ASPECTS OF SQUEEZING IN ATOMIC STATES

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled **Aspects of Squeezing in Atomic States** submitted by **Ram Narayan Deb**, who got his name registered on December 6, 2006 for the award of **Ph.D. (Science) degree** of **Jadavpur University**, is absolutely based upon his own work under the supervision of Dr. Nilkantha Nayak at **S. N. Bose National Centre for Basic Sciences, Kolkata, India** and that neither this thesis nor any part of it has been submitted for any degree / diploma or any other academic award anywhere before.

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List of Publications

[1] P. K. Pathak, R. N. Deb, N. Nayak and B. Dutta Roy *A Spin Squeezing Operator* , arXiv:0711.0995v1 [quant-ph].

[2] N. Nayak, B. Dutta Roy and R. N. deb *On Spin Squeezing* , Book: Quantum Optics-Coherence, Entanglement and Nonlinear Dynamics eds. J. Banerji, P. K. Panigrahi and R. P. Singh, (McMillan, India) 2007, Pp. 93-105.

[3] R. N. Deb, M. Sebawe Abdalla, S. S. Hassan and N. Nayak, *Spin squeezing and entanglement in a dispersive cavity*, Phys. Rev. A **73**, 053817 (2006).

[4] N. Nayak, R. N. Deb and B. Dutta Roy, *Squeezed spin states and pseudo-Hermitian operators* , J. Opt. B: Quantum Semiclass Opt. **7**, S761-S764 (2005).

[5] R. N. Deb, N. Nayak and B. Dutta Roy, *Squeezed "atomic" states, pseudo-Hermitian operators and Wigner D-matrices* , Eur. Phys. J. D **33**, 149-155 (2005).

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Contents

1. Introduction

This thesis is devoted to some novel aspects of squeezing in two-level atomic systems, but before presenting our work in this regard, several preliminaries and basic concepts of the subject are presented in the introduction to the thesis. We, therefore, organize the introduction in the following way. In section 1.1 we discuss the coherent states and squeezed states of the electromagnetic field as these ideas were first concretely developed in the context of the electromagnetic field in 1960's and 1980's respectively. In section 1.2 we introduce the two-level atomic system on which we have made our study regarding its squeezing aspects. We also show the equivalence of a two-level atom with a spin-$\frac{1}{2}$ particle as we describe the physics of a two-level atom with the help of the mathematics of a spin-$\frac{1}{2}$ particle. In section 1.3 we introduce atomic coherent state and atomic squeezed state and also discuss the motivation and interest of the subject. In section 1.4 we discuss the organization of the following chapters of our thesis.

1.1. Coherent States and Squeezed States of Electromagnetic Field

Coherent state or minimum uncertainty state as it is known today was first discovered by E. Schr$\ddot{o}$dinger [1] in the year 1926 in the context of the harmonic oscillator in quantum mechanics. He recognized that it is possible to obtain a particular superposition of the harmonic oscillator wavefunctions such that we obtain a Gaussian wave packet, which does not disperse with time and also exhibits minimum uncertainty in the measurement of position and momentum for the corresponding quantum particle. For these reasons the resultant state was named as a coherent state. But though it was introduced as early as 1926, it was widely recognized in 1960's due to the work of Glauber, Klauder and Sudarshan who applied these notions to the description of electromagnetic radiation. The discovery of laser light made the idea of coherent states, one of the most interesting and important subjects. One aspect of the coherent states is based on the uncertainty principle and hence we begin our discussion with this principle.

1.1.1. The Uncertainty Principle

Every quantum system shows a fundamental and inherent uncertainty in the simultaneous measurements of it's canonically conjugate pair of physical variables that describe the behaviour of the system. This was formulated by W. Heisenberg in 1927 and is known as Heisenberg's uncertainty principle [2]. If q and p are the position and it's canonically conjugate momentum of a quantum particle respectively, then the uncertainties in the simultaneous measurements of q and p represented as

$$\Delta q = \sqrt{\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2} \tag{1.1}$$

and

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2} \tag{1.2}$$

respectively, satisfy

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \tag{1.3}$$

where $\hat{q}$ and $\hat{p}$ are the Hermitian operators coresponding to q and p respectively and $\hbar = \frac{h}{2\pi}$, where h is Planck's constant. The average values $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$ in the expressions of uncertainties in Eqs. (1.1) and (1.2) are calculated over the relevant quantum state of the particle. Here Δq and Δp are measures of quantum noise.

It is to be mentioned here that Schrödinger also presented the uncertainty relation which is much more general and has the form

$$\Delta q \Delta p \geq |\langle [\hat{q}, \hat{p}] \rangle / 2 + \langle \{\hat{q}, \hat{p}\} \rangle / 2|, \tag{1.4}$$

where $[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q}$ and $\{\hat{q}, \hat{p}\} = \hat{q}\hat{p} + \hat{p}\hat{q}$ are called the commutator and anticommutator of $\hat{q}$ and $\hat{p}$ respectively. However, we deal with Eq. (1.3) at present.

Our thesis deals with the squeezing of quantum noise in atomic states, however, we discuss some of the preliminaries in the context of electromagnetic field as the ideas pertinent to the present subject of study originated from there. The concept of coherent states and squeezed states for electromagnetic field is based on it's quantum theory and therefore, it is essential to discuss some basic ideas of this theory. We deal with atom-field interaction in a cavity and from that point of view also the idea of the quantized version of the electromagnetic field is necessary.

1.1.2. Quantization of the Electromagnetic Field in a Cavity

We start with a brief introduction to the quantum mechanics of a simple harmonic oscillator as it is required for our discussion on the quantum theory of electromagnetic field.

The Hamiltonian operator for a simple harmonic oscillator of unit mass and angular frequency ω is given as

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2, \tag{1.5}$$

where $\hat{q}$ and $\hat{p}$ are the position and it's canonically conjugate momentum operators respectively, which satisfy

$$[\hat{q}, \hat{p}] = i\hbar. \tag{1.6}$$

We define two non-Hermitian operators $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{a} = \frac{\omega\hat{q} + i\hat{p}}{\sqrt{2\hbar\omega}} \tag{1.7}$$

and

$$\hat{a}^\dagger = \frac{\omega\hat{q} - i\hat{p}}{\sqrt{2\hbar\omega}}, \tag{1.8}$$

such that they satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{1.9}$$

and the Hamiltonian operator acquires the form

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{1.10}$$

If $|n\rangle$ is the eigenvector of $\hat{a}^\dagger\hat{a}$ with eigenvalue n i.e.

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \tag{1.11}$$

then,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{1.12}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.13}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are called the annihilation and creation operators respectively for the harmonic oscillator states.

An electromagnetic radiation field can be shown to be equivalent to a infinite set of these harmonic oscillators. We now introduce the theory of quantization of the electromagnetic field.

Quantization of the electromagnetic field has been dealt in many text books, but, there it has been done for an unbounded region using vector potential. What we need for the present purpose is the quantized version of the electromagnetic field in a cavity. We do not treat the problem using vector potential and discuss the theory using quantization of electric and magnetic fields directly because this gives the quantized version appropriate for typical problems in quantum optics, in general. We start with the equations of classical electrodynamics also called Maxwell's equations [3]. In a source-free space these equations in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively, are

$$\nabla \cdot \mathbf{E} = 0, \tag{1.14}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.15}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{1.16}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.17}$$

The constants μ_0 and ϵ_0 are the permeability and permitivity of free space respectively. The coupled equations can be decoupled to yield the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{1.18}$$

governing the electric field only and

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \tag{1.19}$$

for the magnetic field. Here we have assumed

$$c^2 = \frac{1}{\mu_0 \epsilon_0}. \tag{1.20}$$

In the interaction between electromagnetic field and atom or molecule, both the electric and the magnetic components interact with the underlying charges and currents. These interactions are known as dipole or, in general, multipole interactions. The electric dipole interaction is 10^{16} times stronger than the corresponding magnetic component counterpart. For this reason the electric field is also known as the optical field and we discuss it's quantization theory [4].

We assume that the electromagnetic radiation field is contained in a source free cavity of volume V. The axis of the cavity is along the z-direction and the length of the cavity along it's axis is L. The electric field is considered to be linearly polarized along x-direction and has the spatial dependence appropriate to the cavity. We show that a single cavity mode of the radiation field is equivalent to a simple harmonic oscillator of unit mass and angular frequency equal to the cavity eigenfrequency and for that we expand the electric field in the normal modes $\mathcal{U}_l(z,t)$ of the cavity [18] as

$$E_x(z,t) = \sum_l A_l \mathcal{U}_l(z,t) \tag{1.21}$$

where

$$\mathcal{U}_l(z,t) = q_l(t) \sin(k_l z). \tag{1.22}$$

q_l is the normal mode amplitude and $k_l = \frac{l\pi}{L}$, with $l = 1, 2, 3....$
The A_l's are given by

$$A_l = \left[\frac{2\omega_l^2}{V\epsilon_0} \right]^{1/2} \tag{1.23}$$

where ω_l is the eigenfrequency of the mode l. The nonvanishing component of the magnetic field B_y can be obtained from Eqs. (1.15), (1.16) and (1.40) as

$$B_y(z,t) = \mu_0\epsilon_0 \sum_l A_l \frac{\dot{q}_l}{k_l}\cos(k_l z). \tag{1.24}$$

The classical Hamiltonian for the field

$$\mathcal{H} = \frac{1}{2}\int_V d\tau(\epsilon_0 E_x^2 + \frac{1}{\mu_0}B_y^2) \tag{1.25}$$

takes the form, in terms of q_l and $\dot{q}_l = p_l$ as,

$$\mathcal{H} = \frac{1}{2}\sum_l(\omega_l^2 q_l^2 + p_l^2). \tag{1.26}$$

We observe that the above Hamiltonian is as if the sum of several simple harmonic oscillator Hamiltonians of unit mass and angular frequency ω_l corresponding to each value of l. This shows that each cavity mode of the electromagnetic field is equivalent to a simple harmonic oscillator. The dynamics is quantized by identifying q_l as operator $\hat{q}_l$ and p_l as operator $\hat{p}_l$ obeying

$$[\hat{q}_l, \hat{p}_s] = i\hbar\delta_{ls} \tag{1.27}$$

and

$$[\hat{q}_l, \hat{q}_s] = [\hat{p}_l, \hat{p}_s] = 0. \tag{1.28}$$

We define two non-Hermitian operators

$$\hat{a}_l = \frac{1}{\sqrt{2\hbar\omega_l}}\big(\omega_l\hat{q}_l + i\hat{p}_l\big) \tag{1.29}$$

and

$$\hat{a}_l^\dagger = \frac{1}{\sqrt{2\hbar\omega_l}}\big(\omega_l\hat{q}_l - i\hat{p}_l\big) \tag{1.30}$$

respectively, by which $\mathcal{H}$ takes the form

$$\hat{\mathcal{H}} = \hbar\sum_l \omega_l(\hat{a}_l^\dagger\hat{a}_l + 1/2). \tag{1.31}$$

From Eqs. $(1.27) - (1.30)$, it follows that $\hat{a}_l$ and $\hat{a}_s^\dagger$ obey the commutation relation

$$[\hat{a}_l, \hat{a}_s^\dagger] = \delta_{ls}. \tag{1.32}$$

If $|n_l\rangle$ is the eigenvector of $\hat{a}_l^\dagger\hat{a}_l$ with eigenvalue n_l that is,

$$\hat{a}_l^\dagger\hat{a}_l|n_l\rangle = n_l|n_l\rangle \tag{1.33}$$

then

$$\hat{a}_l|n_l\rangle = \sqrt{n_l}|n_l - 1\rangle, \tag{1.34}$$

and

$$\hat{a}_l^\dagger|n_l\rangle = \sqrt{n_l + 1}|n_l + 1\rangle. \tag{1.35}$$

The state $|n_l\rangle$ is called the photon number state that is, it infers the number of photons in the mode l which is n_l and due to Eqs. (1.34) and (1.35) the operators $\hat{a}_l$ and $\hat{a}_l^\dagger$ are called the photon annihilation and creation operators respectively. Eq. (1.33) implies that the operator $\hat{a}_l^\dagger\hat{a}_l$ determines the number of photons in the state $|n_l\rangle$ and hence, is called the photon number operator.

The state $|n_l\rangle$ is also an eigenstate of the harmonic oscillator Hamiltonian $\hat{H}_l = \hbar\omega_l(\hat{a}_l^\dagger\hat{a}_l + 1/2)$ with eigenvalue $\hbar\omega_l(n_l + 1/2)$ as

$$\hat{H}_l|n_l\rangle = \hbar\omega_l(\hat{a}_l^\dagger\hat{a}_l + 1/2)\ |n_l\rangle = \hbar\omega_l(n_l + 1/2)\ |n_l\rangle. \tag{1.36}$$

The above derivation establishes the fact that a cavity mode of radiation field is indeed equivalent to a harmonic oscillator. The electric field operator $\hat{E}_{xl}(z,t)$ corresponding to the l-th cavity mode of the radiation field has the form

$$\hat{E}_{xl}(z,t) = A_l\hat{q}_l(t)\sin(k_l z). \tag{1.37}$$

Using the expression of A_l given in Eq. (1.23) the above equation can be written in terms of $\hat{a}_l$ and $\hat{a}_l^\dagger$ as

$$\hat{E}_{xl}(z,t) = \frac{1}{2}\mathcal{E}_l(\hat{a}_l + \hat{a}_l^\dagger)\sin(k_l z) \tag{1.38}$$

where $\mathcal{E}_l = 2\left[\frac{\hbar\omega_l}{V\epsilon_0}\right]^{1/2}$ has the dimension of electric field. We immediately see that $\langle n_l|\hat{E}_{xl}|n_l\rangle = 0$, but,

$$\langle n_l|\hat{E}_{xl}^2(z)|n_l\rangle = \frac{1}{2}\mathcal{E}_l^2\left(n_l + \frac{1}{2}\right)\sin^2 k_i z \tag{1.39}$$

for the cavity mode l in a state $|n_l\rangle$ containing exactly n_l photons. This does not go against quantum mechanics since it is not $\langle\hat{E}\rangle$ but the intensity of the radiation field $I \propto \langle\hat{E}^2\rangle$, which is observed. However, such is not the case for all states of the radiation field. One such example, as we shall see, is the field in a coherent state in which $\langle\hat{E}\rangle \neq 0$. With this information we now describe what is known as a coherent state of electromagnetic field.

1.1.3. Coherent State of Electromagnetic Field

In the remainder of this chapter we restrict ourselves to a single mode of the radiation field with angular frequency ω having annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ respectively. The electric field operator corresponding to the single cavity mode of the radiation field is written by replacing the superscript l from Eq. (1.37) as

$$\hat{E}_x(z,t) = A\hat{q}(t)\sin(kz). \tag{1.40}$$

The coherent state of the electromagnetic radiation field [5–8], is defined as an eigenstate of the annihilation operator $\hat{a}$. Denoting it by $|\alpha\rangle$ we have,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.41}$$

Since $\hat{a}$ is non-Hermitian, α is, in general complex. We can have a commutation relation among $\hat{a}$ and $e^{\alpha\hat{a}^\dagger}$ as

$$[\hat{a}, e^{\alpha\hat{a}^\dagger}] = \alpha e^{\alpha\hat{a}^\dagger}. \tag{1.42}$$

If $|n\rangle$ is the eigenstate of the Hamiltonian operator $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, then the vacuum state is denoted as

$$|n=0\rangle = |0\rangle. \tag{1.43}$$

Operating both sides of the operator equation (1.42) on the vacuum state we obtain

$$\hat{a}e^{\alpha\hat{a}^\dagger}|0\rangle - e^{\alpha\hat{a}^\dagger}\hat{a}|0\rangle = \alpha e^{\alpha\hat{a}^\dagger}|0\rangle.$$

This gives

$$\hat{a}\ e^{\alpha\hat{a}^\dagger}|0\rangle = \alpha\ e^{\alpha\hat{a}^\dagger}|0\rangle. \tag{1.44}$$

We find that $e^{\alpha\hat{a}^\dagger}|0\rangle$ is an eigenvector of $\hat{a}$ and, thus, proportional to $|\alpha\rangle$. With proper normalization we may write

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle. \tag{1.45}$$

In terms of the photon number state, we may write

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.46}$$

If we have two such states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ with $\alpha_1 \neq \alpha_2$ then,

$$\begin{aligned}\langle\alpha_1|\alpha_2\rangle &= e^{-\frac{1}{2}\left(|\alpha_1|^2+|\alpha_2|^2\right)}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\alpha_1^{\star n_1}\alpha_2^{n_2}}{\sqrt{n_1!n_2!}}\langle n_1|n_2\rangle \\ &= e^{-\frac{1}{2}\left(|\alpha_1|^2+|\alpha_2|^2\right)}\sum_{n_1=0}^{\infty}\frac{\alpha_1^{\star n_1}\alpha_2^{n_1}}{n_1!} \\ &= e^{-\frac{1}{2}\left(|\alpha_1|^2+|\alpha_2|^2-\alpha_1{}^\star\alpha_2\right)}.\end{aligned} \tag{1.47}$$

Thus, the states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are not mutually orthogonal and nor do they have to be as they are eigenstates of a non-Hermitian operator. Now,

$$|\langle\alpha_1|\alpha_2\rangle|^2 = e^{-|\alpha_1-\alpha_2|^2}. \tag{1.48}$$

Therefore, the states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ become approximately orthogonal when $|\alpha_1-\alpha_2|^2$ increases.

The set of states $\{|\alpha\rangle\}$ form a complete set. Using Eq. (1.46) we have

$$\int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{|n_1\rangle\langle n_2|}{\sqrt{n_1!n_2!}} \int \alpha^{n_1}\alpha^{\star n_2} e^{-|\alpha|^2} d^2\alpha. \tag{1.49}$$

This integral is over the entire complex plane of α. We evaluate it using polar coordinates $(0 < r \leq \infty, 0 \leq \theta \leq 2\pi)$. We assume

$$\alpha = r\ e^{i\theta} = x + iy. \tag{1.50}$$

Therefore,

$$d^2\alpha = dx\ dy = r\ dr\ d\theta. \tag{1.51}$$

With this change in coordinates Eq. (1.49) becomes,

$$\int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{i(n-m)\theta} r^{n+m+1} e^{-r^2} dr d\theta. \tag{1.52}$$

Now,

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi\delta_{mn}. \tag{1.53}$$

Therefore, Eq. (1.52) becomes,

$$\int |\alpha\rangle\langle\alpha| d^2\alpha = 2\pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_{r=0}^{\infty} e^{-r^2} r^{2n+1} dr. \tag{1.54}$$

We now use the substitution

$$r^2 = s \tag{1.55}$$

in Eq. (1.54) and obtain

$$\begin{aligned} \int |\alpha\rangle\langle\alpha| d^2\alpha &= \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_{s=0}^{\infty} s^n\ e^{-s}\ ds \\ &= \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \Gamma(n+1) \\ &= \pi \sum_{n=0}^{\infty} |n\rangle\langle n| \\ &= \pi, \end{aligned} \tag{1.56}$$

where $\Gamma(n) = (n-1)!$ is the Gamma function. Thus,

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = 1 \tag{1.57}$$

and hence, the coherent states $\{|\alpha\rangle\}$ form a complete set of states.

The variances of q and p over the state $|\alpha\rangle$ satisfy the minimum uncertainty condition of Eq. (1.3). We now show the calculation of these variances Δq and Δp over the state $|\alpha\rangle$. Using Eq. (1.29) and (1.30) we have with $\omega = 1$,

$$\hat{q} = \sqrt{\frac{\hbar}{2}}(\hat{a}^\dagger + \hat{a}) \tag{1.58}$$

and

$$\hat{p} = i\sqrt{\frac{\hbar}{2}}(\hat{a}^\dagger - \hat{a}). \tag{1.59}$$

As $|\alpha\rangle$ is an eigenstate of $\hat{a}$, we have therefore,

$$\begin{aligned} \langle\alpha|\hat{q}|\alpha\rangle &= \sqrt{\frac{\hbar}{2}}\langle\alpha|(\hat{a}^\dagger + \hat{a})|\alpha\rangle \\ &= \sqrt{\frac{\hbar}{2}}\Big(\alpha^\star + \alpha\Big) \end{aligned} \tag{1.60}$$

and

$$\begin{aligned} \langle\alpha|\hat{p}|\alpha\rangle &= i\sqrt{\frac{\hbar}{2}}\langle\alpha|(\hat{a}^\dagger - \hat{a})|\alpha\rangle \\ &= i\sqrt{\frac{\hbar}{2}}\Big(\alpha^\star - \alpha\Big). \end{aligned} \tag{1.61}$$

It can be noted from Eqs. (1.40) and (1.60) that $\langle\alpha|\hat{E}_x|\alpha\rangle \neq 0$. From Eqs. (1.58) and (1.59) we have

$$\hat{q}^2 = \frac{\hbar}{2}\Big(\hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1\Big) \tag{1.62}$$

and

$$\hat{p}^2 = -\frac{\hbar}{2}\Big(\hat{a}^{\dagger 2} + \hat{a}^2 - 2\hat{a}^\dagger\hat{a} - 1\Big) \tag{1.63}$$

respectively. So,

$$\begin{aligned} \langle\alpha|\hat{q}^2|\alpha\rangle &= \frac{\hbar}{2}\langle\alpha|(\hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1)|\alpha\rangle \\ &= \frac{\hbar}{2}\Big(\alpha^{\star 2} + \alpha^2 + 2\alpha^\star\alpha + 1\Big) \\ &= \frac{\hbar}{2}\Big(\alpha^\star + \alpha\Big)^2 + \frac{\hbar}{2}. \end{aligned} \tag{1.64}$$

Similarly,

$$\begin{aligned}\langle\alpha|\hat{p}^2|\alpha\rangle &= -\frac{\hbar}{2}\langle\alpha|(\hat{a}^{\dagger 2}+\hat{a}^2-2\hat{a}^\dagger\hat{a}-1)|\alpha\rangle \\ &= -\frac{\hbar}{2}\Big(\alpha^{\star 2}+\alpha^2-2\alpha^\star\alpha-1\Big) \\ &= -\frac{\hbar}{2}\Big(\alpha^\star-\alpha\Big)^2+\frac{\hbar}{2}. \end{aligned} \tag{1.65}$$

Now from Eqs. (1.60) and (1.64) we have

$$\Delta q^2 = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 = \frac{\hbar}{2} \tag{1.66}$$

and from Eq. (1.61) and (1.65) we have

$$\Delta p^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = \frac{\hbar}{2}. \tag{1.67}$$

Hence, the uncertainty relation for these variances is

$$\Delta q\ \Delta p = \frac{\hbar}{2}. \tag{1.68}$$

Thus, the state $|\alpha\rangle$ possesses the minimum value of the inherent uncertainty as given in Eq. (1.3). In addition, the uncertainties are equally distributed in q and p. Hence they are called minimum uncertainty states. The state $|\alpha\rangle$ in position representation forms a Gaussian wavepacket which retains it's exact shape when it is evolved with respect to time under the action of harmonic oscillator Hamiltanian. Despite these beautiful properties of the coherent state it was not of much practical use even though it was introduced as early as in 1926. But after the invention of LASER, it was found that the coherent state describes very well a laser light.

The state $|\alpha\rangle$ can be obtained from the vacuum state $|0\rangle$ as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{(\alpha\hat{a}^\dagger-\alpha^\star\hat{a})}|0\rangle, \tag{1.69}$$

because, using Baker-Hausdorff lemma [Apendix-I] it follows that

$$\begin{aligned} e^{(\alpha\hat{a}^\dagger-\alpha^\star\hat{a})}|0\rangle &= e^{\alpha\hat{a}^\dagger}e^{-\alpha^\star\hat{a}}e^{-\frac{1}{2}[\alpha\hat{a}^\dagger,-\alpha^\star\hat{a}]}|0\rangle \\ &= e^{\alpha\hat{a}^\dagger}e^{-\alpha^\star\hat{a}}e^{\frac{-|\alpha|^2}{2}[\hat{a}^\dagger,\hat{a}]}|0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}e^{-\alpha^\star\hat{a}}|0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}\Big[1-\alpha^\star\hat{a}+(\alpha^\star\hat{a})^2/2!+...\Big]|0\rangle \\ &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}|0\rangle \end{aligned} \tag{1.70}$$

which is nothing but Eq. (1.45). The operator $\hat{D}(\alpha)$ is called the displacement operator. This is due to the fact that the vacuum state satisfies minimum uncertainty conditions represented by the Eq. (1.68) and hence, $|\alpha\rangle$ is also known as displaced vacuum state. Note here that $\alpha\hat{a}^\dagger-\alpha^\star\hat{a}$ being an anti-Hermitian operator, $e^{\alpha\hat{a}^\dagger-\alpha^\star\hat{a}}$ is unitary and thus retains the normalization when acts on a normalized state.

1.1.4. Squeezed States

When the above mentioned inherent uncertainty of the system is redistributed, so that the uncertainty in either q or p is reduced below the values, as given in Eqs. (1.66) and (1.67), retaining, however, as it must, the product in Eq. (1.68), then we say that squeezing has been introduced in the system. Mathematically we achieve this in this present system through the following scaling transformation on $\hat{q}$ and $\hat{p}$ as,

$$\hat{q} \longrightarrow \hat{q}' = \hat{q}e^{\epsilon} \tag{1.71}$$

and

$$\hat{p} \longrightarrow \hat{p}' = \hat{p}e^{-\epsilon}, \tag{1.72}$$

where ϵ is a real number. The commutation relation between the transformed operators remain intact as

$$[\hat{q}', \hat{p}'] = i\hbar, \tag{1.73}$$

but the uncertainties in q and p are modified as

$$\Delta q' = e^{\epsilon}\Delta q \tag{1.74}$$

and

$$\Delta p' = e^{-\epsilon}\Delta p. \tag{1.75}$$

We see that the uncertainty in p is reduced and that in q is enhanced while retaining the minimum value of the product in Eq. (1.3). At the level of the annihilation and creation operators the above transformation entails a Bogoluibov transformation viz.

$$\hat{a} \longrightarrow \hat{b} = \hat{a}\cosh\epsilon + \hat{a}^{\dagger}\sinh\epsilon \tag{1.76}$$

which is seen to be implemented by a unitary operator

$$\hat{S}(\epsilon) = e^{\frac{\epsilon}{2}\hat{a}^2 - \frac{\epsilon^{\star}}{2}\hat{a}^{\dagger 2}} \tag{1.77}$$

[9]. In other words, we have

$$\hat{a} \rightarrow \hat{b} = e^{[\frac{\epsilon}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})]} \;\; \hat{a} \;\; e^{[-\frac{\epsilon}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})]}. \tag{1.78}$$

The quantum state of the particle in which the above transformation is achieved is called a squeezed state. The squeezed states can be obtained by Yuen's method [10] which he called as two-photon coherent states. Equivalently, the squeezed states can also be obtained by the method given by Caves [11]. The corresponding parameters in the two methods are related to each other. In Caves' method, first, the vacuum state of the electromagnetic field is squeezed by the unitary operator $\hat{S}(\epsilon)$ in Eq. (1.77), also, called as squeeze operator. ϵ is now called the squeeze parameter with

$$\epsilon = se^{i\theta}, 0 \leq s \leq \infty, 0 \leq \theta \leq 2\pi. \tag{1.79}$$

The squeezed vacuum

$$|0, \epsilon\rangle = \hat{S}(\epsilon)|0\rangle \tag{1.80}$$

is then displaced by the displacement operator in Eq. (1.69). Thus the squeezed state can be represented as

$$|\alpha, \epsilon\rangle = \hat{D}(\alpha)\hat{S}(\epsilon)|0\rangle. \tag{1.81}$$

The subject has been widely reviewed in Refs. [12–14]. The term "squeezed state" was first coined in the experiments on gravitational wave detection [15]. In 1980's and onwards many experiments were performed to detect squeezed light and people succeeded in obtaining squeezing by significant amount [16]. With the advanced quantum optics experiments on squeezed states, this field became one of the most interesting subjects.

1.2. A Two-Level Atom

In the above, we have analysed the coherent and squeezed properties of electromagnetic field. Then, a natural question arises as to how these ideas can be implemented in atoms. However, there is a basic difference between the two quantum systems, that is, atom and electromagnetic field. Electromagnetic field has equispaced infinite number of energy levels whereas we know that atomic energy levels are quite different. Thus, one needs a deeper look to define atomic coherent and squeezed states. Before we do so, we need to define what is widely known as a two-level atom in quantum optics. When the electromagnetic field interacts with an atom the resonant (the frequency of the electromagnetic field being equal to the transition frequency between a pair of atomic levels) or near resonant interaction dominates over its interaction with other atomic transitions. So, we concentrate on the resonant pair of levels. The other energy levels are not important for consideration here as their transition frequencies are far away from the frequency of the electromagnetic field. Hence, we use the terminology as "two-level" atoms [17, 18].

Let the upper and lower level states of a two-level atom are denoted as $|u\rangle$ and $|l\rangle$ respectively. We define two operators $\hat{J}_+$ and $\hat{J}_-$ as

$$\hat{J}_+ = |u\rangle\langle l| \tag{1.82}$$

$$\hat{J}_- = |l\rangle\langle u| \tag{1.83}$$

and

$$\hat{J}_z = \frac{1}{2}[|u\rangle\langle u| - |l\rangle\langle l|] \tag{1.84}$$

such that

$$\hat{J}_+|l\rangle = |u\rangle, \tag{1.85}$$

$$\hat{J}_-|u\rangle = |l\rangle. \tag{1.86}$$

Thus, $\hat{J}_+$ converts a lower level state to the upper one and is called as raising operator and $\hat{J}_-$ does the opposite and is the so called lowering operator. Also

$$\hat{J}_z|u\rangle = \frac{1}{2}|u\rangle \tag{1.87}$$

and

$$\hat{J}_z|l\rangle = -\frac{1}{2}|l\rangle \tag{1.88}$$

implying that $\hat{J}_z$ gives the population difference between the upper and the lower levels with the zero of the energy being fixed at the middle such that the state $|u\rangle$ and $|l\rangle$ correspond to the energy $\frac{1}{2}\hbar\omega$ and $-\frac{1}{2}\hbar\omega$ respectively. $\hat{J}_z$ is known as inversion operator in quantum optics. We further define

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \tag{1.89}$$

and

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i} \tag{1.90}$$

such that

$$[\hat{J}_k, \hat{J}_l] = i\epsilon_{klm}\hat{J}_m, \tag{1.91}$$

where the suffixes k, l, m represent any of the three orthogonal components x, y and z [19, 20]. ϵ_{klm} is the Levi Civita symbol representing

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \tag{1.92}$$

and

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \tag{1.93}$$

and other ϵ_{ijk} are zero. The operators $\hat{J}_x$ and $\hat{J}_y$ correspond to the x and y components of the atomic polarization respectively. Since $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ satisfy the same commutation relations as the Pauli spin operators, those are called pseudo-spin operators since the z-component is not the z-component of atomic polarization but the population inversion parameter. Thus a system of N mutually non-interacting two-level atoms with operators

$$\hat{J}_x = \sum_{i=1}^{N} \hat{J}_{x_i}, \tag{1.94}$$

$$\hat{J}_y = \sum_{i=1}^{N} \hat{J}_{y_i} \tag{1.95}$$

and

$$\hat{J}_z = \sum_{i=1}^{N} \hat{J}_{z_i} \tag{1.96}$$

is equivalent to a system with total spin $\frac{N}{2}$ with

$$-\frac{N}{2} \leq \langle J_z \rangle \leq \frac{N}{2}.$$

The collective atomic operators given in Eqs. (1.94) to (1.96) satisfy the same commutation relations as given in (1.91). Thus a two-level atom is described mathematicaly with the help of these pseudo-spin or angular momentum operators. We have studied the atomic coherent states and atomic squeezed states in terms of these operators which we describe in the next section.

1.3. Atomic Coherent State and Atomic Squeezed State

We now consider various atomic states for which the basis vectors are the angular momentum eigenstates, also called the Wigner states.

1.3.1. Wigner State

The collective state of a system of N two-level atoms can be represented by a Wigner state $|j, m\rangle$, which is the simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ operators as $[\hbar = 1]$

$$\hat{J}^2|j, m\rangle = j(j+1)|j, m\rangle \tag{1.97}$$

and

$$\hat{J}_z|j, m\rangle = m|j, m\rangle, \tag{1.98}$$

where $\hat{J}^2$ is the square of the angular momentum operator. The quantum number j here is connected to the number of atoms N as $j = \frac{N}{2}$ and the quantum number m here stands for the population difference between the two atomic levels. The raising and lowering operators when act upon $|j, m\rangle$ give

$$\hat{J}_+|j, m\rangle = \sqrt{(j-m)(j+m+1)}\ |j, m+1\rangle \tag{1.99}$$
$$\hat{J}_-|j, m\rangle = \sqrt{(j+m)(j-m+1)}\ |j, m-1\rangle. \tag{1.100}$$

We know that the variances in J_x and J_y over the state $|j, m\rangle$, that is,

$$\Delta J_{x,y} = \sqrt{\langle \hat{J}_{x,y}^2 \rangle - \langle \hat{J}_{x,y} \rangle^2} \tag{1.101}$$

are given as

$$\Delta J_x = \Delta J_y = \frac{1}{\sqrt{2}}\sqrt{j(j+1)-m^2}. \tag{1.102}$$

It is easy to see that when m takes the value $+j$ or $-j$ the above uncertainties obtain their minimum values as

$$\Delta J_x = \Delta J_y = \sqrt{\frac{j}{2}}. \tag{1.103}$$

Thus, the states $|j, m = \pm j\rangle$ can be called minimum uncertainty states (MUS).

1.3.2. Atomic (Spin) Coherent States

The atomic or spin coherent states were devloped by Bloch [22] and subsequently by Radcliffe [23], Arecchi and others [24]. This state is traditionally developed via

$$\begin{aligned} |j,\chi\rangle &= Ne^{\chi \hat{J}_-}|j, m=+j\rangle & (1.104)\\ &= N\sum_{n=0}^{2j}\frac{\chi^n}{n!}\ \hat{J}_-^n\ |j,m=+j\rangle, & (1.105)\end{aligned}$$

where χ is a complex number. In general the normalization constant N is complex such that,

$$N = |N|\ e^{i\phi_0}, \tag{1.106}$$

where ϕ_0 is the phase factor. But, since the overall phase factor in a state vector has no physical significance, it can be assumed to be zero and hence, we may assume N to be real.

Now,

$$\begin{aligned} \hat{J}_-^n\ \ |j,m=+j\rangle &= \sqrt{2j.1}\ \ \hat{J}_-^{n-1}\ \ |j,m=j-1\rangle \\ &= \sqrt{2j.(2j-1).1.2}\ \ \hat{J}_-^{n-2}\ \ |j,m=j-2\rangle \\ &= \sqrt{2j.(2j-1)...(2j-n+1).1.2...n}\ \ |j,m=j-n\rangle \\ &= \sqrt{\frac{(2j)!n!}{(2j-n)!}}\ \ |j,m=j-n\rangle. \end{aligned} \tag{1.107}$$

Thus,

$$\begin{aligned} |j,\chi\rangle &= N\ \sum_{n=0}^{2j}\chi^n\sqrt{\frac{(2j)!}{n!(2j-n)!}}\ \ |j,j-n\rangle \\ &= N\ \sum_{n=0}^{2j}\sqrt{{}^{2j}C_n}\ \chi^n\ \ |j,j-n\rangle. \end{aligned} \tag{1.108}$$

To find out the normalization constant N we use the condition for normalization as

$$\langle j,\chi|j,\chi\rangle = 1, \tag{1.109}$$

which implies

$$N^2 \sum_{n=0}^{2j} {}^{2j}C_n \left(|\chi|^2\right)^n = 1 \tag{1.110}$$

or,

$$N^2 \left(1+|\chi|^2\right)^{2j} = 1 \tag{1.111}$$

or,

$$N = \frac{1}{\left(1+|\chi|^2\right)^j}. \tag{1.112}$$

Hence the normalized state vector is

$$|j,\chi\rangle = \frac{1}{\left(1+|\chi|^2\right)^j} \sum_{n=0}^{2j} \sqrt{{}^{2j}C_n}\ \chi^n \quad |j, m=j-n\rangle. \tag{1.113}$$

To understand the various properties of this state we need to know the average values of various pseudo-angular momentum operators over this state and hence we now show the calculation of these necessary quantities.

$$\begin{aligned}
\langle j,\chi|\hat{J}_z|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l{}^{2j}C_n} \left(\chi^\star\right)^l \chi^n \ \langle j, j-l|\hat{J}_z|j,j-n\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{n=0}^{2j} {}^{2j}C_n |\chi|^{2n}\ (j-n) \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \left[j \sum_{n=0}^{2j} {}^{2j}C_n\ |\chi|^{2n} - |\chi|^2 \frac{d}{d|\chi|^2} \sum_{n=0}^{2j} {}^{2j}C_n\ |\chi|^{2n} \right] \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \left[j\left(1+|\chi|^2\right)^{2j} - |\chi|^2 \frac{d}{d|\chi|^2} \left(1+|\chi|^2\right)^{2j} \right] \\
&= j - 2j\frac{|\chi|^2}{1+|\chi|^2} \\
&= j\ \frac{1-|\chi|^2}{1+|\chi|^2}\ .
\end{aligned} \tag{1.114}$$

Similarly,

$$\begin{aligned}
\langle j,\chi|\hat{J}_+|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l\ {}^{2j}C_n}\ \left(\chi^\star\right)^l \chi^n\ \langle j,j-l|\hat{J}_+|j,j-n\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l\ {}^{2j}C_n}\ \left(\chi^\star\right)^l \chi^n\ \sqrt{n(2j-n+1)}\delta_{l,n-1} \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j} \sqrt{\frac{2j!\ 2j!\ (l+1)\ (2j-l)}{l!\ (2j-l)!\ (l+1)!\ (2j-l-1)!}}\ \chi^{\star l}\chi^{l+1} \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j} \frac{2j!}{l!\ (2j-l-1)!}\ |\chi|^{2l} \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j} {}^{2j}C_l\ (2j-l)\ |\chi|^{2l} \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \left[2j\sum_{l=0}^{2j} {}^{2j}C_l\ |\chi|^{2l} - \sum_{l=0}^{2j} {}^{2j}C_l\ l\ |\chi|^{2l}\right] \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \left[2j\left(1+|\chi|^2\right)^{2j} - |\chi|^2\frac{d}{d(|\chi|^2)} \sum_{l=0}^{2j} {}^{2j}C_l\ l\ |\chi|^{2l}\right] \\
&= 2j\chi - \frac{\chi|\chi|^2}{\left(1+|\chi|^2\right)^{2j}}\frac{d}{d(|\chi|^2)}\left(1+|\chi|^2\right)^{2j} \\
&= 2j\chi - 2j\chi\frac{|\chi|^2}{\left(1+|\chi|^2\right)} \\
&= \frac{2j\chi}{\left(1+|\chi|^2\right)}\ .
\end{aligned} \tag{1.115}$$

As

$$\langle \hat{J}_-\rangle = \langle \hat{J}_+\rangle^\star. \tag{1.116}$$

Therefore,

$$\langle \hat{J}_-\rangle = \frac{2j\chi^\star}{\left(1+|\chi|^2\right)}. \tag{1.117}$$

Now,

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \tag{1.118}$$

and hence,

$$\langle \hat{J}_x \rangle = \frac{1}{2}\Big(\langle \hat{J}_+ \rangle + \langle \hat{J}_- \rangle\Big) \tag{1.119}$$

and using Eqs. (1.115) and (1.117) we obtain,

$$\langle \hat{J}_x \rangle = j \; \frac{\chi + \chi^\star}{\Big(1 + |\chi|^2\Big)}. \tag{1.120}$$

Similarly,

$$\hat{J}_y = \frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big) \tag{1.121}$$

implying,

$$\begin{aligned} \langle \hat{J}_y \rangle &= \frac{1}{2i}\Big(\langle \hat{J}_+ \rangle - \langle \hat{J}_- \rangle\Big) \\ &= j \; \frac{\chi - \chi^\star}{i\,\Big(1 + |\chi|^2\Big)} \; . \end{aligned} \tag{1.122}$$

Using the substitution

$$\chi = \tan\frac{\theta}{2}\; e^{i\phi} \; , \tag{1.123}$$

Eqs. (1.114), can be casted in the form

$$\begin{aligned} \langle \hat{J}_z \rangle &= j \; \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} \\ &= j\cos\theta \; . \end{aligned} \tag{1.124}$$

Similarly, Eqs. (1.120) and (1.122) can be written as using Eq. (1.123) as

$$\langle j, \chi | \hat{J}_x | j, \chi \rangle = j \sin\theta\cos\phi \tag{1.125}$$

and

$$\langle j, \chi | \hat{J}_y | j, \chi \rangle = j \sin\theta\sin\phi \tag{1.126}$$

respectively. We see that the angles θ and ϕ are as if the polar and azimuthal angles respectively, made by the mean angular momentum vector

$$\langle \hat{\mathbf{J}} \rangle = \langle \hat{J}_x \rangle \mathbf{i} + \langle \hat{J}_y \rangle \mathbf{j} + \langle \hat{J}_z \rangle \mathbf{k} \tag{1.127}$$

with a right handed rectangular cartesian coordinate axes. Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the x, y and z axes respectively. The magnitude of $\langle \hat{\mathbf{J}} \rangle$ is

$$\begin{aligned} |\langle \hat{\mathbf{J}} \rangle| &= \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2 + \langle \hat{J}_z \rangle^2} \\ &= \sqrt{j^2\sin^2\theta\cos^2\phi + j^2\sin^2\theta\sin^2\phi + j^2\cos^2\theta} \\ &= j. \end{aligned} \tag{1.128}$$

To calculate the variances

$$\Delta J_x = \sqrt{\langle \hat{J}_x^2 \rangle - \langle \hat{J}_x \rangle^2} \tag{1.129}$$

and

$$\Delta J_y = \sqrt{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_y \rangle^2} \tag{1.130}$$

over the state $|j, \chi\rangle$, we see that we need to calculate $\langle \hat{J}_x^2 \rangle$ and $\langle \hat{J}_y^2 \rangle$ over that state. We, therefore, now show the calculation of these quantities one by one. We know that

$$\begin{aligned} \hat{J}_x^2 &= \left(\frac{1}{2}\right)^2 \left(\hat{J}_+ + \hat{J}_-\right)\left(\hat{J}_+ + \hat{J}_-\right) \\ &= \frac{1}{4}\left(\hat{J}_+^2 + \hat{J}_-^2 + \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\right). \end{aligned} \tag{1.131}$$

Now,

$$\begin{aligned} \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ &= \left(\hat{J}_x + i\hat{J}_y\right)\left(\hat{J}_x - i\hat{J}_y\right) + \left(\hat{J}_x - i\hat{J}_y\right)\left(\hat{J}_x + i\hat{J}_y\right) \\ &= 2\left(\hat{J}_x^2 + \hat{J}_y^2\right) \\ &= 2\left(\hat{J}^2 - \hat{J}_z^2\right). \end{aligned} \tag{1.132}$$

Therefore, Eq. (1.131) becomes,

$$\hat{J}_x^2 = \frac{1}{4}\left(\hat{J}_+^2 + \hat{J}_-^2\right) + \frac{1}{2}\left(\hat{J}^2 - \hat{J}_z^2\right). \tag{1.133}$$

Therefore,

$$\begin{aligned} \langle \hat{J}_x^2 \rangle &= \frac{1}{4}\left(\langle \hat{J}_+^2 \rangle + \langle \hat{J}_-^2 \rangle\right) + \frac{1}{2}\left(\langle \hat{J}^2 \rangle - \langle \hat{J}_z^2 \rangle\right) \\ &= \frac{1}{4}\left(\langle \hat{J}_+^2 \rangle + \langle \hat{J}_+^2 \rangle^\star\right) + \frac{1}{2}\left(\langle \hat{J}^2 \rangle - \langle \hat{J}_z^2 \rangle\right) \\ &= \frac{1}{2}\left(Re\langle \hat{J}_+^2 \rangle + \langle \hat{J}^2 \rangle - \langle \hat{J}_z^2 \rangle\right). \end{aligned} \tag{1.134}$$

Thus, to calculate $\langle j, \chi|\hat{J}_x^2|j, \chi\rangle$ we need to calculate $\langle j, \chi|\hat{J}_+^2|j, \chi\rangle$, $\langle j, \chi|\hat{J}^2|j, \chi\rangle$ and $\langle j, \chi|\hat{J}_z^2|j, \chi\rangle$.

Now,

$$
\begin{aligned}
\langle j,\chi|\hat{J}_+^2|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l\ {}^{2j}C_n}\ \left(\chi^\star\right)^l \chi^n\ \langle j,j-l|\hat{J}_+^2|j,j-n\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l\ {}^{2j}C_n}\ \left(\chi^\star\right)^l \chi^n \\
&\times\ \sqrt{n(n-1)(2j-n+1)(2j-n+2)}\ \ \delta_{l,n-2} \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j} \sqrt{\frac{2j!\ 2j!\ (l+2)(l+1)\ (2j-l)(2j-l-1)}{l!\ (2j-l)!\ (l+2)!\ (2j-l-2)!}}\ \chi^{\star l}\chi^{l+2} \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j}\ {}^{2j}C_l\ |\chi|^{2l}\chi^2\ (2j-l)(2j-l-1) \\
&= \frac{\chi^2}{\left(1+|\chi|^2\right)^{2j}}\Bigg[(4j^2-2j)\sum_{l=0}^{2j}{}^{2j}C_l\ |\chi|^{2l} - (4j-1)\sum_{l=0}^{2j}{}^{2j}C_l\ l\ |\chi|^{2l} \\
&+\ \sum_{l=0}^{2j}{}^{2j}C_l\ |\chi|^{2l}l^2\ \Bigg] \\
&= \frac{\chi^2}{\left(1+|\chi|^2\right)^{2j}}\Bigg[(4j^2-2j)\left(1+|\chi|^2\right)^{2j} - (4j-1)|\chi|^2\frac{d}{d(|\chi|^2)}\sum_{l=0}^{2j}{}^{2j}C_l\ |\chi|^{2l} \\
&+\ |\chi|^2\frac{d}{d(|\chi|^2)}|\chi|^2\frac{d}{d(|\chi|^2)}\sum_{l=0}^{2j}{}^{2j}C_l\ |\chi|^{2l}\ \Bigg] \\
&= (4j^2-2j)\chi^2 - (4j-1)\frac{\chi^2|\chi|^2}{\left(1+|\chi|^2\right)^{2j}}\frac{d}{d(|\chi|^2)}\left(1+|\chi|^2\right)^{2j} \\
&+\ \frac{\chi^2|\chi|^2}{\left(1+|\chi|^2\right)^{2j}}\frac{d}{d(|\chi|^2)}|\chi|^2\frac{d}{d(|\chi|^2)}\left(1+|\chi|^2\right)^{2j} \\
&= (4j^2-2j)\chi^2 - 2j(4j-1)\frac{\chi^2|\chi|^2}{\left(1+|\chi|^2\right)} + 2j\frac{\chi^2|\chi|^2}{\left(1+|\chi|^2\right)} \\
&+\ 2j(2j-1)\frac{\chi^2|\chi|^4}{\left(1+|\chi|^2\right)^2} \\
&= 2j(2j-1)\frac{\chi^2}{\left(1+|\chi|^2\right)^2}. \qquad (1.135)
\end{aligned}
$$

Using Eq. (1.123) we obtain

$$\langle j,\chi|\hat{J}_+^2|j,\chi\rangle = \frac{1}{2}j(2j-1)\sin^2\theta\ e^{2i\phi}. \tag{1.136}$$

In a similar way

$$\begin{aligned}
\langle j,\chi|\hat{J}_z^2|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\sum_{l,n=0}^{2j}\sqrt{{}^{2j}C_l{}^{2j}C_n}\left(\chi^\star\right)^l\chi^n\ \langle j,j-l|\hat{J}_z^2|j,j-n\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\sum_{n=0}^{2j}{}^{2j}C_n|\chi|^{2n}\ (j-n)^2 \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\Bigg[j^2\sum_{n=0}^{2j}{}^{2j}C_n\ |\chi|^{2n} - 2j|\chi|^2\frac{d}{d|\chi|^2}\sum_{n=0}^{2j}{}^{2j}C_n\ |\chi|^{2n} \\
&+ |\chi|^2\frac{d}{d|\chi|^2}|\chi|^2\frac{d}{d|\chi|^2}\sum_{n=0}^{2j}{}^{2j}C_n\ |\chi|^{2n}\Bigg] \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\Bigg[j^2\left(1+|\chi|^2\right)^{2j} - 2j|\chi|^2\frac{d}{d|\chi|^2}\left(1+|\chi|^2\right)^{2j} \\
&+ |\chi|^2\frac{d}{d|\chi|^2}|\chi|^2\frac{d}{d|\chi|^2}\left(1+|\chi|^2\right)^{2j}\Bigg] \\
&= j^2 - 4j^2\frac{|\chi|^2}{\left(1+|\chi|^2\right)} + 2j\frac{|\chi|^2}{\left(1+|\chi|^2\right)} \\
&+ 2j(2j-1)\frac{|\chi|^4}{\left(1+|\chi|^2\right)^2}.
\end{aligned} \tag{1.137}$$

Using Eq. (1.123) and simplifying we obtain

$$\langle j,\chi|\hat{J}_z^2|j,\chi\rangle = j^2\cos^2\theta + \frac{1}{2}j\sin^2\theta. \tag{1.138}$$

Now,

$$\begin{aligned}
\langle j,\chi|\hat{J}^2|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\sum_{l,n=0}^{2j}\sqrt{{}^{2j}C_l{}^{2j}C_n}\left(\chi^\star\right)^l\chi^n\ \langle j,j-l|\hat{J}^2|j,j-n\rangle \\
&= j(j+1).
\end{aligned} \tag{1.139}$$

Using Eqs. $(1.136), (1.138)$ and (1.139) in Eq. (1.134) we obtain

$$\langle\hat{J}_x^2\rangle = \frac{1}{2}j\left[1 + j\sin^2\theta + \frac{1}{2}(2j-1)\sin^2\theta\cos 2\phi - \frac{1}{2}\sin^2\theta\right]. \tag{1.140}$$

We know that

$$\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \hat{J}^2. \tag{1.141}$$

Therefore,

$$\langle \hat{J}_y^2 \rangle = \langle \hat{J}^2 \rangle - \langle \hat{J}_x^2 \rangle - \langle \hat{J}_z^2 \rangle. \tag{1.142}$$

Hence, using Eqs. $(1.138), (1.139)$ and (1.140) in Eq. (1.142) and simplifying we obtain

$$\langle j, \chi | \hat{J}_y^2 | j, \chi \rangle = \frac{1}{2} j + j^2 \sin^2\theta \sin^2\phi - \frac{1}{2} j \sin^2\theta \sin^2\phi. \tag{1.143}$$

Therefore, using Eqs. (1.125) and (1.140) in Eq. (1.129) we obtain

$$\begin{aligned} \Delta J_x^2 &= \frac{1}{2} j \left[1 + j \sin^2\theta + \frac{1}{2}(2j-1)\sin^2\theta \cos 2\phi - \frac{1}{2}\sin^2\theta \right] - j^2 \sin^2\theta \cos^2\phi \\ &= \frac{1}{2} j \Big(1 - \sin^2\theta \cos^2\phi \Big). \end{aligned} \tag{1.144}$$

Similarly, using Eqs. (1.126) and (1.143) in Eq. (1.130) we obtain

$$\begin{aligned} \Delta J_y^2 &= \frac{1}{2} j + j^2 \sin^2\theta \sin^2\phi - \frac{1}{2} j \sin^2\theta \sin^2\phi - j^2 \sin^2\theta \sin^2\phi \\ &= \frac{1}{2} j \Big(1 - \sin^2\theta \sin^2\phi \Big). \end{aligned} \tag{1.145}$$

We observe that the variances in J_x and J_y are dependent on choice of the coordinate system and can take value less than $\sqrt{\frac{j}{2}}$. Thus the variances given in Eqs. (1.129) and (1.130) do not represent the actual inherent uncertainty of the system. The fundamental and inherent uncertainty of the system is uncovered in a plane perpendicular to $\langle \hat{\mathbf{J}} \rangle$. Therefore, we perform a rotation of the coordinate system such that,

$$\hat{J}'_x = \hat{J}_x \cos\theta \cos\phi + \hat{J}_y \cos\theta \sin\phi - \hat{J}_z \sin\theta, \tag{1.146}$$

$$\hat{J}'_y = -\hat{J}_x \sin\phi + \hat{J}_y \cos\phi, \tag{1.147}$$

$$\hat{J}'_z = \hat{J}_x \sin\theta \cos\phi + \hat{J}_y \sin\theta \sin\phi + \hat{J}_z \cos\theta. \tag{1.148}$$

Thus,

$$\langle \hat{J}'_x \rangle = \langle \hat{J}_x \rangle \cos\theta \cos\phi + \langle \hat{J}_y \rangle \cos\theta \sin\phi - \langle \hat{J}_z \rangle \sin\theta, \tag{1.149}$$

$$\langle \hat{J}'_y \rangle = -\langle \hat{J}_x \rangle \sin\phi + \langle \hat{J}_y \rangle \cos\phi, \tag{1.150}$$

$$\langle \hat{J}'_z \rangle = \langle \hat{J}_x \rangle \sin\theta \cos\phi + \langle \hat{J}_y \rangle \sin\theta \sin\phi + \langle \hat{J}_z \rangle \cos\theta. \tag{1.151}$$

Using Eqs. (1.124), (1.125) and (1.126) in Eq. (1.149) we observe that

$$\begin{aligned} \langle \hat{J}'_x \rangle &= j \sin\theta \cos\theta \cos^2\phi + j \sin\theta \cos\theta \sin^2\phi - j \cos\theta \sin\theta \\ &= 0. \end{aligned} \tag{1.152}$$

Similarly using Eqs. (1.125) and (1.126) in Eq. (1.150) we observe that

$$\begin{aligned}\langle \hat{J}'_y \rangle &= -j\sin\theta\cos\phi\sin\phi + j\sin\theta\sin\phi\cos\phi \\ &= 0 \end{aligned} \tag{1.153}$$

and using Eqs. (1.124), (1.125) and (1.126) in Eq. (1.151) we observe that

$$\begin{aligned}\langle \hat{J}'_z \rangle &= j\sin^2\theta\cos^2\phi + j\sin^2\theta\sin^2\phi + j\cos^2\theta \\ &= j. \end{aligned} \tag{1.154}$$

Thus the mean angular momentum vector is now

$$\begin{aligned}\langle \hat{\mathbf{J}}' \rangle &= \langle \hat{J}'_x \rangle \mathbf{i} + \langle \hat{J}'_y \rangle \mathbf{j} + \langle \hat{J}'_z \rangle \mathbf{k} \\ &= j\ \mathbf{k}, \end{aligned} \tag{1.155}$$

which shows that $\langle \hat{\mathbf{J}}' \rangle$ is along the z-axis. It may be noted that the magnitude of the mean angular momentum vector $\langle \hat{\mathbf{J}}' \rangle$ is

$$|\langle \hat{\mathbf{J}}' \rangle| = j = |\langle \hat{J}'_z \rangle| = |\langle \hat{\mathbf{J}} \rangle|, \tag{1.156}$$

which is expected as a rotation does not change the length of a vector.

We now calculate the variances

$$\Delta J'_x = \sqrt{\langle \hat{J}'^2_x \rangle - \langle \hat{J}'_x \rangle^2} \tag{1.157}$$

and

$$\Delta J'_y = \sqrt{\langle \hat{J}'^2_y \rangle - \langle \hat{J}'_y \rangle^2} \tag{1.158}$$

over the state $|j, \chi\rangle$.

From Eq. (1.146) and (1.147) we get

$$\begin{aligned}\hat{J}'^2_x &= \hat{J}^2_x\cos^2\theta\cos^2\phi + \hat{J}^2_y\cos^2\theta\sin^2\phi + J^2_z\sin^2\theta \\ &+ (\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)\cos^2\theta\sin\phi\cos\phi - (\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)\sin\theta\cos\theta\cos\phi \\ &- (\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y)\sin\theta\cos\theta\sin\phi. \end{aligned} \tag{1.159}$$

and

$$\hat{J}'^2_y = \hat{J}^2_x\sin^2\phi + \hat{J}^2_y\cos^2\phi - (\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)\sin\phi\cos\phi \tag{1.160}$$

respectively.

Therefore,

$$\begin{aligned}\langle \hat{J}'^2_x \rangle &= \langle \hat{J}^2_x \rangle\cos^2\theta\cos^2\phi + \langle \hat{J}^2_y \rangle\cos^2\theta\sin^2\phi + \langle \hat{J}^2_z \rangle\sin^2\theta \\ &+ \langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \rangle\cos^2\theta\sin\phi\cos\phi - \langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x \rangle\sin\theta\cos\theta\cos\phi \\ &- \langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y \rangle\sin\theta\cos\theta\sin\phi \end{aligned} \tag{1.161}$$

and

$$\langle \hat{J}_y'^2 \rangle = \langle \hat{J}_x^2 \rangle \sin^2\phi + \langle \hat{J}_y^2 \rangle \cos^2\phi - \langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \rangle \sin\phi\cos\phi. \tag{1.162}$$

respectively.

Thus to calculate $\langle \hat{J}_x'^2 \rangle$ and $\langle \hat{J}_y'^2 \rangle$ we need to calculate $\langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \rangle$, $\langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x \rangle$ and $\langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y \rangle$ over the state $|j,\chi\rangle$. We now show the calculation of these quantities.

$$\begin{aligned} \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x &= \frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big)\frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big) + \frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big)\frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big) \\ &= \frac{1}{2i}\Big(\hat{J}_+^2 - \hat{J}_-^2\Big). \end{aligned} \tag{1.163}$$

Therefore,

$$\begin{aligned} \langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \rangle &= \frac{1}{2i}\Big(\langle \hat{J}_+^2 \rangle - \langle \hat{J}_-^2 \rangle\Big) \\ &= \frac{1}{2i}\Big(\langle \hat{J}_+^2 \rangle - \langle \hat{J}_+^2 \rangle^\star\Big) \\ &= Im\langle \hat{J}_+^2 \rangle \end{aligned} \tag{1.164}$$

that is the imaginary part of $\langle \hat{J}_+^2 \rangle$. Using Eq. (1.136) we obtain

$$\langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \rangle = \frac{1}{2}j(2j-1)\sin^2\theta\sin 2\phi. \tag{1.165}$$

Now,

$$\begin{aligned} \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x &= \frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big)\hat{J}_z + \hat{J}_z\frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big) \\ &= \frac{1}{2}\Big(\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ + \hat{J}_-\hat{J}_z + \hat{J}_z\hat{J}_-\Big). \end{aligned} \tag{1.166}$$

Thus,

$$\begin{aligned} \langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x \rangle &= \frac{1}{2}\Big(\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ \rangle + \langle \hat{J}_-\hat{J}_z + \hat{J}_z\hat{J}_- \rangle\Big) \\ &= \frac{1}{2}\Big(\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ \rangle + \langle \hat{J}_z\hat{J}_+ + \hat{J}_+\hat{J}_z \rangle^\star\Big) \\ &= Re\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ \rangle, \end{aligned} \tag{1.167}$$

that is the real part of $\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ \rangle$. Similarly,

$$\begin{aligned} \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y &= \frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big)\hat{J}_z + \hat{J}_z\frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big) \\ &= \frac{1}{2i}\Big(\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ - \hat{J}_-\hat{J}_z - \hat{J}_z\hat{J}_-\Big) \end{aligned} \tag{1.168}$$

implying

$$\langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y \rangle = Im\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ \rangle, \tag{1.169}$$

that is the imaginary part of $\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle$. Therefore, we now show the calculation of $\langle \hat{J}_+\hat{J}_z\rangle$ and $\langle \hat{J}_z\hat{J}_+\rangle$ over the state $|j,\chi\rangle$.

$$\begin{aligned}
\langle j,\chi|\hat{J}_+\hat{J}_z|j,\chi\rangle &= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l {}^{2j}C_n}\,\left(\chi^\star\right)^l \chi^n\, \langle j,j-l|\hat{J}_+\hat{J}_z|j,j-n\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}} \sum_{l,n=0}^{2j} \sqrt{{}^{2j}C_l {}^{2j}C_n}\,\left(\chi^\star\right)^l \chi^n (j-n) \\
&\times\ \sqrt{n(2j-n+1)}\delta_{l,n-1} \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \sum_{l=0}^{2j} {}^{2j}C_l (2j-l)(j-l-1)|\chi|^{2l} \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \Bigg[(2j^2-2j)\sum_{l=0}^{2j} {}^{2j}C_l\ |\chi|^{2l} \\
&- (3j-1)\sum_{l=0}^{2j} {}^{2j}C_l\ l\ |\chi|^{2l} + \sum_{l=0}^{2j} {}^{2j}C_l\ l^2\ |\chi|^{2l} \Bigg] \\
&= \frac{\chi}{\left(1+|\chi|^2\right)^{2j}} \Bigg[(2j^2-2j)(1+|\chi|^2)^{2j} \\
&- (3j-1)|\chi|^2 \frac{d}{d|\chi|^2} \sum_{l=0}^{2j} {}^{2j}C_l\ |\chi|^{2l} + |\chi|^2 \frac{d}{d|\chi|^2} |\chi|^2 \frac{d}{d|\chi|^2} \sum_{l=0}^{2j} {}^{2j}C_l\ |\chi|^{2l} \Bigg] \\
&= (2j^2-2j)\chi\ -\ (3j-1)\frac{\chi|\chi|^2}{\left(1+|\chi|^2\right)^{2j}} \frac{d}{d|\chi|^2}\left(1+|\chi|^2\right)^{2j} \\
&+\ \frac{\chi|\chi|^2}{\left(1+|\chi|^2\right)^{2j}} \frac{d}{d|\chi|^2}|\chi|^2 \frac{d}{d|\chi|^2}\left(1+|\chi|^2\right)^{2j} \\
&= (2j^2-2j)\chi - 2j(3j-1)\frac{\chi|\chi|^2}{\left(1+|\chi|^2\right)} \\
&+\ 2j\frac{\chi|\chi|^2}{\left(1+|\chi|^2\right)} + +2j(2j-1)\frac{\chi|\chi|^4}{(1+|\chi|^2)^2} \\
&= \frac{2j\chi}{\left(1+|\chi|^2\right)^2}\Bigg[j(1-|\chi|^2) - 1 \Bigg].
\end{aligned} \tag{1.170}$$

Now,

$$[\hat{J}_z, \hat{J}_+] = \hat{J}_+. \tag{1.171}$$

Therefore,

$$\langle \hat{J}_z \hat{J}_+ \rangle = \langle \hat{J}_+ \rangle + \langle \hat{J}_+ \hat{J}_z \rangle. \tag{1.172}$$

Thus using Eqs. (1.115), (1.170) and (1.172) we obtain,

$$\langle j,\chi|\hat{J}_z \hat{J}_+|j,\chi\rangle = \frac{2j\chi}{\Big(1+|\chi|^2\Big)^2}\Bigg[\, j(1-|\chi|^2)+|\chi|^2 \,\Bigg]. \tag{1.173}$$

Therefore, adding Eqs. (1.170) and (1.173) we obtain

$$\langle j,\chi|\hat{J}_+ \hat{J}_z + \hat{J}_z \hat{J}_+|j,\chi\rangle = \frac{2j\chi}{\Big(1+|\chi|^2\Big)^2}\Bigg[\, 2j+|\chi|^2-2j|\chi|^2-1 \,\Bigg]. \tag{1.174}$$

Using Eq. (1.123) in the above expression we get

$$\begin{aligned}\langle j,\chi|\hat{J}_+ \hat{J}_z + \hat{J}_z \hat{J}_+|j,\chi\rangle &= 2j\frac{\tan(\theta/2)e^{i\phi}}{\Big(1+\tan^2(\theta/2)\Big)^2}\Bigg[\, 2j+\tan^2(\theta/2)-2j\tan^2(\theta/2)-1 \,\Bigg] \\ &= \frac{1}{2}\, j(2j-1)\ \sin 2\theta\ e^{i\phi}.\end{aligned} \tag{1.175}$$

Therefore, from Eqs. (1.167), (1.169) and (1.175) we obtain the correlation terms as

$$\langle j,\chi|\hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x|j,\chi\rangle = \frac{1}{2}j(2j-1)\sin 2\theta \cos\phi \tag{1.176}$$

and

$$\langle j,\chi|\hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y|j,\chi\rangle = \frac{1}{2}j(2j-1)\sin 2\theta \sin\phi. \tag{1.177}$$

Thus, after calculating all the necessary quantities to calculate $\langle \hat{J}'^2_x \rangle$ we now try to obtain its expression. Using Eqs. (1.138), (1.140), (1.143), (1.165), (1.176) and (1.177) in Eq. (1.161) we get

$$\begin{aligned}
\langle \hat{J}_x'^2 \rangle &= \frac{1}{2}j\left[1 + j\sin^2\theta + \frac{1}{2}(2j-1)\sin^2\theta\cos 2\phi - \frac{1}{2}\sin^2\theta \right]\cos^2\theta\cos^2\phi \\
&+ \left[\frac{1}{2}j + j^2\sin^2\theta\sin^2\phi - \frac{1}{2}j\sin^2\theta\sin^2\phi \right]\cos^2\theta\sin^2\phi \\
&+ \left[j^2\cos^2\theta + \frac{1}{2}j\sin^2\theta \right]\sin^2\theta + \frac{1}{2}j(2j-1)\sin^2\theta\sin 2\phi\cos^2\theta\sin\phi\cos\phi \\
&- \frac{1}{2}j(2j-1)\sin 2\theta\cos\phi\sin\theta\cos\theta\cos\phi \\
&- \frac{1}{2}j(2j-1)\sin 2\theta\sin\phi\sin\theta\cos\theta\sin\phi. \qquad (1.178)
\end{aligned}$$

After simplifying the above expression we obtain

$$\langle \hat{J}_x'^2 \rangle = \frac{1}{2}\, j. \qquad (1.179)$$

Similarly using Eqs. (1.140), (1.143) and (1.165) in Eq. (1.162) we obtain

$$\begin{aligned}
\langle \hat{J}_y'^2 \rangle &= \frac{1}{2}j\left[1 + j\sin^2\theta + \frac{1}{2}(2j-1)\sin^2\theta\cos 2\phi - \frac{1}{2}\sin^2\theta \right]\sin^2\phi \\
&+ \left[\frac{1}{2}j + j^2\sin^2\theta\sin^2\phi - \frac{1}{2}j\sin^2\theta\sin^2\phi \right]\cos^2\phi \\
&- \frac{1}{2}j(2j-1)\sin^2\theta\sin 2\phi\sin\phi\cos\phi, \qquad (1.180)
\end{aligned}$$

which after simplification reduces to

$$\langle \hat{J}_y'^2 \rangle = \frac{1}{2}\, j. \qquad (1.181)$$

Since $\langle \hat{J}_x' \rangle = \langle \hat{J}_y' \rangle = 0$ as expressed in Eqs. (1.152) and (1.153), we obtain the variances of $\hat{J}_x'$ and $\hat{J}_y'$ using Eqs. (1.157), (1.158), (1.179) and (1.181) as

$$\Delta {J_x'}^2 = \Delta {J_y'}^2 = \frac{j}{2} = \frac{|\langle \hat{\mathbf{J}} \rangle|}{2}. \qquad (1.182)$$

These are the inherent uncertainties of the system which has the same value as given in Eq. (1.103) and hence, we call the state $|j, \chi\rangle$ as a rotated minimum uncertainty state (MUS) or a coherent state. The uncertainty relation for this kind of systems is formulated as

$$\Delta J_x{}' \Delta J_y{}' \geq \frac{1}{2}|\langle \hat{\mathbf{J}}' \rangle| = \frac{1}{2}|\langle \hat{J}_z' \rangle|. \qquad (1.183)$$

It is to be noted here that, it is not necessary to align $\langle\hat{\mathbf{J}}\rangle$ always along the z-axis and it can be aligned along any direction but then we have to find out the uncertainties in such two mutually perpendicular components of $\langle\hat{\mathbf{J}}\rangle$ which lie in a plane normal to $\langle\hat{\mathbf{J}}\rangle$. As the angles θ and ϕ are varied, the mean angular momentum vector $\langle\hat{\mathbf{J}}\rangle$ traces out a sphere, known as the Bloch sphere. That is, the tip of the vector always remains on the surface of the sphere as it should since the norm of the vector does not change with rotations. Hence, the atomic coherent states are also known as Bloch states [22] and are represented by $|\theta, \phi\rangle$. These are also known as Radcliffe states [23].

It is possible to obtain an elegant expression for the atomic coherent state using Schwinger representation for angular momentum operators [21]. In this representation the angular momentum operators are constructed by defining two kinds of bosonic annihilation operators $\hat{a}_i (i = +, -)$ corresponding to two uncoupled harmonic oscillators, such that

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \tag{1.184}$$

and

$$\left[\hat{a}_i, \hat{a}_j\right] = 0 = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right]. \tag{1.185}$$

With $\hbar = 1$, the raising and lowering operators J_+ and J_- take the form

$$\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_- \tag{1.186}$$

$$\hat{J}_- = \hat{a}_-^\dagger \hat{a}_+ \tag{1.187}$$

and

$$\hat{J}_z = \frac{1}{2}\left(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-\right). \tag{1.188}$$

The atomic coherent state $|j, \chi\rangle$ take the form [details given in Apendix-II]

$$|j, \chi\rangle = \frac{1}{(1 + |\chi|^2)^j} \sum_{n=0}^{2j} \sqrt{{}^{2j}C_n}\, \chi^n \frac{\left(\hat{a}_+^\dagger\right)^{2j-n} \left(\hat{a}_-^\dagger\right)^n}{\sqrt{(2j-n)!n!}} |0_+, 0_-\rangle \tag{1.189}$$

where $|0_+, 0_-\rangle$ is the vacuum state for the two types of oscillators.

With the above definition for an atomic (spin) coherent state, we can now give a definition of atomic (spin) squeezed states.

1.3.3. Atomic (Spin) Squezed States

The squeezed state for the above system is obtained when the variances $\Delta J_{x'}$ or $\Delta J_{y'}$ achieves the value less than $\sqrt{\frac{|\langle\hat{\mathbf{J}}\rangle|}{2}}$ i.e. when

$$\Delta J_{x'} < \sqrt{\frac{|\langle\hat{\mathbf{J}}\rangle|}{2}} \tag{1.190}$$

with

$$\Delta J_{y'} > \sqrt{\frac{|\langle \hat{\mathbf{J}} \rangle|}{2}} \tag{1.191}$$

or vice versa.

A system of atoms goes to a squeezed state when it interacts with a squeezed electromagnetic field. The field establishes quantum correlation among the individual atoms and squeezes them. In fact the quantum correlation is the basis of squeezing in atomic system which is produced via a non linear interaction between the atoms and field. The above idea of spin squeezing was put forward by Kitagawa and Ueda [25]. Wineland and coworkers [26] has also formulated, but slightly differently, and called it spectroscopic squeezing. Agarwal and Puri [27] have studied spectroscopic squeezing in an atomic system interacting with a squeezed radiation field. Since, then the subject has been actively investigated which led to its experimental verification in atomic vapours [28, 29] and in Bose-Einstein condensates [32]. An ensemble of atoms in a cavity driven by a strong coherent field exhibits strong non-linearity in the form of optical bistabitity. The state of the system (consisting of atoms, cavity field and driving field) at the lower turning point of the bistable curve, produces squeezing in the atomic (spin) system [33, 34]. Spin squeezing of the lower two levels of a three-level atomic system has been predicted in systems exhibiting electromagnetically induced transparency [35]. Berman and coworkers [36] have shown that a system of mutually noninteracting atoms in a microwave cavity described by the so-called Tavis-Cummings model are spin squeezed. It has been known that quantum entanglement [38] is at the root of spin squeezing. The relationship between the two outstanding quantum effect has been formulated in Refs [39, 40]. Thus, spin squeezing can offer as a physical measure of quantum entanglement, an abstract quality so far. So, the interest in the study of spin squeezing has been growing further from the quantum information point of view [41]. The interest has also centred around finding a spin squeezing operator in the lines of squeezing operator $\hat{S}(\epsilon)$ for photons defined in Eq. (1.77).

1.4. Organization of the Following Chapters

In Chapter 2 we present our work on the squeezing aspect of the eigenstate of a pseudo-Hermitian operator with real eigenvalues. These days there is a growing interest in pseudo-Hermitian operators as they play vital role in the context of non-unitary quantum mechanics [42]. The eigenstate, which we studied is also very much significant in the context of quantum optics as it represents a collection of two-level atoms interacting with the squeezed vacuum state of the electromagnetic field. We bring out the inherent quantum uncertainty of the system represented by the above state and analyse its squeezing properties.

In Chapter 3 we develop two generic squeezing Hamiltonians, viz.,

$$\hat{H}_1 \propto \hat{J}_z^2 \tag{1.192}$$

and

$$\hat{H}_2 \propto \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x \tag{1.193}$$

having lowest power of nonlinearity with simple structure and try to present the subject of squeezing in two-level atomic systems from a general perspective.

In Chapter 4 we study the spin squeezing dynamics produced when one generic Hamiltonian, that is, $\hat{H}_2$, generates time evolution on an atomic coherent state. We first study a bipartite system as it gives analytical results and then with the knowledge gained from there we proceed to study the system of more than two atoms numerically.

In Chapter 5 we study the spin squeezing dynamics produced when the other generic Hamiltonian, that is, $\hat{H}_1$, generates time evolution on an atomic coherent state. We take our system as N two-level atoms interacting with the single mode of a dispersive cavity having high quality factor and at thermal equilibrium.

In Chapter 6 we present the conclusion of the thesis and in Chapter 7 we present the scope for further investigations.

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2. Spin Squeezing of the Eigenstate of a Pseudo-Hermitian Operator

2.1. Introduction

In quantum mechanics we associate Hermitian operators with the dynamical variables, such that the physically observable or measurable quantities can be represented by the corresponding real eigenvalues or the expectation values of the operator over the relevant quantum state of the system. However, a class of operators are known for a long time [1–6] which though non Hermitian have real eigenvalues. A lot of effort have been made to find out the condition for the reality of the eigenvalue spectrum of those operators and introduce them in quantum mechanics in order to enlarge the scope of the subject [7]. Recently the underlying mathematical structure for the reality of the eigenvalues of the non-Hermitian operators have been shown by Mostafazadeh in his two papers [8, 9]. He introduced the notion of pseudo-Hermiticity and signified it's necessity for the reality of the eigenvalues of a non-Hermitian operator.

An operator $\hat{A}$ is said to be a pseudo-Hermitian operator or more acurately an η-pseudo-Hermitian operator if

$$\hat{\eta}\hat{A}\hat{\eta}^{-1} = \hat{A}^{\dagger}, \tag{2.1}$$

where $\hat{\eta}$ is also an operator which is linear, invertible and Hermitian. The eigenvalues of a pseudo-Hermitian operator are real or come in complex conjugate pairs. Thus, the property of pseudo-Hermiticity is not sufficient for the reality of the eigenvalue spectrum of an operator.

We now present the sufficient condition for a η-pseudo-Hermitian operator with a complete set of discrete biorthonormal eigenvectors to have all the eigenvalues real. The meaning of the existence of a complete set of discrete biorthonormal eigenvectors of an operator say $\hat{A}$ is that, there exists a set of vectors $\{|\psi_n\rangle, |\phi_n\rangle\}$, such that

$$\hat{A}|\psi_n\rangle = E_n|\psi_n\rangle, \tag{2.2}$$

$$\hat{A}^{\dagger}|\phi_n\rangle = E_n^{\star}|\phi_n\rangle, \tag{2.3}$$

$$\langle\phi_m|\psi_n\rangle = \delta_{mn} \tag{2.4}$$

and

$$\sum_n |\psi_n\rangle\langle\phi_n| = 1. \tag{2.5}$$

The necessary and sufficient condition for a η-pseudo-Hermitian operator with a complete set of discrete biorthonormal eigenvectors to have only real eigenvalue spectrum is that it should be possible to express the operator $\hat{\eta}$ as $\hat{O}^\dagger\hat{O}$ where $\hat{O}$ is an invertible linear operator [8, 9]. Details are given in Apendix III.

In this chapter we deal with an eigenstate of such an operator which is of considerable interest in the study of N two-level atoms interacting with the squeezed vacuum state of the electromagnetic field [15]. We show that the above mentioned eigenstate is an atomic squeezed state. We introduce such an operator and the corresponding eigenstate in Section 2.2. In section 2.3 we express that eigenstate by introducing reduced Wigner d-matrix elements as this makes the relevant calculations simple. In section 2.4 we derive the required average values, correlations and the variances of the pseudo angular momentum operators to study the squeezing properties of the above mentioned eigenstate. In section 2.5 we introduce the proper rotation of the system (as mentioned in section 1.3 of chapter 1) represented by the eigenstate and analyse it's squeezing aspect. In section 2.6 we continue our analysis to include the properties that arise due to pseudo-Hermiticity of the operator that we study. In section 2.7 we discuss the physical significance of the above mentioned eigenstate and we conclude the chapter in Section 2.8.

2.2. Pseudo-Hermiticity of the Operator $\hat{\Lambda}$

In this Section, we introduce the pseudo-Hermitian operator

$$\hat{\Lambda} = \hat{J}_x \cosh\xi + i\hat{J}_y \sinh\xi \tag{2.6}$$

with ξ as a real parameter and construct it's eigenstates with real eigenvalues [10, 11].

The rotation of the $x-y$ plane about z-axis in the anticlockwise sense by angle θ changes the x-component of angular momentum operator i.e. $\hat{J}_x$ as

$$\hat{J}_x' = \hat{J}_x \cos\theta + \hat{J}_y \sin\theta = e^{-i\hat{J}_z\theta}\hat{J}_x e^{i\hat{J}_z\theta}. \tag{2.7}$$

If we replace θ by $i\xi$ then

$$\begin{aligned}
\hat{J}_x' &= \hat{J}_x \cos i\xi + \hat{J}_y \sin i\xi \\
&= \hat{J}_x \cosh\xi + i\hat{J}_y \sinh\xi \\
&= \hat{\Lambda} \\
&= e^{\xi\hat{J}_z}\hat{J}_x e^{-\xi\hat{J}_z}.
\end{aligned} \tag{2.8}$$

Thus the operator $\hat{\Lambda}$ is nothing but a "hyperbolically" rotated $\hat{J}_x$ viz. rotation through an imaginary angle $i\xi$ about the z-axis. We now show that $\hat{\Lambda}$ is a pseudo-Hermitian operator. We have,

$$\begin{aligned}\hat{\Lambda}^\dagger &= e^{-\xi\hat{J}_z}\hat{J}_x e^{\xi\hat{J}_z} \\ &= e^{-2\xi\hat{J}_z}e^{\xi\hat{J}_z}\hat{J}_x e^{-\xi\hat{J}_z}e^{2\xi\hat{J}_z} \\ &= e^{-2\xi\hat{J}_z}\hat{\Lambda}e^{2\xi\hat{J}_z} \\ &= \hat{\eta}\hat{\Lambda}\hat{\eta}^{-1} \end{aligned} \tag{2.9}$$

with $\hat{\eta} = e^{-2\xi\hat{J}_z}$. Thus $\hat{\Lambda}$ is η-pseudo-Hermitian . Moreover, $\hat{\eta} = e^{-2\xi\hat{J}_z} = e^{-\xi\hat{J}_z}e^{-\xi\hat{J}_z} = \hat{O}^\dagger\hat{O}$ with $\hat{O} = e^{-\xi\hat{J}_z}$. Therefore $\hat{\eta}$ is decomposable as $\hat{O}^\dagger\hat{O}$ with $\hat{O}$ as a linear and invertible operator. Thus the eigenvalues of $\hat{\Lambda}$ are real.

We now construct it's eigenstates and for that we take a ket vector as $e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle$. When $\hat{J}_x$ operates on it we have

$$\hat{J}_x e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle = e^{-i\frac{\pi}{2}\hat{J}_y}e^{i\frac{\pi}{2}\hat{J}_y}\hat{J}_x e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle. \tag{2.10}$$

Using Campbell-Baker-Hausdorf lemma [see Apendix-I] we obtain

$$\begin{aligned} e^{i\frac{\pi}{2}\hat{J}_y}\hat{J}_x e^{-i\frac{\pi}{2}\hat{J}_y} &= \hat{J}_x + \left(i\frac{\pi}{2}\right)\left[\hat{J}_y, \hat{J}_x\right] + \left(i\frac{\pi}{2}\right)^2\frac{1}{2!}\left[\hat{J}_y, \left[\hat{J}_y, \hat{J}_x\right]\right] + ... \\ &= \hat{J}_x + \frac{\pi}{2}\hat{J}_z - \left(\frac{\pi}{2}\right)^2\frac{1}{2!}\hat{J}_x - \left(\frac{\pi}{2}\right)^3\frac{1}{3!}\hat{J}_z + ... \\ &= \hat{J}_x\left[1 - \left(\frac{\pi}{2}\right)^2\frac{1}{2!} + ...\right] + \hat{J}_z\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^3\frac{1}{3!} + ...\right] \\ &= \hat{J}_x \cos\frac{\pi}{2} + \hat{J}_z \sin\frac{\pi}{2} \\ &= \hat{J}_z. \end{aligned} \tag{2.11}$$

Therefore, Eq. (2.10) becomes,

$$\begin{aligned}\hat{J}_x e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle &= e^{-i\frac{\pi}{2}\hat{J}_y}\hat{J}_z|j,m\rangle \\ &= m e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle. \end{aligned} \tag{2.12}$$

Thus the ket vector $e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle$ is an eigenvector of $\hat{J}_x$ with eigenvalue m. We now construct a state vector $|\Psi_m\rangle$ by operating on $e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle$ by $e^{\xi\hat{J}_z}$ where ξ is a real and positive parameter and by inserting a normalization constant A_m as $e^{\xi\hat{J}_z}$ not being unitary does not preserve the norm. Thus

$$|\Psi_m\rangle = A_m e^{\xi\hat{J}_z}e^{-i\frac{\pi}{2}\hat{J}_y}|jm\rangle. \tag{2.13}$$

We show below that $|\Psi_m\rangle$ is the eigenvector of $\hat{\Lambda}$ with eigenvalue m. Operating on $|\Psi_m\rangle$ by $\hat{\Lambda}$ we observe that,

$$\begin{aligned}
\hat{\Lambda}|\Psi_m\rangle &= e^{\xi\hat{J}_z}\hat{J}_x e^{-\xi\hat{J}_z} A_m e^{\xi\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle \\
&= A_m e^{\xi\hat{J}_z}\hat{J}_x e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle \\
&= m A_m e^{\xi\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle \\
&= m|\Psi_m\rangle,
\end{aligned} \tag{2.14}$$

where we have used Eq. (2.12).

Thus, $|\Psi_m\rangle$ is an eigenvector of $\hat{\Lambda}$ with eigenvalue m [15].

The biorthonormal state vector of $|\Psi_m\rangle$ is constructed by operating on $e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle$ by $e^{-\xi\hat{J}_z}$ and inserting the constant N_m as

$$|\Phi_m\rangle = N_m e^{-\xi\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y}|jm\rangle. \tag{2.15}$$

Proceeding in the same manner it can be shown that

$$\hat{\Lambda}^{\dagger}|\Phi_m\rangle = m|\Phi_m\rangle, \tag{2.16}$$

implying that $|\Phi_m\rangle$ is an eigenvector of $\hat{\Lambda}^{\dagger}$ with eigenvalue m. We can also check that by chosing $N_m = 1/A_m$, we have

$$\langle\Psi_m|\Phi_n\rangle = \delta_{mn}, \tag{2.17}$$

where δ_{mn} is the Kroneker delta symbol. Hence, the states $|\Psi_m\rangle$ and $|\Phi_m\rangle$ are biorthonormal.

For completeness of the above state vectors we see that

$$\sum_{m=-j}^{+j}|\Psi_m\rangle\langle\Phi_m| = \sum_{m=-j}^{+j} A_m \frac{1}{A_m} e^{\xi\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle\langle j,m|e^{i\frac{\pi}{2}\hat{J}_y} e^{-\xi\hat{J}_z}. \tag{2.18}$$

As

$$\sum_{m=-j}^{+j}|j,m\rangle\langle j,m| = 1, \tag{2.19}$$

hence, Eq. (2.18) reduces to

$$\sum_{m=-j}^{+j}|\Psi_m\rangle\langle\Phi_m| = 1. \tag{2.20}$$

Thus the set of state vectors $\{|\Psi_m\rangle, |\Phi_m\rangle\}$ form a discrete set of complete biorthonormal eigenvectors. Therefore, $\hat{\Lambda}$ satisfies the necessary and sufficient condition for having only real eigenvalue spectrum and hence, inspite of the fact that it is non-Hermitian, it has only real eigenvalues.

We now proceed to represent the state vector $|\Psi_m\rangle$ by introducing the reduced Wigner d-matrix elements as this form of $|\Psi_m\rangle$ makes the calculation of moments required to analyse the squeezing aspect of $|\Psi_m\rangle$ very simple.

2.3. Representation of the State $|\Psi_m\rangle$ in Terms of Reduced Wigner d-matrix Elements.

The effect of rotation about y-axis by angle β on the physical system in the state $|j,m\rangle$ is given by the application of the rotation operator $\hat{R}_y(\beta)$ as

$$|j,m\rangle \quad \longrightarrow \quad \hat{R}_y(\beta)|j,m\rangle. \tag{2.21}$$

Using Eq. (2.19) with the magnetic quantum number as m' we can write,

$$\begin{aligned}\hat{R}_y(\beta)|j,m\rangle &= \sum_{m'=-j}^{+j} |j,m'\rangle\langle j,m'|\hat{R}_y(\beta)|j,m\rangle \\ &= \sum_{m'=-j}^{+j} d^j_{m'm}(\beta)|j,m'\rangle.\end{aligned} \tag{2.22}$$

The quantity $d^j_{m'm}(\beta)$ is called the reduced Wigner d-matrix element [12] which is given as (with $\hbar = 1$)

$$\begin{aligned} d^j_{m'm}(\beta) = \langle j,m'|\hat{R}_y(\beta)|j,m\rangle = \langle j,m'|e^{-i\beta J_y}|j,m\rangle \\ = (-1)^{m'-m}\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!} \\ \times \sum_k \frac{(-1)^k(\cos\frac{\beta}{2})^{2j-2k-m'+m}(\sin\frac{\beta}{2})^{2k+m'-m}}{k!(j-m'-k)!(j+m-k)!(m'-m+k)!}.\end{aligned} \tag{2.23}$$

The properties of Wigner d-matrix elements are discussed in Apendix IV.

Now we represent the state $|\Psi_m\rangle$ using the reduced Wigner d-matrix elements.

$$\begin{aligned}|\Psi_m\rangle &= A_m e^{\xi\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle \\ &= A_m e^{\xi\hat{J}_z} \sum_{m'=-j}^{+j} |j,m'\rangle\langle j,m'|e^{-i\frac{\pi}{2}\hat{J}_y}|j,m\rangle \\ &= A_m e^{\xi\hat{J}_z} \sum_{m'=-j}^{+j} d^j_{m'm}(\frac{\pi}{2})|j,m'\rangle \\ &= A_m \sum_{m'=-j}^{+j} d^j_{m'm}(\frac{\pi}{2}) e^{\xi m'}|j,m'\rangle.\end{aligned} \tag{2.24}$$

The normalization constant A_m is obtained by using the normalization condition $\langle\Psi_m|\Psi_m\rangle = 1$ as

$$|A_m|^2 \sum_{m'=-j}^{+j} \sum_{m''=-j}^{+j} e^{\xi(m'+m'')} d^j_{m'm}(\frac{\pi}{2}) d^j_{m''m}(\frac{\pi}{2}) \langle j, m''|j, m'\rangle = 1$$

or,

$$|A_m|^2 \sum_{m'=-j}^{+j} e^{2\xi m'} d^j_{m'm}(\frac{\pi}{2}) d^j_{m'm}(\frac{\pi}{2}) = 1.$$

Using the symmetry property and addition theorem of the reduced Wigner d-matrix elements we obtain (discussed in Apendix IV)

$$|A_m|^2 \sum_{m'=-j}^{+j} e^{2\xi m'} d^j_{mm'}(-\frac{\pi}{2}) d^j_{m'm}(\frac{\pi}{2}) = 1$$

or,

$$|A_m|^2 d^j_{mm}(2i\xi) = 1$$

or,

$$|A_m|^2 = \frac{1}{d^j_{mm}(2i\xi)} = \frac{1}{\Delta}. \tag{2.25}$$

Since the overall phase factor in a state vector is not important hence, we can consider

$$A_m = \frac{1}{\sqrt{d^j_{mm}(2i\xi)}} = \frac{1}{\sqrt{\Delta}}. \tag{2.26}$$

Δ can be obtained by putting $\beta = 2i\xi$ and $m = m'$ in Eq. (2.23) and we get

$$\Delta = d^j_{mm}(2i\xi) = (j+m)!(j-m)! \sum_k \frac{(\cosh\xi)^{2j}(\tanh\xi)^{2k}}{(k!)^2(j+m-k)!(j-m-k)!}. \tag{2.27}$$

It is to be noted here that the quantity $\Delta = d^j_{mm}(2i\xi)$ is the analytic continuation of the reduced Wigner d-matrix elements for imaginary angle. We now proceed to calculate the average values, correlations and the variances of angular momentum operators over the state $|\Psi_m\rangle$ as these are necessary for determining the squeezing aspect of the system.

2.4. Moments and Correlations for a System in State $|\Psi_m\rangle$

As $|\Psi_m\rangle$ is an eigenvector of the operator $\hat{\Lambda}$ with eigenvalue m, we write

$$\hat{\Lambda}|\Psi_m\rangle = (\hat{J}_x \cosh\xi + i\hat{J}_y \sinh\xi)|\Psi_m\rangle = m|\Psi_m\rangle. \tag{2.28}$$

The dual of the above is written as

$$\langle\Psi_m|\hat{\Lambda}^\dagger = \langle\Psi_m|(\hat{J}_x \cosh\xi - i\hat{J}_y \sinh\xi) = \langle\Psi_m|m. \tag{2.29}$$

Taking the scalar product of both sides of Eq. (2.28) by $\langle\Psi_m|$ and equating the real and imaginary parts we get,

$$\langle\Psi_m|\hat{J}_x|\Psi_m\rangle = \frac{m}{\cosh\xi} \tag{2.30}$$

and

$$\langle\Psi_m|\hat{J}_y|\Psi_m\rangle = 0. \tag{2.31}$$

Taking the scalar product of Eq. (2.28) with it's dual i.e. Eq. (2.29) viz.

$$\langle\Psi_m|\hat{\Lambda}^\dagger\hat{\Lambda}|\Psi_m\rangle = m^2, \tag{2.32}$$

we obtain

$$\cosh^2\xi\langle\Psi_m|\hat{J}_x^2|\Psi_m\rangle + \sinh^2\xi\langle\Psi_m|\hat{J}_y^2|\Psi_m\rangle + i\sinh\xi\cosh\xi\langle\Psi_m|[\hat{J}_x,\hat{J}_y]|\Psi_m\rangle = m^2$$

or,

$$\cosh^2\xi\langle\Psi_m|\hat{J}_x^2|\Psi_m\rangle + \sinh^2\xi\langle\Psi_m|\hat{J}_y^2|\Psi_m\rangle - \sinh\xi\cosh\xi\langle\Psi_m|\hat{J}_z|\Psi_m\rangle = m^2. \tag{2.33}$$

Operating with $\hat{\Lambda}$ on both sides of Eq. (2.28) and then taking the scalar product with $\langle\Psi_m|$ we get

$$\langle\Psi_m|\hat{\Lambda}^2|\Psi_m\rangle = m^2. \tag{2.34}$$

More explicitly it is

$$\cosh^2\xi\langle\Psi_m|\hat{J}_x^2|\Psi_m\rangle - \sinh^2\xi\langle\Psi_m|\hat{J}_y^2|\Psi_m\rangle + i\sinh\xi\cosh\xi\langle\Psi_m|\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x|\Psi_m\rangle = m^2.$$

Equating the real and imaginary parts we obtain

$$\cosh^2\xi\langle\Psi_m|\hat{J}_x^2|\Psi_m\rangle - \sinh^2\xi\langle\Psi_m|\hat{J}_y^2|\Psi_m\rangle = m^2 \tag{2.35}$$

and

$$\langle\Psi_m|\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x|\Psi_m\rangle = 0. \tag{2.36}$$

This is one of the correlations needed to analyse the squeezing aspect of the state $|\Psi_m\rangle$. We now calculate the remaining correlations and for that we operate both sides of Eq. (2.28) by $\hat{J}_z$ from left and then take the scalar product with $\langle\Psi_m|$ and obtain

$$\langle\Psi_m|\hat{J}_z(\hat{J}_x\cosh\xi + i\hat{J}_y\sinh\xi)|\Psi_m\rangle = m\langle\Psi_m|\hat{J}_z|\Psi_m\rangle. \tag{2.37}$$

Taking complex conjugate of both sides we obtain

$$\langle\Psi_m|(\hat{J}_x\cosh\xi - i\hat{J}_y\sinh\xi)\hat{J}_z|\Psi_m\rangle = m\langle\Psi_m|\hat{J}_z|\Psi_m\rangle. \tag{2.38}$$

Adding Eqs. (2.37) and (2.38) we have

$$\cosh\xi\langle\Psi_m|\hat{J}_z\hat{J}_x+\hat{J}_x\hat{J}_z|\Psi_m\rangle+i\sinh\xi\langle\Psi_m|[\hat{J}_z,\hat{J}_y]|\Psi_m\rangle=2m\langle\Psi_m|\hat{J}_z|\Psi_m\rangle$$

or,

$$\langle\Psi_m|\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x|\Psi_m\rangle=\frac{2m}{\cosh\xi}\langle\Psi_m|\hat{J}_z|\Psi_m\rangle-\frac{\sinh\xi}{\cosh\xi}\langle\Psi_m|\hat{J}_x|\Psi_m\rangle.$$

Using Eq. (2.30) we have

$$\langle\Psi_m|\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x|\Psi_m\rangle=\frac{2m}{\cosh\xi}\langle\Psi_m|\hat{J}_z|\Psi_m\rangle-m\frac{\sinh\xi}{\cosh^2\xi}. \tag{2.39}$$

Subtracting Eq. (2.38) from Eq. (2.37) we have

$$\langle\Psi_m|[\hat{J}_z,\hat{J}_x]|\Psi_m\rangle\cosh\xi+i\sinh\xi\langle\Psi_m|\hat{J}_y\hat{J}_z+\hat{J}_z\hat{J}_y|\Psi_m\rangle=0$$

or,

$$\langle\Psi_m|\hat{J}_y|\Psi_m\rangle\cosh\xi+\sinh\xi\langle\Psi_m|\hat{J}_y\hat{J}_z+\hat{J}_z\hat{J}_y|\Psi_m\rangle=0.$$

Using Eq. (2.31) we obtain

$$\langle\Psi_m|\hat{J}_y\hat{J}_z+\hat{J}_z\hat{J}_y|\Psi_m\rangle=0. \tag{2.40}$$

To calculate $\langle\Psi_m|\hat{J}_z|\Psi_m\rangle$ we proceed as below.

$$\begin{aligned}\hat{J}_z|\Psi_m\rangle &= \hat{J}_zA_m\sum_{m'=-j}^{+j}d^j_{m'm}(\frac{\pi}{2})e^{\xi m'}|j,m'\rangle\\ &= A_m\sum_{m'=-j}^{+j}d^j_{m'm}(\frac{\pi}{2})e^{\xi m'}m'|j,m'\rangle.\end{aligned}$$

Taking scalar product by $\langle\Psi_m|$ we obtain

$$\begin{aligned}\langle\Psi_m|\hat{J}_z|\Psi_m\rangle &= |A_m|^2\sum_{m'=-j}^{+j}\sum_{m''=-j}^{+j}d^j_{m''m}(\frac{\pi}{2})d^j_{m'm}(\frac{\pi}{2})e^{\xi(m'+m''}m'\langle j,m''|j,m'\rangle\\ &= |A_m|^2\sum_{m'=-j}^{+j}d^j_{m'm}(\frac{\pi}{2})d^j_{m'm}(\frac{\pi}{2})e^{2\xi m'}m'\\ &= |A_m|^2\frac{1}{2}\frac{d}{d\xi}\sum_{m'=-j}^{+j}d^j_{mm'}(-\frac{\pi}{2})d^j_{m'm}(\frac{\pi}{2})e^{2\xi m'}\\ &= |A_m|^2\frac{1}{2}\frac{d}{d\xi}d^j_{mm}(2i\xi).\end{aligned} \tag{2.41}$$

Using Eq. (2.25) we obtain

$$\langle\Psi_m|\hat{J}_z|\Psi_m\rangle=\frac{1}{2\Delta}\frac{d\Delta}{d\xi}. \tag{2.42}$$

With this expression of $\langle \hat{J}_z \rangle$ Eq. (2.39) becomes

$$\langle \Psi_m | \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x | \Psi_m \rangle = \frac{m}{\cosh\xi} \frac{1}{\Delta} \frac{d\Delta}{d\xi} - m \frac{\sinh\xi}{\cosh^2\xi}. \tag{2.43}$$

In the same manner it can be shown that

$$\langle \Psi_m | \hat{J}_z^2 | \Psi_m \rangle = |A_m|^2 \frac{d^2}{d\xi^2} [d^j_{mm}(2i\xi)] = \frac{1}{4\Delta} \frac{d^2\Delta}{d\xi^2}. \tag{2.44}$$

Adding Eqs. (2.33) and (2.35) and using Eq. (2.42) we get

$$\langle \Psi_m | \hat{J}_x^2 | \Psi_m \rangle = \frac{m^2}{\cosh^2\xi} + \frac{1}{4\Delta} \tanh\xi \frac{d\Delta}{d\xi}. \tag{2.45}$$

Subtracting Eq. (2.35) from Eq. (2.33) and using Eq. (2.42) we obtain

$$\langle \Psi_m | \hat{J}_y^2 | \Psi_m \rangle = \frac{1}{4\Delta} \coth\xi \frac{d\Delta}{d\xi}. \tag{2.46}$$

The quantity $\frac{d\Delta}{d\xi}$ is expressed in suitable form. Differentiating once the expression of Δ as given in Eq. (2.27) we obtain

$$\frac{d\Delta}{d\xi} = \tanh\xi\Gamma \tag{2.47}$$

with

$$\Gamma = 2j\Delta + 2\frac{\eta_1}{\cosh^2\xi} \tag{2.48}$$

where η_1 is given as

$$\eta_1 = (\cosh\xi)^{2j}(j+m)!(j-m)! \sum_k \frac{(\tanh\xi)^{2k}}{k!(k+1)!(j+m-1-k)!(j-m-1-k)!}. \tag{2.49}$$

To express $\frac{d^2\Delta}{d\xi^2}$ in suitable form we use the differential equation satisfied by the rotation matrix element $D^j_{m'm}(\alpha,\beta,\gamma)$ familiar from the quantum mechanics of a symmetric top. The detail discussion has been presented in the Apendix IV and here is the final expression as

$$\frac{d^2\Delta}{d\xi^2} = 4j(j+1)\Delta - 4\frac{m^2\Delta}{\cosh^2\xi} + 2\coth 2\xi \frac{d\Delta}{d\xi}. \tag{2.50}$$

Using Eq. (2.47) this can be expressed as

$$\frac{d^2\Delta}{d\xi^2} = 4j(j+1)\Delta - 4\frac{m^2\Delta}{\cosh^2\xi} + \Gamma\frac{\cosh 2\xi}{\cosh^2\xi}. \tag{2.51}$$

Thus we have obtained all the necessary averages, correlations and variances of the angular momentum operators over $|\Psi_m\rangle$. We now proceed to analyse the squeezing aspect of the state $|\Psi_m\rangle$.

2.5. Squeezing Aspect of the State $|\Psi_m\rangle$

From Eqs. (2.30), (2.31) and (2.42) it is evident that the mean angular momentum vector $\langle \hat{\mathbf{J}} \rangle$ is not along the z-axis and lies on the $z-x$ plane making an angle say θ_1 with the z-axis. As mentioned in section 1.3 of Chapter 1, to determine whether the state $|\Psi_m\rangle$ is squeezed or not we first align the vector $\langle \hat{\mathbf{J}} \rangle$ along the z-axis by performing a rotation as below.

$$\hat{J}_x' = \hat{J}_x \cos\theta_1 - \hat{J}_z \sin\theta_1 \tag{2.52}$$

$$\hat{J}_y' = \hat{J}_y \tag{2.53}$$

$$\hat{J}_z' = \hat{J}_x \sin\theta_1 + \hat{J}_z \cos\theta_1 \tag{2.54}$$

with

$$\tan\theta_1 = \frac{\langle \hat{J}_x \rangle}{\langle \hat{J}_z \rangle}. \tag{2.55}$$

This rotation makes $\langle \hat{J}_x' \rangle = 0$ and since $\langle \hat{J}_y \rangle$ is already zero as given in Eq. (2.31), the vector $\langle \hat{\mathbf{J}} \rangle$ is now along the z-axis. We now observe the variances in ${J_x}'$ and ${J_y}'$. As $\hat{J}_y' = \hat{J}_y$, hence the variance in ${J_y}'$ is the same as J_y i.e.

$$\begin{aligned} \Delta {J_y}' &= \sqrt{\langle \hat{J}_y'^2 \rangle - \langle \hat{J}_y' \rangle^2} \\ &= \sqrt{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_y \rangle^2} \\ &= \Delta J_y. \end{aligned} \tag{2.56}$$

As $\langle \hat{J}_y \rangle = 0$ by Eq. (2.31), therefore,

$$\Delta {J_y}^2 = \langle \hat{J}_y^2 \rangle = \frac{1}{4\Delta} \coth\xi \frac{d\Delta}{d\xi} = \Delta {J_y}'^2 \tag{2.57}$$

where we have used Eq. (2.46).

On the other hand since $\langle \hat{J}_x' \rangle = 0$, the square of the variance in ${J_x}'$ is given as

$$\Delta {J_x}'^2 = \langle \hat{J}_x'^2 \rangle. \tag{2.58}$$

Therefore, using Eq. (2.52)

$$\Delta {J_x}'^2 = \langle \hat{J}_x^2 \rangle \cos^2\theta_1 + \langle \hat{J}_z^2 \rangle \sin^2\theta_1 - \langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle \sin\theta_1 \cos\theta_1. \tag{2.59}$$

Using Eqs. (2.43), (2.44), (2.45) and (2.55) etc. we obtain

$$\Delta {J_x'}^2 = \left[\frac{m^2}{\cosh^2\xi} + \frac{\tanh^2\xi}{4}\left(\frac{\Gamma}{\Delta}\right)^2\right]^{-1}\left[\left(\frac{\tanh^2\xi}{4}\right)^2\left(\frac{\Gamma}{\Delta}\right)^3 + \frac{j(j+1)m^2}{\cosh^2\xi} - \frac{m^2}{4\cosh^4\xi}\frac{\Gamma}{\Delta}\right] - \frac{m^2}{\cosh^2\xi}. \quad (2.60)$$

It is to be noted that since $\frac{\Gamma}{\Delta}$ is symmetric about $m = 0$, which can be verified from the Eqs. (2.27), (2.48) and (2.49), the quantity $\Delta {J_x'}^2$ is also symmetric about $m = 0$. For $m = \pm j$ we have

$$\Delta {J_x'}^2 = \Delta {J_y'}^2 = \frac{j}{2}. \quad (2.61)$$

We observe that they are independent of the squeeze parameter ξ and the corresponding state is in a minimum uncertainty state. However, for $|m| < j$, we show that the state $|\Psi_m\rangle$ is a squeezed state. For $\xi = 0$ we observe that

$$\Delta {J_x'}^2 = \Delta {J_y'}^2 = \frac{1}{2}\Big[j(j+1) - m^2\Big], \quad (2.62)$$

which is the same as that for the Wigner state $|j, m\rangle$. This is expected since, $\xi = 0$ represents a thermal field which do not bring any squeezing at all.

To calculate the amount of squeezing we define two parameters S and Q as

$$S = \sqrt{\frac{2}{|\langle\hat{\mathbf{J}}\rangle|}}\Delta J_x' \quad (2.63)$$

and

$$Q = \sqrt{\frac{2}{|\langle\hat{\mathbf{J}}\rangle|}}\Delta J_y'. \quad (2.64)$$

If the system has squeezing in the x'-component then the quantity S goes below 1 and if it has squeezing in the y'-component then Q goes below 1. The quantity $\langle\hat{\mathbf{J}}\rangle$ is the mean angular momentum vector as discussed in section 1.3 of Chapter 1. Using Eqs. (2.30), (2.31) and (2.42) we get the magnitude of $\langle\hat{\mathbf{J}}\rangle$ as

$$|\langle\hat{\mathbf{J}}\rangle| = \sqrt{\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2 + \langle\hat{J}_z\rangle^2} = \left[\frac{m^2}{\cosh^2\xi} + \frac{1}{4}\tanh^2\xi\left(\frac{\Gamma}{\Delta}\right)^2\right]^{\frac{1}{2}}. \quad (2.65)$$

We now analyse the squeezing in y'-component. Using Eqs. (2.57), (2.64 and (2.65) we can write

$$Q = \left[\frac{1}{\sqrt{\frac{4m^2\Delta^2}{\Gamma^2\cosh^2\xi} + \tanh^2\xi}}\right]^{\frac{1}{2}}. \quad (2.66)$$

If squeezing is to be present in the y'-component then for at least some range of ξ, Q should achieve value less than one. For this purpose the denominator in the above expression should go above one in that range of ξ. We therefore, concentrate on the denominator of the above expression which is

$$D = \left[\frac{4m^2\Delta^2}{\Gamma^2 \cosh^2 \xi} + \tanh^2 \xi \right]^{\frac{1}{4}}. \tag{2.67}$$

If $D > 1$ then $D^4 > 1$. Now

$$D^4 = \frac{4m^2\Delta^2}{\Gamma^2 \cosh^2 \xi} + \tanh^2 \xi. \tag{2.68}$$

To find out whether $D^4 > 1$ we see whether the quantity $1 - D^4$ is negative or not.

Now

$$\begin{aligned} 1 - D^4 &= 1 - \frac{4m^2\Delta^2}{\Gamma^2 \cosh^2 \xi} - \tanh^2 \xi \\ &= \frac{1}{\cosh^2 \xi} \left[1 - \frac{4m^2\Delta^2}{\Gamma^2} \right] \\ &= \frac{1}{\Gamma^2 \cosh^2 \xi} \left[\Gamma^2 - 4m^2\Delta^2 \right] \\ &= \frac{1}{\Gamma^2 \cosh^2 \xi} \left[\Gamma - 2m\Delta \right] \left[\Gamma + 2m\Delta \right]. \end{aligned} \tag{2.69}$$

We first deal with the case $m > 0$. Using Eq. (2.48) we get

$$\begin{aligned} \Gamma - 2m\Delta &= 2j\Delta + 2\frac{\eta_1}{\cosh^2 \xi} - 2m\Delta \\ &= 2(j - m)\Delta + 2\frac{\eta_1}{\cosh^2 \xi}. \end{aligned} \tag{2.70}$$

Since $m \leq j$, hence $(j - m) \geq 0$ and also the quantities Δ and η_1 are positive which is prominent from Eqs. (2.27) and (2.49). Therefore, the quantity $(\Gamma - 2m\Delta)$ is positive. The quantity $(\Gamma + 2m\Delta)$ is already positive as we have assumed $m > 0$. Therefore for $m > 0$, $1 - D^4$ is positive.

For $m < 0$ we let

$$m = -m' \tag{2.71}$$

with $m' > 0$ and hence Eq. (2.69) becomes

$$1 - D^4 = \frac{1}{\Gamma^2 \cosh^2 \xi} \left[\Gamma + 2m'\Delta \right] \left[\Gamma - 2m'\Delta \right]. \tag{2.72}$$

As $m \geq -j$, hence $m' \leq j$ and hence like the previous case $\Gamma - 2m'\Delta > 0$, implying $1 - D^4$ as positive. Therefore, for both $m > 0$ and $m < 0$ we have

$$1 - D^4 > 0 \tag{2.73}$$

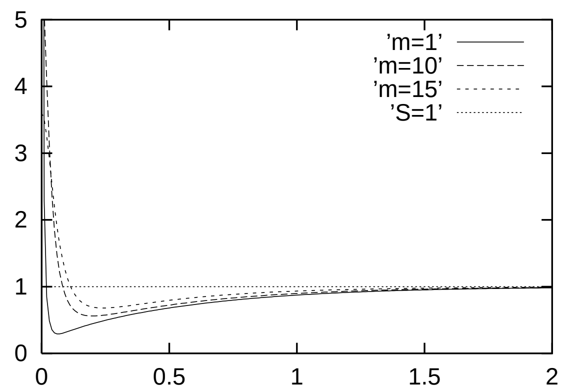

Figure 2.1: Variation of S as a function of the radiation field squeeze parameter ξ with $j = 20$. S and ξ are plotted on the vertical and horizontal axes respectively. Note that $S > 1$ for $\xi = 0$.

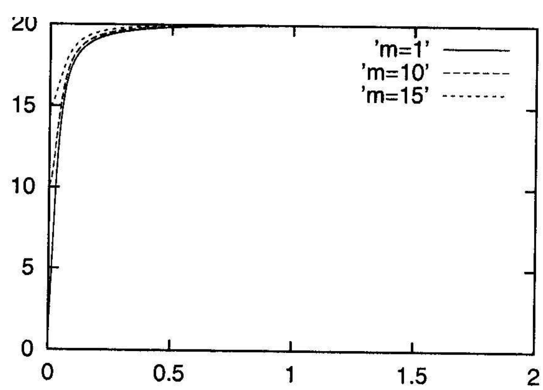

Figure 2.2: Variation of $|\langle\hat{\mathbf{J}}\rangle|$ as a function of the radiation field squeeze parameter ξ with j=20. $|\langle\hat{\mathbf{J}}\rangle|$ and ξ are plotted on the vertical and horizontal axes respectively.

and hence we have

$$Q > 1, \tag{2.74}$$

implying no squeezing in the y'-component.

To detemine whether there is any squeezing in the x'-component we plot S as a function of ξ in Figure 2.1. We notice from the figure that S has a very high value at $\xi = 0$ and then it decreases very rapidly and goes below 1 with increase in ξ. It reaches a minimum and then again increases slowly towards 1 with further increase in ξ. As $\xi \longrightarrow \infty, S \longrightarrow 1$. We have taken $j = 20$ and plotted three graphs corresponding to $m = 1, 10$ and 15. We notice that as the difference $j - |m|$ increases, the minimum value of S also decreases below 1, implying increase in squeezing. That is bigger is the difference between j and m, larger is the squeezing. The reason behind this is that the correlations among the individual atoms (spins) [13] which are responsible for producing squeezing, are proportional to $j^2 - m^2$. As the difference between j and m increases, the factor $j^2 - m^2$ increases and hence the atomic correlations increases and consequently the amount of squeezing also increases. For $m = 0$ the squeezing becomes maximum and then decreases with increase in $|m|$ and finally vanishes at $m = \pm j$.

We have calculated the correlations in primed components also which are as below:

$$\langle \hat{J}'_x \hat{J}'_y + \hat{J}'_y \hat{J}'_x \rangle = 0, \tag{2.75}$$

$$\langle \hat{J}'_y \hat{J}'_z + \hat{J}'_z \hat{J}'_y \rangle = 0 \tag{2.76}$$

and

$$\begin{aligned} \langle \hat{J}'_x \hat{J}'_z + \hat{J}'_z \hat{J}'_x \rangle &= \frac{m}{\cosh\xi} \tanh\xi \left[\frac{\Gamma}{\Delta} + \left\{ \frac{m^2}{\cosh^2\xi} \right.\right. && (2.77) \\ &+ \left. \frac{\tanh^2\xi}{4} \left(\frac{\Gamma}{\Delta} \right)^2 \right\}^{-1} \left\{ \frac{(2\cosh^2\xi - 1)}{4\cosh^2\xi} \right. && (2.78) \\ &\times \left.\left. \left(\frac{\Gamma}{\Delta} \right)^2 - j(j+1)\frac{\Gamma}{\Delta} + \frac{m^2}{\cosh^2\xi} \right\} \right]. && (2.79) \end{aligned}$$

We observe that the rotation of the physical system does not introduce any new correlation or destroy the existing one. It only changes the mathematical form. It is to be also noted that the non-zero correlations both in the primed and unprimed components vanish for $\xi = 0$.

In Figure 2.2 we plot $|\langle \hat{\mathbf{J}} \rangle|$ as a function of ξ and note that as $\xi \longrightarrow \infty$, $|\langle \hat{\mathbf{J}} \rangle| \longrightarrow 2j$. We can observe this from the expression of $|\langle \hat{\mathbf{J}} \rangle|$ given in Eq. (2.65). We see that when $\xi = 0$ $|\langle \hat{\mathbf{J}} \rangle| = m$. As ξ increases the first term in the braket in Eq. (2.65) decreases. From Eq. (2.27), (2.48) and (2.49) we find that as $\xi \longrightarrow \infty$, $\frac{\Gamma}{\Delta} \longrightarrow 2j$ and hence $|\langle \hat{\mathbf{J}} \rangle|$ tends to $2j$ as ξ tends to ∞.

We have studied the variation of the uncertainty product

$$U = \frac{2}{|\langle \hat{\mathbf{J}} \rangle|} \Delta J_x{}' \Delta J_y{}' \tag{2.80}$$

with respect to ξ and found that in the range of ξ where S is much below 1 the quantity U is greater than 1 and as ξ increases towards infinity U tends to 1.

2.6. Squeezing Aspect of the State $|\Phi_m\rangle$

As given in Eq. (2.16), $|\Phi_m\rangle$ is an eigenvector of $\hat{\Lambda}^\dagger$. It is easy to see that we obtain $\hat{\Lambda}^\dagger$ from $\hat{\Lambda}$ and $|\Phi_m\rangle$ from $|\Psi_m\rangle$ by using the transformation $\xi \to -\xi$. Therefore, the average values of the angular momentum operators and their correlations over the state $|\Phi_m\rangle$ are related to those with respect to $|\Psi_m\rangle$ by the same transformation i.e. $\xi \to -\xi$. Therefore we have

$$\begin{aligned}
\langle\Phi_m|\hat{J}_x|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_x|\Psi_m\rangle = \frac{m}{\cosh\xi}, && (2.81)\\
\langle\Phi_m|\hat{J}_y|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_y|\Psi_m\rangle = 0, && (2.82)\\
\langle\Phi_m|\hat{J}_z|\Phi_m\rangle &= -\langle\Psi_m|\hat{J}_z|\Psi_m\rangle = -\frac{1}{2\Delta}\frac{d\Delta}{d\xi} && (2.83)\\
\langle\Phi_m|\hat{J}_x^2|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_x^2|\Psi_m\rangle = \frac{m^2}{\cosh^2\xi} + \frac{1}{4\Delta}\tanh\xi\frac{d\Delta}{d\xi} && (2.84)\\
\langle\Phi_m|\hat{J}_y^2|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_y^2|\Psi_m\rangle = \frac{1}{4\Delta}\coth\xi\frac{d\Delta}{d\xi} && (2.85)\\
\langle\Phi_m|\hat{J}_z^2|\Phi_m\rangle &= -\langle\Psi_m|\hat{J}_z^2|\Psi_m\rangle && (2.86)\\
\langle\Phi_m|\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x|\Psi_m\rangle = 0 && (2.87)\\
\langle\Phi_m|\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y|\Phi_m\rangle &= \langle\Psi_m|\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y|\Psi_m\rangle = 0 && (2.88)
\end{aligned}$$

and

$$\langle\Phi_m|\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x|\Phi_m\rangle = -\langle\Psi_m|\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x|\Psi_m\rangle = -\left[\frac{m}{\cosh\xi}\frac{1}{\Delta}\frac{d\Delta}{d\xi} - m\frac{\sinh\xi}{\cosh^2\xi}\right]. \quad (2.89)$$

From Eq. (2.57) and (2.60) we see that the variances in $\hat{J}'_y$ and $\hat{J}'_x$ remain same under the transformation $\xi \to -\xi$ and hence the variances in the above operators over the state $|\Phi_m\rangle$ are

$$\Delta J_y{}^2 = \langle\hat{J}_y^2\rangle = \frac{1}{4\Delta}\coth\xi\frac{d\Delta}{d\xi} = \Delta J_y{}'^2 \quad (2.90)$$

and

$$\begin{aligned}
\Delta J_x{}'^2 &= \left[\frac{m^2}{\cosh^2\xi} + \frac{\tanh^2\xi}{4}\left(\frac{\Gamma}{\Delta}\right)^2\right]^{-1}\left[\left(\frac{\tanh^2\xi}{4}\right)^2\left(\frac{\Gamma}{\Delta}\right)^3 + \frac{j(j+1)m^2}{\cosh^2\xi} - \frac{m^2}{4\cosh^4\xi}\frac{\Gamma}{\Delta}\right]\\
&- \frac{m^2}{\cosh^2\xi} \quad (2.91)
\end{aligned}$$

respectively. As it is evident from Eq. (2.65) that the magnitude of the mean angular momentum vector also remains same under $\xi \to -\xi$ hence the squeezing aspect of the state $|\Phi_m\rangle$ is same as that of $|\Psi_m\rangle$.

2.7. Physical Significance of the State $|\Psi_m\rangle$

The dynamics of a collection of N-two level atoms interacting with a squeezed vacuum state of the electromagnetic field is given by the master equation [15]

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = & - \gamma(n+1)(\hat{J}_+\hat{J}_-\hat{\rho} - 2\hat{J}_-\hat{\rho}\hat{J}_+ + \hat{\rho}\hat{J}_+\hat{J}_-) \\ & - \gamma n(\hat{J}_-\hat{J}_+\hat{\rho} - 2\hat{J}_+\hat{\rho}\hat{J}_- + \hat{\rho}\hat{J}_-\hat{J}_+) \\ & - \gamma M(\hat{J}_+\hat{J}_+\hat{\rho} - 2\hat{J}_+\hat{\rho}\hat{J}_+ + \hat{\rho}\hat{J}_+\hat{J}_+) \\ & - \gamma M(\hat{J}_-\hat{J}_-\hat{\rho} - 2\hat{J}_-\hat{\rho}\hat{J}_- + \hat{\rho}\hat{J}_-\hat{J}_-) \end{aligned} \quad (2.92)$$

where $\hat{\rho}$ is the reduced density operator for the atomic (spin) system obtained after tracing over the squeezed vacuum. $\hat{J}_+$ and $\hat{J}_-$ are the raising and lowering operators for angular momentum states with the angular momentum quantum number $j = \frac{N}{2}$. γ is the usual atomic decay constant in the ordinary (un-squeezed vacuum). n is known as average photon number of the squeezed vacuum and is given as

$$\bar{n} = \sinh^2 r \quad (2.93)$$

where r has been assumed to be a real and positive parameter, which is called squeeze parameter for the radiation field and

$$M = \bar{n}(\bar{n}+1). \quad (2.94)$$

For $r = 0$, the atomic system interacts with the ordinary vacuum of the radiation field. Defining a non-Hermitian operator $\hat{\Lambda}$ as

$$\hat{\Lambda} = \left[\hat{J}_- \cosh r + \hat{J}_+ \sinh r\right]/\sqrt{2\sinh 2r} \quad (2.95)$$

such that

$$\hat{J}_- = \left[\hat{\Lambda}\cosh r - \hat{\Lambda}^\dagger \sinh r\right]\sqrt{2\sinh 2r} \quad (2.96)$$

the Eq. (2.92) can be casted in the form

$$\frac{d\hat{\rho}}{dt} = -2\gamma\left[\hat{\Lambda}^\dagger\hat{\Lambda}\hat{\rho} - 2\hat{\Lambda}\hat{\rho}\hat{\Lambda}^\dagger + \hat{\rho}\hat{\Lambda}^\dagger\hat{\Lambda}\right]\sinh 2r. \quad (2.97)$$

The steady state solution of this equation is

$$\hat{\rho} = D\hat{\Lambda}^{-1}(\hat{\Lambda}^\dagger)^{-1}, \quad (2.98)$$

where D is a constant. Here we have assumed that $\hat{\Lambda}^{-1}$ exists. Using the substitution

$$e^{2\xi} = \tanh r \tag{2.99}$$

(ξ is a real positive parameter) in Eq. (2.95) we obtain

$$\hat{\Lambda} = \hat{J}_x \cosh\xi + i\hat{J}_y \sinh\xi \tag{2.100}$$

which is Eq. (2.6). We have studied its properties in detail in previous sections. Since $j = \frac{N}{2}$, j is an integer when N is even and one of the value of m is zero. When m is zero the corresponding eigenvector of Λ is $|\Psi_0\rangle$ satisfying

$$\hat{\Lambda}|\Psi_0\rangle = 0|\Psi_0\rangle = 0. \tag{2.101}$$

If we let

$$\hat{\rho} = |\Psi_0\rangle\langle\Psi_0|, \tag{2.102}$$

then we see that $\hat{\rho}$ becomes one of the steady state solution of Eq. (2.97). Thus, a pure state solution of Eq. (2.97) is $|\Psi_0\rangle$. For odd number of atoms the solution of Eq. (2.97) is found from Eq. (2.98) which is as follows. Inserting the identity operator $\sum_{m=-j}^{j} |\Psi_m\rangle\langle\Phi_m| = \sum_{n=-j}^{j} |\Phi_n\rangle\langle\Psi_n| = 1$ into Eq. (2.98) we obtain

$$\hat{\rho} = D\hat{\Lambda}^{-1} \sum_{m=-j}^{+j} |\Psi_m\rangle\langle\Phi_m|(\hat{\Lambda}^\dagger)^{-1} \sum_{n=-j}^{+j} |\Phi_n\rangle\langle\Psi_n|. \tag{2.103}$$

Since

$$\hat{\Lambda}|\Psi_m\rangle = m|\Psi_m\rangle$$

we see that

$$\hat{\Lambda}^{-1}\hat{\Lambda}|\Psi_m\rangle = m\hat{\Lambda}^{-1}|\Psi_m\rangle \tag{2.104}$$

or,

$$\hat{\Lambda}^{-1}|\Psi_m\rangle = \frac{1}{m}|\Psi_m\rangle. \tag{2.105}$$

Similarly since,

$$\hat{\Lambda}^\dagger|\Phi_m\rangle = m|\Phi_m\rangle$$

we see that

$$(\hat{\Lambda}^\dagger)^{-1}|\Phi_m\rangle = \frac{1}{m}|\Phi_m\rangle. \tag{2.106}$$

Using Eq. (2.105) and (2.106) in Eq. (2.103) we obtain

$$\begin{aligned}\hat{\rho} &= D\sum_{m=-j}^{+j}\frac{1}{m}|\Psi_m\rangle\langle\Phi_m|\sum_{n=-j}^{+j}\frac{1}{n}|\Phi_n\rangle\langle\Psi_n| \\ &= D\sum_{m=-j}^{+j}\sum_{n=-j}^{+j}\frac{1}{mn}|\Psi_m\rangle\langle\Phi_m|\Phi_n\rangle\langle\Psi_n| \\ &= D\sum_{m=-j}^{+j}\sum_{n=-j}^{+j}\frac{1}{mn}\langle\Phi_m|\Phi_n\rangle|\Psi_m\rangle\langle\Psi_n|. \end{aligned} \tag{2.107}$$

Thus we see that the steady state solution of Eq. (2.97) is obtained from $|\Psi_m\rangle$. The pure state solution is $|\Psi_0\rangle$ and the mixed state solution is given by Eq. (2.107). Therefore, we say that $|\Psi_m\rangle$ represents a collection of two-level atoms interacting with a squeezed vacuum of the electromagnetic field.

2.8. Conclusion

We have studied the squeezing aspect of the atomic state $|\Psi_m\rangle$ in a proper frame in which the fundamental and inherent uncertainty of the system was extracted out. We observed that the state $|\Psi_m\rangle$ is a squeezed state.

When a system of two level atoms interact with the squeezed vacuum of the electromagnetic field, the system goes to a squeezed state and the resultant collective state vector is given by $|\Psi_m\rangle$ which is the eigenvector of a non-Hermitian operator $\hat{\Lambda} = \hat{J}_x\cosh\xi + i\hat{J}_y\sinh\xi$, having real eigenvalue spectrum. We have given the underlying mathematical reason for the reality of the eignvalue spectrum of $\hat{\Lambda}$. The relevant reason is nothing but the property of pseudo-Hermiticity satisfied by $\hat{\Lambda}$. Since the squeezing of a system of atoms is experimentaly observable now a days , we hope that the concepts of pseudo-Hermitian operators which are widely recognized recently in the context of non unitary quantum mechanics, get a connection with a real physical example from the domain of quantum optics.

We have expressed the state $|\Psi_m\rangle$ involving the reduced Wigner d-matrices for making the necessary calculations simple and as a consequence have introduced the analytical continuation of the d-matrices to imaginary angles.

Wineland et al [14] and Agarwal and Puri [15] investigated the spectroscopic squeezing of the atomic systems interacting with a squeezed radiation field by introducing a parameter R which is related to S in Eq.(2.63) by the relation

$$R = \sqrt{\frac{j}{|\langle\hat{\mathbf{J}}\rangle|}}S. \tag{2.108}$$

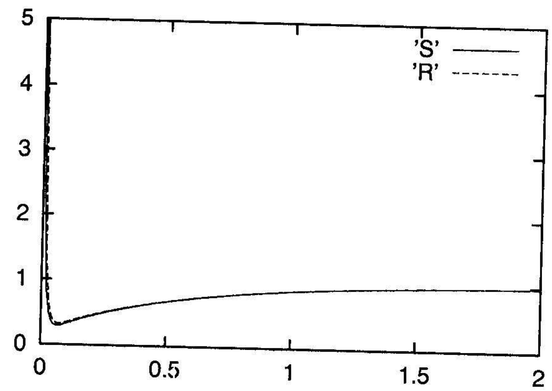

Figure 2.3: Comparison of spin squeezing S with spectroscopic squeezing R. Both S and R are plotted on the vertical axis as a function of ξ on the horizontal axis with j=20 and m=1.

We see that whenever $|\langle\hat{\mathbf{J}}\rangle| = j$, $R = S$. When $j > |\langle\hat{\mathbf{J}}\rangle|$ (which may be the case in dissipative system), the system may not show spectroscopic squeezing even if $S < 1$. We give a comparison in Figure 2.3 between R and S and observe that the difference among them is very small for the present system.

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3. A Generic Spin Squeezing Operator

3.1. Introduction

It is well known that squeezing can be achieved in an atomic system, initially in a coherent state, when the atoms are made to evolve under nonlinear optical interactions. One method of obtaining nonlinear interaction between atoms and radiation field is to use the interaction Hamiltonian, inherently nonlinear in spin operators. It is possible to construct many such nonlinear Hamiltonians which can produce squeezing, however, the interest lies in obtaining a generic spin squeezing Hamiltonian which though nonlinear, has the simplest structure and serves to study the aspects of squeezing from a general point of view.

In the previous chapter we studied a pseudo-Hermitian operator $\hat{\Lambda}$ whose eigenstates are squeezed atomic states. We pointed out that this operator appears in the case of N two level atoms interacting with the squeezed vacuum state of the radiation field. In this chapter we try to develop a generic spin squeezing Hamiltonian which though nonlinear in spin operators has the simplest structure. We proceed in analogy with the squeezing mechanism of the radiation field.

3.2. Spin Squeezing Operator

We begin with a very brief review of the squeezing aspect of the radiation field states as our quest for obtaining a generic squeezing operator for the atomic system goes in analogy with that for the radiation field.

The coherent state for the radiation field, also known as the Glauber-Sudarshan coherent state [1, 2], is given as

$$|\alpha\rangle = \hat{D}(\alpha)\,|0\rangle \tag{3.1}$$

$$= \exp(\alpha\hat{a}^\dagger - \alpha^\star\hat{a})\,|0\rangle\ , \tag{3.2}$$

where α is a complex parameter and $|0\rangle$ is the vacuum state of the radiation field. The operators $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively for the radiation field states satisfying

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{3.3}$$

and $\hat{D}(\alpha)$ is the well known displacement operator.

The squeezing operator in this case is given as [3]

$$\hat{S}(\zeta) = \exp\left[\frac{1}{2}(\zeta\hat{a}^2 - \zeta^\star\hat{a}^{\dagger 2})\right], \tag{3.4}$$

where ζ is another complex parameter. According to Yuen's representation for the squeezed state [4], the operator $\hat{S}(\zeta)$ when acts on $|\alpha\rangle$ produces a squeezed state $|STE\rangle$ (STE stands for squeezed state of the electromagnetic field) of the field as

$$|STE\rangle = \hat{S}(\zeta)\ |\alpha\rangle. \tag{3.5}$$

Our aim is to obtain a generic squeezing operator for the atomic system which can squeeze a system of atoms initialy in a coherent state in the same way as $\hat{S}(\zeta)$ squeezes $|\alpha\rangle$, though the algebra of the spin operators is quite different from that of the annihilation and creation operators for the field states.

We first express the atomic coherent state $|j,\chi\rangle$ in the analogous form as that in Eq. (3.1) for the field coherent state. That is we want to find an operator say $\hat{D}(j,\chi)$ (with χ as some complex parameter) such that $\hat{D}(j,\chi)$ acting on a vacuum state produces the atomic coherent state $|j,\chi\rangle$. For this purpose we take the help of Scwhinger representation of angular momentum operators [5]. In Schwinger representation we define two kinds of bosonic annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger (i = +,-)$ respectively, corresponding to two uncoupled harmonic oscillators, one say is of plus($+$) type and the other say is of minus($-$) type, such that

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \tag{3.6}$$

and

$$[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]. \tag{3.7}$$

If we construct operators $\hat{J}_+$ and $\hat{J}_-$ as

$$\hat{J}_+ = \hbar\hat{a}_+^\dagger\hat{a}_-, \tag{3.8}$$

$$\hat{J}_- = \hbar\hat{a}_-^\dagger\hat{a}_+ \tag{3.9}$$

and

$$\hat{J}_z = \frac{1}{2}\hbar\ \left[\hat{a}_+^\dagger\hat{a}_+ - \hat{a}_-^\dagger\hat{a}_-\right] \tag{3.10}$$

then it is found that $\hat{J}_+, \hat{J}_-$ and $\hat{J}_z$ satisfy the same commutation relations as the angular momentum operators. The atomic coherent state in this representation is written as

$$|j,\chi\rangle = \frac{1}{\sqrt{2j!}(1+|\chi|^2)^j}\sum_{n=0}^{2j} {}^{2j}C_n\chi^n(\hat{a}_+^\dagger)^n(\hat{a}_-^\dagger)^{2j-n}|0_+,0_-\rangle \tag{3.11}$$

$$= \hat{D}(j,\chi)\ \ |0_+,0_-\rangle \tag{3.12}$$

where $|0_+, 0_-\rangle$ is the vacuum (fictitious) state for the two kinds of oscillators. The details have been presented in Apendix II. We see that Eq. (3.12) looks similar to Eq. (3.1) and thus we can say that the operator $\hat{D}(j,\chi)$ for atomic coherent state is analogous to the operator $\hat{D}(\alpha)$ for the radiation field coherent state, though their interpretations are quite different from one another.

We now develop the spin squeezing Hamiltonian for the atomic system. We consider, keeping in mind the quadratic form of the bosonic operators in Eq. (3.4), the most general quadratic form of the angular momentum (pseudo-spin) operators. This has the form $\hat{J}_l\hat{J}_k$ which is a second rank tensor and can be reduced in the following manner.

$$\begin{aligned} \hat{J}_l\hat{J}_k &= \left[\frac{1}{2}(\hat{J}_l\hat{J}_k + \hat{J}_k\hat{J}_l) - \frac{1}{3}\delta_{lk}\hat{J}^2\right] \\ &+ \left[\frac{1}{2}(\hat{J}_l\hat{J}_k - \hat{J}_k\hat{J}_l)\right] + \left[\frac{1}{3}\delta_{lk}\hat{J}^2\right]. \end{aligned} \tag{3.13}$$

The first term in the first square bracket is a symmetric traceless second rank tensor and has five independent components. The first term in the second square braket is a second rank antisymmetric tensor which by virtue of the commutation relation

$$\left[\hat{J}_l, \hat{J}_k\right] = i\epsilon_{lks}\hat{J}_s \tag{3.14}$$

can be expressed in terms of a vector representation. The last term in the square brackets involving $\hat{J}^2$, is a scalar. This is tantamount to the reduction of the direct product of two vectors as

$$3 \otimes 3 = 5 \oplus 3 \oplus 1. \tag{3.15}$$

Thus the quadratic of spin operators $\hat{J}_l\hat{J}_k$ can be decomposed into the sum of three operators with the trace

$$\sum \hat{J}_l\hat{J}_k\delta_{lk} = \hat{J}^2 \tag{3.16}$$

yielding a mere overall phase, the antisymmetric tensor part just a rotation and the symmetric second rank tensor part yielding the nontrivial squeeze. Since the different components of the quadrupole tensor are related to each other by rotations, the generic squeezing Hamiltonians with very simple forms for the atomic system can be chosen, for example, the fiducial quadratic forms

$$\hat{H}_{spin} = g_1(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) = \frac{1}{2i}g_1(\hat{J}_+^2 - \hat{J}_-^2) \tag{3.17}$$

and

$$\hat{H}'_{spin} = g_2\ \hat{J}_z^2\ , \tag{3.18}$$

where g_1 and g_2 are real parameters. Thus the squeezing operators can be constructed out of these Hamiltonians as

$$\begin{aligned}\hat{U}_{spin} = \exp(-i\hat{H}_{spin}t/\hbar) &= \exp\Big[-ig_1t(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)/\hbar\Big] \\ &= \exp\Big[-g_1t(\hat{J}_+^2 - \hat{J}_-^2)/(2\hbar)\Big] \\ &= \exp\Big[\gamma(\hat{J}_+^2 - \hat{J}_-^2)\Big], \end{aligned} \tag{3.19}$$

where $\gamma = -g_1t/(2\hbar)$ and

$$\begin{aligned}\hat{U}'_{spin} = \exp(-i\hat{H}'_{spin}t/\hbar) &= \exp\Big[-ig_2t\hat{J}_z^2/\hbar\Big] \\ &= \exp\Big[-i\beta\hat{J}_z^2\Big], \end{aligned} \tag{3.20}$$

where $\beta = g_2t/\hbar$.

To keep form analogy with the squeezing operator for the radiation field in Eq. (3.4) we consider the generic spin squeezing operators for the atomic systems by slightly modifying $\hat{U}_{spin}$ in Eq. (3.19) by inserting a complex parameter η in place of the real parameter γ as

$$\hat{S}_{spin}(\eta) = \exp\Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big), \tag{3.21}$$

and

$$\hat{S}'_{spin}(\beta) = \hat{U}'_{spin} = \exp\Big[-i\beta\hat{J}_z^2\Big]. \tag{3.22}$$

Thus, if $|SSS\rangle$ represents the spin squeezed state or atomic squeezed state then

$$|SSS\rangle = \hat{S}(\eta)\hat{D}(j,\chi)|0_+,0_-\rangle \tag{3.23}$$
$$= \hat{S}(\eta)|j,\chi\rangle \tag{3.24}$$
$$\tag{3.25}$$

and also

$$|SSS\rangle = \hat{S}(\beta)\hat{D}(j,\chi)|0_+,0_-\rangle \tag{3.26}$$
$$= \hat{S}(\beta)|j,\chi\rangle. \tag{3.27}$$

This is indeed the case which is evident from the following analysis.

The possible types of quadratic Hermitian Hamiltonians which can be constructed out of the three basic spin operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ are

$$\hat{H}_1 \propto \hat{J}_x^2, \tag{3.28}$$
$$\hat{H}_2 \propto \hat{J}_y^2, \tag{3.29}$$
$$\hat{H}_3 \propto \hat{J}_z^2, \tag{3.30}$$
$$\hat{H}_4 \propto \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x, \tag{3.31}$$
$$\hat{H}_5 \propto \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y \tag{3.32}$$

and

$$\hat{H}_6 \propto \hat{J}_z\hat{J}_x + \hat{J}_x\hat{J}_z. \tag{3.33}$$

Since the spin operators $\hat{J}_x$, $\hat{J}_y$ and $\hat{J}_z$ are connected to each other by rotations as

$$e^{i\frac{\pi}{2}\hat{J}_y}\ \hat{J}_x\ e^{-i\frac{\pi}{2}\hat{J}_y} = \hat{J}_z, \tag{3.34}$$

$$e^{i\frac{\pi}{2}\hat{J}_x}\ \hat{J}_z\ e^{-i\frac{\pi}{2}\hat{J}_x} = \hat{J}_y, \tag{3.35}$$

$$e^{i\frac{\pi}{2}\hat{J}_z}\ \hat{J}_y\ e^{-i\frac{\pi}{2}\hat{J}_z} = \hat{J}_x, \tag{3.36}$$

the Hamiltonians $\hat{H}_1$, $\hat{H}_2$ and $\hat{H}_3$ are also related to each other by rotations as

$$e^{i\frac{\pi}{2}\hat{J}_y}\ \hat{H}_1\ e^{-i\frac{\pi}{2}\hat{J}_y} = \hat{H}_3, \tag{3.37}$$

$$e^{i\frac{\pi}{2}\hat{J}_x}\ \hat{H}_3\ e^{-i\frac{\pi}{2}\hat{J}_x} = \hat{H}_2 \tag{3.38}$$

and

$$e^{i\frac{\pi}{2}\hat{J}_z}\ \hat{H}_2\ e^{-i\frac{\pi}{2}\hat{J}_z} = \hat{H}_1. \tag{3.39}$$

Therefore, it is sufficient to study the dynamics produced by any one of the three Hamiltonians $\hat{H}_1$, $\hat{H}_2$ and $\hat{H}_3$. The results of the dynamics produced by any one Hamiltonian, can be used to obtain the results of the dynamics produced by the other Hamiltonians by using the rotations mentioned in Eqs. (3.34) to (3.36).

Regarding the next three remaining Hamiltonians in Eqs. (3.31), (3.32) and (3.33) we see that they are also mutually related to each other by rotations given in Eqs. (3.34), (3.35) and (3.36) as

$$e^{i\frac{\pi}{2}\hat{J}_y}\ \hat{H}_4\ e^{-i\frac{\pi}{2}\hat{J}_y} = \hat{H}_5, \tag{3.40}$$

$$e^{i\frac{\pi}{2}\hat{J}_z}\ \hat{H}_5\ e^{-i\frac{\pi}{2}\hat{J}_z} = \hat{H}_6, \tag{3.41}$$

$$e^{i\frac{\pi}{2}\hat{J}_x}\ \hat{H}_6\ e^{-i\frac{\pi}{2}\hat{J}_x} = \hat{H}_4. \tag{3.42}$$

Therefore, like the earlier case, it is sufficient to study the dynamics produced by any one out of the three Hamiltonians $\hat{H}_4$, $\hat{H}_5$ and $\hat{H}_6$.

We study the dynamics of an initially prepared coherent state under the influence of the Hamiltonian in Eq. (3.31) in the next chapter. The spin squeezing dynamics due to the Hamiltonian in Eq. (3.30) is dealt in Chapter 5.

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4. Spin Squeezing by a Hamiltonian Having the Form $\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x$

4.1. Introduction

In this chapter, we consider spin squeezing produced by the Hamiltonian

$$\begin{aligned}\hat{H}_{spin} &= g_1(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) \\ &= \frac{1}{2i}g_1(\hat{J}_+^2 - \hat{J}_-^2)\end{aligned} \tag{4.1}$$

generating time evolution on a system of N two-level atoms initially in a coherent state [1]

$$|j,\chi\rangle = \frac{1}{\Big(1+|\chi|^2\Big)^j}\sum_{n=0}^{2j}\sqrt{{}^{2j}C_n}\;\chi^n\;\;|j, m=j-n\rangle\;, \tag{4.2}$$

where χ is a complex parameter, refer [Chapter 1, section 1.3, Eq. (1.113)]. This state can be prepared by sending the atoms through a cavity whose single mode is maintained by the radiation from a laser. The time evolution operator corresponding to the above Hamiltonian in Eq. (4.1) is

$$\begin{aligned}\hat{U}_{spin} &= \exp(-i\hat{H}_{spin}t/\hbar) \\ &= \exp\Big[-g_1t(\hat{J}_+^2 - \hat{J}_-^2)/(2\hbar)\Big] \\ &= \exp\Big[\gamma(\hat{J}_+^2 - \hat{J}_-^2)\Big],\end{aligned} \tag{4.3}$$

where $\gamma = -g_1t/(2\hbar)$. To see the spin squeezing dynamics produced by the above Hamiltonian on an initial atomic coherent state, we have to apply the above operator on the state $|j,\chi\rangle$ and then analyse the squeezing aspects. But as the spin squeezing operator, considered by us in Chapter 3, corresponding to the above Hamiltonian in Eq. (4.1) is

$$\hat{S}_{spin}(\eta) = \exp\Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big), \tag{4.4}$$

we concentrate on the action of $\hat{S}_{spin}(\eta)$ on the state $|j,\chi\rangle$.

For the sake of simplicity we first consider the action of $\hat{S}_{spin}$ on a coherent state for two atoms (bipartite system) only in section 4.2. This gives analytical results and presents a deeper insight into the subject. In section 4.3 we study the case of more than two atoms numerically. The possibilities of physical realization of the operator $\hat{S}_{spin}$ has been dealt in section 4.4 and the conclusion of this chapter has been presented in section 4.5.

4.2. A Two Atom System

4.2.1. Derivation of the Expression of Coherent State for Two Atoms

The coherent state for two-atoms $(j = 1)$ is obtained by putting $j = 1$ in Eq. (4.2) which yields

$$|j = 1, \chi\rangle = \frac{1}{(1 + |\chi|^2)}\Big[|1, 1\rangle + \sqrt{2}\chi|1, 0\rangle + \chi^2|1, -1\rangle\Big]. \tag{4.5}$$

This is a linear superposition of the three Wigner states with $j = 1$ and $m = 1, 0,$ and -1 respectively. We apply the operator $\hat{S}_{spin}(\eta)$ given in Eq. $(\mathbf{??})$ on this state vector such that,

$$|1, \chi, \eta\rangle = \hat{S}(\eta)|1, \chi\rangle \tag{4.6}$$
$$= e^{[\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2]}\frac{1}{(1 + |\chi|^2)}\Big[|1, +1\rangle + \sqrt{2}\chi|1, 0\rangle + \chi^2|1, -1\rangle\Big]. \tag{4.7}$$

To evaluate this expression we have to know the action of the operator $\hat{S}(\eta)$ on the states $|1, 1\rangle, |1, 0\rangle$ and $|1, -1\rangle$. We therefore, now proceed to find out the action of $\hat{S}(\eta)$ on these states one by one. First we want to calculate

$$\hat{S}(\eta)|1, -1\rangle = e^{[\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2]}|1, -1\rangle. \tag{4.8}$$

Now,

$$e^{[\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2]}|1, -1\rangle = \Bigg[1 + (\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2) + \frac{(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2)^2}{2!} + ...\Bigg]|1, -1\rangle. \tag{4.9}$$

We can check that

$$\Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big)|1, -1\rangle = 2\eta|1, 1\rangle. \tag{4.10}$$

Using this equation we can get,

$$\Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big)^2|1, -1\rangle = \Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big)2\eta|1, 1\rangle$$
$$= -2^2|\eta|^2|1, -1\rangle. \tag{4.11}$$

Using this we can obtain,

$$\Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big)^3|1, -1\rangle = \Big(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\Big)\Big(-2^2|\eta|^2\Big)|1, -1\rangle$$
$$= -2^3|\eta|^2\eta|1, 1\rangle. \tag{4.12}$$

In the same manner, using the above equation we can obtain

$$\begin{aligned}\Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)^4|1,-1\rangle &= \Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)\Big(-2^3|\eta|^2\eta\Big)|1,1\rangle \\ &= 2^4|\eta|^4|1,-1\rangle \end{aligned} \tag{4.13}$$

and so on. Therefore, using the above equations we can evaluate the action of $\hat{S}(\eta)$ on $|1,-1\rangle$ as shown below.

$$\begin{aligned} e^{(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)}|1,-1\rangle &= \Big[1+(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)+\frac{(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)^2}{2!}+.....\Big]|1,-1\rangle \\ &= |1,-1\rangle+2\eta|1,1\rangle-\frac{2^2|\eta|^2}{(2!)}|1,-1\rangle \\ &- \frac{2^3|\eta|^2}{(3!)}\eta|1,1\rangle+\frac{2^4|\eta^4|}{4!}|1,-1\rangle+.... \\ &= \Big[1-\frac{2^2|\eta|^2}{2!}+\frac{2^4|\eta|^4}{4!}+...\Big]|1,-1\rangle \\ &+ \sqrt{\frac{\eta}{\eta^\star}}\Big[2|\eta|-\frac{2^3|\eta|^3}{3!}+...\Big]|1,1\rangle \\ &= \cos 2|\eta||1,-1\rangle+\sqrt{\frac{\eta}{\eta^\star}}\sin 2|\eta||1,1\rangle. \end{aligned} \tag{4.14}$$

In similar fashion we can now evaluate the action of the operator $\hat{S}(\eta)$ on $|1,1\rangle$. We have,

$$e^{(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)}|1,1\rangle = \Big[1+(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)+\frac{(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2)^2}{2!}+....\Big]|1,1\rangle. \tag{4.15}$$

We can check that,

$$\Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)|1,1\rangle = -2\eta^\star|1,-1\rangle. \tag{4.16}$$

Using this we can have,

$$\begin{aligned}\Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)^2|1,1\rangle &= \Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)(-2\eta^\star)|1,-1\rangle \\ &= -2^2|\eta|^2|1,1\rangle. \end{aligned} \tag{4.17}$$

Using this we have,

$$\begin{aligned}\Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)^3|1,1\rangle &= \Big(\eta\hat{J}_+^2-\eta^\star\hat{J}_-^2\Big)(-2^2|\eta|^2)|1,1\rangle \\ &= 2^3|\eta|^2\eta^\star|1,-1\rangle. \end{aligned} \tag{4.18}$$

Using this we have,

$$\begin{aligned}\left(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\right)^4|1,1\rangle &= \left(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\right)(2^3|\eta|^2\eta^\star)|1,-1\rangle \\ &= 2^4|\eta|^4|1,1\rangle \end{aligned}\tag{4.19}$$

and so on. Using these results we can now write the action of $\hat{S}(\eta)$ on $|1,1\rangle$.

$$\begin{aligned}e^{(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2)}|1,1\rangle &= \left[1 + (\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2) + \frac{(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2)^2}{2!} + ...\right]|1,1\rangle \\ &= \left[|1,1\rangle - 2\eta^\star|1,-1\rangle - \frac{2^2|\eta|^2}{2!}|1,1\rangle + \frac{2^3|\eta|^2\eta^\star}{3!}|1,-1\rangle + \frac{2^4|\eta|^4}{4!}|1,1\rangle + ...\right] \\ &= \left[1 - \frac{2^2|\eta|^2}{2!} + \frac{2^4|\eta|^4}{4!} + ...\right]|1,1\rangle \\ &+ \left[-2\eta^\star + \frac{2^3|\eta|^2\eta^\star}{3!} + ...\right]|1,-1\rangle \\ &= \cos 2|\eta||1,1\rangle - \sqrt{\frac{\eta^\star}{\eta}}\sin 2|\eta||1,-1\rangle. \end{aligned}\tag{4.20}$$

Now as

$$\left(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2\right)|1,0\rangle = 0 \tag{4.21}$$

therefore, the action of $\hat{S}(\eta)$ on $|1,0\rangle$ is as below

$$\begin{aligned}e^{(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2)}|1,0\rangle &= \left[1 + (\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2) + \frac{(\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2)^2}{2!} + ...\right]|1,0\rangle \\ &= |1,0\rangle. \end{aligned}\tag{4.22}$$

We can now add up the action of $\hat{S}(\eta)$ on the states $|1,-1\rangle, |1,0\rangle$ and $|1,1\rangle$ and thus, Eq. (4.7) evolves to

$$\begin{aligned}|1,\chi,\eta\rangle &= \hat{S}(\eta)|1,\chi\rangle \\ &= C_1|1,+1\rangle + C_2|1,0\rangle + C_3|1,-1\rangle \end{aligned}\tag{4.23}$$

where

$$C_1 = \frac{1}{(1+|\chi|^2)}\left[\cos(2|\eta|) + \sqrt{\eta/\eta^\star}\chi^2\sin(2|\eta|)\right], \tag{4.24}$$

$$C_2 = \frac{\sqrt{2}\chi}{(1+|\chi|^2)} \tag{4.25}$$

and

$$C_3 = \frac{1}{(1+|\chi|^2)}\left[\chi^2\cos(2|\eta|) - \sqrt{\eta^\star/\eta}\sin(2|\eta|)\right]. \tag{4.26}$$

That is, the state $|1,\chi,\eta\rangle$ is also a linear superposition of the three Wigner states, $|1,-1\rangle, |1,0\rangle$ and $|1,1\rangle$ with the coefficients given by the above three equations. Since the operator $\hat{S}(\eta)$ is unitary and the state $|1,\chi\rangle$ is normalized the state $|1,\chi,\eta\rangle$ is also normalized i.e.,

$$\langle 1,\chi,\eta|1,\chi,\eta\rangle = 1. \tag{4.27}$$

This implies

$$|C_1|^2 + |C_2|^2 + |C_3|^2 = 1. \tag{4.28}$$

We now proceed to calculate the moments and correlation functions of the angular momentum operators for this state $|1,\chi,\eta\rangle$.

4.2.2. Moments and Correlations for the State $|1,\chi,\eta\rangle$

For simplicity we assume that the quantities χ and η are real and due to this the expressions of the coefficients C_1, C_2 and C_3 given in Eqs. (4.24), (4.25) and (4.26) reduce to

$$\begin{aligned} C_1 &= \frac{1}{(1+\chi^2)}\Big[\cos 2\eta + \chi^2 \sin 2\eta\Big], & (4.29)\\ C_2 &= \frac{\sqrt{2}\chi}{(1+\chi^2)}, & (4.30)\\ C_3 &= \frac{1}{(1+\chi^2)}\Big[\chi^2 \cos 2\eta - \sin 2\eta\Big] & (4.31)\end{aligned}$$

and Eq. (4.28) acquires the form

$$C_1^2 + C_2^2 + C_3^2 = 1. \tag{4.32}$$

4.2.3. Calculation of $\langle 1,\chi,\eta|\hat{J}_x|1,\chi,\eta\rangle = \langle \hat{J}_x\rangle$

We know that

$$\hat{J}_x = \frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big). \tag{4.33}$$

Therefore,

$$\begin{aligned} \langle \hat{J}_x\rangle &= \frac{1}{2}\Big(\langle \hat{J}_+\rangle + \langle \hat{J}_-\rangle\Big)\\ &= \frac{1}{2}\Big(\langle \hat{J}_+\rangle + \langle \hat{J}_+\rangle^\star\Big)\\ &= Re\langle \hat{J}_+\rangle. & (4.34)\end{aligned}$$

As the coefficients C_1, C_2 and C_3 are real the quantity $\langle \hat{J}_+\rangle$ is also real and we have

$$\langle \hat{J}_x\rangle = \langle \hat{J}_+\rangle. \tag{4.35}$$

Hence, we now show the calculation of

$$\langle \hat{J}_+ \rangle = \langle 1, \chi, \eta | \hat{J}_+ | 1, \chi, \eta \rangle. \tag{4.36}$$

$$\begin{aligned}
\langle \hat{J}_+ \rangle &= \Big[C_1 \langle 1, 1| + C_2 \langle 1, 0| + C_3 \langle 1, -1| \Big] \hat{J}_+ \Big[C_1 |1, 1\rangle + C_2 |1, 0\rangle + C_3 |1, -1\rangle \Big] \\
&= \Big[C_1 \langle 1, 1| + C_2 \langle 1, 0| + C_3 \langle 1, -1| \Big] \Big[\sqrt{2} C_2 |1, 1\rangle + \sqrt{2} C_3 |1, 0\rangle \Big] \\
&= \sqrt{2} \Big(C_1 C_2 + C_2 C_3 \Big).
\end{aligned} \tag{4.37}$$

Using the expressions of C_1, C_2 and C_3 as given in Eqs. (4.29), (4.30) and (4.31) we get

$$\langle \hat{J}_+ \rangle = \frac{2\chi}{(1+\chi^2)^2} \Big[(1+\chi^2) \cos 2\eta - (1-\chi^2) \sin 2\eta \Big]. \tag{4.38}$$

Therefore,

$$\langle \hat{J}_x \rangle = \frac{2\chi}{(1+\chi^2)^2} \Big[(1+\chi^2) \cos 2\eta - (1-\chi^2) \sin 2\eta \Big]. \tag{4.39}$$

4.2.4. Calculation of $\langle 1, \chi, \eta | \hat{J}_y | 1, \chi, \eta \rangle = \langle \hat{J}_y \rangle$

We know that

$$\hat{J}_y = \frac{1}{2i} \Big(\hat{J}_+ - \hat{J}_- \Big). \tag{4.40}$$

Therefore,

$$\begin{aligned}
\langle \hat{J}_y \rangle &= \frac{1}{2i} \Big(\langle \hat{J}_+ \rangle - \langle \hat{J}_- \rangle \Big) \\
&= \frac{1}{2i} \Big(\langle \hat{J}_+ \rangle - \langle \hat{J}_+ \rangle^\star \Big) \\
&= Im \langle \hat{J}_+ \rangle.
\end{aligned} \tag{4.41}$$

We have already seen from the previous subsection that due to the choice of the coefficients C_1, C_2 and C_3 as real the quantity $\langle \hat{J}_+ \rangle$ is also real having no imaginary part and therefore,

$$\langle \hat{J}_y \rangle = 0. \tag{4.42}$$

4.2.5. Calculation of $\langle 1, \chi, \eta | \hat{J}_z | 1, \chi, \eta \rangle = \langle \hat{J}_z \rangle$

$$\begin{aligned}
\langle \hat{J}_z \rangle &= \Big[C_1 \langle 1, 1| + C_2 \langle 1, 0| + C_3 \langle 1, -1| \Big] \hat{J}_z \Big[C_1 |1, 1\rangle + C_2 |1, 0\rangle + C_3 |1, -1\rangle \Big] \\
&= \Big[C_1 \langle 1, 1| + C_2 \langle 1, 0| + C_3 \langle 1, -1| \Big] \Big[C_1 |1, 1\rangle - C_3 |1, -1\rangle \Big] \\
&= C_1^2 - C_3^2.
\end{aligned} \tag{4.43}$$

Using the expressions of C_1, C_2 and C_3 as given in Eqs. $(4.29), (4.30)$ and (4.31) we obtain

$$\langle \hat{J}_z \rangle = \frac{1}{(1+\chi^2)^2}\Big[(1-\chi^4)\cos 4\eta + 2\chi^2 \sin 4\eta\Big]. \tag{4.44}$$

4.2.6. Calculation of $\langle 1,\chi,\eta|\hat{J}_z^2|1,\chi,\eta\rangle = \langle \hat{J}_z^2\rangle$

$$\begin{aligned}
\langle \hat{J}_z^2 \rangle &= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\hat{J}_z^2\Big[C_1|1,1\rangle + C_2|1,0\rangle + C_3|1,-1\rangle\Big] \\
&= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\Big[C_1|1,1\rangle + C_3|1,-1\rangle\Big] \\
&= C_1^2 + C_3^2.
\end{aligned} \tag{4.45}$$

Using the expressions of C_1, C_2 and C_3 as given in Eqs. (4.29), (4.30) and (4.31) we obtain

$$\langle \hat{J}_z^2 \rangle = \frac{1+\chi^4}{(1+\chi^2)^2}. \tag{4.46}$$

4.2.7. Calculation of $\langle 1,\chi,\eta|\hat{J}^2|1,\chi,\eta\rangle = \langle \hat{J}^2\rangle$

$$\begin{aligned}
\langle \hat{J}^2 \rangle &= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\hat{J}^2\Big[C_1|1,1\rangle + C_2|1,0\rangle + C_3|1,-1\rangle\Big] \\
&= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\Big[2C_1|1,1\rangle + 2C_2|1,0\rangle + 2C_3|1,-1\rangle\Big] \\
&= 2\Big(C_1^2 + C_2^2 + C_3^2\Big).
\end{aligned} \tag{4.47}$$

Using Eq. (4.32) we obtain

$$\langle \hat{J}^2 \rangle = 2. \tag{4.48}$$

4.2.8. Calculation of $\langle 1,\chi,\eta|\hat{J}_x^2|1,\chi,\eta\rangle = \langle \hat{J}_x^2\rangle$

We know that

$$\begin{aligned}
\hat{J}_x^2 &= \Big(\frac{1}{2}\Big)^2\Big(\hat{J}_+ + \hat{J}_-\Big)\Big(\hat{J}_+ + \hat{J}_-\Big) \\
&= \frac{1}{4}\Big(\hat{J}_+^2 + \hat{J}_-^2 + \hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\Big).
\end{aligned} \tag{4.49}$$

Now,

$$\begin{aligned}
\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+ &= \Big(\hat{J}_x + i\hat{J}_y\Big)\Big(\hat{J}_x - i\hat{J}_y\Big) + \Big(\hat{J}_x - i\hat{J}_y\Big)\Big(\hat{J}_x + i\hat{J}_y\Big) \\
&= 2\Big(\hat{J}_x^2 + \hat{J}_y^2\Big) \\
&= 2\Big(\hat{J}^2 - \hat{J}_z^2\Big).
\end{aligned} \tag{4.50}$$

Therefore, Eq. (4.49) becomes,

$$\hat{J}_x^2 = \frac{1}{4}\Big(\hat{J}_+^2 + \hat{J}_-^2\Big) + \frac{1}{2}\Big(\hat{J}^2 - \hat{J}_z^2\Big). \tag{4.51}$$

Therefore,

$$\begin{aligned}
\langle \hat{J}_x^2\rangle &= \frac{1}{4}\Big(\langle\hat{J}_+^2\rangle + \langle\hat{J}_-^2\rangle\Big) + \frac{1}{2}\Big(\langle\hat{J}^2\rangle - \langle\hat{J}_z^2\rangle\Big)\\
&= \frac{1}{4}\Big(\langle\hat{J}_+^2\rangle + \langle\hat{J}_+^2\rangle^\star\Big) + \frac{1}{2}\Big(\langle\hat{J}^2\rangle - \langle\hat{J}_z^2\rangle\Big)\\
&= \frac{1}{2}\Big(Re\langle\hat{J}_+^2\rangle + \langle\hat{J}^2\rangle - \langle\hat{J}_z^2\rangle\Big). \qquad (4.52)
\end{aligned}$$

From Eq. (4.52) we see that to calculate $\langle\hat{J}_x^2\rangle$ we need to calculate $\langle\hat{J}_+^2\rangle$. As C_1, C_2 and C_3 are real, therefore,

$$Re\langle\hat{J}_+^2\rangle = \langle\hat{J}_+^2\rangle.$$

Hence we now show the calculation of $\langle\hat{J}_+^2\rangle$.

$$\begin{aligned}
\langle\hat{J}_+^2\rangle &= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\hat{J}_+^2\Big[C_1|1,1\rangle + C_2|1,0\rangle + C_3|1,-1\rangle\Big]\\
&= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]2C_3|1,1\rangle\\
&= 2C_1C_3. \qquad (4.53)
\end{aligned}$$

Using the expressions of C_1 and C_3 as given in Eqs. (4.29) and (4.31) we get

$$\langle\hat{J}_+^2\rangle = \frac{1}{(1+\chi^2)^2}\Big[2\chi^2\cos 4\eta + -(1-\chi^4)\sin 4\eta\Big]. \tag{4.54}$$

Using this expression of $\langle\hat{J}_+^2\rangle$ in Eq. (4.52) and using Eqs. (4.46) and (4.48) we obtain

$$\langle\hat{J}_x^2\rangle = \frac{1}{2} + \frac{1}{(1+\chi^2)^2}\Big[\chi^2 + \chi^2\cos 4\eta + \frac{1}{2}(\chi^4-1)\sin 4\eta\Big]. \tag{4.55}$$

4.2.9. Calculation of $\langle 1,\chi,\eta|\hat{J}_y^2|1,\chi,\eta\rangle = \langle\hat{J}_y^2\rangle$

We know that

$$\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \hat{J}^2. \tag{4.56}$$

Therefore,

$$\langle\hat{J}_y^2\rangle = \langle\hat{J}^2\rangle - \langle\hat{J}_x^2\rangle - \langle\hat{J}_z^2\rangle. \tag{4.57}$$

Using Eq. (4.46), (4.48) and (4.55) in the above equation we obtain

$$\langle\hat{J}_y^2\rangle = \frac{1}{2} + \frac{1}{(1+\chi^2)^2}\Big[\chi^2 - \chi^2\cos 4\eta - \frac{1}{2}(\chi^4-1)\sin 4\eta\Big]. \tag{4.58}$$

4.2.10. Calculation of $\langle 1,\chi,\eta|\hat{J}_x\hat{J}_y+\hat{J}_y\hat{J}_x|1,\chi,\eta\rangle=\langle\hat{J}_x\hat{J}_y+\hat{J}_y\hat{J}_x\rangle$

We know that

$$\begin{aligned}\hat{J}_x\hat{J}_y+\hat{J}_y\hat{J}_x &= \frac{1}{4i}\Big[\Big(\hat{J}_++\hat{J}_-\Big)\Big(\hat{J}_+-\hat{J}_-\Big)+\Big(\hat{J}_+-\hat{J}_-\Big)\Big(\hat{J}_++\hat{J}_-\Big)\Big]\\ &= \frac{1}{2i}\Big(\hat{J}_+^2-\hat{J}_-^2\Big).\end{aligned}\tag{4.59}$$

Therefore,

$$\begin{aligned}\langle\hat{J}_x\hat{J}_y+\hat{J}_y\hat{J}_x\rangle &= \frac{1}{2i}\Big(\langle\hat{J}_+^2\rangle-\langle\hat{J}_-^2\rangle\Big)\\ &= \frac{1}{2i}\Big(\langle\hat{J}_+^2\rangle-\langle\hat{J}_+^2\rangle^\star\Big)\\ &= Im\langle\hat{J}_+^2\rangle.\end{aligned}\tag{4.60}$$

As the coefficients C_1, C_2 and C_3 are real hence the quantity $\langle\hat{J}_+^2\rangle$ is also real an so its imaginary part is zero, implying

$$\langle\hat{J}_x\hat{J}_y+\hat{J}_y\hat{J}_x\rangle=0.\tag{4.61}$$

4.2.11. Calculation of $\langle 1,\chi,\eta|\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x|1,\chi,\eta\rangle=\langle\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle$

We know that

$$\begin{aligned}\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x &= \frac{1}{2}\Big[\Big(\hat{J}_++\hat{J}_-\Big)\hat{J}_z+\hat{J}_z\Big(\hat{J}_++\hat{J}_-\Big)\Big]\\ &= \frac{1}{2}\Big(\hat{J}_+\hat{J}_z+\hat{J}_z\hat{J}_++\hat{J}_-\hat{J}_z+\hat{J}_z\hat{J}_-\Big).\end{aligned}\tag{4.62}$$

Therefore,

$$\begin{aligned}\langle\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle &= \frac{1}{2}\Big(\langle\hat{J}_+\hat{J}_z\rangle+\langle\hat{J}_z\hat{J}_+\rangle+\langle\hat{J}_-\hat{J}_z\rangle+\langle\hat{J}_z\hat{J}_-\rangle\Big)\\ &= \frac{1}{2}\Big(\langle\hat{J}_+\hat{J}_z\rangle+\langle\hat{J}_z\hat{J}_+\rangle+\langle\hat{J}_z\hat{J}_+\rangle^\star+\langle\hat{J}_+\hat{J}_z\rangle^\star\Big)\\ &= Re\langle\hat{J}_+\hat{J}_z+\hat{J}_z\hat{J}_+\rangle.\end{aligned}\tag{4.63}$$

As the coefficients C_1, C_2 and C_3 are real the quantity $\langle\hat{J}_+\hat{J}_z+\hat{J}_z\hat{J}_+\rangle$ is also real and we have

$$\langle\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle=\langle\hat{J}_+\hat{J}_z+\hat{J}_z\hat{J}_+\rangle.\tag{4.64}$$

We now show the calculation of $\langle\hat{J}_+\hat{J}_z+\hat{J}_z\hat{J}_+\rangle$.

$$\begin{aligned}
\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle &= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\Big(\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\Big) \\
&\quad \Big[C_1|1,1\rangle + C_2|1,0\rangle + C_3|1,-1\rangle\Big] \\
&= \Big[C_1\langle 1,1| + C_2\langle 1,0| + C_3\langle 1,-1|\Big]\Big[-\sqrt{2}C_3|1,0\rangle + \sqrt{2}C_2|1,1\rangle\Big] \\
&= \sqrt{2}\Big(C_1C_2 - C_2C_3\Big). \qquad (4.65)
\end{aligned}$$

Using the expressions of C_1, C_2 and C_3 as given in Eqs. (4.29), (4.30) and (4.31) we obtain

$$\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle = \frac{2\chi}{(1+\chi^2)^2}\Big[(1-\chi^2)\cos 2\eta + (1+\chi^2)\sin 2\eta\Big]. \qquad (4.66)$$

Therefore,

$$\langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle = \frac{2\chi}{(1+\chi^2)^2}\Big[(1-\chi^2)\cos 2\eta + (1+\chi^2)\sin 2\eta\Big]. \qquad (4.67)$$

4.2.12. Calculation of $\langle 1,\chi,\eta|\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y|1,\chi,\eta\rangle = \langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle$

We know that

$$\begin{aligned}
\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y &= \frac{1}{2i}\Big[\Big(\hat{J}_+ - \hat{J}_-\Big)\hat{J}_z + \hat{J}_z\Big(\hat{J}_+ - \hat{J}_-\Big)\Big] \\
&= \frac{1}{2i}\Big[\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ - \hat{J}_z\hat{J}_- - \hat{J}_-\hat{J}_z\Big]. \qquad (4.68)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle &= \frac{1}{2i}\Big[\langle \hat{J}_+\hat{J}_z\rangle + \langle \hat{J}_z\hat{J}_+\rangle - \langle \hat{J}_+\hat{J}_z\rangle^\star - \langle \hat{J}_z\hat{J}_+\rangle^\star\Big] \\
&= Im\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle. \qquad (4.69)
\end{aligned}$$

We have already seen in the previous subsection that due to the choice of the coefficients C_1, C_2 and C_3 as real the quantity $\langle \hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle$ is also real having no imaginary part and therefore,

$$\langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle = 0. \qquad (4.70)$$

After deriving all the relevant moments and correlations for the state $|1,\chi,\eta\rangle$ we now proceed to analyse it's squeezing aspect.

4.2.13. Squeezing Aspect of the State $|1, \chi, \eta\rangle$

We observe from Eqs. $(4.39), (4.42)$ and (4.44) that the mean angular momentum vector

$$\langle \hat{\mathbf{J}} \rangle = \langle \hat{J}_x \rangle \mathbf{i} + \langle \hat{J}_y \rangle \mathbf{j} + \langle \hat{J}_z \rangle \mathbf{k} \tag{4.71}$$

is not along the z-axis, where $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the x, y and z axes respectively. To analyse the squeezing aspect of the state $|1, \chi, \eta\rangle$ we have to go to a proper coordinate frame in which the mean angular momentum vector $\langle \hat{\mathbf{J}} \rangle$ is along the z-axis. As mentioned earlier, we choose the direction of alignment of $\langle \hat{\mathbf{J}} \rangle$ along the z-axis by convention and to achieve this we perform a rotation as below.

$$\begin{aligned}
\hat{J}'_x &= \hat{J}_x \cos\theta_R + \hat{J}_z \sin\theta_R && (4.72)\\
\hat{J}'_y &= \hat{J}_y && (4.73)\\
\hat{J}'_z &= -\hat{J}_x \sin\theta_R + \hat{J}_z \cos\theta_R. && (4.74)
\end{aligned}$$

The rotation angle θ_R is determined from the condition

$$\langle \hat{J}'_x \rangle = 0, \tag{4.75}$$

which is necessary to ensure that $\langle \hat{\mathbf{J}}' \rangle$ is along the z-axis as we already have

$$\langle \hat{J}'_y \rangle = \langle \hat{J}_y \rangle = 0. \tag{4.76}$$

Thus, Eq. (4.75) yields

$$\tan\theta_R = -\frac{\langle \hat{J}_x \rangle}{\langle \hat{J}_z \rangle}. \tag{4.77}$$

It can be noted that under the above mentioned rotation the magnitude of the mean angular momentum vector remains invariant, that is,

$$\begin{aligned}
|\langle \hat{\mathbf{J}}' \rangle| &= \left[\langle \hat{J}'_x \rangle^2 + \langle \hat{J}'_y \rangle^2 + \langle \hat{J}'_z \rangle^2 \right]^{1/2} \\
&= \Big[\langle \hat{J}_x \rangle^2 \cos^2\theta_R + \langle \hat{J}_z \rangle^2 \sin^2\theta_R + \langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle \cos\theta_R \sin\theta_R \\
&+ \langle \hat{J}_y \rangle^2 + \langle \hat{J}_x \rangle^2 \sin^2\theta_R + \langle \hat{J}_z \rangle^2 \cos^2\theta_R - \langle \hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \rangle \cos\theta_R \sin\theta_R \Big]^{1/2} \\
&= \left[\langle \hat{J}_x \rangle^2 + \langle \hat{J}_y \rangle^2 + \langle \hat{J}_z \rangle^2 \right]^{1/2} \\
&= |\langle \hat{\mathbf{J}} \rangle|. && (4.78)
\end{aligned}$$

This is obvious as a rotation does not change the magnitude of a vector. We now proceed to calculate the variances $(\Delta J'_x)^2$ and $(\Delta J'_y)^2$. As

$$(\Delta J'_x)^2 = \langle \hat{J}'^2_x \rangle - \langle \hat{J}'_x \rangle^2, \tag{4.79}$$

we, therefore, show the calculation of $\langle \hat{J}'^2_x \rangle$ and $\langle \hat{J}'_x \rangle^2$ one after the other. From Eq. (4.72) we get

$$\hat{J}'^2_x = \hat{J}^2_x \cos^2\theta_R + \hat{J}^2_z \sin^2\theta_R + (\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)\sin\theta_R\cos\theta_R. \tag{4.80}$$

Therefore,

$$\langle \hat{J}'^2_x \rangle = \langle \hat{J}^2_x \rangle \cos^2\theta_R + \langle \hat{J}^2_z \rangle \sin^2\theta_R + \langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x \rangle \sin\theta_R\cos\theta_R. \tag{4.81}$$

From Eq. (4.77) we have

$$\cos\theta_R = \langle \hat{J}_z \rangle / \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2} \tag{4.82}$$

and

$$\sin\theta_R = -\langle \hat{J}_x \rangle / \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2}. \tag{4.83}$$

As $\langle J_y \rangle = 0$, we obtain

$$|\langle \hat{\mathbf{J}} \rangle| = \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2} \tag{4.84}$$

and therefore,

$$\begin{aligned} \cos\theta_R &= \langle \hat{J}_z \rangle / \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2} \\ &= \langle \hat{J}_z \rangle / |\langle \hat{\mathbf{J}} \rangle|. \end{aligned} \tag{4.85}$$

Similarly

$$\begin{aligned} \sin\theta_R &= -\langle \hat{J}_x \rangle / \sqrt{\langle \hat{J}_x \rangle^2 + \langle \hat{J}_z \rangle^2} \\ &= -\langle \hat{J}_x \rangle / |\langle \hat{\mathbf{J}} \rangle|. \end{aligned} \tag{4.86}$$

Using Eqs. (4.44), (4.55) and (4.85) we obtain

$$\begin{aligned} \langle \hat{J}^2_x \rangle \cos^2\theta_R &= \left[\frac{1}{2} + \frac{1}{(1+\chi^2)^2} \left\{ \chi^2 + \chi^2\cos 4\eta + \frac{1}{2}(\chi^4 - 1)\sin 4\eta \right\} \right] \\ &\times \frac{1}{|\langle \hat{\mathbf{J}} \rangle|^2 (1+\chi^2)^4} \left[(1-\chi^4)\cos 4\eta + 2\chi^2 \sin 4\eta \right]^2, \end{aligned} \tag{4.87}$$

which after simplification yields

$$\begin{aligned}\langle \hat{J}_x^2\rangle \cos^2\theta_R &= \frac{1}{2|\langle \hat{\mathbf{J}}\rangle|^2(1+\chi^2)^4}\Bigg[4\chi^4+(1-6\chi^4+\chi^8)\cos^2 4\eta+2\chi^2(1-\chi^4)\sin 8\eta\Bigg]\\ &+ \frac{1}{|\langle \hat{\mathbf{J}}\rangle|^2(1+\chi^2)^6}\Bigg[4\chi^6-\chi^2(1-8\chi^4+\chi^8)\cos 4\eta\\ &- \frac{1}{2}(1-11\chi^4+11\chi^8-\chi^{12})\sin 4\eta+(\chi^2-6\chi^6+\chi^{10})\cos^2 4\eta\\ &+ 2\chi^2(1-5\chi^4+\chi^8)\cos^3 4\eta+\frac{1}{2}(1-15\chi^4+15\chi^8-\chi^{12})\sin^3 4\eta\\ &+ 2\chi^4(1-\chi^4)\sin 8\eta\Bigg]. \end{aligned} \tag{4.88}$$

Similarly Eqs. (4.39), (4.46) and (4.86) give

$$\langle \hat{J}_z^2\rangle \sin^2\theta_R = \frac{1+\chi^4}{(1+\chi^2)^2}\times\frac{4\chi^2}{|\langle \hat{\mathbf{J}}\rangle|^2(1+\chi^2)^4}\Big[(1+\chi^2)\cos 2\eta-(1-\chi^2)\sin 2\eta\Big]^2, \tag{4.89}$$

which after simplification yields

$$\begin{aligned}\langle \hat{J}_z^2\rangle \sin^2\theta_R &= \frac{1}{|\langle \hat{\mathbf{J}}\rangle|^2(1+\chi^2)^6}\Bigg[4\chi^2(1+\chi^4)^2+8\chi^4(1+\chi^4)\cos 4\eta\\ &+ 4\chi^2(\chi^8-1)\sin 4\eta\Bigg]. \end{aligned} \tag{4.90}$$

Now the third term on the right hand side of Eq. (4.81) is $\langle \hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle \sin\theta_R\cos\theta_R$. We, therefore, now show the calculation of the above quantity. Using Eqs. (4.67), (4.39), (4.44), (4.85) and (4.86) we obtain

$$\begin{aligned}\langle \hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle \sin\theta_R\cos\theta_R &= \frac{2\chi}{(1+\chi^2)^2}\Bigg[(1-\chi^2)\cos 2\eta+(1+\chi^2)\sin 2\eta\Bigg]\\ &\times \frac{2\chi}{|\langle \hat{\mathbf{J}}\rangle|(1+\chi^2)^2}\Bigg[(1+\chi^2)\cos 2\eta+(\chi^2-1)\sin 2\eta\Bigg]\\ &\times \frac{1}{|\langle \hat{\mathbf{J}}\rangle|(1+\chi^2)^2}\Bigg[(1-\chi^4)\cos 4\eta+2\chi^2\sin 4\eta\Bigg]. \end{aligned} \tag{4.91}$$

Simplifying we get

$$\begin{aligned}\langle \hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x\rangle \sin\theta_R\cos\theta_R &= \frac{1}{|\langle \hat{\mathbf{J}}\rangle|^2(1+\chi^2)^6}\Bigg[16\chi^6+4\chi^2(1-6\chi^4+\chi^8)\cos^2 4\eta\\ &+ 8\chi^4(1-\chi^4)\sin 8\eta\Bigg]. \end{aligned} \tag{4.92}$$

We now add the expressions of $\langle \hat{J}_x^2\rangle \cos^2\theta_R$, $\langle \hat{J}_z^2\rangle \sin^2\theta_R$ and $\langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle \sin\theta_R\cos\theta_R$ and obtain the expression of $(\Delta J'_x)^2$ which after simplification has the form

$$\begin{aligned}(\Delta J_{x'})^2 \;=\; & \frac{1}{|\langle\hat{\mathbf{J}}\rangle|^2}\Bigg[\frac{1}{2(1+\chi^2)^4}\Bigg\{(\chi^8-2\chi^4+1)\cos^2(4\eta)- \\ & 2\chi^2(\chi^4-1)\sin(8\eta)+4\chi^4\sin^2(4\eta)\Bigg\}+ \\ & \frac{1}{(1+\chi^2)^6}\Bigg\{(3\chi^{10}-10\chi^6+3\chi^2)\cos^3(4\eta)- \\ & 3(\chi^{10}-2\chi^6+\chi^2)\cos^2(4\eta)-2(\chi^{10}-4\chi^8- \\ & 4\chi^6-4\chi^4+\chi^2)\cos(4\eta)-\frac{1}{2}(\chi^{12}-15\chi^8+ \\ & 15\chi^4-1)\sin^3(4\eta)-12\chi^6\sin^2(4\eta)+\frac{1}{2}(\chi^{12}+ \\ & 8\chi^{10}-11\chi^8+11\chi^4-8\chi^2-1)\sin(4\eta)+6\chi^4\times \\ & (\chi^4-1)\sin(8\eta)+4\chi^{10}+8\chi^6+4\chi^2\Bigg\}\Bigg].\end{aligned} \quad (4.93)$$

The presence of $|\langle\hat{\mathbf{J}}\rangle|^2$ in the denominator of the above equation is due to the expression of $\cos\theta_R$ and $\sin\theta_R$ as given in Eqs. (4.85) and (4.86) respectively.

To calculate $\Delta {J'_y}^2$ we note from Eq. (4.73) that $\hat{J}'_y = \hat{J}_y$ and hence we have,

$$(\Delta J'_y)^2 = \langle \hat{J}_y^2\rangle - \langle \hat{J}_y\rangle^2. \quad (4.94)$$

As $\langle \hat{J}_y\rangle = 0$ therefore, we have

$$(\Delta J'_y)^2 = \langle \hat{J}_y^2\rangle. \quad (4.95)$$

Using Eqs. (4.58) and (4.94) we get,

$$\begin{aligned}(\Delta J'_y)^2 \;=\; & \frac{1}{2}\Bigg[1+\frac{1}{(1+\chi^2)^2}\Bigg\{2\chi^2(1-\cos(4\eta)) \\ + \; & (1-\chi^4)\sin(4\eta)\Bigg\}\Bigg].\end{aligned} \quad (4.96)$$

To derive the expression of $|\langle\hat{\mathbf{J}}\rangle|^2$ we use Eqs. (4.39), (4.42) and (4.44) and get,

$$\begin{aligned}|\langle\hat{\mathbf{J}}\rangle|^2 \;&=\; \langle \hat{J}_x\rangle^2+\langle \hat{J}_y\rangle^2+\langle \hat{J}_z\rangle^2 \\ &=\; \left[\frac{2\chi}{(1+\chi^2)^2}\Big\{(1+\chi^2)\cos 2\eta-(1-\chi^2)\sin 2\eta\Big\}\right]^2 \\ &+\; \left[\frac{1}{(1+\chi^2)^2}\Big\{(1-\chi^4)\cos 4\eta+2\chi^2\sin 4\eta\Big\}\right]^2,\end{aligned} \quad (4.97)$$

which after simplification yields,

$$|\langle\hat{\mathbf{J}}\rangle|^2 = \frac{1}{(1+\chi^2)^4}\left[4\chi^2\left\{1+\chi^4+2\chi^2\cos(4\eta)-(1-\chi^4)\sin(4\eta)\right\}\right.$$
$$\left. + (1-\chi^4)^2\cos^2(4\eta)+4\chi^4\sin^2(4\eta)-2\chi^2(\chi^4-1)\sin(8\eta)\right]. \quad (4.98)$$

We can verify that for $\eta = 0$ the results reduce to those for an atomic coherent state.

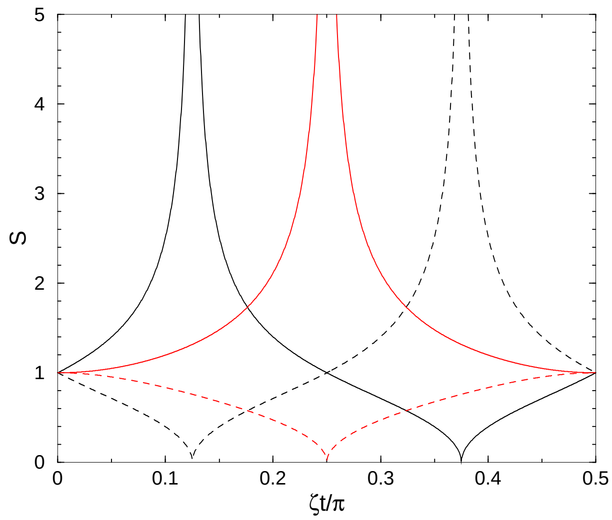

FIGURE 4.1: Variation of S_x and S_y with respect to $\eta = \zeta t$ in units of π. S_x and S_y are plotted on the vertical axis and $\zeta t/\pi$ is plotted on the horizontal axis. The solid and broken lines represent S_x and S_y respectively. The black and red curves are for $\theta = 0(\chi = 0)$ and $\theta = \pi/2(\chi = 1)$ respectively. Note that for $\theta = \pi/2$, curves are same as for $\theta = 0$ but the S_x and S_y are interchanged.

To visualize the amount of squeezing for this system we define two parameters

$$S_x = \sqrt{\frac{2}{|\langle\hat{\mathbf{J}}\rangle|}}\Delta J_x{}' \quad (4.99)$$

and

$$S_y = \sqrt{\frac{2}{|\langle\hat{\mathbf{J}}\rangle|}}\Delta J_y{}' \quad (4.100)$$

and plot these with respect to $\eta = \zeta t$ in Figure 4.1. In this figure S_x and S_y are plotted numerically for $\theta = 0$ and $\theta = \pi/2$ where

$$\chi = \tan(\theta/2) \quad (4.101)$$

as discussed in Chapter 1.

We notice that both S_x and S_y goes much below 1 in alternate fashion implying squeezing in the system. We observe from Eqs. (4.93) and (4.96) that the arguments of all sine and cosine functions appearing there are in the form of 4η and 8η. Therefore, the variances repeat themselve with a period of $\eta = \frac{\pi}{2}$ and they return to their initial condition, that is a coherent state, for $\eta = n\frac{\pi}{2}$, with n as an integer.

4.3. System With More Than Two Atoms $(j > 1)$

As we increase the number of atoms in the system from two to three and so on, the analytical calculations of the moments and correlations get more and more cumbersome and for a general value of j the problem becomes too complicated to study it analytically. However, we can tackle the problem somewhat numerically.

We are concerned with the state

$$|SSS\rangle = \hat{S}(\eta)|j,\chi\rangle \tag{4.102}$$

with $j > 1$ and study it's squeezing aspect. For that we need to calculate various moments and correlations over this state. Now quantum mechanical average of any operator say $\hat{O}$ over the state $|SSS\rangle$, is given by

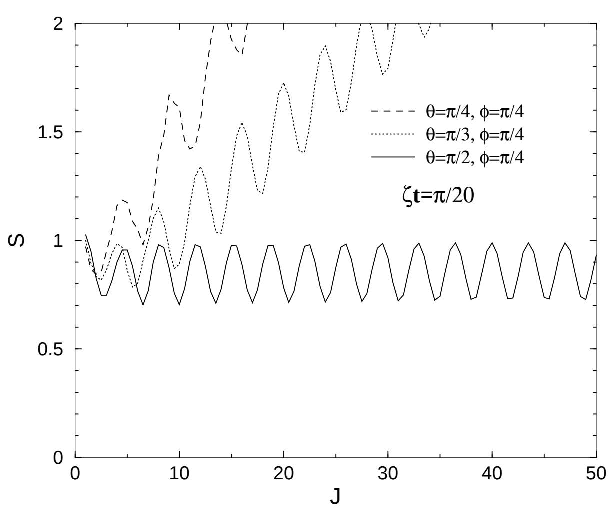

Figure 4.2: Variation of S_y as a function of j. S_y is plotted on the vertical axis and j is plotted on the horizontal axis. S_y oscillates with j below the line $S_y = 1$ indicating squeezing in the y' quadrature. Note that there is no squeezing in the other quadrature for such interaction times.

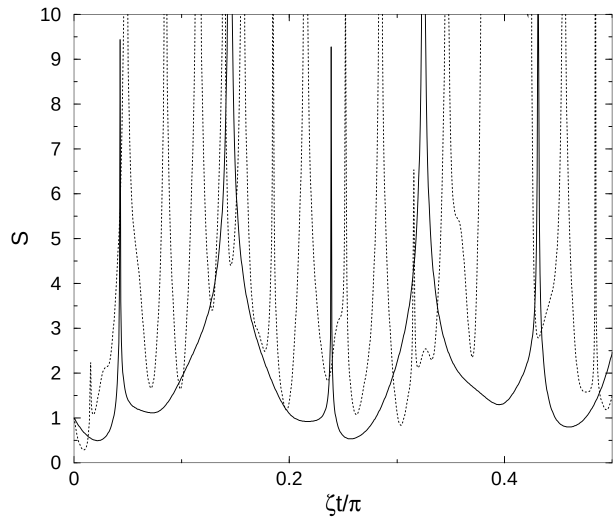

Figure 4.3: Variation of S_x as a funtion of $\zeta t/\pi$ and for $\chi = 0$. S_x and $\zeta t/\pi$ are plotted on the vertical and the horizontal axes respectively. S_x goes below 1 indicating squeezing in x' quadrature. The solid and dotted lines are for $j = 4$ and 15 respectively.

$$\langle \hat{O} \rangle = \langle SSS|\hat{O}|SSS\rangle = \sum_{m'=-j}^{j} \sum_{m''=-j}^{j} \langle SSS|j, m'\rangle \times \langle j, m'|\hat{O}|j, m''\rangle \langle j, m''|SSS\rangle. \tag{4.103}$$

The inner product $\langle j, m^{|}SSS\rangle$ is given by

$$\begin{aligned} \langle j, m|SSS\rangle &= \langle j, m|\hat{S}(\eta)|j, \chi\rangle \\ &= \sum_k \langle j, m|SSS_k\rangle \langle SSS_k|j, \chi\rangle e^{-i\lambda_k t} \end{aligned} \tag{4.104}$$

where we have used

$$\hat{S}(\eta) = e^{-i\hat{H}t} \tag{4.105}$$

with $\hat{H}$ given by Eq. (4.1). The $|SSS_k\rangle$ are the eigenvectors of $\hat{H}$ corresponding to eigenvalue λ_k.

In Figure 4.2 we show the variation of S_y with j for short interaction time, that is, $\eta = \zeta t = \pi/20$. The interesting feature is that, for $\theta = \pi/2 (\chi = 1)$, S_y oscillates just below the line $S_y = 1$ indicating squeezing in the y' quadrature for all values of j. Thus the operator $\hat{S}(\eta)$, operating on $|j, \chi\rangle$ produces squeezing of a large number of atoms which are prepared initially in a coherent state $|j, \chi\rangle$.

The distribution function for the initial condition $\theta = \pi/2(\chi = 1)$ takes the form

$$P(j,m) = |\langle j,m|j,\chi = 1\rangle| = \frac{1}{2^{2j}}\frac{(2j)!}{(j+m)!(j-m)!}. \tag{4.106}$$

It can be easily shown that $P(j,m)$ peaks at $m = 0$ by using the expression of digamma function, the derivative of the factorial function. The state $|j,m = 0\rangle$ has the property of maximum correlation among individual spins [2] which is exploited by the operator $\hat{S}_{spin}$ to squeeze out noise displayed in Figure 1. This type of behaviour has also been noticed in the spin squeezing properties of the eigenstate of a pseudo-Hermitian operator discussed in Chapter 2. This interesting property persists for long interaction time also. The only difference is that, for this case S_y oscillates more vigorously as a function of j, but, still below the line $S_y = 1$, implying squeezing.

In Figure 4.3 we show the variation of S_x as a function of $\zeta t/\pi$. We observe that S_x goes below 1, indicating squeezing in x' quadrature.

4.4. Possibilities of Physical Realization of the Hamiltoian $\hat{H}_{spin}(g_1) = g_1\left(\hat{J}_+^2 - \hat{J}_-^2\right)$

The quadratic forms of spin operators have been discussed in the literature for quite sometime. A widely known Hamiltonian studied in nuclear physics is the Lipkin-Meshkov and Glick (LMG) Hamiltonian [3] given as

$$\hat{H}_{LMG} = G_1(\hat{J}_+^2 + \hat{J}_-^2) + G_2(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)\ , \tag{4.107}$$

where G_1 and G_2 are the coupling constants representing the two body interactions. We see that by a suitable choice of the constants G_1 and G_2 the LMG-Hamiltonian is very close to the generic spin squeezing Hamiltonian $\hat{H}_{spin}$ for an atomic system.

The $\hat{H}_{spin}$ also appears in the Hamiltonian of a complex magnetic molecule in a static magnetic field [4, 5].

4.5. Conclusion

We have shown theoretically the squeezing of an atomic coherent state by using a Hamiltonian which is quadratic in spin operators. We took the squeezing operator as $\hat{S}(\eta) = e^{\eta\hat{J}_+^2 - \eta^\star\hat{J}_-^2}$ to keep analogy with the squeezing operator for the electromagnetic field, that is $\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta\hat{a}^2 - \zeta^\star\hat{a}^{\dagger 2})}$. We note that by choosing η as real the operator $\hat{S}(\eta)$ becomes $\hat{S}(\eta) = e^{\eta(\hat{J}_+^2 - \hat{J}_-^2)}$. We performed our study on a two atom (bipartite) system analytically. We operated $\hat{S}(\eta)$ (with η real) on the atomic coherent state for $j = 1$ and calculated the relevant moments and correlations. The presence of correlation among the individual atoms is necessary for the existence of squeezing [6]. We found that one of the correlations is non-zero for the bipartite system and due to that the system was found in a squeezed state. The case of N atoms was dealt numerically and in that case also we observed squeezing by significant amount.

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5. Squeezing of an Atomic Coherent State with the Hamiltonain Quadratic in Population Inversion Operator

5.1. Introduction

In this chapter we present our work on squeezing of an atomic coherent state when it evolves under the action of a Hamiltonian proportional to the quadratic of the population inversion operator $\hat{J}_z$ that is,

$$\hat{H} \propto \hat{J}_z^2. \tag{5.1}$$

This is one of the generic spin squeezing Hamiltonians developed in Chapter 3. Here, in particular, we shall consider $\hat{H}$ as the effective Hamiltonian $\hat{H}_{eff}$ governing the interaction of two level atoms with the radiation field in a dispersive cavity where $\hat{H}_{eff}$ has the form

$$\hat{H}_{eff} = \Delta_0\Big(\hat{J}^2 - \hat{J}_z^2\Big). \tag{5.2}$$

The physical significance of $\hat{H}_{eff}$, in detail, has been discussed in section 5.3. Since $\hat{J}^2$ is invariant under rotation and $\hat{J}_z^2$ is related to the other two quadratic forms of spin operators, namely, $\hat{J}_x^2$ and $\hat{J}_y^2$ by rotations, as discussed in Chapter 3, section 3.2, the study of the dynamics of $\hat{H}_{eff}$ alongwith the investigation of Chapter 4 exhaust the quadratic forms of spin operators for the production of spin squeezing.

5.2. Squeezing of an Atomic Coherent State

5.2.1. The Initial Condition

We assume a system of N two-level atoms put in a coherent state before [4] they enter the cavity. The state vector of the atomic system is given as

$$|j, \chi\rangle = \frac{1}{(1+|\chi|^2)^j} \sum_{n=0}^{2j} \sqrt{{}^{2j}C_n}\,\chi^n |j, j-n\rangle. \tag{5.3}$$

This state has already been introduced in Section 1.3 of Chapter 1 and is called an atomic coherent state. If we put

$$\chi = e^{i\phi}\tan(\theta/2) \tag{5.4}$$

then we know from Section 1.3 of Chapter 1 that

$$\begin{aligned} \langle j,\chi|\hat{J}_x|j,\chi\rangle &= j\sin\theta\cos\phi, & (5.5)\\ \langle j,\chi|\hat{J}_y|j,\chi\rangle &= j\sin\theta\sin\phi & (5.6)\end{aligned}$$

and

$$\langle j,\chi|\hat{J}_z|j,\chi\rangle = j\cos\theta. \tag{5.7}$$

θ and ϕ are the polar and azimuthal angles respectively, made by the mean angular momentum vector

$$\langle\hat{\mathbf{J}}\rangle = \langle\hat{J}_x\rangle\mathbf{i} + \langle\hat{J}_y\rangle\mathbf{j} + \langle\hat{J}_z\rangle\mathbf{k} \tag{5.8}$$

with the right handed rectangular cartesian coordinate axes, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors along the x, y and z axes respectively. The number of atoms N is related to j as $j = \frac{N}{2}$. For $\chi = 0$ we see that the state vector $|j,\chi\rangle$ reduces to $|j,+j\rangle$. This means that for $\chi = 0$ all the atoms are in the upper state $|j,+j\rangle$. For the sake of simplicity we assume ϕ to be zero. We achieve the state $|\theta,0\rangle$ first by sending the atoms in their upper states through an auxillary cavity. The duration of the atom field interaction decides the angle θ. These atoms then are made to enter the cavity where their evolution is governed by the Hamiltonian proportional to $\hat{O}$ in Eq. (5.2).

5.2.2. Derivation of Moments and Correlation Functions

We now investigate the squeezing aspect of the resulting state when the atomic coherent state represented in Eq. (5.3) evolves with respect to time under the action of the Hamiltonian $\hat{H}_{eff}$. We write the time evolved resultant state (with $\hbar = 1$) as

$$|j,\chi,t\rangle = e^{-i\hat{H}_{eff}t}|j,\chi\rangle. \tag{5.9}$$

We first calculate the expectation values of the pseudo angular momentum or spin operators and their correlations over the state $|j,\chi,t\rangle$. Since

$$\left[\hat{H}_{eff},\hat{J}_z^2\right] = 0 = \left[\hat{H}_{eff},\hat{J}_z\right] \tag{5.10}$$

the dynamical variables corresponding to the operators $\hat{J}_z$ and $\hat{J}_z^2$ are constants of motion. That is

$$\langle j,\chi,t|\hat{J}_z|j,\chi,t\rangle = \langle j,\chi|e^{i\hat{H}_{eff}t}\hat{J}_z e^{-i\hat{H}_{eff}t}|j,\chi\rangle = \langle j,\chi|\hat{J}_z|j,\chi\rangle \tag{5.11}$$

and

$$\langle j,\chi,t|\hat{J}_z^2|j,\chi,t\rangle = \langle j,\chi|e^{i\hat{H}_{eff}t}\hat{J}_z^2e^{-i\hat{H}_{eff}t}|j,\chi\rangle = \langle j,\chi|\hat{J}_z^2|j,\chi\rangle. \tag{5.12}$$

We have already shown the calculation of $\langle\hat{J}_z\rangle$ and $\langle\hat{J}_z^2\rangle$ over the state $|j,\chi\rangle$ in Chapter1 which are

$$\langle\hat{J}_z\rangle = j\cos\theta \tag{5.13}$$

and

$$\langle\hat{J}_z^2\rangle = j^2\cos^2\theta + \frac{1}{2}j\sin^2\theta. \tag{5.14}$$

We now proceed to calculate the remaining moments and correlations. We know from Chapter 4 that

$$\begin{aligned}
\hat{J}_x &= \frac{1}{2}\Big(\hat{J}_+ + \hat{J}_-\Big), && (5.15)\\
\hat{J}_y &= \frac{1}{2i}\Big(\hat{J}_+ - \hat{J}_-\Big), && (5.16)\\
\hat{J}_x^2 &= \frac{1}{4}\Big(\hat{J}_+^2 + \hat{J}_-^2\Big) + \frac{1}{2}\Big(\hat{J}^2 - \hat{J}_z^2\Big), && (5.17)\\
\hat{J}_y^2 &= -\frac{1}{4}\Big(\hat{J}_+^2 + \hat{J}_-^2\Big) + \frac{1}{2}\Big(\hat{J}^2 - \hat{J}_z^2\Big), && (5.18)\\
\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x &= \frac{1}{2i}\Big(\hat{J}_+^2 - \hat{J}_-^2\Big), && (5.19)\\
\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x &= \frac{1}{2i}\Big[\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ + \hat{J}_-\hat{J}_z + \hat{J}_z\hat{J}_-\Big], && (5.20)
\end{aligned}$$

and

$$\hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y = \frac{1}{2i}\Big[\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+ - \hat{J}_-\hat{J}_z - \hat{J}_z\hat{J}_-\Big]. \tag{5.21}$$

Therefore,

$$\begin{aligned}
\langle\hat{J}_x\rangle &= Re\langle\hat{J}_+\rangle, && (5.22)\\
\langle\hat{J}_y\rangle &= Im\langle\hat{J}_+\rangle, && (5.23)\\
\langle\hat{J}_x^2\rangle &= \frac{1}{2}\Big[Re\langle\hat{J}_+^2\rangle + \langle\hat{J}^2\rangle - \langle\hat{J}_z^2\rangle\Big], && (5.24)\\
\langle\hat{J}_y^2\rangle &= \frac{1}{2}\Big[-Re\langle\hat{J}_+^2\rangle + \langle\hat{J}^2\rangle - \langle\hat{J}_z^2\rangle\Big], && (5.25)\\
\langle\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x\rangle &= Im\langle\hat{J}_+^2\rangle, && (5.26)\\
\langle\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle &= Re\langle\hat{J}_+\hat{J}_z + \hat{J}_z\hat{J}_+\rangle && (5.27)
\end{aligned}$$

and

$$\langle \hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y \rangle = Im \langle \hat{J}_+ \hat{J}_z + \hat{J}_z \hat{J}_+ \rangle. \tag{5.28}$$

We now show the calculation of $\langle \hat{J}_+ \rangle$ over the state given in Eq. (5.9).

$$\langle j, \chi, t | \hat{J}_+ | j, \chi, t \rangle = \langle j, \chi | e^{i\hat{H}_{eff}t} \hat{J}_+ e^{-i\hat{H}_{eff}t} | j, \chi \rangle. \tag{5.29}$$

Using Campbell-Baker-Hausdorff lemma [see Apendix-I] it can be shown that

$$e^{i\hat{H}_{eff}t} \hat{J}_+ e^{-i\hat{H}_{eff}t} = \hat{J}_+ e^{-i\Delta_0(2\hat{J}_z+1)t} = \langle \hat{J}_+(t) \rangle. \tag{5.30}$$

We can also calculate the above quantity by shifting to Heisenberg picture of quantum mechanics in which the operators are time dependent whereas the states are not. According to Heisenberg's equation of motion we know that

$$\frac{d}{dt} \hat{J}_+(t) = i[\hat{H}_{eff}, \hat{J}_+(t)] = -i\Delta_0 \hat{J}_+(t)[2\hat{J}_z + 1]. \tag{5.31}$$

Therefore,

$$\hat{J}_+(t) = \hat{J}_+(0) e^{-i\Delta_0(2\hat{J}_z+1)t}. \tag{5.32}$$

Hence,

$$\begin{aligned} \langle \hat{J}_+(t) \rangle &= \langle j, \chi | \hat{J}_+ e^{-i\Delta_0(2\hat{J}_z+1)t} | j, \chi \rangle \\ &= \frac{1}{(1+|\chi|^2)^{2j}} \sum_{m,n=0}^{2j} \sqrt{{}^{2j}C_n {}^{2j}C_m} \chi^{\star m} \chi^n \langle j, j-m | \hat{J}_+ e^{-i\Delta_0(2\hat{J}_z+1)t} | j, j-n \rangle \\ &= \frac{\chi}{(1+|\chi|^2)^{2j}} e^{-i\Delta_0(2j+1)t} \sum_{n=0}^{2j} {}^{2j}C_n (|\chi|^2)^{n-1} n e^{2i\Delta_0 nt} \\ &= \frac{\chi}{(1+|\chi|^2)^{2j}} e^{-i\Delta_0(2j+1)t} \frac{d}{d|\chi|^2} \sum_{n=0}^{2j} {}^{2j}C_n |\chi|^{2n} e^{2i\Delta_0 nt} \\ &= \frac{2j\chi}{(1+|\chi|^2)} e^{-i\Delta_0(2j-1)t}. \end{aligned} \tag{5.33}$$

Using Eq. (5.4) with $\phi = 0$ and simplifying, we obtain

$$\begin{aligned} \langle \hat{J}_+(t) \rangle &= j \sin\theta \left[\cos^4(\theta/2) + \sin^4(\theta/2) + \frac{1}{2} \sin^2\theta \cos 2\Delta_0 t \right]^{j-\frac{1}{2}} \\ &\times \; e^{i[(2j-1)\Theta_1(t) - \Delta_0(2j-1)t]} \end{aligned} \tag{5.34}$$

where

$$\tan \Theta_1(t) = \frac{\sin^2(\theta/2) \sin 2\Delta_0 t}{\cos^2(\theta/2) + \sin^2(\theta/2) \cos 2\Delta_0 t}. \tag{5.35}$$

After obtaining $\langle \hat{J}_+(t)\rangle$, we now show the calculation of $\langle \hat{J}_+^2(t)\rangle$. Using Heisenberg's equation of motion we have,

$$\frac{d}{dt}\hat{J}_+^2(t) = -4i\Delta_0 \hat{J}_+^2(t)\Big[1+\hat{J}_z(0)\Big] \tag{5.36}$$

implying,

$$\hat{J}_+^2(t) = \hat{J}_+^2(0)e^{-4i\Delta_0(\hat{J}_z+1)t}. \tag{5.37}$$

Thus, using Eq. (5.3) we get,

$$\begin{aligned}
\langle j,\chi|\hat{J}_+^2(t)|j,\chi\rangle &= \langle j,\chi|\hat{J}_+^2 e^{-4i\Delta_0(\hat{J}_z+1)t}|j,\chi\rangle \\
&= \frac{1}{\Big(1+|\chi|^2\Big)^{2j}}\sum_{m,n=0}^{2j}\sqrt{{}^{2j}C_n{}^{2j}C_m}\chi^{\star m}\chi^n \\
&\times \langle j,j-m|\hat{J}_+^2 e^{-i\Delta_0(2\hat{J}_z+1)t}|j,j-n\rangle \\
&= \frac{\chi^2}{\Big(1+|\chi|^2\Big)^{2j}}e^{-4i\Delta_0(j+1)t}\sum_{n=0}^{2j}{}^{2j}C_n n(n-1)\Big(|\chi|^2\Big)^{n-2}e^{4in\Delta_0 t} \\
&= \frac{\chi^2}{\Big(1+|\chi|^2\Big)^{2j}}e^{-4i\Delta_0(j+1)t}\frac{d^2}{d|\chi|^2}\sum_{n=0}^{2j}{}^{2j}C_n|\chi|^{2n}e^{4in\Delta_0 t} \\
&= \frac{2j(2j-1)\chi^2}{\Big(1+|\chi|^2\Big)^{2j}}e^{-4i\Delta_0(j-1)t}\Big(1+|\chi|^2e^{4i\Delta_0 t}\Big)^{2j-2}.
\end{aligned} \tag{5.38}$$

Using Eq. (5.4) with $\phi=0$ and simplifying, we obtain

$$\begin{aligned}
\langle \hat{J}_+^2(t)\rangle &= \frac{1}{2}j(2j-1)\sin^2\theta\Big[\cos^4(\theta/2)+\sin^4(\theta/2)+\frac{1}{2}\sin^2\theta\cos 4\Delta_0 t\Big]^{j-1} \\
&\times e^{i[(2j-2)\Theta_2(t)-4\Delta_0(j-1)t]}
\end{aligned} \tag{5.39}$$

where,

$$\tan\Theta_2(t) = \frac{\sin^2(\theta/2)\sin 4\Delta_0 t}{\cos^2(\theta/2)+\sin^2(\theta/2)\cos 4\Delta_0 t}. \tag{5.40}$$

We now show the calculation of $\langle \hat{J}_+(t)\hat{J}_z(t)\rangle$ over $|j,\chi\rangle$. Using Heisenberg's equation of motion we have,

$$\frac{d}{dt}\Big[\hat{J}_+(t)\hat{J}_z(t)\Big] = -i\Delta_0\hat{J}_+(t)\hat{J}_z(t)(1+2\hat{J}_z(0)). \tag{5.41}$$

Hence,

$$\hat{J}_+(t)\hat{J}_z(t) = \hat{J}_+(0)\hat{J}_z(0)e^{-i\Delta_0(2\hat{J}_z+1)t}. \tag{5.42}$$

Thus, using Eq. (5.3) we get,

$$
\begin{aligned}
\langle j,\chi|\hat{J}_+(t)\hat{J}_z(t)|j,\chi\rangle &= \langle j,\chi|\hat{J}_+\hat{J}_z e^{-i\Delta_0(2\hat{J}_z+1)t}|j,\chi\rangle \\
&= \frac{1}{\left(1+|\chi|^2\right)^{2j}}\sum_{n=0}^{2j}\sum_{m=0}^{2j}\sqrt{{}^{2j}C_n{}^{2j}C_m}\chi^{\star m}\chi^n \\
&\times \langle j,j-m|\hat{J}_+\hat{J}_z e^{-i\Delta_0(2\hat{J}_z+1)t}|j,j-n\rangle \\
&= \chi\frac{e^{-i\Delta_0(2j+1)t}}{\left(1+|\chi|^2\right)^{2j}}\sum_{n=0}^{2j}{}^{2j}C_n(jn-n^2)(|\chi|^2)^{n-1}e^{2i\Delta_0 nt} \\
&= \chi\frac{e^{-i\Delta_0(2j+1)t}}{\left(1+|\chi|^2\right)^{2j}}\Bigg[j\frac{d}{d|\chi|^2}\sum_{n=0}^{2j}{}^{2j}C_n|\chi|^{2n}e^{2i\Delta_0 nt} \\
&- \frac{d}{d|\chi|^2}\left\{|\chi|^2\frac{d}{d|\chi|^2}\sum_{n=0}^{2j}{}^{2j}C_n|\chi|^{2n}e^{2i\Delta_0 nt}\right\}\Bigg] \\
&= \chi\frac{e^{-i\Delta_0(2j+1)t}}{\left(1+|\chi|^2\right)^{2j}}\Bigg[j\frac{d}{d|\chi|^2}\left(1+|\chi|^2e^{2i\Delta_0 t}\right)^{2j} \\
&- \frac{d}{d|\chi|^2}\left\{|\chi|^2\frac{d}{d|\chi|^2}\left(1+|\chi|^2e^{2i\Delta_0 t}\right)^{2j}\right\}\Bigg] \\
&= 2j\chi\frac{e^{-i\Delta_0(2j-1)t}}{\left(1+|\chi|^2\right)^{2j}}\left[j\left(1-|\chi|^2e^{2i\Delta_0 t}\right)-1\right] \\
&\times \left(1+|\chi|^2e^{2i\Delta_0 t}\right)^{(2j-2)}. \qquad (5.43)
\end{aligned}
$$

We now show the calculation of $\langle\hat{J}_z(t)\hat{J}_+(t)\rangle$ over the state given in Eq. (5.3). We know that

$$\left[\hat{J}_z,\hat{J}_+\right]=\hat{J}_+. \qquad (5.44)$$

Since in going from Schr$\ddot{o}$dinger to Heisenberg picture the commutation relations remain preserved therefore, we have

$$\langle\hat{J}_z(t)\hat{J}_+(t)\rangle=\langle\hat{J}_+(t)\hat{J}_z(t)\rangle+\langle\hat{J}_+(t)\rangle. \qquad (5.45)$$

Using Eqs. (5.33) and (5.43) and simplifying we obtain,

$$
\begin{aligned}
\langle\hat{J}_z(t)\hat{J}_+(t)\rangle &= \chi\frac{e^{-i\Delta_0(2j-1)t}}{(1+|\chi|^2)^{2j}}\left[2j^2(1-|\chi|^2e^{2i\Delta_0 t})+2j|\chi|^2e^{2i\Delta_0 t}\right] \\
&\times \left(1+|\chi|^2e^{2i\Delta_0 t}\right)^{(2j-2)}. \qquad (5.46)
\end{aligned}
$$

To calculate $\langle \hat{J}_+(t)\hat{J}_z(t) + \hat{J}_z(t)\hat{J}_+(t)\rangle$ we add the two Eqs. (5.43) and (5.46) and obtain,

$$\begin{aligned}\langle \hat{J}_+(t)\hat{J}_z(t) + \hat{J}_z(t)\hat{J}_+(t)\rangle &= 2j(2j-1)\chi\frac{e^{-i\Delta_0(2j-1)t}}{(1+|\chi|^2)^{2j}}\left(1+|\chi|^2 e^{2i\Delta_0 t}\right)^{(2j-2)} \\ &\times \left(1-|\chi|^2 e^{2i\Delta_0 t}\right). \end{aligned} \quad (5.47)$$

Using Eq. (5.4) with $\phi = 0$ and simplifying we obtain,

$$\begin{aligned}\langle \hat{J}_+(t)\hat{J}_z(t) &+ \hat{J}_z(t)\hat{J}_+(t)\rangle = j(2j-1)\sin\theta\Big[\Lambda_1(t)\Big]^{j-1} \\ &\times \Bigg[\Bigg(\cos^2(\theta/2)\cos\Big\{(2j-2)\Theta_1(t) - (2j-1)\Delta_0 t\Big\} \\ &- \sin^2(\theta/2)\cos\Big\{(2j-2)\Theta_1(t) - (2j-3)\Delta_0 t\Big\}\Bigg) \\ &+ i\Bigg(\cos^2(\theta/2)\sin\Big\{(2j-2)\Theta_1(t) - (2j-1)\Delta_0 t\Big\} \\ &- \sin^2(\theta/2)\sin\Big\{(2j-2)\Theta_1(t) - (2j-3)\Delta_0 t\Big\}\Bigg)\Bigg] \end{aligned} \quad (5.48)$$

where,

$$\Lambda_1(t) = \cos^4(\theta/2) + \sin^4(\theta/2) + \frac{1}{2}\sin^2\theta\cos 2\Delta_0 t. \quad (5.49)$$

As in Eq. (5.10), $\hat{J}^2$ is a constant of motion. Hence, $\langle \hat{J}^2\rangle$ over the state given in Eq. (5.3) is

$$\begin{aligned}\langle j,\chi|\hat{J}^2|j,\chi\rangle &= \frac{1}{(1+|\chi|^2)^{2j}}\sum_{n=0}^{2j}\sum_{m=0}^{2j}\sqrt{{}^{2j}C_n{}^{2j}C_m}\chi^{\star m}\chi^n\langle j, j-m|\hat{J}^2|j, j-n\rangle \\ &= \frac{j(j+1)}{(1+|\chi|^2)^{2j}}\sum_{n=0}^{2j}{}^{2j}C_n|\chi|^{2n} \\ &= j(j+1). \end{aligned} \quad (5.50)$$

Using Eqs. (5.22) and (5.34) we obtain,

$$\begin{aligned}\langle \hat{J}_x(t)\rangle &= j\sin\theta\Big[\cos^4(\theta/2) + \sin^4(\theta/2) + \frac{1}{2}\sin^2\theta\cos 2\Delta_0 t\Big]^{(2j-1)/2} \\ &\quad \cos\big[(2j-1)\Theta_1(t) - \Delta_0(2j-1)t\big]. \end{aligned} \quad (5.51)$$

Using Eqs. (5.23) and (5.34) we have,

$$\begin{aligned}\langle \hat{J}_y(t)\rangle &= j\sin\theta\Big[\cos^4(\theta/2) + \sin^4(\theta/2) + \frac{1}{2}\sin^2\theta\cos 2\Delta_0 t\Big]^{(2j-1)/2} \\ &\quad \sin\big[(2j-1)\Theta_1(t) - \Delta_0(2j-1)t\big]. \end{aligned} \quad (5.52)$$

Using Eqs. (5.24), (5.14), (5.39) and (5.50) we get,

$$\langle \hat{J}_x^2(t)\rangle = \frac{j}{2} + \frac{1}{4}j(2j-1)\sin^2\theta\Big[1 + [\Lambda_2(t)]^{(j-1)}\cos 2[2\Delta_0(j-1)t - (j-1)\Theta_2(t)]\Big]. \quad (5.53)$$

where

$$\Lambda_2(t) = \cos^4(\theta/2) + \sin^4(\theta/2) + \frac{1}{2}\sin^2\theta\cos 4\Delta_0 t. \quad (5.54)$$

Using Eqs. (5.25), (5.14), (5.39) and (5.50) we get,

$$\langle \hat{J}_y^2(t)\rangle = \frac{j}{2} + \frac{1}{4}j(2j-1)\sin^2\theta\Big[1 - [\Lambda_2(t)]^{(j-1)}\cos 2[2\Delta_0(j-1)t - (j-1)\Theta_2(t)]\Big]. \quad (5.55)$$

Using Eqs. (5.26) and (5.39) we obtain

$$\langle \hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x\rangle = \frac{1}{2}j(2j-1)\sin^2\theta\big[\Lambda_2(t)\big]^{(j-1)}\sin\big[(2j-2)\Theta_2(t) - 4\Delta_0(j-1)t\big]. \quad (5.56)$$

Using Eqs. (5.27) and (5.48) we obtain,

$$\begin{aligned}\langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle &= j(2j-1)\sin\theta\big[\Lambda_1(t)\big]^{(j-1)}\\ &\Big[\cos^2(\theta/2)\cos\Big\{(2j-2)\Theta_1(t) - (2j-1)\Delta_0 t\Big\}\\ &-\sin^2(\theta/2)\cos\Big\{(2j-2)\Theta_1(t) - (2j-3)\Delta_0 t\Big\}\Big]. \quad (5.57)\end{aligned}$$

Using Eqs. (5.28), (5.48) we obtain,

$$\begin{aligned}\langle \hat{J}_y\hat{J}_z + \hat{J}_z\hat{J}_y\rangle &= j(2j-1)\sin\theta\big[\Lambda_1(t)\big]^{(j-1)}\\ &\Big[\cos^2(\theta/2)\sin\Big\{(2j-2)\Theta_1(t) - (2j-1)\Delta_0 t\Big\}\\ &-\sin^2(\theta/2)\sin\Big\{(2j-2)\Theta_1(t) - (2j-3)\Delta_0 t\Big\}\Big]. \quad (5.58)\end{aligned}$$

These correlation functions are responsible for spin squeezing or squeezing of the atomic state [2].

5.2.3. Analysis of the Amount of Squeezing for the State $|j,\chi,t\rangle$

To analyse the squeezing aspect of the state represented in Eq. (5.9) we perform a rotation of the coordinate frame $\{x,y,z\}$ to a new frame $\{x',y',z'\}$ so that the mean angular momentum vector $\langle\hat{\mathbf{J}}\rangle$ is along the z' axis. The magnitude of the mean angular momentum vector for this system is

$$\begin{aligned}|\langle\hat{\mathbf{J}}\rangle| &= \sqrt{\langle\hat{J}_x\rangle^2 + \langle\hat{J}_y\rangle^2 + \langle\hat{J}_z\rangle^2}\\ &= j\sqrt{[\Lambda_1(t)]^{2j-1}\sin^2\theta + \cos^2\theta}. \quad (5.59)\end{aligned}$$

It is easy to verify that for $j = \frac{1}{2}$ i.e. for a single two-level atom all the three correlations vanish, implying that there is no squeezing. We now look for the squeezing aspect for $j > 1$. We first aim at the variance in the y' quadrature. It has the form

$$\begin{aligned}\langle \Delta {J'_y}^2 \rangle &= \frac{j}{2} + \frac{j}{4}(2j-1)\sin^2\theta\Bigg[1 - \left(1 - \frac{1}{2}\sin^2\theta(1-\cos 4\Delta_0 t)\right)^{j-1} \\ &\quad \cos 2\Big\{(2j-1)\Theta_1(t) - (j-1)\Theta_2(t) - \Delta_0 t\Big\}\Bigg]. \end{aligned} \tag{5.60}$$

We notice that

$$\langle \Delta {J'_y}^2 \rangle > \frac{j}{2}. \tag{5.61}$$

We define a quantity S_y as

$$S_y = \sqrt{\frac{2\langle \Delta J^2_{y'} \rangle}{|\langle \hat{\mathbf{J}} \rangle|}}. \tag{5.62}$$

If S_y is less than one, we say that the atomic system has squeezing in the y' quadrature. Now from Eq. (5.59) we can see that the maximum value of $|\langle \mathbf{J} \rangle|$ is j and thus from Eq. (5.61) we can conclude that

$$S_y > 1. \tag{5.63}$$

Thus the y' quadrature is never squeezed.

We now look for the squeezing aspect in x' quadrature. We define

$$S_x = \sqrt{\frac{2\langle \Delta J^2_{x'} \rangle}{|\langle \hat{\mathbf{J}} \rangle|}}. \tag{5.64}$$

If S_x is less than 1, we say that the system has squeezing in the x' quadrature. We plot S_x in Figure 5.1.

We notice that for $\theta = \pi/16$, representing about 2% of the atoms in their upper states, there is about 21% squeezing [when $S_x = 0.79$, we say that there is 21% squeezing] when the interaction time is set at $\Delta_0 t = \pi/2$. We also notice that an atomic system with 140 atoms can be squeezed at this interaction time. The maximum squeezing takes place for $N = 26$. As we deviate from this interaction time, the degree of squeezing and also the number of atoms that can be squeezed are reduced. For $\Delta_0 t = \frac{9\pi}{16}$, the squeezing reduces to 15% and for $\Delta_0 t = \frac{5\pi}{16}$, it reduces to 10%. As we deviate further from $\Delta_0 t = \frac{\pi}{2}$, the squeezing reduces more and finally vanishes at $\Delta_0 t = \pi$.

The degree of squeezing also decreases as we increase the number of atoms in their upper states, that is, as we increase θ. For $\theta = \frac{\pi}{2}$, that is, when we have equal number of atoms in the upper and lower levels, we have $\langle \hat{J}_z \rangle = 0$. For $\theta = \frac{\pi}{2}$ the variance in the x' quadrature reduces to

$$\langle \Delta {J'_x}^2 \rangle = \frac{j}{2} = \langle \hat{J}^2_z \rangle. \tag{5.65}$$

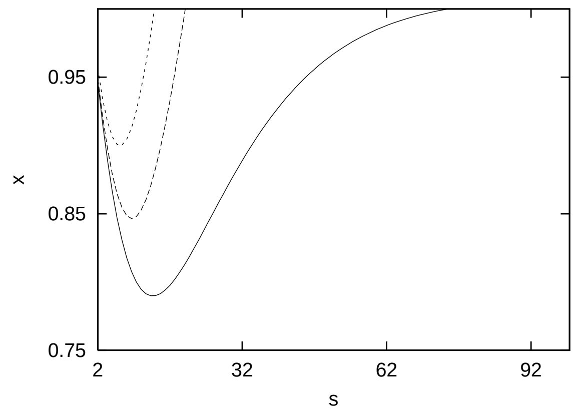

Figure 5.1: Degree of squeezing S_x as a function of $j \geq 2$ for $\theta = \pi/16$. Here in the figure the quantity X on the vertical axis is S_x and the quantity s on the horizontal axis is j. The dimensionless interaction time $\Delta_0 t = \pi/2,\ 9\pi/16,\ 5\pi/16$ for the full, broken and dotted curves, respectively. Note that $S_x < 1$ indicates squeezing.

Since at this value of θ

$$|\langle \hat{\mathbf{J}} \rangle| = j \cos^{(2j-1)} \Delta_0 t \tag{5.66}$$

we have,

$$S_x^2 = \frac{1}{\cos^{(2j-1)} \Delta_o t} > 1 \tag{5.67}$$

implying that the system does not show any squeezing for $\theta = \frac{\pi}{2}$. This is in contrast to the results in Ref. [2] where the squeezing has been studied in the $y' - z'$ plane, whereas we have studied it in the $x' - y'$ plane.

We have studied the squeezing aspect of a two-atom or a bipartite system for which $j = 1$ as it reveals interesting properties. We set $\Delta_0 t = \pi/2$ as the maximum squeezing or lowest noise level is obtained for this interacion time for all values of j. We observe from Eqs. (5.51) to (5.58) that the moments and correlations take very simple form as

$$\langle \hat{J}_x(t) \rangle = -\frac{1}{2}\sin 2\theta, \tag{5.68}$$

$$\langle \hat{J}_y(t) \rangle = 0, \tag{5.69}$$

$$\langle \hat{J}_z(t) \rangle = -\cos\theta, \tag{5.70}$$

$$\langle \hat{J}_x^2(t) \rangle = \frac{1}{2}, \tag{5.71}$$

$$\langle \hat{J}_y^2(t) \rangle = \frac{1}{2}(1 + \sin^2\theta) \tag{5.72}$$

$$\langle \hat{J}_z^2(t) \rangle = 1 - \frac{1}{2}\sin^2\theta \tag{5.73}$$

and

$$\langle \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\rangle = \sin\theta. \tag{5.74}$$

The other two correlations are zero at $\Delta_0 t = \frac{\pi}{2}$. With these values the expression of S_x for $0 < \theta < \frac{\pi}{2}$ has the form

$$S_x = \left(\frac{\cos\theta}{\sqrt{1+\sin^2\theta}}\right)^{1/2} < 1. \tag{5.75}$$

We observe that there is sufficient squeezing in the x' quadrature. and it increases with increase in θ. As $\theta \to \frac{\pi}{2}$ the squeezing approaches 100%. This behaviour is just opposite to that of a multipartite system that is for $j > 1$. However, we may notice from Eq. (5.59), that $|\langle\hat{\mathbf{J}}\rangle| = 0$ at $\theta = \Delta_0 t = \frac{\pi}{2}$, implying that squeezing is not defined at these values. In the interval $\frac{\pi}{2} < \theta < \pi$, S_x has the form

$$S_x = \left(\frac{-\cos\theta}{\sqrt{1+\sin^2\theta}}\right)^{1/2} < 1 \tag{5.76}$$

which implies squeezing. Thus squeezing increases as θ increases from 0 towards $\frac{\pi}{2}$ and decreases as θ increases from $\frac{\pi}{2}$ towards π. We can interpret this result from the expression of the only non-zero correlation given Eq. (5.74). We observe that as θ increases from 0 the correlation factor also increases and hence the squeezing also increases. The correlation decreases when θ increases from $\frac{\pi}{2}$ to π and thus it decreases the squeezing.

5.3. Physical Significance of the Hamiltonian $\hat{H}_{eff}$

We now show that the dynamics of a system of N two-level atoms interacting with the single mode of a cavity having high quality factor Q and at thermal equilibrium is governed by the effective Hamiltonian $\hat{H}_{eff}$ as given in Eq. (5.2). The cavity quality factor is related to cavity bandwidth κ by $\kappa = \frac{1}{2Q}$ and then the cavity photon lifetime is represented by $\frac{1}{2\kappa}$. The average number of thermal photons $\bar{n}_{th}$ present inside the cavity is related to its temperature T by the relation

$$\bar{n}_{th} = \frac{1}{e^{\hbar\omega_c/k_B T} - 1}, \tag{5.77}$$

where ω_c is the frequency of the single mode cavity. We consider a highly detuned cavity, that is, ω_c is very large compared to the atomic transition frequency ω_a which makes the cavity dispersive in nature. Since we are dealing with a high-Q cavity (small k), the system easily satisfies $\delta >> k$ where $\delta = \omega_c - \omega_a$. Further, if $|i\delta + k| >> g\sqrt{N}$, where g is atom-field coupling constant, is satisfied, the field remains almost stationary in the time scale of changes in atomic observables. Under this condition, the time evolution of atomic density matrix can be written as [1]

$$\frac{d\hat{\rho}_a}{dt} = \frac{g^2}{k^2+\delta^2}\Bigg[-i\delta\Big\{[\hat{J}_+\hat{J}_-,\hat{\rho}_a] + 2n[\hat{J}_z,\hat{\rho}_a]\Big\}$$
$$+k\Big\{(n+1)(2\hat{J}_-\hat{\rho}_a\hat{J}_+ - \hat{J}_+\hat{J}_-\hat{\rho}_a - \hat{\rho}_a\hat{J}_+\hat{J}_-)$$
$$+n(2\hat{J}_+\hat{\rho}_a\hat{J}_- - \hat{J}_-\hat{J}_+\hat{\rho}_a - \hat{\rho}_a\hat{J}_-\hat{J}_+)\Big\}\Bigg] \tag{5.78}$$

the details are given in the Apendix V. Since $\delta >> k$, we notice in Eq. (5.78) that the damping terms contribute negligibly to the atomic dynamics. This reduces the equation of motion to

$$\frac{d\hat{\rho}_a}{dt} = -i\Delta_0\Big[\hat{J}_+\hat{J}_- + 2\bar{n}\hat{J}_z \ , \ \hat{\rho}_a\Big]. \tag{5.79}$$

As

$$\hat{J}^2 = \frac{1}{2}\Big(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+\Big) + \hat{J}_z^2 \ , \tag{5.80}$$

the effective Hamiltonian describing the time evolution of the system is

$$\hat{H}_{eff} = \Delta_0\hat{J}_+\hat{J}_- + \Delta_1\hat{J}_z = \Delta_0(\hat{J}^2 - \hat{J}_z^2) + (\Delta_1 + \Delta_0)\hat{J}_z \tag{5.81}$$

where

$$\Delta_0 = \frac{g^2\delta}{k^2+\delta^2} \tag{5.82}$$

and

$$\Delta_1 = 2\bar{n}_{th}\Delta_0. \tag{5.83}$$

In the effective Hamiltonian the cavity temperature appears in the term linear in $\hat{J}_z$. We know that such terms produces simply a rotation which does not change the final results. Thus we drop the terms linear in atomic operators making the effective Hamiltonian

$$\hat{H}_{eff} = \Delta_0(\hat{J}^2 - \hat{J}_z^2). \tag{5.84}$$

Thus we find that the Hamiltonian is a function of $\hat{J}_z^2$ [2]. In this chapter, in addition to probing deeper into the spin squeezing dynamics of $\hat{H}_{eff}$, we have shown that, when the system is in a bipartite state ($N = 2$), the time evolution of the interaction has quite interesting properties. It is interesting to note that $\hat{H}_{eff}$ is a special case of the so called Lipkin-Meshkov-Glick (LMG) Hamiltonian for a many-body fermionic system [3].

$$\hat{H}_{LMG} = G_1(\hat{J}_+^2 + \hat{J}_-^2) + G_2(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) \tag{5.85}$$

where G_1 and G_2 are the parameters specifying interaction strengths there. They are quite different from the interactions we consider in the thesis. What we are pointing out is the degree of similarity between $\hat{H}_{eff}$ and $\hat{H}_{LMG}$.

5.4. Conclusion

We have shown that a system of atoms, initially in a coherent state, can be squeezed by an Hamiltonian proportional to $\hat{J}_z^2$. Also, this operator can represent atom-field interactions in cavity QED as discussed above. It may be noted that this operator was first considered in Ref. [2] to establish the phenomenon of spin squeezing. We, however, has examined the dynamics in detail.

The LMG-Hamiltonian [3] in Eq. (5.85) has this form if we set $G_1 = 0$ there. Thus the system, described by the LMG Hamiltonian and if the values of G_2 is set to the numerical values of Δ_0, will display spin squeezing properties.

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6. Conclusion

In our thesis we dealt with the squeezing aspects of two level atomic systems. We have analysed the subject from various perspectives to get a deeper insight. In Chapter 1, containing the introduction to the thesis, we developed the necessary basic concepts and preliminaries for the subject. We first discussed the coherent states and squeezed states of electromagnetic radiation field as these ideas were first developed in that context. We then introduced a two level atomic system and discussed its coherent states and squeezed states subsequently. A brief review of the work done so far has been given there. We presented our work in Chapter 2 to Chapter 5.

In Chapter 2 we considered an atomic state, denoted as $|\Psi_m\rangle$, representing a two level atomic system interacting with the squeezed vacuum state of the radiation field [1]. This state is an eigenstate of a non-Hermitian operator, termed in our thesis as $\hat{\Lambda}$, with real eigenvalues [2]. We showed the underlying mathematical reason for the reality of the eigenvalue spectrum of that operator by introducing the notion of pseudo-Hermiticity. The necessary condition for a non-Hermitian operator with a complete set of biorthonormal eigenvectors to have a real eigenvalue spectrum is that the operator must be pseudo-Hermitian [3]. A pseudo-Hermitian operator with a complete set of biorthonormal eigenvectors possesses either real eigenvalues or complex conjugate pair of eigenvalues. The condition under which all the eigenvalues are real has been shown in Reference [4] and also discussed in Apendix-III. We proved that the operator $\hat{\Lambda}$ is pseudo-Hermitian and satisfies the necessary condition to have real eigenvalues and thus connected the recently developed ideas of pseudo-Hermiticity with a real physical example from the domain of quantum optics.

We analysed the squeezing aspect of the state $|\Psi_m\rangle$ by calculating the relevant moments, correlations and variances of the spin operators over this state and observed a significant amount of squeezing. In the course of our study we introduced reduced Wigner d-matrix elements [5] with their analytic continuation to imaginary angles, as this makes the relevant calculations much simple.

We also discussed the squeezing aspect of the biorthonormal state $|\Phi_m\rangle$ of $|\Psi_m\rangle$ and concluded that the state $|\Phi_m\rangle$ has the same squeezing properties as that of $|\Psi_m\rangle$.

After studying the squeezing properties of the eigenstate of a pseudo-Hermitian operator we tried to develop a generic but simplest squeezing operator to produce squeezing in two level atomic systems put initially in coherent state. We know that a Hamiltonian nonlinear in spin operators when generates time evolution on an atomic coherent state, produces squeezing in the resultant state. Many such Hamiltonians can be constructed which serve this purpose, however, it is better to construct a generic Hamiltonian which reveals the subject from general aspect. In Chapter 3 we developed such a Hamiltonian keeping analogy with the case of electromagnetic radiation field. We took all possible Hermitian combinations of spin operators to form Hamiltonians with lowest power of nonlinearity, that is, the quadratics and got six such entities. Those six Hamiltonians can be divided into two groups each containing three, such that the Hamiltonians of the same group are connected to each other by rotations of the coordinate axes. Therefore, we argued that it is not needed to study the squeezing properties of each of the six Hamiltonians individually but, it is sufficient to study the squeezing properties of only one from each group. The knowledge gained from the study made on any one Hamiltonian from each group can be used to derive the squeezing properties of the other group members by using the rotations of the coordinate axes. Therefore, we took the generic squeezing Hamiltonians as

$$\hat{H}_{spin} = g_1(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) \tag{6.1}$$

and

$$\hat{H}'_{spin} = g_2\hat{J}_z^2, \tag{6.2}$$

with g_1 and g_2 as real parameters. We thus constructed the generic spin squeezing operators as

$$\hat{S}_{spin} = \exp(-i\hat{H}_{spin}t/\hbar) = \exp\Big[2i\gamma(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)\Big] \tag{6.3}$$

and

$$\hat{S}'_{spin} = \exp(-i\hat{H}'_{spin}t/\hbar) = \exp\Big[-i\beta\hat{J}_z^2\Big], \tag{6.4}$$

where $\gamma = -g_1t/(2\hbar)$ and $\beta = g_2t/\hbar$.

After constructing the generic spin squeezing operators $\hat{S}_{spin}$ and $\hat{S}'_{spin}$ we study their squeezing properties when they act on an atomic coherent state. In Chapter 4 we study the spin squeezing dynamics produced when the operator $\hat{S}_{spin}$ acts on an atomic coherent state. We studied the case in two parts, one when $\hat{S}_{spin}$ acts on a two-atom system and the other when the operator acts on a system of more than two atoms. The case of two-atom system was studied analytically. We calculated the relevant moments, correlations and the variances of the spin operators over the state produced when $\hat{S}_{spin}$ acts on a two atom coherent state. We graphically showed the variation of squeezing in two quadratures x and y and observed significant amount of squeezing in both the quadratures in alternate fashion.

The case of more than two atoms was dealt numerically as it is difficult to make analytical study. The striking feature in this case is that for short interaction time the squeezing parameter, denoted as S_y oscillates just below the line $S_y = 1$ indicating spin squeezing for all values of j. Thus the operator $\hat{S}_{spin}$ is capable of squeezing a large number of atoms if they are initially prepared in a coherent state. It is also noted that the oscillations in squeezing with time increases with increase in the value of j.

In Chapter 5 we studied the spin squeezing dynamics when the Hamiltonian $\hat{H}'_{spin}$ generates time evolution on an atomic coherent state, that is the operator $\hat{S}'_{spin}$ acts on an atomic coherent state. The dynamics of a system of N two-level atoms interacting with the single mode of a cavity having high quality factor and at thermal equilibrium is governed by the effective Hamiltonian which has the same form as that of $\hat{H}'_{spin}$. We observed that there is no squeezing in the y-quadrature however, x-quadrature shows significant amount of squeezing.

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7. Scope for Further Investigations

It has been stressed in the literature that spin squeezing is induced by entanglement [1]. We know that entanglement forms the basis of the emerging field of quantum information. This is definitely the driving factor in the continuing interest in the study of spin squeezing. The other reasons for the progressive interest in spin squeezing are as follows:

(i) Spin entangled (spin squeezed) states after their measurements for extracting information, for example, can be used again and again. This property of spins stands out when compared with photonic systems. Entangled photons are destroyed for ever after their measurements. Thus an ensemble of two level atoms (spins) can be used as an unit in quantum information processing.

(ii) We have shown in this thesis that spin squeezing is possible for number of spins $N > 2$. In other words multipartite entangled states in the spin systems are possible. Multipartite entanglement is a subject of vigorous study in its identification and quantification.

(iii) The measure of quantum entanglement has so far been mathematical [2]. Spin squeezing gives a physical (laboratory) measurement of quantum entanglement. Work has already began in this direction, but, for a bipartite (two spin) system only [3]. We have also indicated work in this direction [4]. It can be stated that a system with $j = 1$ (a bipartite system) is entangled if

$$S_x^2 \; |\langle \hat{\mathbf{J}} \rangle| < 1 \tag{7.1}$$

in the notations of parameters in Eq. (5.64) of Chapter 5. We know that a system is spin squeezed, that is, $S_x < 1$ for dimensionless interaction time $\Delta_0 t = \pi/2$ and the atomic excitation parameter $0 < \theta < \pi/2$. The involved parameters Δ_0 and θ have been defined in Chapter 5. In this situation, we have shown

$$|\langle \hat{\mathbf{J}} \rangle| = \sqrt{1 - \sin^2\theta} < 1. \tag{7.2}$$

Thus the system evolving under the Hamiltonian $\hat{H}_{eff} = \Delta_0(\hat{J}^2 - \hat{J}_z^2)$ in Eq. (5.84) of Chapter 5 is entangled.

We plan to extend the study for a multipartite system, that is, $j > 1$.

In Chapter 3, we identified two Hamiltonians nonlinear in spin operators, namely

$$\hat{H}_{spin} = \frac{1}{2i} g_1 \big(\hat{J}_+^2 - \hat{J}_-^2\big) \tag{7.3}$$

and

$$\hat{H}'_{spin} = g_2 \hat{J}_z^2 \tag{7.4}$$

which are capable of squeezing out noise from a spin system in a coherent state. We have shown their characteristics in the subsequent Chapters 4 and 5. We discussed there various systems whose Hamiltonians include the above nonlinear operators. In addition, the operator in Eq. (7.4) appears in a system, the so called 'driven Dicke model' in quantum optics. The Dicke system [5] consists of an ensemble of mutually non-interacting two level atoms confined to a cube of volume λ^3, where λ is the wavelength of the radiation field with which the atoms are interacting. The laser cooling technology can be used to cool down a system of atoms sufficiently to get a Dicke system. When this system is driven by an external laser, we get the system known as 'driven Dicke system'. Such a system in a cavity can be described by a model Hamiltonian [6] in which one of the terms has the form in Eq. (7.4). This term arises due to atom-atom cooperation through the common radiation field they are interacting. We plan to study this system to analyze its squeezing capabilities. In addition to this, we also plan to study various systems discussed in Chapters 4 and 5.

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A. Apendix-I

A.1. Baker Hausdorff Lemma

A.1.1. Theorem 1

Let $\hat{A}$ and $\hat{B}$ are two non-commutating operators and λ be a complex parameter. Then

$$e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}} \;=\; \hat{B} + \lambda\Big[\hat{A},\hat{B}\Big] + \frac{\lambda^2}{2!}\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big] + \frac{\lambda^3}{3!}\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]\Big] + ... \qquad (A.1)$$

To prove this theorem we follow the method outlined in Ref. [1].

Proof

To prove the above theorem we let

$$\hat{f}(\lambda) = e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}. \qquad (A.2)$$

Therefore,

$$\hat{f}(0) = \hat{B}. \qquad (A.3)$$

Expanding $\hat{f}(\lambda)$ in Maclaurin's series we obtain,

$$\hat{f}(\lambda) = \hat{f}(0) + \lambda\frac{d\hat{f}}{d\lambda}\bigg|_{\lambda=0} + \frac{\lambda^2}{2!}\frac{d^2\hat{f}}{d\lambda^2}\bigg|_{\lambda=0} + ... \qquad (A.4)$$

From Eq. $(A.2)$ we see that

$$\begin{aligned}\frac{d\hat{f}}{d\lambda} &= \hat{A}e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}} - e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}\hat{A} \\ &= \Big[\hat{A},\hat{f}(\lambda)\Big]. \qquad (A.5)\end{aligned}$$

Hence,

$$\frac{d\hat{f}}{d\lambda}\bigg|_{\lambda=0} = \Big[\hat{A},\hat{B}\Big]. \qquad (A.6)$$

Now,

$$\begin{aligned}\frac{d^2\hat{f}}{d\lambda^2} &= \hat{A}^2 e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}} - \hat{A}e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}\hat{A} - \hat{A}e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}\hat{A} + e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}\hat{A}^2 \\ &= \Big[\hat{A}, \Big[\hat{A}, \hat{f}(\lambda)\Big]\Big]. \end{aligned} \tag{A.7}$$

Therefore,

$$\left.\frac{d^2\hat{f}}{d\lambda^2}\right|_{\lambda=0} = \Big[\hat{A}, \Big[\hat{A}, \hat{B}\Big]\Big]. \tag{A.8}$$

Using Eqs. $(A.3)$, $(A.6)$ and $(A.8)$ in Eq. $(A.4)$ we obtain Eq. $(A.1)$.

A.1.2. Theorem 2

If $\hat{A}$ and $\hat{B}$ are two non commutating operators that satisfy

$$\Big[\hat{A}, \Big[\hat{A}, \hat{B}\Big]\Big] = \Big[\hat{B}, \Big[\hat{A}, \hat{B}\Big]\Big] = 0, \tag{A.9}$$

then

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\ e^{\hat{B}}\ e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{\hat{B}}\ e^{\hat{A}}\ e^{\frac{1}{2}[\hat{A},\hat{B}]}. \tag{A.10}$$

This is a special case of the Baker-Hausdorff theorem.

Proof

Let $\hat{g}(\lambda)$ be a operator function given as,

$$\hat{g}(\lambda) = e^{\lambda\hat{A}} e^{\lambda\hat{B}}, \tag{A.11}$$

where λ is a complex parameter. Differentiating with respect to λ we obtain

$$\begin{aligned}\frac{d\hat{g}}{d\lambda} &= \hat{A}\ e^{\lambda\hat{A}}\ e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\ e^{\lambda\hat{B}}\ \hat{B} \\ &= \Big(\hat{A} + e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}}\Big)\hat{g}(\lambda). \end{aligned} \tag{A.12}$$

The second term in the parentheses can be written using Eq. $(A.1)$ and Eq. $(A.9)$ as

$$e^{\lambda\hat{A}}\ \hat{B}\ e^{-\lambda\hat{A}} = \hat{B} + \lambda\Big[\hat{A}, \hat{B}\Big]. \tag{A.13}$$

Thus, Eq. $(A.12)$ may be written as

$$\frac{d\hat{g}}{d\lambda} = \Big\{(\hat{A}+\hat{B}) + \lambda\Big[\hat{A}, \hat{B}\Big]\Big\}\hat{g}(\lambda). \tag{A.14}$$

From Eq. $(A.9)$ we see that the quantity $\hat{A}+\hat{B}$ and $\left[\hat{A},\hat{B}\right]$ commute with each other and hence, we may treat these quantities as ordinary commuting variables and integrate Eq. $(A.14)$ subject to the condition

$$\hat{g}(0)=1. \tag{A.15}$$

The operator function which satisfies Eq. $(A.14)$ subject to the condition of Eq. $(A.15)$ is

$$\hat{g}(\lambda)=e^{(\hat{A}+\hat{B})\lambda+(\lambda^2/2)[\hat{A},\hat{B}]}. \tag{A.16}$$

Since $(\hat{A}+\hat{B})$ commute with $[\hat{A},\hat{B}]$, we can write the above equation as

$$\hat{g}(\lambda)=e^{(\hat{A}+\hat{B})\lambda}e^{(\lambda^2/2)[\hat{A},\hat{B}]}. \tag{A.17}$$

Equating the right hand side of Eq. $(A.11)$ with that of Eq. $(A.17)$ we obtain

$$e^{\lambda\hat{A}}e^{\lambda\hat{B}}=e^{(\hat{A}+\hat{B})\lambda}e^{(\lambda^2/2)[\hat{A},\hat{B}]}. \tag{A.18}$$

If we let $\lambda=1$ and multiply both sides of the above equation from right by $e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, then we obtain the first part of Eq. $(A.10)$. The proof of the second part of Eq. $(A.10)$ is obtained by assuming an operator function as

$$\hat{u}(\lambda)=e^{\lambda\hat{B}}e^{\lambda\hat{A}} \tag{A.19}$$

and proceeding in the same manner as above.

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B. Apendix-II

B.1. Atomic Coherent States in Schwinger Representation

Atomic coherent states are traditionally developed via

$$|j,\chi\rangle = Ne^{\chi\hat{J}_-}|j,m=j\rangle \tag{B.1}$$

$$= N\sum_{n=0}^{2j}\frac{\chi^n}{n!}\hat{J}_-^n|j,m=j\rangle. \tag{B.2}$$

Now

$$\hat{J}_-^n|j,m=j\rangle = \sqrt{\frac{(2j)!n!}{(2j-n)!}}|j,j-n\rangle. \tag{B.3}$$

Thus

$$|j,\chi\rangle = N\sum_{n=0}^{2j}\sqrt{{}^{2j}C_n}\chi^n|j,j-n\rangle. \tag{B.4}$$

To find the normalization constant N we use the condition

$$\langle j,\chi|j,\chi\rangle = 1 \tag{B.5}$$

and therefore, we obtain

$$\begin{aligned}\langle j,\chi|j,\chi\rangle &= N^2\sum_{n=0}^{2j}{}^{2j}C_n(|\chi|^2)^n \\ &= N^2(1+|\chi|^2)^{2j}.\end{aligned}$$

Accordingly the normalized state is

$$|j,\chi\rangle = \frac{1}{(1+|\chi|^2)^j}\sum_{n=0}^{2j}\sqrt{{}^{2j}C_n}\chi^n|j,j-n\rangle. \tag{B.6}$$

We now represent the atomic coherent state using Schwinger construction for angular momentum operators [1]. Let us introduce two bosonic annihilation operators $\hat{a}_i(i=+,-)$ such that

$$[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{ij} \tag{B.7}$$

and

$$[\hat{a}_i, \hat{a}_j] = 0 = [\hat{a}_i^\dagger, \hat{a}_j^\dagger]. \tag{B.8}$$

Out of these operators we construct

$$\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_-, \tag{B.9}$$
$$\hat{J}_- = \hat{a}_-^\dagger \hat{a}_+ \tag{B.10}$$

and

$$\hat{J}_z = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-). \tag{B.11}$$

It can be noted that

$$[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z; \tag{B.12}$$
$$[\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm, \tag{B.13}$$

so that these satisfy the same algebra as the angular momentum operators.

Now the angular momentum states $|j, m\rangle$ may be elegantly constructed out of these fundamental oscillator operators vide

$$|j, m\rangle = \frac{(\hat{a}_+^\dagger)^{j+m} (\hat{a}_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}} |0_+, 0_-\rangle \tag{B.14}$$

as this satisfies

$$\hat{J}_z|j, m\rangle = m|j, m\rangle \tag{B.15}$$
$$\hat{J}_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1}|j, m \pm 1\rangle. \tag{B.16}$$

Here $|0_+, 0_-\rangle$ is the vacuum state defined as

$$\hat{a}_+|0_+, 0_-\rangle = 0; \hat{a}_-|0_+, 0_-\rangle = 0. \tag{B.17}$$

It is as if a symmetric state is built out of $2j$ spin one-half quantities, $(j+m)$ of them 'up' and $(j-m)$ of them 'down' giving

$$m = \frac{1}{2}(j+m) + (-\frac{1}{2})(j-m). \tag{B.18}$$

$(j+m)!$ and $(j-m)!$ stands for the number of permutations giving the normalization. Now let us take the atomic coherent state given in Eq. $(B.6)$ and express it in terms of the Schwinger construction.

$$|j, \chi\rangle = \frac{1}{(1+|\chi|^2)^j} \sum_{n=0}^{2j} \sqrt{^{2j}C_n} \chi^n \frac{(\hat{a}_+^\dagger)^{2j-n} (\hat{a}_-^\dagger)^n}{\sqrt{(2j-n)!n!}} |0_+, 0_-\rangle \tag{B.19}$$

$$= \frac{1}{\sqrt{(2j)!}} \sum_{n=0}^{2j} {}^{2j}C_n \frac{\chi^n}{(1+|\chi|^2)^j} (\hat{a}_+^\dagger)^{2j-n} (\hat{a}_-^\dagger)^n |0_+, 0_-\rangle \tag{B.20}$$

$$= \frac{1}{\sqrt{(2j)!}} \left(\frac{\chi}{\sqrt{1+|\chi|^2}} \hat{a}_-^\dagger + \frac{1}{\sqrt{1+|\chi|^2}} \hat{a}_+^\dagger \right)^{2j} |0_+, 0_-\rangle. \tag{B.21}$$

Putting

$$\chi = \tan(\theta/2)e^{i\phi}, \tag{B.22}$$

we obtain

$$|j,\chi\rangle = |j,\theta,\phi\rangle = e^{i\phi j}\frac{\left(\cos\frac{\theta}{2}e^{-i\phi/2}\hat{a}_+^\dagger + \sin\frac{\theta}{2}e^{i\phi/2}\hat{a}_-^\dagger\right)^{2j}}{\sqrt{(2j)!}}|0_+,0_-\rangle. \tag{B.23}$$

Removing the overall phase factor $e^{i\phi j}$ we obtain

$$|j,\chi\rangle = |j,\theta,\phi\rangle = \frac{\left(\cos\frac{\theta}{2}e^{-i\phi/2}\hat{a}_+^\dagger + \sin\frac{\theta}{2}e^{i\phi/2}\hat{a}_-^\dagger\right)^{2j}}{\sqrt{(2j)!}}|0_+,0_-\rangle. \tag{B.24}$$

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C. Apendix-III

C.1. A Pseudo-Hermitian Operator

An operator $\hat{A}$ is said to be non-Hermitian if

$$\hat{A} \neq \hat{A}^{\dagger}, \tag{C.1}$$

where $\hat{A}^{\dagger}$ is the Hermitian adjoint of $\hat{A}$. A non-Hermitian operator, in general, possesses non real eigenvalues. However, a class of non-Hermitian operators exist which possess only real eigenvalue spectrum. The necessary condition for an operator to have real eigenvalue spectrum is that it should be pseudo-Hermitian [1]. An operator $\hat{A}$ is said to be pseudo-Hermitian if there exists a Hermitian, linear and invertible operator $\hat{\eta}$ such that

$$\hat{\eta}\hat{A}\hat{\eta}^{-1} = \hat{A}^{\dagger}. \tag{C.2}$$

The operator $\hat{A}$ is then said to be η-pseudo-Hermitian. An operator $\hat{L}$ is said to be linear if for any two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ and any two complex numbers c_1 and c_2 we have

$$\hat{L}\Big[c_1|\psi_1\rangle + c_2|\psi_2\rangle\Big] = c_1\hat{L}|\psi_1\rangle + c_2\hat{L}|\psi_2\rangle. \tag{C.3}$$

It is invertible if there exists $\hat{L}^{-1}$. The eigenvalues of a pseudo-Hermitian operator are either real or in the form of complex conjugate pairs.

The necessary and sufficient condition for an operator admitting a complete set of discrete biorthonormal eigenvectors to have only real eigenvalues is that the operator must be $\hat{O}^{\dagger}\hat{O}$-pseudo-Hermitian, where $\hat{O}$ is a linear and invertible operator [2].

The existence of a complete set of discrete biorthonormal eigenvectors for an operator $\hat{A}$ means that there exists a set of vectors $\{|\psi_n\rangle, |\phi_n\rangle\}$ such that

$$\hat{A}|\psi_n\rangle = E_n|\psi_n\rangle \tag{C.4}$$

and

$$\hat{A}^{\dagger}|\phi_n\rangle = E_n^{\star}|\phi_n\rangle. \tag{C.5}$$

and

$$\begin{aligned} \langle \phi_m | \psi_n \rangle &= \delta_{mn}, & \text{(C.6)} \\ \sum_n |\psi_n\rangle\langle\phi_n| &= \sum_n |\phi_n\rangle\langle\psi_n| = \mathbf{1}, & \text{(C.7)} \end{aligned}$$

where, $\mathbf{1}$ in the equation above is the identity operator.

We now present the proof of the condition mentioned above.

Proof

Let there be an operator $\hat{A}$, which acts in a Hilbert space $\mathcal{H}$. We assume that the spectrum of $\hat{A}$ is discrete and it admits a complete set of discrete biorthonormal eigenvectors $\{|\psi_n\rangle, |\phi_n\rangle\}$. Let the Hilbert space is spanned by a complete set of basis vectors $\{|n\rangle\}$. Therefore,

$$\langle m|n\rangle = \delta_{mn} \tag{C.8}$$

and

$$\sum_n |n\rangle\langle n| = \mathbf{1}. \tag{C.9}$$

We define two invertible and linear operators $\hat{O}_1$ and $\hat{H}_0$ in the Hilbert space $\mathcal{H}$ as

$$\hat{O}_1 = \sum_n |\psi_n\rangle\langle n| \tag{C.10}$$

and

$$\hat{H}_0 = \sum_n E_n |n\rangle\langle n|. \tag{C.11}$$

Then the inverse of $\hat{O}$ is given as

$$\hat{O}_1^{-1} = \sum_n |n\rangle\langle\phi_n|. \tag{C.12}$$

Now,

$$\begin{aligned} \hat{O}_1^{-1}\ \hat{A}\ \hat{O}_1 &= \sum_n |n\rangle\langle\phi_n|\hat{A}\sum_m |\psi_m\rangle\langle m| \\ &= \sum_{n,m} E_m |n\rangle\langle\phi_n|\psi_m\rangle\langle m| \\ &= \sum_n E_n |n\rangle\langle n| \\ &= \hat{H}_0. & \text{(C.13)} \end{aligned}$$

Therefore,

$$\hat{O}_1^{-1}\ \hat{A}\ \hat{O}_1 = \hat{H}_0, \tag{C.14}$$

where we have used Eq. $(C.6)$ in the third step. Now suppose that the spectrum of $\hat{A}$ is real. Then E_n are real implying that $\hat{H}_0$ is Hermitian. Now taking the adjoint of both sides of Eq. $(C.14)$ we obtain,

$$\begin{aligned} \hat{O}_1^\dagger\ \hat{A}^\dagger\ \left(\hat{O}_1^{-1}\right)^\dagger &= \hat{H}_0^\dagger \\ &= \hat{H}_0. \end{aligned} \tag{C.15}$$

Therefore,

$$\hat{O}_1^\dagger\ \hat{A}^\dagger\ \left(\hat{O}_1^{-1}\right)^\dagger = \hat{O}_1^{-1}\ \hat{A}\ \hat{O}_1. \tag{C.16}$$

Operating by $\left(\hat{O}_1^\dagger\right)^{-1}$ from left on both sides of the above equation we obtain

$$\hat{A}^\dagger\left(\hat{O}_1^{-1}\right)^\dagger = \left(\hat{O}_1^\dagger\right)^{-1}\hat{O}_1^{-1}\hat{A}\hat{O}_1. \tag{C.17}$$

As

$$\left(\hat{O}_1^{-1}\right)^\dagger = \left(\hat{O}_1^\dagger\right)^{-1}, \tag{C.18}$$

therefore, Eq. $(C.17)$ becomes

$$\hat{A}^\dagger\left(\hat{O}_1^\dagger\right)^{-1} = \left(\hat{O}_1^\dagger\right)^{-1}\hat{O}_1^{-1}\hat{A}\hat{O}_1. \tag{C.19}$$

Operating by $\hat{O}_1^\dagger$ from right on both sides of the above equation we obtain

$$\begin{aligned} \hat{A}^\dagger &= \left(\hat{O}_1^\dagger\right)^{-1}\hat{O}_1^{-1}\ \hat{A}\ \hat{O}_1\hat{O}_1^\dagger \\ &= \left(\hat{O}_1^{-1}\right)^\dagger\hat{O}_1^{-1}\ \hat{A}\ \hat{O}_1\hat{O}_1.^\dagger \end{aligned} \tag{C.20}$$

Since $\hat{O}_1$ is invertible, we let

$$\hat{O}_1^{-1} = \hat{O}. \tag{C.21}$$

Therefore, we can write Eq. $(C.20)$ as

$$\begin{aligned} \hat{A}^\dagger &= \hat{O}^\dagger\hat{O}\ \hat{A}\ \hat{O}^{-1}\left(\hat{O}^{-1}\right)^\dagger \\ &= \hat{O}^\dagger\hat{O}\hat{A}\hat{O}^{-1}\left(\hat{O}^\dagger\right)^{-1} \\ &= \hat{O}^\dagger\hat{O}\ \hat{A}\ \left(\hat{O}^\dagger\hat{O}\right)^{-1}. \end{aligned} \tag{C.22}$$

Thus, the operator $\hat{A}$ is $\hat{O}^\dagger\hat{O}$ pseudo-Hermitian. This completes the proof.

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D. Apendix-IV

D.1. Reduced Wigner D-Matrix Elements

We present here the derivation of the reduced Wigner D-matrix elements using Schwinger's algebra for angular momentum [1]. We also discuss some properties of these matrix elements [2].

The angular momentum state can be represented using Schwinger algebra as [Apendix II],

$$|j,m\rangle = \frac{(\hat{a}_+^\dagger)^{j+m}(\hat{a}_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0_+,0_-\rangle. \tag{D.1}$$

We apply the rotaion operator

$$\hat{D}(\beta) = \exp(-i\frac{\hat{J}_y\beta}{\hbar}), \tag{D.2}$$

on $|j,m\rangle$ and obtain

$$\hat{D}(\beta)|j,m\rangle = \hat{D}(\beta)\frac{(\hat{a}_+^\dagger)^{j+m}(\hat{a}_-^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0_+,0_-\rangle. \tag{D.3}$$

Inserting the identity operator

$$\hat{D}(\beta)^{-1}\hat{D}(\beta) = \hat{1} \tag{D.4}$$

we obtain,

$$\hat{D}(\beta)|j,m\rangle = \frac{\left[\hat{D}(\beta)\hat{a}_+^\dagger\hat{D}^{-1}(\beta)\right]^{j+m}\left[\hat{D}(\beta)\hat{a}_-^\dagger\hat{D}^{-1}(\beta)\right]^{j-m}}{\sqrt{(j+m)!(j-m)!}}\hat{D}(\beta)|0_+,0_-\rangle. \tag{D.5}$$

Now

$$\hat{D}(\beta)|0_+,0_-\rangle = \left[1 - i\frac{\hat{J}_y\beta}{\hbar} - \frac{\hat{J}_y^2\beta^2}{2!\hbar^2} + ...\right]|0_+,0_-\rangle. \tag{D.6}$$

We know from Apendix-II that,

$$\hat{J}_y = \frac{\hbar}{2i}\left(\hat{a}_+^\dagger\hat{a}_- - \hat{a}_-^\dagger\hat{a}_+\right). \tag{D.7}$$

As,

$$\hat{a}_+|0_+,0_-\rangle = 0 \tag{D.8}$$

and

$$\hat{a}_-|0_+,0_-\rangle = 0, \tag{D.9}$$

therefore,

$$\begin{aligned}\hat{J}_y|0_+,0_-\rangle &= \frac{1}{2i}\left(\hat{a}_+^\dagger\hat{a}_- - \hat{a}_-^\dagger\hat{a}_+\right)|0_+,0_-\rangle \\ &= 0.\end{aligned} \tag{D.10}$$

Using these results we can write Eq. $(D.6)$ as

$$\hat{D}(\beta)|0_+,0_-\rangle = |0_+,0_-\rangle. \tag{D.11}$$

Now,

$$\hat{D}(\beta)\ \ \hat{a}_\pm^\dagger\ \ \hat{D}^{-1}(\beta) = \exp(-i\frac{\hat{J}_y\beta}{\hbar})\ \ \hat{a}_\pm^\dagger\ \ \exp(i\frac{\hat{J}_y\beta}{\hbar}). \tag{D.12}$$

According to Baker-Hausdorff lemma, presented in Apendix-I, we can write

$$\begin{aligned}\exp(-i\frac{\hat{J}_y\beta}{\hbar})\ \ \hat{a}_\pm^\dagger\ \ \exp(i\frac{\hat{J}_y\beta}{\hbar}) &= \hat{a}_\pm^\dagger + (-i\beta)\left[\frac{\hat{J}_y}{\hbar},\hat{a}_\pm^\dagger\right] + \frac{(-i\beta)^2}{2!}\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\hat{a}_\pm^\dagger\right]\right] + \ldots\ldots \\ &+ \ldots\ldots\frac{(-i\beta)^n}{n!}\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\ldots\left[\frac{\hat{J}_y}{\hbar},\hat{a}_\pm^\dagger\right]\right]\right]\right]\ldots\ldots\end{aligned} \tag{D.13}$$

Now

$$\begin{aligned}\left[\frac{\hat{J}_y}{\hbar},\hat{a}_+^\dagger\right] &= -\frac{1}{2i}\left[\hat{a}_-^\dagger\hat{a}_+,\hat{a}_+^\dagger\right] \\ &= -\frac{1}{2i}\hat{a}_-^\dagger\end{aligned} \tag{D.14}$$

$$\begin{aligned}\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\hat{a}_+^\dagger\right]\right] &= -\left[\frac{\hat{J}_y}{\hbar},\frac{\hat{a}_-^\dagger}{2i}\right] \\ &= \frac{1}{4}\hat{a}_+^\dagger\end{aligned} \tag{D.15}$$

$$\begin{aligned}\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\hat{a}_+^\dagger\right]\right]\right] &= \left[\frac{\hat{J}_y}{\hbar},\frac{\hat{a}_+^\dagger}{4}\right] \\ &= -\frac{1}{8i}\hat{a}_-^\dagger\end{aligned} \tag{D.16}$$

$$\begin{aligned}\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\left[\frac{\hat{J}_y}{\hbar},\hat{a}_+^\dagger\right]\right]\right]\right] &= -\left[\frac{\hat{J}_y}{\hbar},\frac{\hat{a}_-^\dagger}{8i}\right] \\ &= \frac{1}{16}\hat{a}_+^\dagger\end{aligned} \tag{D.17}$$

and so on. Thus using these results we obtain from Eq. $(D.13)$ as

$$\begin{aligned}
\hat{D}(\beta)\ \ \hat{a}_+^\dagger\ \ \hat{D}^{-1}(\beta) &= \exp(-i\frac{\hat{J}_y\beta}{\hbar})\ \ \hat{a}_+^\dagger\ \ \exp(i\frac{\hat{J}_y\beta}{\hbar}) \\
&= \hat{a}_+^\dagger + \frac{\beta}{2}\hat{a}_-^\dagger - \frac{1}{2!}\left(\frac{\beta}{2}\right)^2 \hat{a}_+^\dagger \\
&- \frac{1}{3!}\left(\frac{\beta}{2}\right)^3 \hat{a}_-^\dagger + \frac{1}{4!}\left(\frac{\beta}{2}\right)^4 \hat{a}_+^\dagger + ... \\
&= \hat{a}_+^\dagger \cos\left(\frac{\beta}{2}\right) + \hat{a}_-^\dagger \sin\left(\frac{\beta}{2}\right). \qquad (D.18)
\end{aligned}$$

Similarly,

$$\hat{D}(\beta)\ \ \hat{a}_-^\dagger\ \ \hat{D}^{-1}(\beta) = \hat{a}_-^\dagger \cos\left(\frac{\beta}{2}\right) - \hat{a}_+^\dagger \sin\left(\frac{\beta}{2}\right). \qquad (D.19)$$

Substituting Eqs. $(D.18), (D.19)$ and $(D.11)$ into Eq. $(D.5)$ we obtain

$$\hat{D}(\beta)|j,m\rangle = \frac{\left(\hat{a}_+^\dagger \cos\frac{\beta}{2} + \hat{a}_-^\dagger \sin\frac{\beta}{2}\right)^{j+m} \left(\hat{a}_-^\dagger \cos\frac{\beta}{2} - \hat{a}_+^\dagger \sin\frac{\beta}{2}\right)^{j-m}}{\sqrt{(j+m)!(j-m)!}} |0_+, 0_-\rangle. \qquad (D.20)$$

Recalling the binomial theorem

$$\left(x+y\right)^r = \sum_k \frac{r! x^{r-k} y^k}{(r-k)!k!} \qquad (D.21)$$

we can write Eq. $(D.20)$ as

$$\begin{aligned}
\hat{D}(\beta)|j,m\rangle &= \sum_k \sum_l \frac{(j+m)!(j-m)!}{(j+m-k)!k!(j-m-l)!l!} \\
&\times \frac{\left[\hat{a}_+^\dagger \cos(\beta/2)\right]^{j+m-k} \left[\hat{a}_-^\dagger \sin(\beta/2)\right]^k}{\sqrt{(j+m)!(j-m)!}} \\
&\times \left[-\hat{a}_+^\dagger \sin(\beta/2)\right]^{j-m-l} \left[\hat{a}_-^\dagger \cos(\beta/2)\right]^l |0_+, 0_-\rangle. \qquad (D.22)
\end{aligned}$$

Now using the identity operator

$$\sum_{m'=-j}^{j} |j,m'\rangle\langle j,m'| = \hat{1} \qquad (D.23)$$

we can write,

$$
\begin{aligned}
\hat{D}(\beta)|j,m\rangle &= \sum_{m'=-j}^{j} |j,m'\rangle\langle j,m'|\hat{D}(\beta)|j,m\rangle \\
&= \sum_{m^{\dagger}=-j}^{j} |j,m'\rangle d^{j}_{m'm}(\beta) \\
&= \sum_{m'=-j}^{j} d^{j}_{m'm}(\beta)\frac{(\hat{a}^{\dagger}_{+})^{j+m'}(\hat{a}^{\dagger}_{-})^{j-m'}}{\sqrt{(j+m')!(j-m')!}}|0_{+},0_{-}\rangle \qquad \text{(D.24)}
\end{aligned}
$$

where we have used Eq. $(D.1)$ in the last step.

Now comparing Eqs. $(D.22)$ and $(D.24)$ we see that

$$
\begin{aligned}
\textstyle\sum_k \sum_l \quad & \frac{(j+m)!(j-m)!}{(j+m-k)!k!(j-m-l)!l!} \times \frac{\Big[\hat{a}^{\dagger}_{+}\cos(\beta/2)\Big]^{j+m-k}\Big[\hat{a}^{\dagger}_{-}\sin(\beta/2)\Big]^{k}}{\sqrt{(j+m)!(j-m)!}} \\
\times \quad & \Big[-\hat{a}^{\dagger}_{+}\sin(\beta/2)\Big]^{j-m-l}\Big[\hat{a}^{\dagger}_{-}\cos(\beta/2)\Big]^{l} \\
= \quad & \sum_{m'=-j}^{j} d^{j}_{m'm}(\beta)\frac{(\hat{a}^{\dagger}_{+})^{j+m'}(\hat{a}^{\dagger}_{-})^{j-m'}}{\sqrt{(j+m')!(j-m')!}}. \qquad \text{(D.25)}
\end{aligned}
$$

We can obtain an explicit form for $d^{j}_{m'm}(\beta)$ by equating the coefficient of $(\hat{a}^{\dagger}_{+})^{j+m'}(\hat{a}^{\dagger}_{-})^{j-m'}$ from both sides of the above equation. We first equate the coefficients of $(\hat{a}^{\dagger}_{+})^{j+m'}$ for a fixed m' from both sides. The power of $\hat{a}^{\dagger}_{+}$ on the left hand side of the above equation takes the value $j+m'$ when the existing power that is $2j-k-l$ is equal to $j+m'$, that is when we have,

$$
\begin{aligned}
j+m' &= 2j-k-l \\
or, \qquad l &= j-k-m'. \qquad \text{(D.26)}
\end{aligned}
$$

We can verify that when the above equation is satisfied the power of $\hat{a}^{\dagger}_{-}$ on the left hand side of Eq. $(D.25)$ matches with that in the right hand side. For a fixed value of m' Eq. $(D.26)$ indicates that the k-sum and the l-sum on the left hand side of Eq. $(D.25)$ are not independent. To find out the coefficient of $(\hat{a}^{\dagger}_{+})^{j+m'}(\hat{a}^{\dagger}_{-})^{j-m'}$ from the left hand side of Eq. $(D.25)$ we eliminate l in the exponents of $\cos(\beta/2), \sin(\beta/2)$ and (-1) with the help of Eq. $(D.26)$ and obtain,

$$
\begin{aligned}
j+m-k+l &= 2j-2k+m-m' \qquad \text{(D.27)} \\
k+j-m-l &= 2k-m+m' \qquad \text{(D.28)} \\
j-m-l &= k-m+m'. \qquad \text{(D.29)}
\end{aligned}
$$

Using these equations we can now write the coefficient of $(\hat{a}^{\dagger}_{+})^{j+m'}(\hat{a}^{\dagger}_{-})^{j-m'}$ from the left hand side of Eq. $(D.25)$ as

$$\begin{aligned}\sum_k \quad & (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!}}{k!(j+m-k)!(j-m'-k)!(m'-m+k)!} \\ \times \quad & \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k+m'-m}. \end{aligned} \tag{D.30}$$

The coefficient of $(\hat{a}_+^\dagger)^{j+m'}(\hat{a}_-^\dagger)^{j-m'}$ on the right hand side of Eq. (20) is

$$d^j_{m'm}(\beta)\frac{1}{\sqrt{(j+m')!(j-m')!}}. \tag{D.31}$$

Equating the two expressions given in $(D.30)$ and $(D.31)$ we can write

$$\begin{aligned} d^j_{m'm}(\beta) &= (-1)^{m'-m}\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!} \\ &\times \sum_k \frac{(-1)^k(\cos\frac{\beta}{2})^{2j-2k-m'+m}(\sin\frac{\beta}{2})^{2k+m'-m}}{k!(j-m'-k)!(j+m-k)!(m'-m+k)!}, \end{aligned} \tag{D.32}$$

where the sum over k is taken until none of the arguments of factorials in the denominator are negative.

D.2. Symmetry Property

We can check that

$$d^j_{mm}(-\beta) = d^j_{mm}(\beta). \tag{D.33}$$

D.3. Addition Theorem

The addition theorem for the $d^j_{mm'}(\beta)$ is

$$\sum_{m'} d^j_{mm'}(\beta_1)d^j_{m'm''}(\beta_2)e^{-im'\phi} = e^{-im\alpha}d^j_{mm''}(\beta)e^{-im''\gamma} \tag{D.34}$$

with α, β and γ given by

$$\cot\alpha = \cos\beta_1\cot\phi + \cot\beta_2\frac{\sin\beta_1}{\sin\phi}, \tag{D.35}$$

$$\cos\beta = \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2\cos\phi, \tag{D.36}$$

$$\cot\gamma = \cos\beta_2\cot\phi + \cot\beta_1\frac{\sin\beta_2}{\sin\phi}. \tag{D.37}$$

D.4. Second Derivative

To handle the second derivative of the Wigner reduced matrix occuring in the calculation of the expectation value of $\hat{J}_z^2$ in the squeezed state we use the differential equation satisfied by $D^j_{mm'}(\alpha,\beta,\gamma)$ familiar from the quantum mechanics of an anisotropic rotor

$$\left[-\frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\beta}+\frac{m^2-2mm'\cos\beta+m'^2}{\sin^2\beta}\right]D^j_{mm'}(\alpha,\beta,\gamma)=j(j+1)D^j_{mm'}(\alpha,\beta,\gamma). \tag{D.38}$$

Putting $m'=m$ and $\alpha=\gamma=0$ we obtain

$$\frac{d^2}{d\beta^2}\,d^j_{mm}(\beta)=-j(j+1)d^j_{mm}(\beta)+m^2\sec^2(\beta/2)\,d^j_{mm}(\beta)-\cot\beta\frac{d}{d\beta}\,d^j_{mm}(\beta), \tag{D.39}$$

wherein putting $\beta=2i\xi$ and using the definition of Δ as

$$\Delta=d^j_{mm}(2i\xi), \tag{D.40}$$

we obtain

$$\frac{d^2\Delta}{d\xi^2}=4j(j+1)\Delta-4\frac{m^2\Delta}{\cosh^2\xi}+2\coth 2\xi\frac{d\Delta}{d\xi}. \tag{D.41}$$

References

[1] J. J. Sakurai in *Modern Quantum Mechanics*, *Modern Quantum Mechanics* Addison-Wesley Publishing Company, Inc.

[2] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, in *"Quantum Theory of Angular Momentum"*, (World Scientific, 1988).

E. Apendix-V

E.1. Derivation of the Effective Hamiltonian for the Atom Field Interaction in a Highly Detuned Cavity

We consider a system of N two-level atoms interacting collectively with a single mode electromagnetic field in a cavity [1]. Let $\hbar\omega_0$ be the energy gap between the two atomic energy levels where ω_0 is the angular frequency of a photon emitted or absorbed when the transition among the two levels take place. The characteristic frequency of the electromagnetic field is ω_c. If the raising and lowering operators for the transition among the two atomic levels are $\hat{J}_+$ and $\hat{J}_-$ respectively with $\hat{J}_z$ as the inversion operator (already intrduced in Chapter-1) and $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively for the photon numbers in the Fock state of the electromagnetic field then the atom-field interaction is governed by the Hamiltonian operator

$$\hat{H} = \hbar\omega_0 \hat{J}_z + \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar g(\hat{J}_+ \hat{a} + \hat{a}^\dagger \hat{J}_-). \tag{E.1}$$

Here g is the atom field coupling constant. The irreversible loss of the electromagnetic field due to leakage out of the cavity is described by the operator

$$\hat{\Lambda}_f \hat{\rho} = \kappa(\bar{n}+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \kappa\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \tag{E.2}$$

where $\hat{\rho}$ is the density operator for the whole system, 2κ represents the loss of the photons and $\bar{n}$ is the average number of thermal photons in the cavity. The dynamics of the system in a frame rotating with ω_0 is described by the master equation

$$\frac{d\hat{\rho}}{dt} = \hat{L}_{af}\hat{\rho} + \hat{L}_f\hat{\rho}. \tag{E.3}$$

The operators $\hat{L}_{af}\hat{\rho}$ and $\hat{L}_f\hat{\rho}$ are

$$\hat{L}_{af}\hat{\rho} = -ig\Big[\hat{J}_+\hat{a} + \hat{a}^\dagger_-, \rho\Big] \tag{E.4}$$

and

$$\hat{L}_f\hat{\rho} = -i\delta_c\Big[\hat{a}^\dagger\hat{a}, \hat{\rho}\Big] + \hat{\Lambda}_f\hat{\rho} \tag{E.5}$$

with

$$\delta_c = \omega_c - \omega_0. \tag{E.6}$$

We now concentrate on the atomic system and for that we derive an equation for the reduced density matrix ρ_a for the atomic system. We write the formal solution of Eq. $(E.3)$ as

$$\begin{aligned} \hat{\rho}(t) &= e^{\hat{L}_f t}\hat{\rho}(0) + \int_0^t dt_1 e^{[\hat{L}_f(t-t_1)]}\hat{L}_{af}\hat{\rho}(0) \\ &+ \int_0^t dt_1 \int_0^{t_1} dt_2 e^{[\hat{L}_f(t-t_1)]}\hat{L}_{af} e^{[\hat{L}_f(t_1-t_2)]}\hat{L}_{af}\hat{\rho}(t_2). \end{aligned} \tag{E.7}$$

The operator $\hat{L}_{af}$ determines the rate of exchange of energy between the atoms and field which for large N is of the order of $g\sqrt{N}$. The rate of the process described by the operator $\hat{L}_f$ is of the order of $|i\delta_c + \kappa|$. If $|i\delta_c + \kappa| >> g\sqrt{N}$ then the evolution of the field is dominated by $\hat{L}_f$ and we may assume that the field remains for all times adiabatically in the state determined by $\hat{L}_f$. If the time scale of observation is much longer than $|i\delta_c + \kappa|$ then that state is the steady state $\hat{\rho}_f^{ss}$ of $\hat{L}_f$. This state is the steady state of thermal equilibrium and is given by

$$\hat{\rho}_f^{ss} = e^{-\beta\hat{a}^\dagger\hat{a}}/Tr\Big[e^{-\beta\hat{a}^\dagger\hat{a}}\Big], \tag{E.8}$$

where

$$e^{-\beta} = \frac{\bar{n}}{\bar{n}+1}. \tag{E.9}$$

We write the density operator $\hat{\rho}(t)$ in Eq. $(E.7)$ as the outer product of the density matrix $\hat{\rho}_a(t)$ of the atomic system and steady state density matrix $\hat{\rho}_f^{ss}$ for the field. After taking the trace of the two sides of Eq. $(E.7)$ over the electromagnetic field states we get

$$\frac{d\hat{\rho}_a}{dt} = \int_0^t dt_1 Tr_f\Big\{\hat{L}_{af} e^{[\hat{L}_f(t-t_1)]}\hat{L}_{af}(t_1)\hat{\rho}_a(t_1)\hat{\rho}_f^{ss}\Big\}. \tag{E.10}$$

Here $\hat{L}_{af}(t_1)$ is the Liouville operator $\hat{L}_{af}$ in the interaction picture. We evaluate the above integral using the results

$$\begin{aligned} e^{\hat{L}_f t}\hat{a}\hat{\rho} &= e^{i\delta_c t}\Big[\{(\bar{n}+1)e^{\kappa t} - \bar{n}e^{-\kappa t}\}\hat{a}\hat{\rho} \\ &- \bar{n}\{e^{\kappa t} - e^{-\kappa t}\}\hat{\rho}\hat{a}\Big] \end{aligned} \tag{E.11}$$

and

$$\begin{aligned} e^{\hat{L}_f t}\hat{\rho}\hat{a} &= e^{i\delta_c t}\Big[(\bar{n}+1)\{e^{\kappa t} - e^{-\kappa t}\}\hat{a}\hat{\rho} \\ &- \{\bar{n}e^{\kappa t} - (\bar{n}+1)e^{-\kappa t}\}\hat{\rho}\hat{a}\Big] \end{aligned} \tag{E.12}$$

along with their Hermitian conjugates and obtain

$$\begin{aligned} \frac{d\hat{\rho}_a}{dt} &= \frac{g^2}{\kappa^2+\delta_c^2}\Big[-i\delta_c\{[\hat{J}_+\hat{J}_-,\hat{\rho}_a]+2\bar{n}[\hat{J}_z,\hat{\rho}_a]\} \\ &+ \kappa\{(\bar{n}+1)(2\hat{J}_-\hat{\rho}_a\hat{J}_+-\hat{J}_+\hat{J}_-\hat{\rho}_a-\hat{\rho}_a\hat{J}_+\hat{J}_-) \\ &+ \bar{n}(2\hat{J}_+\hat{\rho}_a\hat{J}_--\hat{J}_-\hat{J}_+\hat{\rho}_a-\hat{\rho}_a\hat{J}_-\hat{J}_+)\}\Big]. \end{aligned} \tag{E.13}$$

We also have assumed that the atomic density matrix $\hat{\rho}_a(t)$ evolves slowly on the scale $1/\omega_0$. If $\delta_c >> \kappa$ then the contribution due to damping in Eq. $(E.13)$ is very small and hence negligible and therefore, the above equation reduces to

$$\frac{d\hat{\rho}_a}{dt} = -i\eta\Big[\hat{J}_+\hat{J}_- + 2\bar{n}\hat{J}_z, \hat{\rho}_a\Big] \tag{E.14}$$

where,

$$\eta = \frac{g^2\delta_c}{\kappa^2+\delta_c^2} \tag{E.15}$$

and $Ng^2 << \kappa^2+\delta_c^2$. For a given value of $\sqrt{N}g$ and κ, the adiabatic condition can be satisfied by increasing the detuning $|\omega_0-\omega_c|$. Thus in a cavity, highly detuned from the atomic transition frequency, the evolution of the atomic system is governed by the effective Hamiltonian

$$\hat{H}_{eff} = \hbar\eta\Big[\hat{J}_+\hat{J}_- + 2\bar{n}\hat{J}_z\Big]. \tag{E.16}$$

References

[1] G. S. Agarwal, R. R. Puri and R. P. Singh *Phys. Rev. A* **56**, 2249 (1997).