The brackets with $A_1^a(\mathbf{x})$ vanish as expected from the gauge condition. Note that the second relation preserves the antisymmetry of the bracket under the simultaneous exchange $x \leftrightarrow y, a \leftrightarrow b$.

Let us now compare the Dirac algebra with the corresponding symplectic algebra. Although the bracket (4.82) agrees with (4.71), the bracket (4.83) has a different structure from (4.69). Thus, at the level of the basic algebra, we find a distinction between the two approaches. It now takes only a little effort to show that the difference between (4.69) and (4.83) is just the boundary term in the l.h.s. of (4.77). Using (4.65), the bracket (4.69) reduces to the following,

$$\begin{aligned}\{A_2^a(\mathbf{x}), A_2^b(\mathbf{x}')\}i &= -\frac{1}{k}\int d^2\mathbf{y} G(\mathbf{x}-\mathbf{y})G(\mathbf{x}'-\mathbf{y}) f^{abc}\partial_1 A_{2c}(\mathbf{y}) \\ &= \frac{f^{abc}}{k}\int d^2\mathbf{y}\partial_1^y[G(\mathbf{x}-\mathbf{y})G(\mathbf{x}'-\mathbf{y})A_{2c}(\mathbf{y})]\end{aligned}$$

$$+ \quad \frac{1}{k} f^{abc} G(\mathbf{x}' - \mathbf{x})(A_{2c}(\mathbf{x}) - A_{2c}(\mathbf{x}'))$$

$$= \{A_2^a(\mathbf{x}), A_2^b(\mathbf{x}')\}^* \quad + \quad \frac{f^{abc}}{k} \int d^2\mathbf{y} \partial_1^y [G(\mathbf{x} - \mathbf{y}) G(\mathbf{x}' - \mathbf{y}) A_{2c}(\mathbf{y})] \qquad (4.84)$$

where, in going from the first to the second line, we have used (4.67). Thus, as announced, the difference between the symplectic and Dirac algebras is proportional to the boundary term in (4.77). If we impose the condition (4.77) then the two results agree. Finally, using the Dirac brackets (4.82,4.83), it is possible to establish the validity of the complete Galilean algebra (without any restrictions) including the bracket $\{J, H_c\}^*$ which previously yielded an anomalous structure (4.76) in the symplectic approach. This is not surprising because the anomalous structure in (4.76) is precisely compensated by the difference in the basic bracket $\{A_2^a, A_2^b\}$, equation (4.84), in the two approaches.

4.5 Quantum analysis

In this section the Galilean symmetry will be studied by taking into account the ordering ambiguities of relevant operators. Quantum effects, if any, will therefore be manifestated by additional terms. For an abelian theory this aspect has been analysed in [59] and [48]. An unusual ordering was devised in the former to preserve Galilean covariance. In [48], on the other hand, the gauge fixed computations done by solving the Gauss constraint revealed an anomaly which was absent in the gauge independent approach. The authors of [48] attributed this to inequivalent quantisation prescriptions. Here we shall show work with a conventional normal ordering of operators. The closure of the Galilean algebra is modified by $O(h^2)$ quantum corrections. These corrections are boundary terms which, interestingly, are of similar nature as already found in the classical analysis and hence can be ignored by invoking identical conditions on the Green function. There also appears a term proportional to G(0) which vanishes provided the self interaction is ignored. Algebraic arguments show that it is indeed feasible to take a vanishing self interaction.

4.5.1 The Gauge independent analysis

According to the usual method of quantisation the fields $\psi_m(x)$, its conjugate and $A_i(x)$ are considered as operators acting on some Hilbert space of state vectors $|\eta\rangle$. The brackets (4.6) and (4.7) are elevated to commutators following the prescription, $[,] \to i\hbar\{,\}$,i.e.;

$$\begin{aligned} [\psi_n(\mathrm{x}), \psi_m^\dagger(\mathrm{x}')] &= \hbar\delta_{nm}\delta(\mathrm{x}-\mathrm{x}'), \\ \left[A_i^a(\mathrm{x}), A_j^b(\mathrm{x}')\right] &= \frac{i\hbar}{k}\epsilon_{ij}g^{ab}\delta(\mathrm{x}-\mathrm{x}'), \end{aligned} \tag{4.85}$$

while complex conjugation is replaced by hermitian conjugation. All other commutators vanish. Note that we have taken $A_0 = 0$ thereby eliminating it and its conjugate π_0 from the phase space. This can always be done when the gauge is not fixed, as in the present context. For explicit gauge fixing which imposes conditions on A_i, the situation is different as will be elaborated in the forthcoming sub-section.

We now consider the definition of the various space-time generators. An ordering is taken such that these operators are (i) hermitian and (ii) their vacuum expectation values vanish. The latter condition means that the operators are to be normal ordered. From (4.85) we can interpret $\psi(\psi^\dagger)$ as the destruction and creation operators, respectively. Hence ψ is placed to the right of $\psi^\dagger$. Since the Chern-Simons gauge field is photonless, no extra care is needed for its normal ordering. However, on account of (4.85), there is an ambiguity proportional to $\delta(0)$ in the product of identical components of the gauge field at the same space-time point. We chose to work with a regularisation that enforces $\delta(0) = 0$ so that there is no ambiguity in this ordering. The mixed sector is already commuting. With this ordering the Gauss constraint (4.5) can be written in the desired form. the physical space is annihilated by this constraint,

$$G^a |\eta\rangle = (i\psi^\dagger T^a \psi + \frac{k}{2}\epsilon_{ij}(2\partial_i A_j^a + f^a_{bc} A_i^b A_j^c)) |\eta\rangle = 0. \tag{4.86}$$

This equation is the quantum analogue of (4.8).

Following the above prescription the quantised versions of the Galilean generators are easily written. For instance, the canonical Hamiltonian obtained from (4.19) is given by

$$\begin{aligned} H &= \frac{1}{2}\int d^2\mathbf{x}(D_i\psi)^\dagger(D_i\psi) \\ &= \frac{1}{2}\int d^2\mathbf{x}(\partial_i\psi^\dagger - \psi^\dagger A_{ia}T^a\psi)(\partial_i\psi + A_{ia}T^a\psi). \end{aligned} \tag{4.87}$$

This is both normal ordered and hermitian. Although the ordering in the mixed sector is immaterial in this case, it will be essential for the gauge fixed computations presented subsecuently. The ordeering in (4.87) anticipates this fact. Likewise the other operators are bodily written from their classical expressions preserving exactly the same order.

To verify the Galilean algebra, it is convenient to first give the equations of motion,

$$\begin{aligned} i\hbar\partial_0\psi &= [\psi, H] = -\frac{\hbar}{2}D_iD_i\psi, \\ i\hbar\partial_0 A_{ia} &= [A_{ia}(x), H] = -i\frac{\hbar}{k}\epsilon_{ij}J_{ja}(x), \end{aligned} \tag{4.88}$$

obtained from the basic commutators. These equations are the quantum analogues of (4.20) and (4.21), respectively. No extra terms are generated due to ordering effects so that the classical equations of motion are preserved. Since the classical Galilean algebra was already shown by the gauge independent method, it is expected that the same would now also hold. Indeed an explicit check reveals that the algebra closes in a straightforward manner except for the relation,

$$[K_i, H] = \hbar \int d^2\mathbf{x}\psi^\dagger \partial_i \psi + \hbar \int d^2\mathbf{x} A_{ia} \psi^\dagger T^a \psi. \tag{4.89}$$

The first and second terms are simplified by using the definitions of the translation generators and the Gauss constraint (4.5), respectively,

$$[K_i, H] = -i\hbar P_i - i\frac{\hbar k}{2} \int d^2\mathbf{x} \epsilon_{lm} (A_{la} \partial_i A_m^a - 2A_{ia} \partial_l A_m^a) - i\hbar \int d^2\mathbf{x} A_{ia} G^a \tag{4.90}$$

Acting on the physical states the last term vanishes. The term involving the integral reduces to a pure boundary. In particular for i = 1 it simplifies to, apart from an overall normalisation

$$\int d^2\mathbf{x} \{\partial_1 (A_{1a} A_2^a) - \partial_2 (A_{1a} A_1^a) + [A_{2a}(x), \partial_1^x A_1^a(x)]\} \tag{4.91}$$

The boundary terms can be dropped while the commutator has to be regularised to make it meaningful since it involves the derivative of a delta function at identical space-time points. This term also appears in the quantum formulation of an abelian theory. Using a point splitting regularuisation technique, it can be shown to vanish [48]. The occurence of such a term in the abelian context is not surprising since it is related to an ordering effect and not to the nonabelian nature of the problem. In the physical Hilbert space we therefore have,

$$[K_i, H] = -i\hbar P_i, \tag{4.92}$$

which establishes the closure of the complete galilean algebra.

4.5.2 The Gauge Fixed Analysis

We next discuss the gauge fixed analysis by following the Dirac procedure. While we are free to choose any admissible gauge, let us restrict the analysis to the axial gauge. this helps in compairing with a corresponding analysis [74] that has been done using the symplectic formalism. In the gauge independent approach we were able to set $A_0 = 0$ and eliminate the (A_0, π_0) pair from the phase space. For the gauge fixed analysis, however , the time conservation of the gauge fixes the multiplier. Demanding, therefore,

$$[A_{1a}, H_c] = 0 \tag{4.93}$$

where H_c is defined in (4.19), and using the basic commutators (4.85), leads to,

$$k\partial_1 A_0 + J_2 = 0. \tag{4.94}$$

The above relation, which is just the 2-component of the equation of motion (4.13) in the axial gauge, is the second gauge condition. The axial gauge together with this condition completely eliminates the gauge freedom associated with the two first class constraints of the theory. The Dirac brackets remain unchanged from (4.71) and (4.83). The corresponding commutators are,

$$[\psi_n(\mathbf{x}), A_2^a(\mathbf{y})] = \frac{i\hbar}{k} G(\mathbf{x}-\mathbf{y})[T^a\psi(\mathbf{x})]_n, \tag{4.95}$$

$$[A_2{}^a(\mathbf{x}), A_2{}^b(\mathbf{y})] = \frac{i\hbar}{k} f^{abc} G(\mathbf{x}-\mathbf{y})(A_{2c}(\mathbf{x}) - A_{2c}(\mathbf{y})). \tag{4.96}$$

The commutators in the pure matter sector are identical to those given in the gauge independent formulation. As before it is useful to deduce the basic equations of motion which are then exploited to verify the Galilean algebra. Using the above commutators, the equation of motion for ψ is obtained from

$$i\hbar\partial_0\psi = [\psi, H]$$

$$
\begin{aligned}
&= -\frac{\hbar}{2}D_i(D_i\psi(x)) + \frac{i\hbar}{k}\int d^2\mathbf{y}G(\mathbf{x}-\mathbf{y})J_2^a(y)T^a\psi(x) \\
&+ \frac{\hbar^2}{2k^2}\int d^2\mathbf{y}G(\mathbf{x}-\mathbf{y})G(\mathbf{x}-\mathbf{y})\psi^\dagger(y)T^aT^b\psi(y)T_aT_b\psi(x).
\end{aligned} \tag{4.97}
$$

The O($\hbar$) term involving the connection can be simplified by using the gauge condition (4.94). A simple rearrangement then yields,

$$
\begin{aligned}
i\hbar D_0\psi(x) &= -\frac{\hbar}{2}D_i(D_i\psi(x)) \\
&+ \frac{\hbar^2}{2k^2}\int d^2\mathbf{y}G(\mathbf{x}-\mathbf{y})G(\mathbf{x}-\mathbf{y})\psi^\dagger(y)T^aT^b\psi(y)T_aT_b\psi(x).
\end{aligned} \tag{4.98}
$$

Comparision with the classical equation of motion clearly reveals that the $O(\hbar^2)$ term is a quantum correction. This correction is a peculiarity of the axial gauge. For instance, in the gauge independent analysis, such a term is nonexistent. It is not difficult to explain the difference. The $O(\hbar^2)$ term comes from the nonvanishing commutator of ψ with the gauge field. In the gauge independent analysis this commutator vanishes so that there is no correction. Following a symplectic approach, this term was also obtained in [74]. In an abelian context the above equation of motion simplifies considerably,

$$
i\hbar D_0\psi(x) = -\frac{\hbar}{2}D_i(D_i\psi(x)) + \frac{\hbar^2}{2k^2}\int d^2\mathbf{y}G(\mathbf{x}-\mathbf{y})G(\mathbf{x}-\mathbf{y})J_0(y)\psi, \tag{4.99}
$$

where J_0 is the charge density. Strongly imposing the Gauss constraint eliminates this charge density in favour of the gauge field A_2. Finally, dropping a boundary term which is identical to that already encountered in (4.76), we obtain,

$$
i\hbar D_0\psi(x) = -\frac{\hbar}{2}D_i(D_i\psi(x)) + \frac{\hbar^2}{k}G(0)A_2(x)\psi(x). \tag{4.100}
$$

If the self interaction is ignored so that G(0) = 0, the quantum correction vanishes. Later on we shall return to this point.

Proceeding similarly it is also possible to compute the equation of motion for A_2. A rather lengthy algebra yields,

$$
\begin{aligned}
i\hbar\partial_0 A_{2a} &= [A_{2a}, H] \\
&= \frac{i\hbar}{2k}(\psi^\dagger T_a \partial_1 \psi - \partial_1 \psi^\dagger T_a \psi) + \frac{i\hbar}{k}\int d^2\mathbf{y}\partial_2 G(\mathbf{y}-\mathbf{x})J_{2a}(y) \\
&+ \frac{i\hbar}{2k} f_{abc}\int d^2\mathbf{y} G(\mathbf{y}-\mathbf{x})\{\partial_2\psi^\dagger(y)A_2^c(x)T^b\psi(y) - \psi^\dagger(y)A_2^c(x)T^b\partial_2\psi(y) \\
&- \psi^\dagger(y)A_2^c(x)T^b A_2(y)\psi(y) - \psi^\dagger(y)A_2(y)A_2^c(x)T^b\psi(y)\}, \qquad (4.101)
\end{aligned}
$$

where the ordering of operators is important. It is this equation that will be used in the ensuing analysis of the Galilean algebra. Before proceeding, however, it is worthwhile to compare it with the classical equation of motion (4.13), whose $\alpha = 1$ component is given by,

$$J_1^a = kF_{20}^a, \qquad (4.102)$$

with

$$J_1^a = -\frac{1}{2}(\psi^\dagger T^a \partial_1 \psi - \partial_1 \psi^\dagger T^a \psi), \qquad (4.103)$$

where the axial gauge has been implemented in obtaining (4.103) from (4.16). Let us next identify the various terms in the R.H.S. of (4.101). The first term is (4.103). The second term is simplified by eliminating J_2 in favour of A_0 using the gauge condition (4.94). The definition of the connection then yields a term proportional to $\partial_2 A_0$. Finally, the piece involving the structure constant is manipulated so that $A_2(x)$ is taken outside the integral, both through the left and right sides. The remaining integral can be shown, using the gauge condition (4.94), to be identical to A_0.. However the passage of $A_2(x)$ through the various terms will generate $O(\hbar^2)$ factors coming from the nonvanishing commutators. Collecting everything togather, we find,

$$i\hbar J_{1a} = i\hbar k(\partial_2 A_{0a} - \partial_0 A_{2a} - \frac{1}{2}f_{abc}\{A_0^b, A_2^c\})$$

$$
\begin{aligned}
&+ \quad \frac{\hbar^2}{4k^2} f_{abc} \int d^2\mathbf{y} G(\mathbf{y}-\mathbf{x})(2\partial_2 G(\mathbf{y}-\mathbf{x})\psi^\dagger(y) T^c T^b \psi(y) \\
&+ \quad f^c_{de} G(\mathbf{y}-\mathbf{x})\psi^\dagger(y) A^e_2(x)[T^d, T^b]\psi(y)). \qquad (4.104)
\end{aligned}
$$

Note that a Weyl ordering has been used to define the product among the noncommuting variables A_0 and A_2. In the classical limit the above equation reproduces (4.102). Once again the quantum correctios is of $O(\hbar^2)$ which explicitly appears in (4.104). It may be remarked that in contrast to (4.100), the quantum correction is absent for an abelian theory, independent of the self interaction. We shall precisely exploit this fact later on to show, from algebraic consistency arguments, that G(0) must vanish.

Using these basic equations of motion the algebra of the Galilean generators is computed. Apart from the bracket involving the angular momentum with the Hamiltonian, the others close in the standard way. The nontrivial bracket is written as,

$$
[J, H] = [M, H] + [G, H], \qquad (4.105)
$$

where the matter (M) and gauge (G) contributions have explicitly separated,

$$
\begin{aligned}
M &= \int d^2\mathbf{x} \epsilon_{ij} x_i (i\psi^\dagger \partial_j \psi) \\
G &= \frac{k}{2} \int d^2\mathbf{x} A_{2a} A^a_2. \qquad (4.106)
\end{aligned}
$$

Although the equation of motion for ψ involves an $O(\hbar^2)$ correction, these are cancelled when computing the bracket of M with H. The final result is given by,

$$
\begin{aligned}
[M, H] &= \frac{i\hbar}{2} \int d^2\mathbf{x}(\partial_1\psi^\dagger A_2 \psi - \psi^\dagger A_2 \partial_1 \psi) \\
&- \frac{i\hbar}{2} \int d^2\mathbf{x} d^2\mathbf{y} G(\mathbf{x}-\mathbf{y})\{\partial_2\psi^\dagger(y)\partial_2 A_2(x)\psi(y) - \psi^\dagger(y)\partial_2 A_2(x)\partial_2\psi(y) \\
&- \psi^\dagger(y)\partial_2 A_2(x) A_2(y)\psi(y) - \psi^\dagger(y) A_2(y)\partial_2 A_2(x)\psi(y)\}. \qquad (4.107)
\end{aligned}
$$

In order to generate identical terms from the other bracket,it is essential to use the commutation relations in the mixed sector. Consequently $O(\hbar^2)$ corrections are found and one obtains

$$
\begin{aligned}
[G,H] &= -[M,H]-\frac{\hbar^2}{k}G(0)\int d^2\mathbf{x}J_1(x) \\
&- \frac{\hbar^2}{8k}\int d^2\mathbf{y}\psi^\dagger(\mathbf{y})\int d^2\mathbf{x}\partial_2^x[G^2(\mathbf{x}-\mathbf{y})f_{abc}f^{abd}(A_2^c(x)-A_2^c(y))]T_d\psi(y) \\
&+ \frac{\hbar^2}{4k}\int d^2\mathbf{y}(J_2+\psi^\dagger\{A_2,T^a\}T_a\psi+2f_{abc}f^{abd}\psi^\dagger A_2^c T_d\psi) \\
&\times \left(\int d^2\mathbf{x}\partial_2^x(G^2(\mathbf{x}-\mathbf{y}))\right), \qquad (4.108)
\end{aligned}
$$

where the charge density has been eliminated in favour of the gauge field. It is evident that the anomalous piece in the commutator of J with H involves $O(\hbar^2)$ quantum corrections. These are boundary terms which are of an identical nature as already found in the classical analysis of the model. Employing similar restrictions on the Green functions these terms can therefore be dropped. Finally there remains a term proportional to the self interaction G(0). In [74], for instance, a regularisation was assumed which enforced G(0) = 0. An alternative point of view [59] was to take a nonvanishing self interaction, but choose an unconventional ordering prescription so that the Galilean algebra in an abelian model was preserved. In the present case certain definite arguments are now provided which ensure G(0) = 0, thereby proving the closure of the Galilean algebra (4.108) using, as we have done, the usual normal ordering of operators.

The first point to note is that the presence of terms like G(0) is related to the algebra of operators at identical space time points. The symplectic algebra (4.69), at identical points, can be written as,

$$
\{A_2^a(\mathbf{x}),A_2^b(\mathbf{x})\} = -\frac{1}{k}\int d^2\mathbf{y}G(\mathbf{x}-\mathbf{y})G(\mathbf{x}-\mathbf{y})f^{abc}\partial_1 A_{2c}(\mathbf{y}), \qquad (4.109)
$$

where the charge density has been eliminated in favour of the gauge field by imposing the Gauss constraint in the axial gauge. Dropping the familiar boundary term and using the definition (4.67) of the Green function, this simplifies to,

$$\{A_2^a(\mathbf{x}), A_2^b(\mathbf{x})\} = -\frac{2}{k}G(0)f^{abc}A_{2c}(x). \tag{4.110}$$

As can be seen the corresponding Dirac algrbra (4.83), on the other hand, vanishes. Since the conditions on the Green function were such as to preserve equivalence among these algebras, it is clear that G(0) must vanish. In other words the restriction on the Green function satisfying (4.77) also implies the equivalent condition G(0) = 0. Yet another way of visualizing this result is to reconsider equations (4.100) and (4.104). For an abelian theory the quantum correction in the latter vanishes, while in the former it is true provided G(0) = 0. Now in the abelian case the Gauss constraint in the axial gauge can be used to unambiguously eliminate the gauge field in terms of the matter variables. The equation of motion for the gauge field is determined purely from the latter. Consequently a lack of quantum correction in one implies the same in other, thereby requiring G(0) = 0. The result is therefore a consequence of algebraic consistency.

4.6 Conclusions

We have investigated by different approaches the (classical) Galilean symmetry in a nonrelativistic model involving the coupling of nonabelian Chern-Simons term to matter fields [74]. Since the model is a constrianed system there are different formulations depending on how one accounts for the constraints. A conceptually clean and elegant way of doing this is to work in the gauge independent formalism [12, 13, 62]. The various Galilean generators are defined from first principles. It is also verified that on the constraint surface, these generators are gauge invariant. The basic fields are found to transform covariantly under the different space-time generators. The classical Galilean algebra is reproduced on the constraint surface. Since this algebra involves (physical) gauge invariant quantities, it implies that the algebra should be preserved in any gauge fixed compuatation. However this is not always true [48] so that explicit gauge fixed computations are necessary. Two distinct approaches to gauge fixing have been considered here. In the symplectic approach [20] the Gauss constraint is explicitly solved in the axial gauge. The gauge degrees of freedom are eliminated in favour of the matter variables. Since the process involves the inversion of a derivative, the solution for the gauge field is nonlocal. It is found that, except for $\{J, H_c\}$, the classical Galilean algebra is preserved. The bracket $\{J, H_c\}$ is anomalous; it is in fact proportional to a boundary term involving the square of the connection. Compatibility with the Galilean covariance established gauge invariantly is therefore preserved provided this term is constrained to vanish. Conditions are therby imposed on the connection which recall similar findings based on the consistency of the Schrodinger equation.

The results of the gauge fixed analysis are also given in the Dirac [14] formalism. The Dirac brackets for the mixed sector coincides with the symplectic result, but it does not for the pure gauge sector. This is an important subtlety in the analysis and recalls a similar finding [58] for nonabelian relativistic Chern - Simons models. Interestingly, however, this difference in the algebras is shown to be a boundary term that is exactly identified with the boundary term occuring in the anomalous bracket $\{J, H_c\}$ found in the symplectic approach. This provides an alternative interpretation whereby the conditions on the Green function follow by simply demanding the equivalence between the Dirac and symplectic reductions. These conditions will then naturally lead to the validity of the Galilean algebra.

We have next made a detailed analysis of the quantum Galilean algebra by including the effects of ordering among the operators. A standard normal ordering has been chosen such that the hermiticity of the various Galileo generators is respected. The closure of the algebra in the gauge independent scheme mimics the classical analysis. The gauge fixed analysis, on the contrary, reveals subtle and interesting aspects. The equations of motion for both the matter and gauge fields acquire quantum corrections of $O(\hbar^2)$. It is not surprising, therefore, that the closure of the quantum Galilean algebra is modified by the presesnce of such corrections. Interestingly, these are boundary terms which can be dropped by imposing restrictions on the Green function that are precisely identical to those found in the classical analysis. There also appears an anomalous term that is proportional to the self interaction G(0). In the literature [74] recourse is taken to a regularisation that enforces the vanishing of this term. Alternatively, it was suggested [59] to keep this term but modify the conventional normal ordering to preserve the Galilean symmetry in an abelian model. We have however, shown that the vanishing of the self interaction is naturally required from conditions of algebraic consistency. Thus, with the usual normal ordering, the complete quantum Galilean algebra is preserved.

A key point to be stressed is the distinctive roles played by Dirac and symplectic reductions. While it may be appreciated that these yield equivalent descriptions, the present analysis shows that by exploiting this fact, it is possible to obtain conditions on the Green function which are precisely required to preserve the Galilean algebra in the classical or quantal framework. No other input is necessary. Besides this it is interesting to observe that the symplectic method, already at the classical level, anticipates the ordering ambiguities since the corresponding Galilean algebra does not close unless restrictions are imposed on the Green function. This is an important point of distinction of the symplectic formalism from the Dirac method.

Chapter 5

Concluding Remarks

Our works on certain aspects of three dimensional field theories have been reported in this dissertation. The Chern - Simons (C - S) interaction [1, 2] was the focal theme of the thesis. We have investigated the fractional spin of the solitonic excitations of the C - S coupled theories. A new type of self duality in connection with the gauged O(3) nonlinear sigma model was then discussed. Finally the issue of space time symmetries of a nonabelian C - S coupled matter system was analysed from alternative approaches of the symplectic [20] and Dirac [14] methods, both at the classical and the quantum level. Before concluding the thesis we will briefly discuss some possible elaborations and new directions of research which follow from our works presented here.

The analysis of fractional spin was based on a new field theoretic method [16] advanced recently in connection with the C - S coupled nonlinear O(3) sigma model. We have applied this method to a number of C - S coupled relativistic field theories [38] and also found a generalisation to include the nonrelativistic models in the same framework [39]. Our approach was semiclassical which, interestingly, reproduces the same results obtained from the path integral method of Wilczek and Zee [10]. Connection between these methods lies in the topological properties of the C - S interaction as has been indicated in our work. This correspondence remains to be explored in detail. A full fledged quantum treatment using coherent state representation [11, 80] technique may be suggested as a possible extension of the work reported here, which is expected to exhibit the correspondence more clearly.

The models we have discussed in the analysis of fractional spin are known to give rise to self dual theories [3]. The quantisation of the soliton sector of a model requires the classical solutions. Since the theories are nonlinear, these solutions are very difficult to find. The self - dual theories provide a set of first order equations which automatically satisfies the second order Euler - Lagrange equations. This explains the importance of the self dual theories in both classical and quantum field theories. We have discusssed a new type of self duality in connection with the gauged nonlinear O(3) sigma models. The motivation was to lift the degeneracy of the solutions in a particular topological sector observed earlier [54, 55]. The topology of the vaccua was shown to be responsible for this degeneracy. We have demonstrated that the degeneracy may be lifted by considering symmetry breaking vaccua [49, 50]. Models with either Maxwell or C - S coupling was discussed. The corresponding C - S coupled theory admits nontopological solitons in addition to topological ones. In the present dissertation we have not discussed the nontopological solutions elaborately. The simultaneous presence of both the C - S and the Maxwell term was also not explored. These issues may constitute the subject of future work.

So far our analysis was classical. Also the calculations were performed in a gauge independent [12, 13] setting. Subtleties of gauge fixing were then considered in relation to the studies of space time symmetries of a nonabelian C - S coupled matter system [60]. The study was carried out at both the classical and the quantum level. The gauge fixed computations were done from alternative approaches of symplectic and Dirac method of phase space reduction. We have worked in the axial gauge which linearises the Gauss constraint thereby offering some simplifications. In the radiation gauge the nonlinearity of the Gauss constraint prevents an exact solution of the problem [15]. Nevertheless, it will be interesting to pursue the work in this gauge since a violation of Galilean algebra was reported in the quantum level analysis of the corresponding abelian model [48].

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