TURBULENCE IN ROTATING FLUIDS

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by

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"Turbulence in Rotating Fluids"**, submitted by **Sagar Chakraborty** — a Post-B.Sc-Integrated-Research (PBIR, 2003) student with registration number: 138081511004 — for the award of Ph.D.(Science) degree of West Bengal University of Technology, is absolutely based upon his own work under the joint supervision of **Prof. Jayanta Kumar Bhattacharjee** and **Dr. Partha Guha** and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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To

My Friend

Ashish Bakshi

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LIST OF PUBLICATIONS

This thesis is based on the following papers and preprints:

1. arXiv:physics/0701201, *Signatures of two-dimensionalisation of 3D turbulence in presence of rotation*

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   Europhysics Letters **79**, 14002 (2007)

2. arXiv:physics/0701052, *On the third order structure function for rotating 3D homogeneous turbulent flow*

   Sagar Chakraborty and J.K. Bhattacharjee

   Physical Review E **76**, 036304 (2007)

3. arXiv:cond-mat/0611520, *On the use of Kolmogorov-Landau approach in deriving various correlation functions in 2-D incompressible turbulence*

   Sagar Chakraborty

   Physics of Fluids **19**, 085110 (2007)

4. arXiv:0801.0346, *Two point third order correlation functions for quasi-geostrophic turbulence: Kolmogorov-Landau approach*

   Sagar Chakraborty

   Physics of Fluids **20**, 075106 (2008)

5. arXiv:0809.0377, *Role of third-order structure function in studying two-dimensionalisation of turbulence*

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6. *Dominance of rare events in some problems in statistical physics*

   Arnab Saha, **Sagar Chakraborty** and J. K. Bhattacharjee

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7. arXiv:0806.2906, *A critical note on one-eighth law in 2D turbulence*

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8. arXiv:0803.2273, *Shell model for rotating turbulence and two-dimensionalisation effect*

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9. arXiv:0811.0265, *Is turbulence as simple as tossing a coin?*

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Chapter 1

INTRODUCTION

1.1 What is turbulence?

Turbulence is everywhere in the natural world from blood rushing through our veins to galaxy formation in outer space. It can be seen when we pour milk into a cup of tea or when oil flows through a pipeline. But despite its ubiquitous nature, turbulence has stymied some of the greatest minds in physics and is only just beginning to be understood by scientists. But what exactly is turbulence? Well, according to Webster's "New International Dictionary", turbulence means: agitation, commotion, disturbance..... This definition is however too general. Actually, turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and position, so that statistically distinct average values can be discerned. By the way, fluid means anything that flows and it necessarily doesn't mean only the liquids; even air is a fluid. Osborne Reynolds, one of the pioneers in the study of turbulent flows, named this type of motion 'sinuous motion'. Basically, turbulence initiates from an instability in a laminar or streamlined flow; by this we mean that in a flow which is devoid

Figure 1.1: **Cigarette smoke**. Note how a turbulent region is formed above the burning tip.

of any randomness in its velocity field (e.g., water flowing slowly through a pipe), if one somehow introduces a perturbation, the perturbation grows and the streamlined nature of the flow is eventually destroyed *in toto* and what remains is a turbulent flow that is random, or in more fashionable terms chaotic. One may recall such a transition in the case of the smoke from the tip of a live cigarette placed quietly in an ashtray; though the smoke is streamlined upto a certain height up in the air from the tip of the cigarette, it does become random afterwards at a larger distance (see figure-1.1).

The turbulent velocity field can be thought of as being made of many eddies of different sizes. By the size of an eddy we mean the order of magnitude of the distances over which the velocity varies appreciably. The input energy is usually fed into the system in a way to produce the largest eddies. Kolmogorov in 1941 had the great intuition to realise that these large eddies can feed energy to the smaller eddies and these in turn feed the still smaller eddies, resulting in a cascade of energy from the larger eddies to the smaller ones. This modern concept about turbulence in very low viscous fluid started to evolve

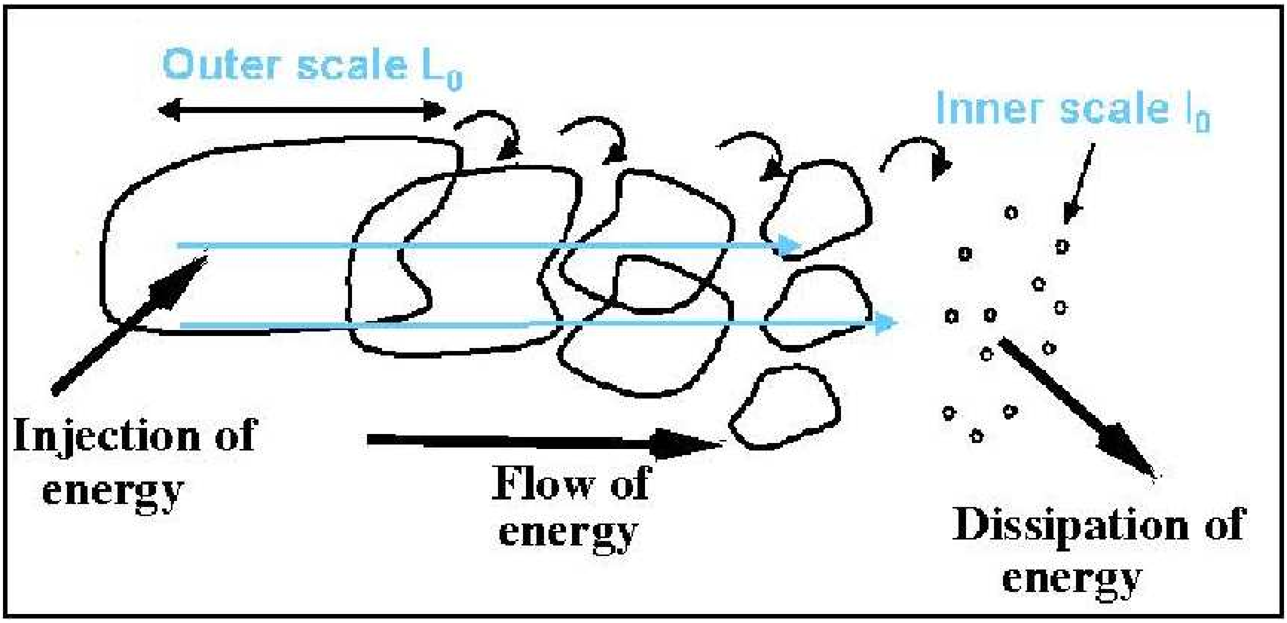

Figure 1.2: **Richardson's cascade**. A schematic diagram depicting how a stationary turbulence is maintained in the cascade model.

with the eminent scientist Richardson's insightful contributions which contained the famous poem that paraphrased Jonathan Swift, the author of Gulliver's Travels: *Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity — in the molecular sense*. In this way Richardson conveyed an image of the creation of turbulence by large scale forcing, setting up a cascade of energy transfers to smaller and smaller scales by the nonlinearities of fluid motion, until the energy dissipates at small scales by viscosity, turning into heat. This picture (see figure-1.2) led in time to innumerable theoretical cascade models that tried to capture the statistical physics of turbulence by assuming some thing or other about the cascade process. Indeed, no one in their right mind is interested in the full solution of the turbulent velocity field at all points in space-time. The interest is in the statistical properties of the turbulent flow. Interestingly enough, L. da Vinci is credited with the first studies of turbulence — the unstable flow of a liquid or gas — after observing eddies in a stream in the sixteenth century. The sketch (see figure-2.3) is drawn by him about 500

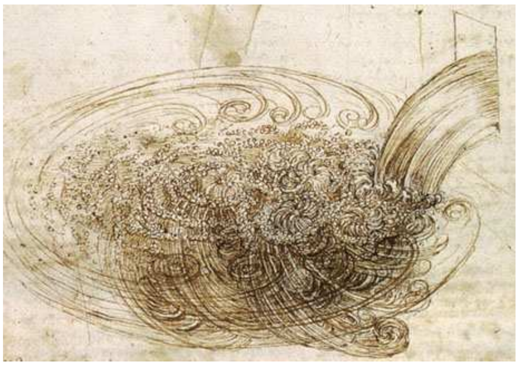

Figure 1.3: **L. da Vinci's sketch.** The sketch of a turbulent region as observed and drawn by Leonardo di ser Piero da Vinci (April 15, 1452 – May 2, 1519), an Italian polymath: scientist, mathematician, engineer, inventor, anatomist, painter, sculptor, architect, musician, and writer.

years ago — a free water jet issuing from a square hole into a pool — represents perhaps the world's first use of visualization as a scientific tool to study a turbulent flow. Some experts even argue that the already had the vision necessary to propose the cascade process in turbulence which was (re-?)established by Richardson in, as late as, 1922.

1.1.1 A Definition

Now if the above paragraph was too simple for some of the readers, they might get a more technical idea in what follows. The difficulty in solving the problem of turbulence starts with the fact that till now no one has even given a strict mathematical definition of turbulence; following humouristic, probably sarcastic, remark of the noted physicist Lesieur (1987) speaks volume for this: "*Turbulence is a dangerous topic which is at the origin of serious fights in scientific meetings since it represents extremely different points of view, all or which have in common their complexity, as well as an inability to solve the problem. It is even difficult to agree on what exactly is the problem to be solved.*" Anyways, some of the defining characteristics of the turbulent flows are the following:

- **Randomness**: Turbulent flows seem irregular, chaotic, and unpredictable.
- **Vorticity**: Turbulence is characterized by high levels of fluctuating vorticity. Vorticity, roughly speaking, is a mathematical concept giving a measure of swirl if present in a flow. The identifiable whirling structures in a turbulent flow are vaguely called eddies. Technique for flow visualization of turbulent flows show various structures — coalescing, dividing, stretching, and above all spinning. A characteristic feature of turbulence is the existence of an enormous range of eddy sizes. The largest of the eddies have a size of order of the width of the region of turbulent flow. The large eddies contain most of the energy. The energy is handed down from large to small eddies by nonlinear interactions.
- **Nonlinearity**: Turbulent flows are highly nonlinear. The nonlinearity serves two purposes. First, it causes the relevant nonlinearity parameter (say the Reynolds number), to exceed a critical value so that the flow eventually reaches a chaotic state. Second, the nonlinearity of a turbulent flow results in the interaction between various aforementioned eddies; it is due to this very property that the small eddies are created from the bigger ones and hence, the input energy is cascaded down to smaller and smaller eddies until it is dissipated into heat by viscous diffusion in the smallest eddies, whose size is of the order of millimeters. Non-linearity can be intuitively understood by comparing the cases of two light-beams (which are governed by linear equations) falling on each other with two streams of fluid flowing into each others path. In the first case the light-beams will simply pass through each other as if the other had not been present, whereas in the second case the streams would collide and become chaotic in the region of overlap.
- **Diffusivity**: Due to the macroscopic mixing of fluid particles, turbulent flows are characterized by a rapid rate of diffusion of momentum and heat. In this context,

it is worth recalling the simple observation: how quickly in a rapidly stirred tea (thus, turbulent tea!), a spoonful of sugar placed at the base of the cup mixes compared to the heap of sugar kept unstirred in a cup of tea.

- **Dissipation**: In a real fluid, there always is viscosity of the fluid to act on the smaller eddies in order to dissipate the kinetic energy of the eddies into heat energy. Turbulent flows therefore require a continuous supply of energy to make up for the viscous losses. This should be apparent, for, if we shake a pot of water making it turbulent and leave it to itself, turbulence will die out. However, a continuously shaken pot of water will remain turbulent i.e., a stationarity in the turbulence is maintained only by dint of continuous supply of energy.

These features of turbulence suggest that many flows that seem random, such as gravity waves in the ocean or in the atmosphere, are not turbulent because they are not dissipative, vortical, and nonlinear. Physicists more or less agree on such an elaborate definition of turbulence.

1.1.2 We And Turbulence

We now shall reflect on how deeply our daily life, industries and our entire existence is connected to turbulence making the study of this ubiquitous phenomenon so important. The combustion of fossil fuels remains a key technology for the foreseeable future. It is therefore important that we understand the mechanisms of combustion and, in particular, the role of turbulence within this process. Combustion always takes place within a turbulent flow field for two reasons: turbulence increases the mixing process and enhances combustion, but at the same time combustion releases heat which generates flow instability through buoyancy, thus enhancing the transition to turbulence.

The flow conditions in many industrial equipment (such as pipes, ducts, precipitators, gas scrubbers, etc.) and machines (for instance, internal combustion engines and gas turbines) is generally turbulent. When designing piping systems, turbulent flow requires a higher input of energy from a pump (or fan) than laminar flow. However, for applications such as heat exchangers and reaction vessels, turbulent flow is essential for good heat transfer and mixing.

The external flow, producing drag or friction, over all kind of vehicles such as cars, airplanes, ships and submarines is also turbulent; designing of vehicles is dependent on it. Again, it might be of surprise for the readers to know that it actually is the turbulence generated around the fast moving balls that causes some types of swing of cricket ball, tennis ball, football, baseball *etc*.

Turbulence in the atmosphere may be caused by many factors e.g., storms that generate powerful and unexpected gusts, high winds blowing past mountains and other obstructions, heating off the surface that produces plumes of rising air, interaction of strong winds in the jet stream and weather fronts, etc. Because of this many of us are all too familiar with the occasional bumps and lurches that come with flying on an airplane. Turbulence, the stealthiest of weather hazards, can strike from a sky that is literally clear and blue. It costs airlines in astronomical units each year because of injuries and operational disruptions, such as delays and rerouting.

The amount of turbulence in the ambient atmosphere has a major effect on the dispersion of air pollution plumes because turbulence increases the entrainment and mixing of unpolluted air into the plume and thereby acts to reduce the concentration of pollutants in the plume (i.e, enhances the plume dispersion). It is therefore important to categorize the amount of atmospheric turbulence present at any given time. Everyone knows that air-pollution is a burning problem of the present era.

Earth's oceans form a complex web of physical processes. The actions of the oceans, which cover over 70 percent of its surface, regulate the planets climate. In order for the ocean model, designed to predict oceanic behaviour i.e., to reproduce the behaviour of the real ocean, these models must realistically capture the ocean physics. To generate a realistic model one must take into account the turbulent transport of heat, salt and momentum. Also, weather-forecasters can hope to be dead accurate only if they have full understanding of the turbulence prevalent in the atmosphere.

Last but not the least, the phenomenon of turbulence lends its helping hand to the rotating molten core of the earth for the production of the magnetic field armouring the entire living world of the earth against the impact of many harmful charged cosmic particles.

1.1.3 Turbulence: The Unsolved Problem

Such an omnipresent phenomenon of fluid turbulence, in spite of the fact that best of the minds on earth have been scratching their head over it for many years, still remains an unsolved problem in the world of physics, aeronautics, mechanical engineering, combustion, meteorology, etc. According to an apocryphal story Nobel laureate Werner Heisenberg was asked what he would ask God, given the opportunity. His reply was: "*When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first.*" A similar witticism has been attributed to Horace Lamb (who had published a noted text book on Hydrodynamics) - his choice being quantum mechanics (instead of relativity) and turbulence. Lamb was quoted as saying in a speech to the British Association for the Advancement of Science, "*I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.*" Turbulence is also, in the words of the legendary physicist Richard Feynman, "*the last great unsolved* *problem of classical physics*".

Mathematicians and physicists believe that an explanation for the turbulence can be found through an understanding of solutions to the Navier-Stokes equations. Navier-Stokes equations is nothing but a continuum form of the famous Newtons laws which today is a secondary-school level topic. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations. In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven Prize Problems, one of them is about the solutions of the Navier-Stokes equations. The Scientic Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7, 000, 000 prize fund for the solution to these problems, with $1, 000, 000 allocated to each. Announcement of such a prize itself suggests how hard it is to solve turbulence although one hopes that one day this elusive, although down-to-earth, problem will be solved to the hilt.

1.2 TWO-DIMENSIONALISATION EFFECT

Rotation, in the face of the discovery of two-dimensionalisation effect, has emerged as a parameter that can progressively make a three-dimensional (3D) turbulent flow look like a two-dimensional (2D) or a quasi-2D turbulent flow. The phrase 'look like' basically means that certain properties of 3D turbulence, such as wavenumber dependence of energy spectrum[1], direction of energy cascade *etc.*, become such that they give impression that the flow is getting two-dimensionalised. In view of the fact that the

[1]Appendix-A discusses, defines and 'derives' energy spectrum of homogeneous isotropic 3D turbulence.

dynamics of oceans, atmospheres, liquid planetary cores, fluid envelopes of stars and, other bodies of astrophysical and geophysical interest do require an understanding of inherent properties of turbulence in the rotating frame of reference, the problem of two-dimensionalisation is of central interest to any serious scientist; turbulence in rotating bodies is even of some industrial and engineering interest.

In the steady non-turbulent flow, for the case of low Rossby number ($Ro = U/2L\Omega$; U is typical velocity scale, L is typical length scale and Ω is the rotation rate *i.e.*, the angular velocity) and high Reynolds number ($Re = UL/\nu$; ν is the kinematic viscosity of the fluid), Taylor-Proudman theorem[1] argues that rotation 'two-dimensionalises' the flow. This argument is often carelessly extended to turbulent flows to explain the rotation induced two-dimensionalisation arising therein. The two-dimensionalisation of the 3D turbulent flow in presence of rotation has begun to be understood as a subtle non-linear effect which is distinctly different from Taylor-Proudman effect.

Cambon *et al.*[2] have showed that in the presence of rotation, the transfer of energy from small to high wavenumbers is inhibited; at the same time, the strong angular dependence of this effect leads to a draining of the spectral energy from the parallel to the normal wave vectors (w.r.t. the rotation axis) showing a trend towards two-dimensionalisation. Waleffe[3] has used helical decomposition of the velocity field to study the nature of triad interactions in homogeneous turbulence and coupling it with the instability assumption predicted a transfer of energy toward wave vectors perpendicular to the rotation axis under rapid rotation. The helical decomposition turns out to be very handy to deal with rapidly rotating turbulent flow. In that case the linear eigensolutions of the problem, the so-called inertial waves, have the structure of helical modes. The assumption about the triadic transfers, coupled with resonance condition for non-linear interaction between inertial waves, show that there will be a tendency toward non-linear two-dimensionalisation of the flow.

Simulations by Smith *et al.*[4] speak volumes for the two-dimensionalisation effect. They show the coexistence of inverse cascade (a typical feature of 2D turbulence) and forward cascade in forced rotating turbulence within a periodic box of small aspect ratio. In the simulations, the ratio of the mean rates of energy dissipated to the energy injected decreased almost linearly, for Ro less than a critical value, with decrease in Ro (increase in angular velocity $|\vec{\Omega}|$). By the way, a very recent numerical study[5] shows similar transition from stratified to quasi-geostrophic turbulence, manifested by the emergence of an inverse cascade – a conclusion that agrees with that of Lindborg[6].

Although recent experiments by Baroud *et al.*[7, 8] and Morize *et al.*[9, 10] have shed some light on the two-dimensionalisation effect, the scaling of two-point statistics and energy spectrum in rotating turbulence remains a controversial topic. Zhou[11], in analogy with MHD turbulence, has proposed an energy spectrum $E(k) \sim k^{-2}$ for rapidly rotating 3D turbulent fluid (also see [12]) and this does seem to be validated by some experiments[7, 8] and numerical simulations[13, 14, 15, 16]. But some experiments[9] do not tally with this proposed spectrum. They predict steeper than k^{-2} spectrum and this again seem to be drawing some support from numerical results[17, 18] and analytical results found using wave turbulence theory[19, 20].

Unbiasedly speaking, if one wishes angular velocity to become a relevant parameter in constructing the energy spectrum $E(k)$, simple dimensional analysis would lead one to:

$$E(k) \propto \Omega^{\frac{3m-5}{2}} \varepsilon^{\frac{3-m}{2}} k^{-m} \tag{1.1}$$

where m is a real number and ε is the rate of dissipation of energy per unit mass. m should be restricted within the range 5/3 to 3 to keep the exponents of Ω and ε in relation (1.1) positive. The two limits $m = 5/3$ and $m = 3$ corresponds to isotropic homogeneous 3D turbulence and 2D turbulence respectively. The spectrum due to Zhou — $E(k) \sim k^{-2}$

— is due an intermediate value of $m = 2$. So, as far as the present state of the literature on rotating turbulence goes, two-dimensionalisation of 3D turbulence would mean the dominance of a spectrum which goes towards $E(k) \sim k^{-3}$ and which may choose to settle at $E(k) \sim k^{-2}$ — an issue yet to be fully resolved.

1.3 OUTLINE OF THE THESIS

As a turbulent flow can be treated as the manifestation of a random velocity field, one actually hopes to unveil the statistical properties of the flow rather than every other detail of the flow. This means that we basically are after some probability distribution for the flow: Even today, this remains a tough nut to crack. However, the knowledge of the structure functions (also called velocity correlation functions defined in the next chapter) assists one to take the first step towards finding the distribution. The structure functions are experimentally measurable and hence are of extreme practical importance. The scaling relations of the structure functions, thus, are the lynchpins of turbulence theory although uncertainty lingers as to their general validity and the details of the derivation as far as the present status of research in turbulence is concerned. Therefore, naturally a lot of time and effort are spent by the scientists working in the field of turbulence to determine the exact forms for these functions and to study various phenomena in their light.

Sturcture functions have their own reasons to be preferred over corresponding spectral quantities. Besides the fact that they are more readily evaluable from an experimental data, they also facilitate rather direct connection between the concept of eddy or scale and the result of the experimental measurement: One might note, by reflecting on the fact that the link between wavenumber and length is not direct, the disadvantage which the spectral analysis harbours in this respect. Again, only a complete specification of the velocity field allows one to measure the fluxes of energy and enstrophy from the spectral analysis. Naturally, one would prefer to measure the structure functions, especially third-order structure function, that can yield the value of the fluxes — thanks to certain rare rigorous relations — bypassing the requirement of knowing the entire velocity field. In this thesis, we shall deal with the phenomenon of two-dimensionalisation of 3D incompressible high Reynold's number fluid turbulence and try to see what can be said about it from the study of structure functions, especially S_3 (to be defined soon). Basically, herein we shall comprehensively review the works[21, 22, 23, 24, 25, 26] done in this direction by the author and observe how the method of calculating structure functions, developed by Kolmogorov and subsequently Landau, serves as the cornerstone for studying the two-dimensionalisation effect from the angle adopted by the author.

In chapter-2, we shall look at various correlation functions, which include those that involve both the velocity and the vorticity fields, in two-dimensional (2D) isotropic homogeneous unforced turbulence. We shall adopt the more intuitive approach due to Kolmogorov (and subsequently, Landau in his text on fluid dynamics) and show how the 2D turbulence's results, obtainable using other methods, may be established in a simpler way. Same method will be used again in chapter-3 to calculate some third-order structure functions for quasi-geostrophic (QG) turbulence for the forward cascade of pseudo-potential enstrophy and the inverse energy cascade in quasi-geostrophic turbulence. These results will motivate us to study the two-point third order structure function in the context of the two-dimensionalisation effect. Consequent studies (reported in chapter-4) shall enable us to give a reason for the inverse energy cascade in the two-dimensionalised rapidly rotating three dimensional (3D) incompressible turbulence. For such a system, literature shows a possibility of the exponent of wavenumber in the energy spectrum's relation to lie between -2 and -3. We shall argue the existence of a stricter range of -2 to -7/3 for the exponent in the case of rapidly rotating turbulence which is in accordance with the recent experiments. Also, a derivation for the two point third order structure function will be provided helping one to argue that even with slow rotation one gets, although dominated, a spectrum with the exponent -2.87, thereby hinting at the initiation of the two-dimensionalisation effect with rotation. Moreover, using the Gledzer-Ohkitani-Yamada (GOY) shell model, modified appropriately for rotation, these signatures of two-dimensionalisation effect will be verified in chapter-5. The concluding chapter (chapter-6) will contain relevant discussions and conclusions. One of the two appendices accompanying these chapters will assist the readers to grasp the idea of energy spectrum (used widely throughout this dissertation) in the rather simple context of homogeneous, isotropic, incompressible high Reynolds number turbulence. This very appendix will also help one to recall the Kolmogorov phenemenology. In the last appendix, we shall delve into an elegant way of looking into the intermittency in turbulence and witness that a mere coin toss can model it to an astonishing extent.

Chapter 2

2D TURBULENCE

*I*t may be unanimously accepted that Kolmogorov's four-fifths law[27] is a landmark in the theory of turbulence because it is an exact non-trivial result. In three spatial dimensions, this law says that the two-point third order velocity correlation function behaves as:

$$S_3 \equiv \left\langle \left[\left\{ \vec{v}(\vec{x}+\vec{l}) - \vec{v}(\vec{x}) \right\} . \frac{\vec{l}}{|\vec{l}|} \right]^3 \right\rangle = -\frac{4}{5}\varepsilon l \tag{2.1}$$

where ε is the rate per unit mass at which energy is being transferred through the inertial range. $\vec{v}$ is the velocity field. The inertial range is the intermediate spatial region postulated by Kolmogorov where the large scale disturbances (flow maintaining mechanisms) and the molecular scale viscous dissipation play no part. This result is of such central significance that attempts are regularly made to understand it afresh and to extend it in other situations involving turbulence. There appears to be following two primary methods[28] of obtaining this result:

1. The original Kolmogorov method put forward in details in the fluid dynamics text due to Landau and Lifshitz[29]. There is no external forcing in this approach, and

the equality of dissipation rate and forcing rate for the energy is never enforced.

2. A field-theoretic technique that invokes the so-called 'dissipation anomaly' in the high Reynold's number fluid turbulence. In this approach, there is an external forcing that maintains a steady state turbulence.

The two approaches yield the same important results as they should.

If one goes by the standard procedure given in the book by Frisch[30] to derive the form of the correlation function in d-D turbulence with the assumption of forward energy cascade, one would land up on[31]:

$$S_3 = -\frac{12}{d(d+2)}\varepsilon l \tag{2.2}$$

where ε is the mean rate of dissipation of energy per unit mass. This result is not quite true for the two-dimensional case since it gives for $d = 2$: $S_3 = -(3/2)\varepsilon l$ and not $S_3 = +(3/2)\varepsilon l$. This is so because the calculation doesn't take into account the conservation of enstrophy in 2D turbulence which causes the reverse cascade of energy[32]. It might be noted that $S_3 = (3/2)\varepsilon l$ for $d = 2$ is for the regime of scales larger than the forcing scale[21, 33].

Actually, if we consider the two-dimensional turbulence, then in the inviscid limit, we have two conserved quantities – energy and enstrophy. This gives rise to two fluxes with the enstrophy flux occurring from the larger to the smaller spatial scales. The energy flux goes in the reverse direction. Recently, Bernard[33] and Lindborg[34] have used the above mentioned techniques to obtain the third order structure function for both the energy and the enstrophy cascade regions in forced 2D turbulence. We believe that the issue is important enough that a derivation of these results using the Kolmogorov-Landau approach should be useful. This is what we have attempted here and our results do come out in agreement with them. Besides, we also have derived some other correlation functions which deal with vorticity fields in the inertial region and also some two-point second order correlation functions in the dissipative region following the arguments of Landau, thereby consolidating the equivalence between the two approaches mentioned in the beginning.

2.1 SECOND ORDER VELOCITY CORRELATION FUNCTION

It is a well-established fact that there exists a direct-cascade of enstrophy in 2D turbulence. One defines total enstrophy as $\Gamma = \frac{1}{2}\int_{\text{all space}} \omega^2 d^2\vec{\rho}$ where $\omega = \partial_x v_y - \partial_y v_x$ is the vorticity in the Cartesian coordinates; $\vec{v}$ being the velocity field. As we shall consider incompressible fluids only ($\vec{\nabla}.\vec{v} = 0$), we shall take density to be unity and let $\vec{\rho}$ take over the task of representing position vector in 2D plane. The enstrophy flows through the inertial range and gets dissipated near dissipation scale. Using the antisymmetric symbol $\varepsilon_{\alpha\beta}$ that has four components, viz. $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\varepsilon_{12} = -\varepsilon_{21} = 1$, one may define the mean rate of dissipation of enstrophy per unit mass as:

$$\eta \equiv \nu\langle\vec{\nabla}\omega.\vec{\nabla}\omega\rangle \tag{2.3}$$

$$\Rightarrow \qquad \eta = \nu\varepsilon_{\tau\alpha}\varepsilon_{\theta\beta}\langle(\partial_\tau\partial_\gamma v_\alpha)(\partial_\theta\partial_\gamma v_\beta)\rangle \tag{2.4}$$

Here, angular brackets denote an averaging procedure which averages over all possible positions of points 1 and 2 at a given instant of time and a given separation. Now, if $\vec{v}_1$ and $\vec{v}_2$ represent the fluid velocities at the two neighbouring points at $\vec{\rho}_1$ and $\vec{\rho}_2$ respectively, one may define rank two correlation tensor:

$$B_{\alpha\beta} \equiv \langle(v_{2\alpha} - v_{1\alpha})(v_{2\beta} - v_{1\beta})\rangle \tag{2.5}$$

For simplicity, we shall take a rather idealised situation of turbulence flow which is homogeneous and isotropic on every scale — a case achievable in practice in a vigorously-shaken-fluid left to itself. The component of the correlation tensor will obviously, then, be dependent on time, a fact which won't be shown explicitly in what follows. As the features of local turbulence is independent of averaged flow, the result derived below is applicable also to the local turbulence at a distance ρ much smaller than the fundamental scale. Isotropy and homogeneity suggests following general form for $B_{\alpha\beta}$

$$B_{\alpha\beta} = A_1(\rho)\delta_{\alpha\beta} + A_2(\rho)\rho^o_\alpha\rho^o_\beta \tag{2.6}$$

where A_1 and A_2 are functions of time and ρ. The Greek subscripts can take two values ρ and $\perp$ which respectively mean the component along the radial vector ρ and the component in the transverse direction. Einstein's summation convention will be used extensively. Also,

$$\vec{\rho} = \vec{\rho}_2 - \vec{\rho}_1, \qquad \rho^o_\alpha \equiv \rho_\alpha/|\vec{\rho}|, \qquad \rho^o_\rho = 1, \qquad \rho^o_\perp = 0 \tag{2.7}$$

using which in the relation (2.6), one gets:

$$B_{\alpha\beta} = B_{\perp\perp}(\delta_{\alpha\beta} - \rho^o_\alpha\rho^o_\beta) + B_{\rho\rho}\rho^o_\alpha\rho^o_\beta \tag{2.8}$$

One may break the relation (2.5) as

$$B_{\alpha\beta} = \langle v_{1\alpha}v_{1\beta}\rangle + \langle v_{2\alpha}v_{2\beta}\rangle - \langle v_{1\alpha}v_{2\beta}\rangle - \langle v_{2\alpha}v_{1\beta}\rangle \tag{2.9}$$

and defining $b_{\alpha\beta} \equiv \langle v_{1\alpha} v_{2\beta} \rangle$, one may proceed, keeping in mind the isotropy and the homogeneity, to write

$$B_{\alpha\beta} = \langle v^2 \rangle \delta_{\alpha\beta} - 2b_{\alpha\beta} \tag{2.10}$$

Again, having assumed incompressibility, one may write:

$$\partial_\beta B_{\alpha\beta} = 0 \tag{2.11}$$

$$\Rightarrow \quad B_{\perp\perp} = \rho B'_{\rho\rho} + B_{\rho\rho} \tag{2.12}$$

where the equation (2.8) has been used and prime (′) is allowed to denote derivative w.r.t. ρ. Near the dissipation region the flow is regular and its velocity varies smoothly which allows to expand v in a series of power of ρ. One must take $v \sim \rho^2$ neglecting the higher powers ($v \sim \rho$ is not taken because it leads to the contradictory result that $\eta = 0$ as can be seen from the relation (2.4)). So, treating a as a proportionality constant, let $B_{\rho\rho} = a\rho^4$, which means $B_{\perp\perp} = 5a\rho^4$ (using equation (2.12)) and hence,

$$\langle v_{1\alpha} v_{2\beta} \rangle = \frac{1}{2}\langle v^2 \rangle \delta_{\alpha\beta} - \frac{5}{2} a\rho^4 \delta_{\alpha\beta} + 2a\rho^2 \rho_\alpha \rho_\beta \tag{2.13}$$

$$\Rightarrow \quad \langle (\partial_{1\tau}\partial_{1\gamma} v_{1\alpha})(\partial_{2\theta}\partial_{2\gamma} v_{2\beta}) \rangle = -72a\delta_{\theta\tau}\delta_{\alpha\beta} + 24a\delta_{\beta\theta}\delta_{\alpha\tau} + 24a\delta_{\alpha\theta}\delta_{\beta\tau} \tag{2.14}$$

$$\Rightarrow \quad \varepsilon_{\tau\alpha}\varepsilon_{\theta\beta} \langle (\partial_\tau \partial_\gamma v_\alpha)(\partial_\theta \partial_\gamma v_\beta) \rangle = -192a \tag{2.15}$$

$$\Rightarrow \quad B_{\rho\rho} = -\frac{\eta\rho^4}{192\nu} \tag{2.16}$$

In the equation (2.16), we have put $\vec{\rho}_1 \approx \vec{\rho}_2$, for these relations are assumed to be valid for arbitrarily small ρ. While writing the relation (2.16), equation (2.4) has been recalled. This $B_{\rho\rho}$ is the two-point second order correlation function for enstrophy cascade in dissipation range.

2.2 Third order velocity correlation function

Now, we shall focus thoroughly on the inertial range. Let's again define:

$$b_{\alpha\beta,\gamma} \equiv \langle v_{1\alpha} v_{1\beta} v_{2\gamma} \rangle$$

Invoking homogeneity and isotropy once again along with the symmetry in the first pair of indices, one may write the most general form of the third rank Cartesian tensor for $b_{\alpha\beta,\gamma}$ as

$$b_{\alpha\beta,\gamma} = C(\rho)\delta_{\alpha\beta}\rho_\gamma^o + D(\rho)(\delta_{\gamma\beta}\rho_\alpha^o + \delta_{\alpha\gamma}\rho_\beta^o) + F(\rho)\rho_\alpha^o\rho_\beta^o\rho_\gamma^o \quad (2.17)$$

where, C, D and F are functions of ρ. Yet again, incompressibility dictates:

$$\frac{\partial}{\partial \rho_{2\gamma}} b_{\alpha\beta,\gamma} = \frac{\partial}{\partial \rho_\gamma} b_{\alpha\beta,\gamma} = 0 \quad (2.18)$$

$$\Rightarrow C'\delta_{\alpha\beta} + \frac{C}{\rho}\delta_{\alpha\beta} + \frac{2D}{\rho}\delta_{\alpha\beta} + \frac{2D'}{\rho^2}\rho_\alpha\rho_\beta - \frac{2D}{\rho^3}\rho_\alpha\rho_\beta + \frac{F'}{\rho^2}\rho_\alpha\rho_\beta + \frac{F}{\rho^3}\rho_\alpha\rho_\beta = 0 \quad (2.19)$$

Putting $\alpha = \beta$ in equation (2.19) one gets:

$$2C + 2D + F = \frac{\text{constant}}{\rho} = 0 \quad (2.20)$$

where, it as been imposed that $b_{\alpha\beta,\gamma}$ should remain finite for $\rho = 0$. Again, using equation (2.19), putting $\alpha \neq \beta$ and manipulating a bit one gets:

$$D = -\frac{1}{2}(\rho C' + C) \quad (2.21)$$

using which in relation (2.20), one arrives at the following expression for F:

$$F = \rho C' - C \tag{2.22}$$

Defining

$$\begin{aligned} B_{\alpha\beta\gamma} &\equiv \langle (v_{2\alpha} - v_{1\alpha})(v_{2\beta} - v_{1\beta})(v_{2\gamma} - v_{1\gamma}) \rangle \\ &= 2(b_{\alpha\beta,\gamma} + b_{\gamma\beta,\alpha} + b_{\alpha\gamma,\beta}) \end{aligned} \tag{2.23}$$

and putting relations (2.21) and (2.22) in the equation (2.23) and using relation (2.17), one gets:

$$B_{\alpha\beta\gamma} = -2\rho C'(\delta_{\alpha\beta}\rho^o_\gamma + \delta_{\gamma\beta}\rho^o_\alpha + \delta_{\alpha\gamma}\rho^o_\beta) + 6(\rho C' - C)\rho^o_\alpha\rho^o_\beta\rho^o_\gamma \tag{2.24}$$

$$\Rightarrow \qquad S_3 \equiv B_{\rho\rho\rho} = -6C \tag{2.25}$$

which along with relations (2.21), (2.22) and (2.17) yields the following expression:

$$b_{\alpha\beta,\gamma} = -\frac{S_3}{6}\delta_{\alpha\beta}\rho^o_\gamma + \frac{1}{12}(\rho S_3' + S_3)(\delta_{\gamma\beta}\rho^o_\alpha + \delta_{\alpha\gamma}\rho^o_\beta) - \frac{1}{6}(\rho S_3' - S_3)\rho^o_\alpha\rho^o_\beta\rho^o_\gamma \tag{2.26}$$

Navier-Stokes equation gives:

$$\frac{\partial}{\partial t}v_{1\alpha} = -v_{1\gamma}\partial_{1\gamma}v_{1\alpha} - \partial_{1\alpha}p_1 + \nu\partial_{1\gamma}\partial_{1\gamma}v_{1\alpha} \tag{2.27}$$

$$\frac{\partial}{\partial t}v_{2\beta} = -v_{2\gamma}\partial_{2\gamma}v_{2\beta} - \partial_{2\beta}p_2 + \nu\partial_{2\gamma}\partial_{2\gamma}v_{2\beta} \tag{2.28}$$

Multiplying equations (2.27) and (2.28) with $v_{2\beta}$ and $v_{1\alpha}$ respectively and adding subsequently, one gets the following after averaging the consequent result:

$$\begin{aligned}\frac{\partial}{\partial t}\langle v_{1\alpha}v_{2\beta}\rangle &= -\partial_{1\gamma}\langle v_{1\gamma}v_{1\alpha}v_{2\beta}\rangle - \partial_{2\gamma}\langle v_{2\gamma}v_{1\alpha}v_{2\beta}\rangle \\ &\quad -\partial_{1\alpha}\langle p_1 v_{2\beta}\rangle - \partial_{2\beta}\langle p_2 v_{1\alpha}\rangle\rangle \\ &\quad +\nu\partial_{1\gamma}\partial_{1\gamma}\langle v_{1\alpha}v_{2\beta} + \nu\partial_{2\gamma}\partial_{2\gamma}\langle v_{1\alpha}v_{2\beta}\rangle \qquad (2.29)\end{aligned}$$

Due to isotropy, the correlation function for the pressure and velocity, ($\langle p_1\vec{v}_2\rangle$), should have the form $f(\rho)\vec{\rho}/|\vec{\rho}|$. But since $\partial_\alpha\langle p_1 v_{2\alpha}\rangle = 0$ due to solenoidal velocity field, $f(\rho)\vec{\rho}/|\vec{\rho}|$ must have the form constant $\times$ $(\vec{\rho}/|\vec{\rho}|^2)$ that in turn must vanish to keep correlation functions finite even at $\rho = 0$. Thus, equation (2.29) can be written as:

$$\frac{\partial}{\partial t}b_{\alpha\beta} = \partial_\gamma(b_{\alpha\gamma,\beta} + b_{\beta\gamma,\alpha}) + 2\nu\partial_\gamma\partial_\gamma b_{\alpha\beta} \qquad (2.30)$$

2.2.1 One-Eighth Law

For isotropic and homogeneous turbulence, the condition of incompressibility gives the easily derivable well-known result:

$$4\partial_\gamma b_{\alpha\gamma,\alpha} = \partial_\gamma B_{\alpha\alpha\gamma} \qquad (2.31)$$

Defining $W \equiv \langle\omega_1\omega_2\rangle$ and noting that $W = -\partial_\delta\partial_\delta b_{\alpha\alpha}$, we get from relations (2.30) and (2.31):

$$-\frac{\partial W}{\partial t} = \frac{1}{2}\partial_\delta\partial_\delta(\partial_\gamma B_{\alpha\alpha\gamma}) - 2\nu\partial_\delta\partial_\delta W \qquad (2.32)$$

Again, if one defines $\Omega \equiv \langle(\omega_2 - \omega_1)(\omega_2 - \omega_1)\rangle$ (which is not to be confused with the rotation rate discussed earlier), for homogeneous isotropic turbulence one may write $\Omega = 2\langle\omega^2\rangle - 2W$. So, equation (2.32) can be manipulated into the following:

$$\frac{1}{2}\frac{\partial\Omega}{\partial t} - \frac{\partial\langle\omega^2\rangle}{\partial t} = \frac{1}{2}\partial_\delta\partial_\delta\partial_\gamma B_{\alpha\alpha\gamma} + \nu\partial_\delta\partial_\delta\Omega - 2\nu\partial_\delta\partial_\delta\langle\omega^2\rangle \tag{2.33}$$

$$\Rightarrow \quad \partial_\delta\partial_\delta\partial_\gamma B_{\alpha\alpha\gamma} = 4\eta \tag{2.34}$$

$$\Rightarrow \quad B_{\alpha\alpha\rho} = \frac{1}{4}\eta\rho^3 \tag{2.35}$$

Here, we have assumed $\frac{\partial\Omega}{\partial t}$ to be relatively negligible and let $\nu \to 0$ so that the terms proportional to ν vanish. Also, we have recalled that $\frac{1}{2}\frac{\partial\langle\omega^2\rangle}{\partial t} = -\eta$. From equations (2.17) and (2.23), and the condition of incompressibility, it readily follows that $B_{\perp\perp\rho} = \frac{\rho}{3}\frac{\partial}{\partial\rho}B_{\rho\rho\rho}$ putting which in expression (2.35) and integrating subsequently (keeping in mind that $B_{\rho\rho\rho}$ shouldn't blow up at $\rho = 0$), we arrive at:

$$B_{\rho\rho\rho} = +\frac{1}{8}\eta\rho^3 \tag{2.36}$$

This is the one-eighth law for the unforced 2D incompressible turbulence proved using the Kolmogorov-Landau approach.

Perhaps, it would be quite instructive to provide a justification for the assumption: $|(1/2)(\partial\Omega/\partial t)| \ll 2\eta$. Setting $\partial\Omega/\partial t \approx 0$ in equation (2.33), needed to arrive at equation (2.34), is sometimes recognized as imposing the condition of quasi-stationarity. The velocity field in the 2D turbulent fluid system may be decomposed as $\vec{v} = \langle\vec{v}\rangle + \vec{v}'$; $\vec{v}'$ being the fluctuating part. This irregular random motion may be regarded as the superposition of so-called eddies of different sizes (or scales). The largest eddies (in case of enstrophy-cascade-dominated region they are of the size $\sim l_i$, say) have the largest amplitudes. The velocity in them is of the order of the variation of $\langle v\rangle$, Δ, over the length

l_i. The frequency corresponding to such an eddy, obviously, is $f \sim \langle v \rangle / l_i$. Smaller eddies that may be associated with larger frequencies ($> f$) are known to have much smaller amplitudes ($< \Delta$). Based on this picture, we may conclude what follows.

Whereas, over time intervals shorter than f^{-1}, the velocity does not vary too much, over longer intervals ($\sim f^{-1}$) velocity variations of the order Δ may be realised. This, in turn, implies that the two-point vorticity correlation function (which, by definition, can be directly linked with the velocity field) varies appreciably only over a temporal interval f^{-1}. Therefore, within the inertial range one can safely ignore variation of Ω with respect to t; this, by the way, can also be treated as an implicit assumtion that assists one to define the forward inertial range.

2.2.2 Yet Another Relation

Let us go back to equation (2.30). Using expressions (2.10) and (2.26), one can rewrite the equation as:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle v^2 \rangle - \frac{1}{2}\frac{\partial}{\partial t} B_{\rho\rho} = \nu \partial_\gamma \partial_\gamma \langle v^2 \rangle + \frac{1}{6\rho^3}\frac{\partial}{\partial \rho}\left(\rho^3 B_{\rho\rho\rho}\right) - \frac{\nu}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial B_{\rho\rho}}{\partial \rho}\right) \tag{2.37}$$

As we are interested in the enstrophy cascade, the first term in the R.H.S. is zero due to homogeneity and the first term in the L.H.S. is zero because of energy remains conserved in 2D turbulence in the inviscid limit (and of course, it is the high Reynolds number regime that we are interested in); it cannot be dissipated at smaller scales. Also, as we are interested in the forward cascade which is dominated by enstrophy cascade, on the dimensional grounds in the inertial region $B_{\rho\rho}$ (as it may depend only on η and ρ) may be written as:

$$\frac{\partial}{\partial t} B_{\rho\rho} = A \eta \rho^2 \tag{2.38}$$

where A is a numerical proportionality constant. Hence, using the relation (2.38), the equation (2.37) reduces to the following differential equation:

$$\frac{1}{6\rho^3}\frac{\partial}{\partial\rho}\left(\rho^3 B_{\rho\rho\rho}\right) = \frac{\nu}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial B_{\rho\rho}}{\partial\rho}\right) - \frac{A}{2}\eta\rho^2 \tag{2.39}$$

which when solved using expression (2.25) in the limit of infinite Reynolds number ($\nu \to 0$), one gets

$$B_{\rho\rho\rho} = -\frac{A\eta}{2}\rho^3 \tag{2.40}$$

Comparing it with the expression (2.36) for the two-point third order velocity correlation function for the isotropic and homogeneous 2D unforced turbulence in the inertial range of the forward enstrophy cascade, we determine the value of A to be -1/4. Therefore, relation (2.38) yields

$$\frac{\partial}{\partial t}B_{\rho\rho} = -\frac{1}{4}\eta\rho^2 \tag{2.41}$$

This, to the best of our knowledge, is yet another exact new result that has to be verified experimentally and numerically to test its validity.

2.2.3 Forward Inertial Range

We shall now try, rather quantatively, to define the range for which the one-eighth law has been derived.

In forced turbulence the length scales can be partitioned into following five ranges: (i) the energy dissipation range where the infrared viscosity, if it exists, drains away the energy from the system, (ii) the inertial range for the inverse energy cascade, (iii) the forcing scale l_i wherein the energy and the enstrophy are injected into the system, (iv) the forward inertial range for enstrophy cascade, and (v) the enstrophy dissipation range ($\leq l_d$, say) where molecular viscosity acts to dissipate enstrophy.

The case of decaying or unforced turbulent fluid system is essentially the same as for the forced one described above — the only difference being that the forcing has been stopped for good at a past moment of time. The ranges herein exist till the fluid comes to rest to become a non-turbulent state. The one-eighth law in the unforced 2D turbulence is expected to be valid in the fourth range: the forward inertial range for enstrophy cascade.

Let us extend the standard phenemenology meant for 3D turbulence into the subject of 2D turbulence. Let l denote the scale under consideration: $l_i \gg l \gg l_d$. Also, let v_l be the typical velocity associated with the scale l. Naturally, the eddy turnover time should be $t_l \sim l/v_l$ and the typical vorticity $\omega_l \sim v_l/l$. So, the enstrophy flux which flows from a scale l to smaller scales is $\Pi_l \sim \omega_l^2/t_l \sim v_l^3/l^3$ and this has been assumed equal to η. Therefore, $v_l \sim \eta^{1/3} l$ and this implies $t_l \sim \eta^{-1/3}$ — a constant, very different from what one gets in 3D turbulence. Now, typical time for molecular viscosity to attenuate any exitation present at the scale l is $t_l^{(\nu)} \sim l^2/\nu$. If $t_l^{(\nu)} < t_l$, then viscous diffusion will turn out to be relevant and hence ν acts at scales $l \leq l_d$, where

$$l_d \sim \left(\frac{\nu^3}{\eta}\right)^{\frac{1}{6}} \tag{2.42}$$

So far, so good. Now, it might be recalled that to get equation (2.33) from equation (2.34), we had actually, without stating, assumed that

$$|\nu \nabla^2 \langle \omega^2 \rangle| \ll \eta \tag{2.43}$$

This assumption allows one to locate the lower end of the inertial range. Note that

$$|\nu\nabla^2\langle\omega^2\rangle| \ll \frac{\nu}{l^2}\langle\omega^2\rangle = \frac{2\nu\overline{\Gamma}}{l^2} \tag{2.44}$$

where $\overline{\Gamma}$ is the mean enstrophy which, one hopes, exists. From expressions (2.42), (2.43) and (2.44), we arrive at:

$$l \gg \left(\frac{2\nu\overline{\Gamma}}{\eta}\right)^{\frac{1}{2}} \sim l_d\left(\frac{2\overline{\Gamma}^{\frac{1}{2}}}{\eta^{\frac{1}{3}}}\right) \tag{2.45}$$

Therefore, if we want the range, wherein the one-eighth law is valid, to be unaffected by the mechanisms of forcing and dissipation so as to award the law the status of universality, then such a forward inertial range may be written down as

$$l_i \gg l \gg l_d'; \qquad l_d' \equiv l_d\left(\frac{2\overline{\Gamma}^{\frac{1}{2}}}{\eta^{\frac{1}{3}}}\right) \tag{2.46}$$

It is in this very range where the one-eighth law is, supposedly, applicable.

2.2.4 Three-Halves Law

Again let us suppose that in the homogeneous isotropic fully-developed turbulence in two-dimensional space, energy is being supplied and the mean rate of injection of energy per unit mass is denoted by ε. Let us concentrate on the inverse energy cascade. Then technically we have to proceed as before and on doing so one would re-arrive at the differential equation (2.37); only that now the arguments would differ. In the larger scales viscosity is not as significant and anyway we shall be interested in the infinite Reynolds number case which would mean that the last term in the R.H.S. of equation (2.37) would go to zero. One obviously would also set $\frac{1}{2}\frac{\partial}{\partial t}\langle v^2\rangle = \varepsilon$ and lets assume $\frac{\partial}{\partial t}B_{\rho\rho} \approx$

0 in the inverse cascade regime, justification of which can be sought from the fact that the ultimate result that is obtained has been experimentally and numerically verified. So, we are left with the following differential equation:

$$\frac{1}{6\rho^3}\frac{\partial}{\partial\rho}\left(\rho^3 B_{\rho\rho\rho}\right) = \varepsilon \tag{2.47}$$

$$\Rightarrow \qquad B_{\rho\rho\rho} = +\frac{3}{2}\varepsilon\rho \tag{2.48}$$

where in the last step the integration constant has been set to zero to prevent $B_{\rho\rho\rho}$ from blowing up at $\rho = 0$. This equation (2.48) is the expression for two-point third order velocity correlation function of the energy cascade in the inverse inertial range.

2.3 SECOND ORDER VORTICITY CORRELATION FUNCTION

Recall that:

$$W \equiv \langle\omega_1\omega_2\rangle \tag{2.49}$$

$$\text{and,} \qquad \Omega \equiv \langle(\omega_2 - \omega_1)(\omega_2 - \omega_1)\rangle \tag{2.50}$$

and that due to homogeneity, Ω may be expressed as:

$$\Omega = 2\langle\omega^2\rangle - 2W \tag{2.51}$$

In the dissipation range: $v \sim \rho^2$ so, $\omega \sim \rho$ and hence we may, choosing a proportionality constant b (say), assume:

$$\Omega = b\rho^2 \tag{2.52}$$

Using relations (2.49), (2.51) and (2.52), one gets:

$$\langle \omega_1 \omega_2 \rangle = \langle \omega^2 \rangle - \frac{b}{2}\rho^2 \tag{2.53}$$

$$\Rightarrow \quad \langle (\partial_{1\alpha}\omega)(\partial_{2\alpha}\omega) \rangle = 2b \tag{2.54}$$

But we know,

$$\eta = \nu \langle (\partial_\alpha \omega)(\partial_\alpha \omega) \rangle \tag{2.55}$$

So, relation (2.54) would yield:

$$\eta = 2\nu b \tag{2.56}$$

where, we have put $\vec{\rho}_1 \approx \vec{\rho}_2$ in the relation (2.54), for, being in the dissipation range, these relations are assumed to be valid for arbitrarily small ρ. Combining relations (2.52) and (2.56), one arrives at a experimentally verifiable result for two-point second order vorticity correlation function in the dissipation range:

$$\Omega = \frac{\eta}{2\nu}\rho^2 \tag{2.57}$$

2.4 THIRD ORDER MIXED CORRELATION FUNCTION

Now, we wish to find two-point third order mixed correlation function in the inertial range of enstrophy cascade. We start by defining a two-point third order mixed correlation tensor in inertial range:

$$\Omega_\beta \equiv \langle (v_{2\beta} - v_{1\beta})(\omega_2 - \omega_1)(\omega_2 - \omega_1)\rangle \tag{2.58}$$

$$\Rightarrow \quad \Omega_\beta = 2M_\beta + 4W_\beta \tag{2.59}$$

where, $W_\beta \equiv \langle v_{1\beta}\omega_1\omega_2\rangle$ and $M_\beta \equiv \langle \omega_1\omega_1 v_{2\beta}\rangle$. Due due to isotropy and homogeneity, we can write following form for M_β:

$$M_\beta = M(\rho)\rho^o_\beta \tag{2.60}$$

$$\Rightarrow \quad \frac{\partial}{\partial \rho_{2\beta}} M_\beta = \langle \omega_1\omega_1\partial_{2\beta}v_{2\beta}\rangle = 0 \tag{2.61}$$

$$\Rightarrow \quad \frac{\partial}{\partial \rho} M(\rho) + \frac{M(\rho)}{\rho} = 0 \tag{2.62}$$

$$\Rightarrow \quad M(\rho) = \frac{\text{constant}}{\rho} = 0 \tag{2.63}$$

In the relation (2.61), we are assuming incompressibility and in writing the relation (2.63) we have taken into account the fact that M_β should remain finite when $\rho = 0$. Relations (2.60) and (2.63) imply that:

$$M_\beta = 0 \tag{2.64}$$

using which in the relation (2.59), we get:

$$\Omega_\beta = 4W_\beta \tag{2.65}$$

From the equations (2.27) and (2.28), we may write respectively:

$$\frac{\partial}{\partial t}\omega_1 = -v_{1\gamma}\partial_{1\gamma}\omega_1 + \nu\partial_{1\gamma}\partial_{1\gamma}\omega_1 \tag{2.66}$$

$$\frac{\partial}{\partial t}\omega_2 = -v_{2\gamma}\partial_{2\gamma}\omega_2 + \nu\partial_{2\gamma}\partial_{2\gamma}\omega_2 \tag{2.67}$$

Multiplying equations (2.66) and (2.67) by ω_2 and ω_1 respectively and adding subsequently, we get the following differential equation after averaging:

$$\frac{\partial}{\partial t}W = 2\partial_\beta W_\beta + 2\nu\partial_\beta\partial_\beta W \tag{2.68}$$

where we have used the fact $\partial_\beta = -\partial_{1\beta} = \partial_{2\beta}$. Using relations (2.51) and (2.65) in the equation (2.68), one gets for the inertial range for the enstrophy cascade in homogeneous, isotropic and fully-developed freely decaying turbulence in two-dimensional space in the infinite Reynolds number limit (*i.e.*, $\nu \to 0$) following differential equation:

$$\frac{\partial}{\partial t}\langle\omega^2\rangle - \frac{1}{2}\frac{\partial}{\partial t}\Omega = \frac{1}{2\rho}\frac{\partial}{\partial\rho}\left(\rho\Omega_\rho\right) \tag{2.69}$$

$$\Rightarrow \qquad \Omega_\rho = -2\eta\rho \tag{2.70}$$

In getting relation (2.70) from the equation (2.69), we have used the facts: $\frac{1}{2}\frac{\partial}{\partial t}\langle\omega^2\rangle = -\eta$ and $\frac{1}{2}\frac{\partial}{\partial t}\Omega \approx 0$ as it may be supposed that the value of Ω varies considerably with time only over an interval corresponding to the fundamental scale of turbulence and in relation to local turbulence the unperturbed flow may be regarded as steady which mean that for local turbulence one can afford to neglect $\frac{\partial}{\partial t}\Omega$ in comparison with the enstrophy dissipation rate η. This result (relation (2.70)) has gained importance by serving as the starting point in deriving various rigorous inequalities for short-distance scaling exponents in 2D incompressible turbulence[35].

Chapter 3

QG TURBULENCE

Quasi-geostrophic (QG) turbulence is a rather more realistic class of turbulent flow than the isotropic homogeneous 3D turbulence. It can be seen in the large scale flows on oceans and atmosphere; thus having profound geophysical and astrophysical significance. QG turbulence[36] stands somewhere in between 2D and 3D turbulences. Thus, naturally it is very appealing candidate that deserves study if one is interested in the two-dimensionalisation effect. In the inviscid limit, besides total energy, QG flows enjoy the possession of yet another conserved quantity which is conserved at the horizontal projection of the particle motion. We shall call this pseudo-potential vorticity to distinguish it from the potential vorticity that is conserved at a particle in a homentropic fluid. Defining pseudo-potential enstrophy as half the square of the pseudo-potential vorticity, one would say that like 2D turbulence there are two cascades — forward cascade of pseudo-potential vorticity and inverse cascade of energy — in QG turbulence which, however, is inherently three dimensional in nature.

Recently, a paper[37] has calculated some structure functions in QG turbulence and has made illuminating revelation that isotropy in the sense of Charney[36] is useless in deriving the structure functions for QG turbulence. It has gone on to show that formulation of QG turbulence under the constraint of axisymmetry is productive. However, it criticized (though somewhat rightly) the ineffectiveness of use of tensorial quantities in the case of QG turbulence in deriving the results. Now, manipulating the tensorial quantities are at the heart of the derivation of many important two-point velocity correlation functions and other ones[29]. The technique is very intuitive and straightforward. It has, recently, also been thoroughly used to find out various correlation functions for 2D turbulence[21]. In this chapter, we shall closely (and trickily) follow the original Kolmogorov method put forward in details in the fluid dynamics text due to Landau and Lifshitz[29]; and repeated in the ref.-([21]), to derive structure functions in QG turbulence. The method has the extra advantage to being able to probe into the form for the two-point third order velocity correlation function in the forward pseudo-potential enstrophy cascade regime — this has remained uninvestigated earlier in ref.-([37]).

3.1 THIRD ORDER MIXED CORRELATION FUNCTION

First of all, we shall briefly introduce the necessary equations (see ref.-([38]) for details). Let $\vec{u}$ be the three dimensional velocity field of the fluid in a frame rotating with constant angular velocity $\vec{\Omega}$. The fluid body (such as ocean) is assumed to be of uniform density with free surface at $z = \xi(x, y, t)$. Suppose the bottom $z = -H(x, y)$ is rigid. The shallow-water equations, then, are:

$$\frac{\partial h}{\partial t} + \vec{\nabla}.(\vec{v}h) = 0 \tag{3.1}$$

and,
$$\frac{D\vec{v}}{Dt} + \vec{f} \times \vec{v} = -g\vec{\nabla}\xi \tag{3.2}$$

Here, $h(x,y,t) \equiv \xi(x,y,t) + H(x,y)$, $\vec{v} \equiv (u_x, u_y)$, $\vec{\nabla} \equiv (\partial_x, \partial_y)$, $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}$ and $\vec{v} = \vec{v}(x,y,t)$. f is Coriolis parameter that is Taylor-expanded to write $f = f_0 + \beta y$. Using the equations (3.1) and (3.2), one gets the relation:

$$\frac{D}{Dt}\left[\frac{\hat{z}.(\mathbf{curl}\vec{u}) + f}{h}\right] = 0 \tag{3.3}$$

Let us assume: a) Rossby number $Ro \ll 1$, b) Fractional changes in h are small, and c) $\beta L/f_0 \ll 1$ where L is the horizontal scale of the flow. Imposing these three assumptions on the shallow-water equations one can modify the relation (3.3) to yield

$$\frac{\partial q}{\partial t} + \vec{v}.\vec{\nabla} q = 0 \tag{3.4}$$

where $q = \nabla^2\psi + f - f_0^2\psi/gH_0 + f_0(H_0 - H)/H_0$ (ψ being $g\xi/f_0$) may be called pseudo-potential vorticity. Under the same assumptions, for QG flow, one also has the condition:

$$\vec{\nabla}.\vec{v} = 0 \tag{3.5}$$

Now, the trick is to select an arbitrary two-dimensional plane in the QG turbulent flow such that the plane's normal is along $\vec{f}$ and impose the properties of homogeneity and isotropy in the plane only. By the way, one must keep in mind that the so-called fundamental scale of 3D turbulence has its analogy as the horizontal length scale L for the case of QG turbulence. The correlation functions to be derived for the forward cascade in this thesis are valid in the range (which we shall call inertial range) that is much smaller than L but quite larger than the scale at which the dissipation is effective. Whereas, the structure function to be derived for the inverse cascade is valid in the range whose scale is larger than the scale at which energy had been fed in. As we shall consider fluid bodies of uniform density only, we shall take density to be unity and let $\vec{\rho}$, as usual in this dissertation, take over the task of representing position vector in the 2D plane. The Greek subscripts used herein can take two values ρ and $\perp$ which respectively mean the component along the radial vector ρ and the component in the transverse direction. Whenever we shall use the Latin subscript (*e.g.*, a), it should mean that it can take one more value apart from the ones mentioned above: the third value 'z' would signify the vertical direction. As before, Einstein's summation convention will be used extensively. Also,

$$\vec{\rho} = \vec{\rho}_2 - \vec{\rho}_1, \qquad \rho^o_\alpha \equiv \rho_\alpha / |\vec{\rho}|, \qquad \rho^o_\rho = 1, \qquad \rho^o_\perp = 0 \tag{3.6}$$

Now, if $\vec{v}_1$ and $\vec{v}_2$ represent the horizontal fluid velocities at the two neighbouring points at $\vec{\rho}_1$ and $\vec{\rho}_2$ respectively then with similar meaning for q_1 and q_2, one may define, just for the sake of notational convenience:

$$K \equiv \langle q_1 q_2 \rangle \tag{3.7}$$

$$\text{and,} \qquad Q \equiv \langle (q_2 - q_1)(q_2 - q_1) \rangle \tag{3.8}$$

The angular brackets denote an averaging procedure which averages over all possible positions of points 1 and 2 at a given instant of time and a given separation. Due to homogeneity, Q may be re-expressed as:

$$Q = 2\langle q^2 \rangle - 2K \tag{3.9}$$

For simplicity, we shall take a rather idealised situation of QG turbulence which is homogeneous and isotropic on every scale in the plane. For the unforced case, the component of the correlation tensor will obviously be dependent on time, a fact which we won't be showing explicitly in what follows. As the features of local QG turbulence should

be independent of averaged flow, the result derived below is applicable also to the local turbulence in the plane at scale ρ much smaller than the fundamental scale.
Again, we define a two-point third order mixed correlation tensor in inertial range:

$$Q_\beta \equiv \langle (v_{2\beta} - v_{1\beta})(q_2 - q_1)(q_2 - q_1) \rangle \tag{3.10}$$

$$\Rightarrow \quad Q_\beta = 4K_\beta + 2L_\beta \tag{3.11}$$

where just to reduce the effort of writing, we have defined:

$$K_\beta \equiv \langle v_{1\beta} q_1 q_2 \rangle \tag{3.12}$$

$$\text{and,} \quad L_\beta \equiv \langle q_1 q_1 v_{2\beta} \rangle \tag{3.13}$$

Obviously, isotropy, homogeneity and the condition (3.5) compels L_β to vanish. Hence, equation (3.11) reduces to:

$$Q_\beta = 4K_\beta \tag{3.14}$$

From the equation (3.4), we may write for the points 1 and 2 respectively:

$$\frac{\partial}{\partial t} q_1 = -v_{1\gamma} \partial_{1\gamma} q_1 \tag{3.15}$$

$$\frac{\partial}{\partial t} q_2 = -v_{2\gamma} \partial_{2\gamma} q_2 \tag{3.16}$$

Multiplying equations (3.15) and (3.16) by q_2 and q_1 respectively and averaging subsequently after adding, we get the following differential equation:

$$\frac{\partial K}{\partial t} = 2\partial_\beta K_\beta \tag{3.17}$$

Using relations (3.9) and (3.14) in the equation (3.17), one gets for the inertial range for the pseudo-potential enstrophy cascade in homogeneous and isotropic QG turbulence (forced at an intermediate scale or unforced) in inviscid limit the following differential equation:

$$\frac{\partial}{\partial t}\langle q^2\rangle - \frac{1}{2}\frac{\partial}{\partial t}Q = \frac{1}{2\rho}\frac{\partial}{\partial \rho}\left(\rho Q_\rho\right) \quad (3.18)$$

$$\Rightarrow \quad Q_\rho = -2\varepsilon_q \rho \quad (3.19)$$

In getting relation (3.19) from the equation (3.18), we have assumed the following:

1. Q_ρ does not blow up at $\rho = 0$. This sets the integration constant as zero.

2. $\frac{1}{2}\frac{\partial}{\partial t}\langle q^2\rangle = -\varepsilon_q$, *i.e.,* there exists a pseudo-potential enstrophy sink at small scales due to some dissipative force such as viscosity and ε_q is the finite and constant dissipation rate of the mean pseudo-potential enstrophy.

3. $\frac{\partial}{\partial t}Q \approx 0$ due to quasi-stationarity. It may be supposed that the value of Q varies considerably with time only over an interval corresponding to the fundamental scale of turbulence and in relation to local turbulence the unperturbed flow may be regarded as steady which mean that for local turbulence one can afford to neglect $\frac{\partial}{\partial t}Q$ in comparison with the pseudo-potential enstrophy dissipation rate ε_q.

3.2 THIRD ORDER VELOCITY CORRELATION FUNCTION

Having explored the form for two-point third order mixed correlation function in the preceding discussion, we now proceed to find the scaling for the two-point third order velocity correlation function. For this motive, one may define a rank two correlation tensor:

$$B_{\alpha\beta} \equiv \langle (v_{2\alpha} - v_{1\alpha})(v_{2\beta} - v_{1\beta}) \rangle \tag{3.20}$$

Isotropy and homogeneity in the plane suggests following general form for $B_{\alpha\beta}$

$$B_{\alpha\beta} = A_1(\rho)\delta_{\alpha\beta} + A_2(\rho)\rho^o_\alpha\rho^o_\beta \tag{3.21}$$

where A_1 and A_2 are functions of time and ρ. Making use of the relations (3.6) in the equation (B.2), one gets:

$$B_{\alpha\beta} = B_{\perp\perp}(\delta_{\alpha\beta} - \rho^o_\alpha\rho^o_\beta) + B_{\rho\rho}\rho^o_\alpha\rho^o_\beta \tag{3.22}$$

One may expand the R.H.S. of the relation (B.1) and defining $b_{\alpha\beta} \equiv \langle v_{1\alpha}v_{2\beta} \rangle$, one may proceed, keeping in mind the isotropy and the homogeneity, to arrive at:

$$B_{\alpha\beta} = \langle v^2 \rangle \delta_{\alpha\beta} - 2b_{\alpha\beta} \tag{3.23}$$

Let's concentrate on the following statistically averaged quantity that will prove to be of crucial importance for deriving the desired results:

$$b_{\alpha\beta,\gamma} \equiv \langle v_{1\alpha}v_{1\beta}v_{2\gamma} \rangle$$

Invoking homogeneity and isotropy in the plane once again along with the symmetry in the first pair of indices, one may write the most general form of the third rank Cartesian tensor for this case as

$$b_{\alpha\beta,\gamma} = C(\rho)\delta_{\alpha\beta}\rho^o_\gamma + D(\rho)(\delta_{\gamma\beta}\rho^o_\alpha + \delta_{\alpha\gamma}\rho^o_\beta) + F(\rho)\rho^o_\alpha\rho^o_\beta\rho^o_\gamma \tag{3.24}$$

where, C, D and F are functions of ρ. Imposing the condition (3.5) on the expression (3.24), one can get (in the same way as done earlier for the 2D case) the following relations:

$$D = -\frac{1}{2}(\rho C' + C) \tag{3.25}$$

$$\text{and,} \qquad F = \rho C' - C \tag{3.26}$$

Here, prime ($'$) denotes derivative w.r.t. ρ. Defining

$$\begin{aligned} B_{\alpha\beta\gamma} &\equiv \langle (v_{2\alpha} - v_{1\alpha})(v_{2\beta} - v_{1\beta})(v_{2\gamma} - v_{1\gamma}) \rangle \\ &= 2(b_{\alpha\beta,\gamma} + b_{\gamma\beta,\alpha} + b_{\alpha\gamma,\beta}) \end{aligned} \tag{3.27}$$

and putting relations (3.25) and (3.26) in the equation (3.27) and using relation (3.24), one gets:

$$B_{\alpha\beta\gamma} = -2\rho C'(\delta_{\alpha\beta}\rho^o_\gamma + \delta_{\gamma\beta}\rho^o_\alpha + \delta_{\alpha\gamma}\rho^o_\beta) + 6(\rho C' - C)\rho^o_\alpha\rho^o_\beta\rho^o_\gamma \tag{3.28}$$

which along with relations (3.24), (3.25) and (3.26) yields the following expression:

$$b_{\alpha\beta,\gamma} = -\frac{B_{\rho\rho\rho}}{6}\delta_{\alpha\beta}\rho^o_\gamma + \frac{1}{12}(\rho B'_{\rho\rho\rho} + B_{\rho\rho\rho})(\delta_{\gamma\beta}\rho^o_\alpha + \delta_{\alpha\gamma}\rho^o_\beta) - \frac{1}{6}(\rho B'_{\rho\rho\rho} - B_{\rho\rho\rho})\rho^o_\alpha\rho^o_\beta\rho^o_\gamma \tag{3.29}$$

The equation (3.2) suggests:

$$\frac{\partial}{\partial t}v_{1\alpha} = -v_{1\gamma}\partial_{1\gamma}v_{1\alpha} + f_{1a}\epsilon_{a\alpha\gamma}v_{1\gamma} - g\partial_{1\alpha}\xi_1 \tag{3.30}$$

$$\frac{\partial}{\partial t}v_{2\beta} = -v_{2\gamma}\partial_{2\gamma}v_{2\beta} + f_{1a}\epsilon_{a\beta\gamma}v_{2\gamma} - g\partial_{2\beta}\xi_2 \tag{3.31}$$

multiplying equations (3.30) and (3.31) with $v_{2\beta}$ and $v_{1\alpha}$ respectively and adding subsequently, one gets the following:

$$\begin{aligned}\frac{\partial}{\partial t}\langle v_{1\alpha}v_{2\beta}\rangle &= -\partial_{1\gamma}\langle v_{1\gamma}v_{1\alpha}v_{2\beta}\rangle - \partial_{2\gamma}\langle v_{2\gamma}v_{1\alpha}v_{2\beta}\rangle \\ &\quad +\epsilon_{a\alpha\gamma}\langle f_{1a}v_{1\gamma}v_{2\beta}\rangle + \epsilon_{a\beta\gamma}\langle f_{2a}v_{2\gamma}v_{1\alpha}\rangle \\ &\quad -g\partial_{1\alpha}\langle \xi_1 v_{2\beta}\rangle - g\partial_{2\beta}\langle \xi_2 v_{1\alpha}\rangle \end{aligned} \tag{3.32}$$

Due to isotropy, the correlation function $\langle \xi_1 \vec{v}_2\rangle$ should have the form $f(\rho)\vec{\rho}/|\vec{\rho}|$. This f should not be confused with the Coriolis parameter. But since, $\partial_\alpha\langle \xi_1 v_{2\alpha}\rangle = 0$ owing to the relation (3.5), $f(\rho)\vec{\rho}/|\vec{\rho}|$ must have the form $k\vec{\rho}/|\vec{\rho}|^2$, where k is a constant. Now, k must vanish to keep correlation functions finite even at $\rho = 0$. Thus, equation (3.32) can be written as:

$$\frac{\partial}{\partial t}b_{\alpha\beta} = \partial_\gamma(b_{\alpha\gamma,\beta} + b_{\beta\gamma,\alpha}) + f_0\epsilon_{z\alpha\gamma}b_{\gamma\beta} + f_0\epsilon_{z\beta\gamma}b_{\alpha\gamma} \tag{3.33}$$

Here we have used the approximation: $\vec{f} = f_0\hat{z}$. Using equations (3.23) and (3.29), one can rewrite equation (3.33) as:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle v^2\rangle - \frac{1}{2}\frac{\partial}{\partial t}B_{\rho\rho} = \frac{1}{6\rho^3}\frac{\partial}{\partial \rho}\left(\rho^3 B_{\rho\rho\rho}\right) \tag{3.34}$$

Note that the terms containing the Levi-Civita symbol vanish by virtue of the joint effect of the expressions (B.3) and (3.23), and the antisymmetry property of Levi-Civita symbol.

3.2.1 Forward Cascade Regime

If we are interested in the pseudo-potential enstrophy cascade, the first term in the L.H.S. is zero because of energy remains conserved in QG turbulence in the inviscid limit: it cannot be dissipated at smaller scales. Also, as we are interested in the forward cascade which is dominated by pseudo-potential enstrophy cascade, on the dimensional grounds in the inertial range $B_{\rho\rho}$ (if it is assumed to depend only on ε_q and ρ) may be written as:

$$\frac{\partial}{\partial t} B_{\rho\rho} = \Gamma \varepsilon_q \rho^2 \tag{3.35}$$

where Γ is a numerical proportionality constant. Hence, using the relation (3.35), the equation (3.34) reduces to the following differential equation:

$$\frac{1}{6\rho^3} \frac{\partial}{\partial \rho} \left(\rho^3 B_{\rho\rho\rho} \right) = -\frac{\Gamma}{2} \varepsilon_q \rho^2 \tag{3.36}$$

which when solved, imposing finiteness of $B_{\rho\rho\rho}$ for $\rho = 0$, gives:

$$B_{\rho\rho\rho} = -\frac{\Gamma \varepsilon_q}{2} \rho^3 \tag{3.37}$$

The relation (3.37) is the expression for the two-point third order correlation function in the isotropic and homogeneous plane of QG turbulence (forced or unforced) in the range of the forward cascade where there is no overlapping with energy cascade. Since Γ has not been determined, one must confess that the equation (3.37) is just a scaling law at this stage.

3.2.2 Inverse Cascade Regime

Now suppose the fluid body is being forced at small scales *i.e.*, energy is being supplied and the mean rate of injection of energy per unit mass is denoted by ε_u (assumed finite and constant). Let us focus on the inverse energy cascade. Then technically we have to proceed just as before to finally arrive at the differential equation (3.34). One obviously would set $\frac{1}{2}\frac{\partial}{\partial t}\langle v^2\rangle = \frac{2}{3}\varepsilon_u$ invoking the hypothesis[36] that there should be equipartition of energy between potential energy and the energy content in each of the two horizontal velocity components in the plane. This equipartition had been proposed in view of the assumption that at sufficiently small scales the interaction of the mean flow with the eddies (and thus the eddy-energies) diminishes; as a result, for increasingly smaller vertical and horizontal scales the energies will tend to become homogeneous and equally distributed among the perturbations. By the way, the concept of equipartition of energy is very old and wide-spread in the literature of statistical mechanics. Historically, equilibrium statistical mechanics had been used to justify many aspects of turbulence *e.g.,* the dual cascades in 2D turbulence *etc.* A detailed discussion may be found in the books by Chorin[39] and Lim *et al.*[40]. Now, lets also assume that $\frac{\partial}{\partial t}B_{\rho\rho} \approx 0$ in the inverse cascade regime supposing the forced QG turbulence to be in the state of quasi-stationarity. So we are left with the following differential equation:

$$\frac{1}{6\rho^3}\frac{\partial}{\partial\rho}\left(\rho^3 B_{\rho\rho\rho}\right) = \frac{2}{3}\varepsilon_u \tag{3.38}$$

$$\Rightarrow \qquad B_{\rho\rho\rho} = +\varepsilon_u\rho \tag{3.39}$$

where in the last step the integration constant has been set to zero to prevent $B_{\rho\rho\rho}$ from blowing up at $\rho = 0$. The expression (3.39) is the expression for the two-point third order velocity correlation function in the isotropic and homogeneous plane of forced QG turbulence for the inverse energy cascade.

The fact that the structure functions for the inherently three-dimensional QG turbulence are more like that for the 2D turbulence than that for the 3D turbulence speaks volumes for the importance of study of third order structure functions for demystifying the two-dimensionalisation effect of the 3D turbulent fluid due to rapid rotation. This serves as the motivation for jumping into the subject of rotating flows and to attempt finding the form of S_3 therein.

Chapter 4

ROTATING TURBULENCE

All the studies on the two-dimensionalisation effect are mainly for low Ro high Re limit while the high Ro and high Re limit has been rather less ventured in relation to the two-dimensionalisation effect of turbulence, although the second case, we believe, should be analytically more tractable. If, using calculations of structure functions, in the limit of high Ro and high Re, one wishes to see whether a trend towards two-dimensionalisation of 3D homogeneous isotropic turbulence occurs or not, then basically one would have to check (a) if $S_3 = -(4/5)\varepsilon l$ at small scales for 3D turbulence shows a tilt towards $S_3 = (3/2)\varepsilon l$ at large scales for the 2D turbulence and (b) if the forward energy cascade is depleted at the smaller scales. As we shall show, in the lowest order calculation this is what one may get, hinting at the initiation of the effect of two-dimensionalisation of 3D turbulence owing to the small anisotropy induced by slow rotation.

4.1 RELEVANT SCALES

Let us look in to the various length scales that have to be taken into consideration while talking about a homogeneous rotating turbulence which basically satisfies following version of Navier-Stoke's equation:

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}.\vec{\nabla}\right)\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\Omega}\times\left(\vec{\Omega}\times\vec{x}\right) - 2\vec{\Omega}\times\vec{v} + \nu\nabla^2\vec{v} + \vec{f} \tag{4.1}$$

In this context, $\vec{f}$ is external force and $\vec{\Omega}$ is angular velocity. Various parameters to be considered are: ν (kinematic viscosity), ε (finite mean rate of dissipation of energy per unit mass), Ω (angular velocity) and l_0 (integral scale which typically is the system-size). The three important time-scales involved in the system are: $t_l \sim \varepsilon^{-1/3} l^{2/3}$ (eddy-turnover time or circulation time for the eddy of scale l; $l \leq l_0$), $t_\Omega \sim \Omega^{-1}$ and $t_d \sim l^2/\nu$ (diffusion time scale). It is well-known that a length scale $l_\Omega = \sqrt{(\varepsilon/\Omega^3)}$ is what responsible for the estimation of the anisotropy introduced by the rotation. The competition between the time-scales t_l and t_d gives rise to what is known as dissipation length scale l_d, defined as $l_d = (\nu^3/\varepsilon)^{1/4}$ and a similar competition between the time-scales t_d and t_Ω allows us to define a length scale $l_{\Omega d} = \sqrt{(\nu/\Omega)}$. Now, lets look at the typical scenario when Ro is moderate. The four vital length scales are typically arranged according to the order : $l_0 > l_\Omega > l_{\Omega d} > l_d$. Thus, the regime $l_0 > l > l_\Omega$ is the regime where effect of rotation is important and anisotropy reigns. The scales $l \in (l_\Omega, l_d)$ may be considered to have isotropy, though to be precise, probably l_d here should be replaced by $l_{\Omega d}$ since rotation seems to be bringing the effect of viscosity to rather larger length scales. So, now what happens when the Ro is decreased by increasing the angular velocity is interesting. Both the scales l_Ω and $l_{\Omega d}$ rush towards the dissipation length scale, thereby increasing the anisotropic regime and at the angular velocity $\Omega = \Omega_a \equiv \sqrt{(\varepsilon/\nu)}$ one has $l_\Omega = l_{\Omega d} = l_d$

and the turbulence is fully anisotropic.

Strictly speaking, even a small rotation introduces anisotropy (however small) at all scales and the isotropic regime does have a degree of anisotropy in it as we shall see shortly. In the fully anisotropic limit, *i.e.* for $\Omega = \Omega_a$, one expects full decoupling of the plane perpendicular to the rotation axis from the direction of the rotation axis. However, even in the partially anisotropic limit (*e.g.* when we have slow rotation imparted on the turbulent fluid), l_z should still be given a special status for being in the direction of the rotation axis, by which we mean that the structure functions should no longer depend on l but rather on l_z and $\vec{l}_\perp$ (where $l^2 = l_z^2 + l_\perp^2$ and $|\vec{\Omega}| = \Omega_z$).

We shall see how this decoupling sets in, in the limit of low angular velocity and try to study in that very limit, the two-point third order structure function in the first approximation and see how the effect of two-dimensionalisation is all set to sneak in with the switching on of rotation.

4.2 S_3 FOR SMALL Ω

In this section, we shall work in the low Ω-limit. With this statement we mean, as discussed in the previous section, $\Omega \ll \Omega_a$.

4.2.1 First Line Of Attack

In this limit, the entire fluid may still be treated as isotropic but since rotation should play a role, we assume that $\langle v_i v_j v_k' \rangle$ (where angular brackets mean ensemble average and $v_i = v_i(\vec{x}, t)$ is the i-th component of velocity and similarly, $v_i' = v_i(\vec{x} + \vec{l}, t)$) should depend on $\vec{\Omega}$ as well. $\vec{\Omega}$ would take care of the mild anisotropy. Since, physically speaking, S_3 should not depend on which way the rotation axis is and since we are interested in low values of Ω, we shall let $\langle v_i v_j v'_k \rangle$ depend only on the terms quadratic in Ω and not bother about higher order terms in Ω. As a result, we write the following most general tensorial form for $\langle v_i v_j v'_k \rangle$:

$$\begin{aligned} b_{ij,k} &\equiv \langle v_i v_j v'_k \rangle & (4.2) \\ &= C(l)\delta_{ij} l^o_k + D(l)(\delta_{ik} l^o_j + \delta_{jk} l^o_i) + F(l) l^o_i l^o_j l^o_k \\ &\quad + G(l)[(\epsilon_{imk} l^o_j + \epsilon_{jmk} l^o_i) l^o_m] + H(l)\Omega_i \Omega_j l^o_k \\ &\quad + I(l)[(\epsilon_{imk}\Omega_j + \epsilon_{jmk}\Omega_i)\Omega_m] + K(l)(\Omega_i \Omega_k l^o_j + \Omega_j \Omega_k l^o_i) & (4.3) \end{aligned}$$

where l^o_i is the i-th component of the unit vector along $\vec{l}$. We have assumed that the coefficients are dependent only on l and it is the $\vec{\Omega}$ which is taking care of the mild anisotropy which the turbulent fluid might have. We must accept that the assumption of letting coefficients depend only on l is rather crude in the light of the complex forms that the two-point tensors in a fully anisotropic turbulence flow take[41]. The justification, and hence solace, for the assumption can be drawn from the fact that very simple revealing results matching with recent experiments are arrived at in the long run. As we are considering incompressible fluid, we must have:

$$\partial'_k b_{ij,k} = 0 \qquad (4.4)$$

which when applied to relation (4.3), yields relationships between various coefficients. As usual Einstein summation convention has been extensively followed in these calculations unless otherwise specified. Using relations (4.3) and (4.4), one lands up in the end on the following:

$$
\begin{aligned}
B_{ijk} &\equiv \langle (v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \rangle && (4.5)\\
&= 2(b_{ij,k} + b_{jk,i} + b_{ki,j}) && (4.6)\\
&= -2(lC' + C)(\delta_{ij} l^o_k + \delta_{ik} l^o_j + \delta_{jk} l^o_i) + 6(lC' - C) l^o_i l^o_j l^o_k \\
&\quad +4Jl(\Omega_i\Omega_j l^o_k + \Omega_i\Omega_k l^o_j + \Omega_j\Omega_k l^o_i) && (4.7)
\end{aligned}
$$

Here, in expression (4.7), prime ("′") denotes derivative w.r.t. l and J is a constant which, curiously enough, is of the same dimension $[L^2T^{-1}]$ as that of the kinematic viscosity and therefore, probably calls for a physical interpretation. Now we can see that using the relation (4.7), two-point third order structure function (S_3) can be extracted from B_{ijk} in the following way:

$$
\begin{aligned}
S_3(l) &\equiv \langle (\delta v_{\parallel}(\vec{l}))^3 \rangle \equiv \left\langle \left[\left\{ \vec{v}(\vec{x} + \vec{l}) - \vec{v}(\vec{x}) \right\} . \frac{\vec{l}}{l} \right]^3 \right\rangle && (4.8)\\
\Rightarrow S_3(l) &= \langle [(v'_i - v_i) l^o_i][(v'_j - v_j) l^o_j][(v'_k - v_k) l^o_k] \rangle && (4.9)\\
\Rightarrow S_3(l) &= B_{ijk} l^o_i l^o_j l^o_k && (4.10)\\
\Rightarrow S_3(l) &= -12C + \frac{12J}{l}(\vec{\Omega}.\vec{l})^2 && (4.11)
\end{aligned}
$$

where we have used relation (4.7). One may define physical space energy flux ($\varepsilon(\vec{l})$) as:

$$
\begin{aligned}
&\varepsilon(l) \equiv -\frac{1}{4}\vec{\nabla}_l.\langle |\delta\vec{v}(\vec{l})|^2 \delta\vec{v}(\vec{l}) \rangle && (4.12)\\
\Rightarrow \quad &\varepsilon(l) = lC'' + 7C' + \frac{8C}{l} + 3J\Omega^2 + \frac{6J}{l^2}(\vec{\Omega}.\vec{l})^2 && (4.13)
\end{aligned}
$$

To get relation (4.13), we have again made use of the relation (4.7). The energy flux through the wave number K (Π_K) for the isotropic homogeneous turbulence may be calculated to be:

$$\Pi_K = \frac{2}{\pi}\int_0^{\infty} dl \frac{\sin(Kl)}{l}(1 + l\partial_l)\varepsilon(l) \tag{4.14}$$

Now if one makes the standard assumption (often made during the derivation of S_3) that as $Re \to \infty$, the mean energy dissipation per unit mass $\varepsilon(\nu)$ tends to a positive finite value (*i.e.,* $\lim_{\nu\to 0}\varepsilon(\nu) = \varepsilon > 0$), then $\lim_{\nu\to 0}\Pi_K = \varepsilon$ in the inertial regime. Therefore, in the inertial range, putting $x = Kl$, one has

$$\Pi_K = \frac{2}{\pi}\int_0^{\infty} dx \frac{\sin(x)}{x} f\left(\frac{x}{K}\right) = \varepsilon \tag{4.15}$$

where,

$$f\left(\frac{x}{K}\right) = f(l) = (1 + l\partial_l)\varepsilon(l) \tag{4.16}$$

For small l (large K), the integral in relation (4.15) yields

$$f(l) \approx \varepsilon \tag{4.17}$$

Now using relations (4.13), (4.16) and (4.17), we form a differential equation which when solved, keeping in mind that S_3 should not blow up at $l = 0$, one gets following form for S_3 in slowly rotating homogeneous turbulent fluid.

$$S_3(l) = -\frac{4}{5}\varepsilon l + \frac{12}{5}Jl[\Omega^2 + 7(\Omega_k l_k^o)^2] \tag{4.18}$$

One may note from the relation (4.18) how magically Ω has brought up the anisotropic effects even for small Ω though for the entire calculation we followed the procedure

meant for the homogeneous isotropic turbulence. Thus, the form for S_3 is plausible.
One may ask: Does the effect of two-dimensionalisation shows up in the relation (4.18)? As one may note from the relation (4.18) this is quite a possibility but the only catch being that J should be positive — an issue which we have not been able to resolve. If J is positive, it means if we increase Ω the value of S_3 would distort away from the usual $-(4/5)\varepsilon l$ for the non-rotating case to more positive values. This apparently shows that the effective value of ε is decreased depicting that the forward energy transfer is depleted which is in keeping with what is expected and hence the tendency of the rotating 3D turbulence to show the effect the two-dimensionalisation is being highlighted. That the sign of J should be positive is a question remains to be addressed.

4.2.2 Second Line Of Attack

By the way, the relation (4.18) also suggests that the coefficients in the tensorial form for $b_{ij,k}$ should have dependence on l_z and $l_\perp$ separately effecting a mild decoupling of directions. So taking hint from it, we proceed to rewrite $b_{ij,k}$ for slowly rotating 3D turbulent fluid but now introducing anisotropy directly into the coefficients and not letting Ω take care of anisotropy explicitly. Of course, the coefficients will now depend on Ω.
For completely isotropic homogeneous turbulence, one would write following general form (relation (4.19)) for $b_{ij,k}$ which is made up of Kronecker delta and components of the unit vectors $\vec{l}/|\vec{l}|$.

$$b_{ij,k} = C(l)\delta_{ij}l_k^o + D(l)(\delta_{ik}l_j^o + \delta_{jk}l_i^o) + F(l)l_i^o l_j^o l_k^o \tag{4.19}$$

The expression is symmetric in i and j and the coefficients are dependent on l only. As discussed earlier, with rotation coming into effect, anisotropy comes into effect. If this effects in the possible decoupling (even if partial) of the direction along the rotation axis (which we shall take along the z-axis), then mathematically we may introduce this effect by modifying the form (4.19) of $b_{ij,k}$ to the following:

$$b_{ij,k} = C(l,l_z,\Omega)\delta_{ij}l_k^o + D(l,l_z,\Omega)(\delta_{ik}l_j^o + \delta_{jk}l_i^o) + F(l,l_z,\Omega)l_i^o l_j^o l_k^o \quad (4.20)$$

If one uses the incompressibility condition (relation (4.4)), one gets:

$$D = \frac{l}{2}\left(-C' - \frac{\dot{C}l_z}{l}\right) - C \quad (4.21)$$

$$\text{and} \qquad \dot{D} = 0 \quad (4.22)$$

where dot represents the derivative w.r.t. l_z and prime, as before, the derivative w.r.t. l. Using equation (4.21) in the equation (4.22), one land up on:

$$\ddot{C}l_z + l\dot{C}' + 3\dot{C} = 0 \quad (4.23)$$

$$\Rightarrow \quad C = \sum_n A_n l^{-n-2} l_z^n \quad (4.24)$$

$$\Rightarrow \quad C \neq 0 \ \text{ for } \ n \in (-\infty, -2] \cap [0, \infty) \quad (4.25)$$

$$\Rightarrow \quad C = D = F = 0 \quad (4.26)$$

In arriving at the result (4.26), we have taken care of the fact that C can not be allowed to blow up for either for $l_z = 0$ or for $l = 0$. Thus, relation (4.20) vanishes trivially. So, we are left with the following choice:

$$b_{ij,k} = C(l_\perp,l_z,\Omega)\delta_{ij}l_k^o + D(l_\perp,l_z,\Omega)(\delta_{ik}l_j^o + \delta_{jk}l_i^o) + F(l_\perp,l_z,\Omega)l_i^o l_j^o l_k^o \quad (4.27)$$

Using equations (4.4) and (4.27), we arrive at following relationship between the coefficients:

$$D = -\frac{l_\perp}{2}\tilde{C} - \frac{l_z}{2}\dot{C} - C \tag{4.28}$$

$$F = \frac{l^2}{2}\tilde{\tilde{C}} + \frac{l^2 l_z}{2l_\perp}\dot{\tilde{C}} + \left(\frac{3l^2}{2l_\perp} - \frac{l_\perp}{2}\right)\tilde{C} - \frac{l_z}{2}\dot{C} - C \tag{4.29}$$

Here tilde and dot define derivatives w.r.t. $l_\perp$ and l_z respectively. Proceeding rather monotonously as before we get

$$B_{ijk} = 2(b_{ij,k} + b_{jk,i} + b_{ki,j}) \tag{4.30}$$

$$= -2(l_\perp\tilde{C} + l_z\dot{C} + C)(\delta_{ij}l_k^o + \delta_{ik}l_j^o + \delta_{jk}l_i^o) + 6Fl_i^o l_j^o l_k^o \tag{4.31}$$

And hence,

$$S_3 = B_{ijk}l_i^o l_j^o l_k^o = 6[F - (l_\perp\tilde{C} + l_z\dot{C} + C)] \tag{4.32}$$

The definition for the physical space energy flux ($\varepsilon(\vec{l})$) has to be obviously modified. Natural choice would be:

$$\langle|\delta\vec{v}(\vec{l})|^2\delta\vec{v}(\vec{l})\rangle = B_{ii\alpha}l_\alpha^o\frac{\vec{l}_\perp}{l_\perp} + B_{iiz}l_z^o\frac{\vec{l}_z}{l_z} \tag{4.33}$$

where α takes two values: x and y only. Now, using relations (4.12), (4.29), (4.31) and (4.33) and performing tedious algebra one gets:

$$\begin{aligned}
\varepsilon(l_\perp, l_z) \;=\;& \frac{-1}{4(l_\perp^2 + l_z^2)^2}\Big[(3l_\perp^6 + 6l_\perp^4 l_z^2 + 3l_\perp^2 l_z^4)\tilde{\tilde{\tilde{C}}} \\
&+ \left(3l_\perp^5 l_z + 6l_\perp^3 l_z^3 + 3l_\perp^4 l_z^2 + 6l_\perp^2 l_z^4 + 3l_\perp l_z^5 + 3l_z^6\right)\dot{\tilde{\tilde{C}}} \\
&+ \left(3l_\perp^3 l_z^3 + 6l_\perp l_z^5 + 3l_\perp^{-1} l_z^7\right)\ddot{\tilde{C}} \\
&+ \left(5l_\perp^5 + 6l_\perp^4 l_z + 23l_\perp^3 l_z^2 + 12l_\perp^2 l_z^3 + 18l_\perp l_z^4 + 6l_z^5\right)\tilde{\tilde{C}} \\
&+ \left(-7l_\perp^4 l_z + 5l_\perp^3 l_z^2 - l_\perp^2 l_z^3 + 23l_\perp l_z^4 + 6l_z^5 + 18l_\perp^{-1} l_z^6\right)\dot{\tilde{C}} \\
&+ \left(-12l_z^4 - 8l_\perp^3 l_z - 20l_\perp^2 l_z^2 + 36l_\perp l_z^3 + 18l_z^4 + 8l_\perp^{-1} l_z^5\right)\tilde{C} \\
&+ \left(-13l_\perp^3 l_z - 43l_\perp^2 l_z^2 - 39l_\perp l_z^3 - 17l_z^4\right)\dot{C} \\
&+ \left(-4l_\perp^3 - 8l_\perp^2 l_z - 12l_\perp l_z^2\right)C\Big] \qquad (4.34)
\end{aligned}$$

The energy flux (Π_K) through the wave number K for the homogeneous (not necessarily isotropic) turbulence may be shown to be:

$$\Pi_K = \frac{1}{2\pi^2}\int_{\mathbb{R}^3} d^3l \frac{\sin(Kl)}{l}\vec{\nabla}_l.\left[\varepsilon(\vec{l})\frac{\vec{l}}{l^2}\right] \qquad (4.35)$$

Using cylindrical polar coordinates we reduce the relation (4.35) to:

$$\Pi_K \;=\; \frac{1}{\pi}\int\!\!\int l_\perp dl_\perp dl_z \left\{\frac{\sin(Kl)}{l}\left[\frac{l_\perp}{l^2}\frac{\partial}{\partial l_\perp} + \frac{l_z}{l^2}\frac{\partial}{\partial l_z} + \frac{1}{l^2}\right]\varepsilon(\vec{l})\right\} \qquad (4.36)$$

Now, we introduce the variables $y = Kl_\perp$ and $z = Kl_z$ in relation (4.36) to get:

$$\Pi_K = \frac{1}{\pi}\int_{z=-\infty}^{\infty}\int_{y=0}^{\infty} dydz\frac{\sin(y^2+z^2)^{\frac{1}{2}}}{y^2+z^2}\left[f\left(\frac{y}{K},\frac{z}{K}\right)\right] \qquad (4.37)$$

Now, let's probe small l behaviour. Because $\int_{z=-\infty}^{\infty}\int_{y=0}^{\infty} dydz[\sin(y^2+z^2)^{1/2}]/(y^2+z^2) = \pi^2/2$, we have

$$f(l_\perp, l_z) \approx \frac{2\varepsilon}{\pi} \tag{4.38}$$

Obviously, ε has the meaning of finite positive mean rate of dissipation of energy per unit mass. Using the expressions (4.34) and (4.38), we look for the $l_z = 0$ limit. One then has the result:

$$\left[l_\perp \frac{\partial}{\partial l_\perp} + 1\right]\left(3l_\perp^2 \tilde{\tilde{\tilde{C}}} + 5l_\perp \tilde{\tilde{C}} - 12\tilde{C} - 4\frac{C}{l_\perp}\right) = -\frac{8\varepsilon}{\pi} \tag{4.39}$$

$$\Rightarrow \quad 3l_\perp^4 \tilde{\tilde{\tilde{\tilde{C}}}} + 14l_\perp^3 \tilde{\tilde{\tilde{C}}} - 2l_\perp^2 \tilde{\tilde{C}} - 16 l_\perp \tilde{C} = -\frac{8\varepsilon}{\pi} l_\perp \tag{4.40}$$

$$\Rightarrow \quad C = \left(A_1 + A_2 l_\perp^{-1} + A_3 l_\perp^{\frac{7-\sqrt{97}}{6}} + A_4 l_\perp^{\frac{7+\sqrt{97}}{6}}\right) + \frac{\varepsilon l_\perp}{2\pi} \tag{4.41}$$

Relations (4.29), (4.32) and (4.41) together yield following expression for S_3:

$$S_3|_{l_z=0} = -\frac{6}{\pi}\varepsilon l_\perp + A_4\left[3\left(\frac{7+\sqrt{97}}{6}\right)\left(\frac{1+\sqrt{97}}{6}\right) - 12\right] l_\perp^{\frac{7+\sqrt{97}}{6}} \tag{4.42}$$

$$\Rightarrow \quad S_3|_{l_z=0} = -\frac{6}{\pi}\varepsilon l_\perp + A l_\perp^{\frac{7+\sqrt{97}}{6}} \tag{4.43}$$

where, A is a constant which for obvious reason depends on Ω and ε. Using dimensional arguments and introducing a non-dimensional constant c, we may set

$$A = c\Omega^{\frac{1+\sqrt{97}}{4}} \varepsilon^{\frac{11-\sqrt{97}}{12}} \tag{4.44}$$

From relations (4.43) and (4.44), we may write finally

$$S_3|_{l_z=0} = -\frac{6}{\pi}\varepsilon l_\perp + c\Omega^{\frac{1+\sqrt{97}}{4}} \varepsilon^{\frac{11-\sqrt{97}}{12}} l_\perp^{\frac{7+\sqrt{97}}{6}} \tag{4.45}$$

This (relation (4.45)) is the final form for two-point third order structure function in the plane whose normal is parallel to the rotation axis for slowly rotating homogeneous 3D turbulence.

4.3 ENERGY SPECTRUM FOR SMALL Ω

If we for the time being forget about the issue of anomalous scaling, then a mere inspection of the relation (4.45) from the point of view of dimensional analysis would tell that in the directions perpendicular to the axis of rotation, there are two possible energy spectrums *viz.*

$$E(k) \sim k^{-\frac{5}{3}} \tag{4.46}$$

and,

$$E(k) \sim k^{-\frac{16+\sqrt{97}}{9}} \tag{4.47}$$

which are respectively due to the first term and the second term in the R.H.S. of the relation (4.45). It is very interesting to note that the exponent of k in the relation (4.47), *i.e.* $-(16+\sqrt{97})/9$, equals -2.87 which is in between -3 (for 2D turbulence) and -2 (for rapidly rotating 3D turbulence as proposed by Zhou). Obviously, the spectrum (4.46) will be dominant compared to the spectrum (4.47). But as the Ω is increased (of course, remaining within a range so that the anisotropy is not strong enough to breakdown the arguments used to calculate the S_3 of the relation (4.45)), the spectrum (4.47) becomes more and more prominent; thereby two-dimensionalisation of the 3D homogeneous turbulent fluid is initiated which then carries over to high rotation regime as is being extensively studied now-a-days. This signature of two-dimensionalisation is, of course, in agreement with what present literature on turbulence hails as the two-dimensionalisation of turbulence. Thus, the third order structure function has proved to be very handy in studying this effect because the phenomenon of two-dimensionalisation is reflected as a change in the scaling law of the third order structure function. We pause here for a moment and ponder upon the signatures of two-dimensionalisation effect in a rather more intuitive, though a bit non-rigorous, way.

4.4 THE SIGNATURES: AN INTUITIVE PICTURE

4.4.1 Why Inverse Cascade?

Let us first concentrate on why at all there should be an inverse cascade of energy. Inverse cascade of energy is a trademark of 2D turbulence where a second conserved quantity — enstrophy — besides energy plays the defining role behind it. One might be tempted to search for this conserved quantity in the case of rapidly rotating 3D turbulence, for, there in the limit of infinite rotation the axes of all the vortices are expected to point up towards the direction of angular velocity. Hence, looking at the every section perpendicular to the axis one might tend to think that 2D turbulence is being shown by each transverse section. This obviously is not a correct inference because of the non-zero axial velocity may depend on the coordinates on the plane. Searching for the enstrophy conservation seems to be a dead end as far as explaining the inverse cascade in rapidly rotating turbulence is concerned. In such an unfortunate scenario, helicity (defined as $\int \vec{v}.\vec{\omega}d^3\vec{r}$) which remains conserved in a 3D inviscid unforced flow comes to our rescue. It has been long known that helicity is introduced into a rotating turbulent flow[42]. Kraichnan[43] argued that both the helicity and energy cascade in 3D turbulence would proceed from lower to higher wave numbers and went on to remark that forward helicity cascade would pose a hindrance for the energy cascade — a fact validated by numerical simulations[44, 45]. He also showed that in presence of helicity two-way cascade is

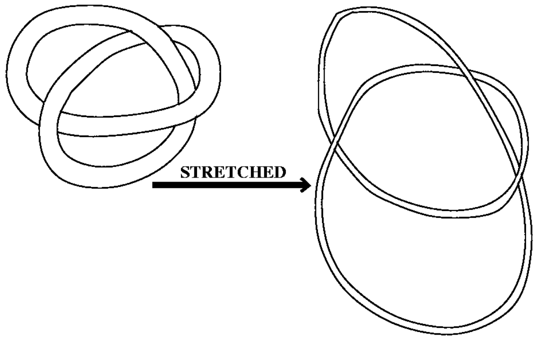

Figure 4.1: **A knotted vortex tube.** When it is stretched the tube thins out to create smaller eddies but the entire structure occupies a larger volume.

possible. Lets see topologically why this should be so. It is well known that a knotted vortex tube is capable of introducing helicity in fluid[46]. Consider a knotted vortex tube (see fig-4.1) in a turbulent flow. Due to the vortex stretching phenomenon in turbulence, the vortex line stretches and as a result owing to the assumed incompressibility of the fluid the tube thins out keeping the volume inside it preserved and smaller scales are created; in a sense, this is what is meant by the flow of energy to the smaller scales. But now this also means that the "scale" of the knotted structure would in general increase *i.e.,* the knot would now reach out to farther regions in the fluid. Evidently, if we wanted this scale to reduce, we must let the stretched knotted tube fold in such a way so that the scale becomes smaller; such a neat arrangement seems to be a far cry in a turbulent flow which is inherently chaotically random causing the separation of two nearby particles of fluid on an average. Thus, as the degree of knottedness measures helicity, the aforementioned argument suggests that if one forces energy to go to smaller scales, helicity would tend to go to larger scale and vice-versa. This topological argument gives an intuitive way of comprehending how the forward helicity cascade can inhibit the forward cascade of energy. The point is that in presence of forward helicity cascade, reverse cascade of energy is not impossible.

Waleffe[47], with the help of detailed helicity conservation by each triad, showed that helicity indeed affects the turbulence dynamics even in isotropic turbulence; this is a kind of catalytic effect. One can thus take inspiration to make the argument in the previous paragraph more concrete by playing around with a simplified triad using logic in the line suggested by Fjortoft's theorem[48] in 2D turbulence. Let the helicity spectrum be $H(k)$ and the energy spectrum be $E(k)$. It may be shown that

$$|H(k)| \leq kE(k) \tag{4.48}$$

Consider 3D Euler equation in Fourier space truncated in order to retain only three parallel wave vectors $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ and suppose it is possible for these three particular wave vectors to be such that $|H(k)| = nkE(k)$, where n is a positive number lesser than 1 to be in consistence with the relation (4.48). Assume $\vec{k}_2 = 2\vec{k}_1$ and $\vec{k}_3 = 3\vec{k}_1$. Conservation of energy and helicity imply that between two times t_1 and t_2, the variation $\delta E_i = E(k_i, t_2) - E(k_i, t_1)$ satisfies two constraints

$$\delta E_1 + \delta E_2 + \delta E_3 = 0 \tag{4.49}$$

$$\text{and, } nk_1\delta E_1 + nk_2\delta E_2 + nk_3\delta E_3 = 0 \tag{4.50}$$

solving which in terms of δE_2, we get:

$$\delta E_1 = \delta E_3 = -\frac{\delta E_2}{2} \tag{4.51}$$

$$\text{and, } nk_1\delta E_1 = -\frac{n}{4}k_2\delta E_2; \qquad nk_3\delta E_3 = -\frac{3n}{4}k_2\delta E_2 \tag{4.52}$$

If one assumes that the wave vector k_2 is losing energy, *i.e.* $\delta E_2 < 0$, then the results (4.51) and (4.52) show that as more helicity goes into the higher wavenumber, the energy is equally transferred to both the lower and the higher wave numbers suggesting a possibility of the coexistence of reverse and forward energy cascades.

4.4.2 Energy Spectrum For Lagre Ω

Now let us come to the point. In the case of 3D isotropic and homogeneous turbulence rotation can input helicity in it when there is a mean flow in the inertial frame and this value of input helicity increases with the increase in angular velocity. Experiments on rotating turbulence invariably introduce helicity. As the angular velocity is increased the helicity increases enough to inhibit the energy cascade appreciably so that a reverse cascade is seen. This consistently explains the reason behind the existence of the reverse energy cascade in a rapidly rotating turbulent flow. Hence, the argued existence of a direct helicity cascade in such experiments turns out to be an interesting (however not rigourously proven) assumption.

As discussed earlier, the next important signature of the two-dimensionalisation of turbulence that remains to be pondered upon is the exponent of the wave vector in the energy spectrum relation. To be precise, if one wishes angular velocity becomes a relevant parameter in the energy spectrum $E(k)$, simple dimensional analysis would give:

$$E(k) \propto \Omega^{\frac{3m-5}{2}} \varepsilon^{\frac{3-m}{2}} k^{-m} \tag{4.53}$$

where m is a real number. m must be restricted within the range 5/3 to 3 to keep the exponents of Ω and ε in relation (4.53) non-negative. The two limits $m = 5/3$ and $m = 3$ corresponds to isotropic homogeneous 3D turbulence and 2D turbulence respectively. The spectrum due to Zhou — $E(k) \sim k^{-2}$ — is due to an intermediate value of $m = 2$. So, as far as the present state of the literature on rotating turbulence is concerned, two-dimensionalisation of 3D turbulence would mean the dominance of a spectrum which goes towards $E(k) \sim k^{-3}$ and which may choose to settle at $E(k) \sim k^{-2}$.

Lets give a twist to the tale. In general, the energy spectrum[42] in the inertial range will be determined by both the helicity cascade and the energy cascade which simply means that the energy spectrum from the dimensional arguments should be written as

$$E(k) \propto \varepsilon^{\frac{7}{3}-m} h^{m-\frac{5}{3}} k^{-m} \tag{4.54}$$

where h is the rate of helicity dissipation per unit mass. Demanding positivity of the exponents of ε and h, one fixes the possible values for k within the closed range [5/3,7/3], imposing which on the arguments given in the previous paragraph, one can easily propound the range

$$2 \leq m \leq \frac{7}{3} \tag{4.55}$$

for the rapidly rotating 3D turbulent flow. Direct experiments[9] by Morize *et al.* have found energy spectrum for rapidly rotating turbulence going as $k^{-2.2}$ which is as predicted by the relation (4.55).

One may note that the scaling exponent derived as expression (4.47) has not fallen into the more strict range $[-7/3, -5/3]$ obviously because Ω is too low and may be because to maintain isotropy to a certain extent for the sake of hiccup-free calculations we have chosen not to include terms involving ϵ_{ijk} in the relation (4.27) which could grab the effect of helicity explicitly; thereby again showcasing the need for the helicity to be effective to give the right exponent for the rotating turbulence.

4.5 YET ANOTHER SIGNATURE

Having explained the two signatures of the two-dimensionalisation effect, we search for another possible signature of the effect. The advection of a passive scalar θ may serve the purpose since the Yaglom's law[49] in d-D incompressible turbulent fluid may be written as $\langle \delta v_{\parallel} (\delta\theta)^2 \rangle = -(4/d)\varepsilon_\theta l$, where $\varepsilon_\theta \equiv \kappa \langle (\partial_{l_i}\theta)(\partial_{l_i}\theta) \rangle = -\partial_t \langle \theta^2 \rangle$ and κ being the diffusivity. This law distinguishes between a 2D and a 3D turbulence and hence it is worth getting a form for it for a rotating 3D turbulence and find if in a plane perpendicular to the rotation axis it reduces to the form for 2D turbulence and thereby bringing in the effect of two-dimensionalisation. Since we have witnessed earlier that small Ω could bring in anisotropy in the otherwise isotropic scales, one would look out for the effect of small Ω on the passive scalar which follows the equation:

$$\frac{\partial\theta}{\partial t} + \vec{\nabla}.(\vec{v}\theta) = \kappa\nabla^2\theta - \epsilon_{ijk}\Omega_j \frac{\partial}{\partial x_i}(x_k\theta) \tag{4.56}$$

If one goes by the procedure given in the reference [50] to find out a value for $\langle \delta v_{\parallel} (\delta\theta)^2 \rangle$ for small l in this case assuming very small Ω (and hence isotropy), one arrives back at the Yaglom's law. We can however land up on a very neat experimentally and numerically verifiable correlation which can serve the purpose of a signature of the two-dimensionalisation effect if we treat equation (4.56) anisotropically as follows.
Defining $\vec{l} \equiv \vec{x'} - \vec{x}$ and $\partial_{l_i} \equiv \nabla_i = \partial'_i = -\partial_i$, one can manipulate the equation (4.56) to get:

$$\partial_t \langle (\delta\theta)^2 \rangle + \nabla_i \langle \delta v_i (\delta\theta)^2 \rangle = 2\kappa \nabla_{ii} \langle \theta^2 \rangle - 4\kappa \langle \nabla_i\theta \nabla_i\theta \rangle - \epsilon_{ijk}\Omega_j \nabla_i \langle l_k (\delta\theta)^2 \rangle \tag{4.57}$$

Now, owing to the anisotropy caused by the rapid rotation, we may write $\langle \delta\vec{v}(\delta\theta)^2 \rangle = \langle \delta v_\perp (\delta\theta)^2 \rangle \vec{l}_\perp / l_\perp + \langle \delta v_z (\delta\theta)^2 \rangle \vec{l}_z / l_z$ and as $\langle (\delta\theta)^2 \rangle$ is proportional to terms quadratic in $l_\perp$ and

l_z, in the limit $\kappa \to 0$ and small scales, one can easily reach at the following relation:

$$\langle \delta v_\perp (\delta\theta)^2 \rangle |_{l_\perp = 0} = 0 \tag{4.58}$$

This relation predicts that in the presence of rapid rotation, and hence anisotropy, on the small line segment parallel to axis of rotation the correlation in the L.H.S. of (4.58) vanishes. This may be readily used in numerics to check if the two-dimensionalisation has been achieved and hence may be treated as a signature of the effect.

Chapter 5

GOY TURBULENCE

*S*hell models of turbulence are simplified caricatures of the equations of fluid mechanics — it includes the Navier-Stokes equations which we are interested in — in wave-vector representation; typically they exhibit anomalous scaling even though their non-local interactions are local in wave-number space. The main advantage is that they can be studied via fast and accurate numerical simulations, in which the values of the scaling exponents can be determined very precisely. Need for such a model arises due to the necessity of the reserachers to be able to define a model that can describe the analogue of the phenomenological Richardson cascade but possessing a deterministic time evolution.

Such a typical shell model has many advantages that the researchers of turbulence die for. Not only it can boast of moderate number of degrees of freedom enabling one to investigate many aspects of turbulence with moderate computational power but also it gets rid of the sweeping effects[1]. Thus, shell models are the ideal place where non-trivial time properties of the energy-cascade mechanism can be studied, measured and hope-

[1]Sweeping effects, in rather simple terms, refer to direct coupling between the integral scales and the inertial scales.

fully, analytically calculated, because time fluctuations are not hidden by the large scale sweeping, which is quite different from what happens in any Eulerian measurement of 3D turbulence.

5.1 NEED FOR A NEW SHELL MODEL

As might have been guessed by now, in this chapter, we shall use a typical shell model — GOY shell model[51, 52] (modified appropriately for rotation) — to investigate the behaviour of the structure functions and, thus, the signatures of the two-dimensionalisation effect. One may ask immediately why one needs another shell model though Hattori *et. al.*[14] have already proposed a shell model for rotating turbulence — an improved version of shell model by L'vov — couples of years back. To answer this question, let us collect the main results of that model: (i) the exponent of the energy spectrum in the inertial range changes from $-5/3$ to -2, (ii) no inverse cascade is detected with the increase in rotation rate, and (iii) the PDF's of the longitudinal velocity difference doesn't match with the experiments. Well, this field of studying the two-dimensionalisation effect is growing rapidly. It has been confirmed well beyond doubt that the exponent overshoots the value -2 quite comfortably. One may refer to the experiments by Morize *et. al.*[9]. The model justifies its results by invoking weak-wave-turbulence-theory in which inverse cascade is not really shown. This theory is a highly successful theory but one must be open-minded while dealing with problems as complex as turbulence and therefore, should take the experimental results at their face value. That some experiments and numerics do show inverse cascade with increase in the rotation rate should motivate one to construct shell models that can mimic this effect. As mentioned above, Hattori *et. al.*'s model finds PDF which mismatches with experiments and also, it requires a fluctuating part in the rotation rate to arrive at various results while in experiments and numerics there's no such part. This again should make it clear that why at all we need another model. Moreover, the numerical experiments done here are for unforced turbulence whereas Hattori *et. al.*'s model deals with forced turbulence. Hence, with due respect to the Hattori *et. al.*'s work, in this chapter we have tried to look at other possible shell model that can mimic the signatures of the two-dimensionalisation effect more closely.

5.2 THE MODEL

We have adopted the following strategy[53, 54] for the numerical experiments. A specific form of GOY shell model for non-rotating decaying 3D turbulence is:

$$\left[\frac{d}{dt} + \nu k_n^2\right] u_n = ik_n \left[u_{n+2}u_{n+1} - \frac{1}{4}u_{n+1}u_{n-1} - \frac{1}{8}u_{n-1}u_{n-2}\right]^* \tag{5.1}$$

This may be thought as a time evolution equation for complex scalar shell velocities $u_n(k_n)$ that depends on k_n — the scalar wavevectors labeling a logarithmic discretised Fourier space ($k_n = k_0 2^n$). We choose: $k_0 = 1/6, \nu = 10^{-7}$ and $n = 1$ to 22. The initial condition imposed is: $u_n = k^{1/2}e^{i\theta_n}$ for $n = 1, 2$ and $u_n = k^{1/2}e^{-k_n^2}e^{i\theta_n}$ for $n = 3$ to 22 where $\theta_n \in [0, 2\pi]$ is a random phase angle. The boundary conditions are: $u_n = 0$ for $n < 1$ and $n > 22$. In the inviscid limit ($\nu \to 0$), equation (5.1) owns two conserved quantities *viz.*, $\sum_n |u_n|^2$ (energy) and $\sum_n (-1)^n k_n |u_n|^2$ (helicity). If the fluid is rotating then one may modify equation (5.1) by adding a term $R_n = -i\left[\omega + (-1)^n h\right] u_n$ in the R.H.S. ω and h are real numbers. It may be noted that this term, as is customary of Coriolis force, wouldn't add up to the energy. The $(-1)^n h$ term part in R_n has been introduced[15] to have non-zero mean level of helicity that otherwise has a stochastic temporal behaviour and zero mean level. Therefore, the appropriate shell model for rotating 3D turbulent fluid is:

$$\left[\frac{d}{dt} + \nu k_n^2\right] u_n = ik_n \left[u_{n+2}u_{n+1} - \frac{1}{4}u_{n+1}u_{n-1} - \frac{1}{8}u_{n-1}u_{n-2}\right]^* - i\left[\omega + (-1)^n h\right] u_n \qquad (5.2)$$

We fix $h = 0.1$ in our numerical experiments and test for $\omega = 0.01, 0.1, 1.0$ and 10.0. We shall henceforth refer ω as rotation strength. All the data points reported here are averaged over 500 independent initial conditions and the error-bars reported herein are the corresponding standard deviations obtained using 40 different statistically independent runs. Data have been recorded only after cascade completion (see fig-5.1) for each case has been attained. Inertial range has been taken as $n = 4$ to 15 — the range we

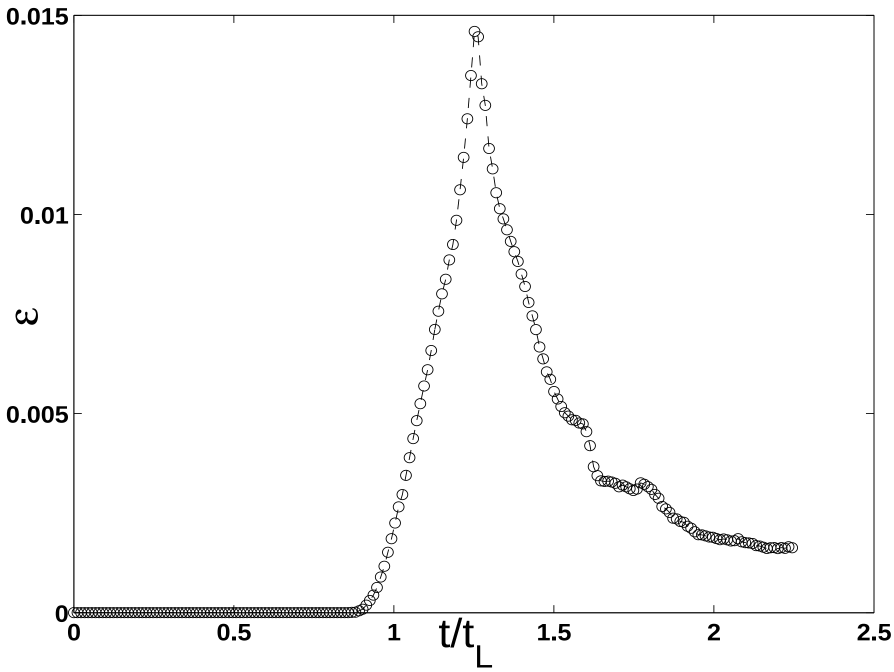

Figure 5.1: **Cascade completion.** A representative curve ($\omega = 0.0, h = 0.0$) of mean rate of dissipation of energy ε vs. time (in units of eddy-circulation time: $t_L \equiv \left[k_1(\langle\sum_n |u_n|^2\rangle)^{1/2}\right]^{-1} = 8.865$; initial complex shell scalar velocity has been used to calculate the r.m.s.). The attainment of the peak suggests the completion of cascade.

are interested in. We have, by the by, adopted slaved second order Adam-Bashforth

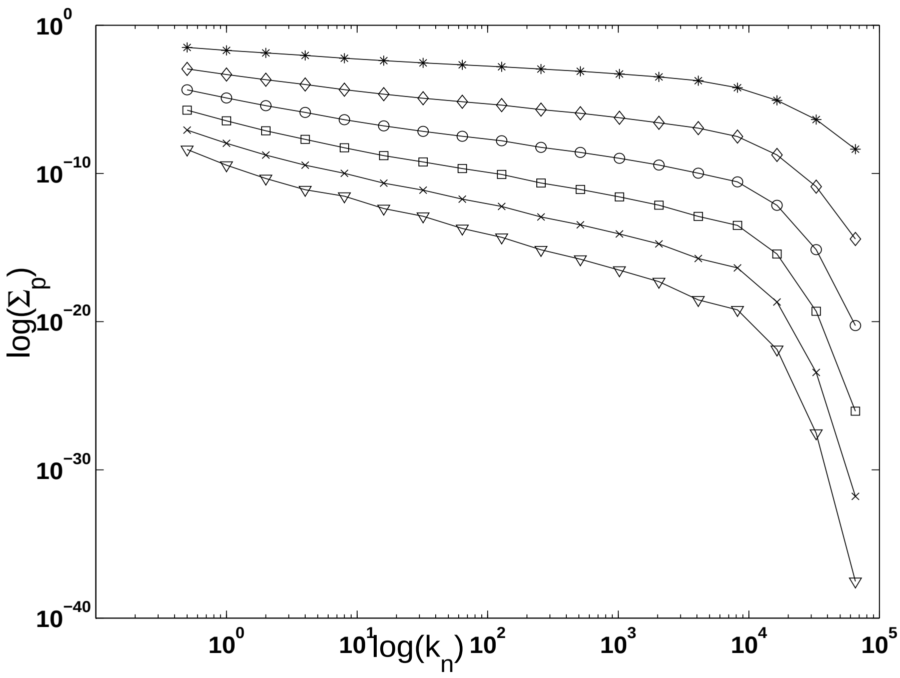

Figure 5.2: **Structure functions.** A representative log-log plot $((\omega = 0.01, h = 0.1))$ for Σ_p vs. k_n. From the topmost curve to the bottommost curve p increases from 1 to 6. We plot for $n = 3$ to 20.

scheme[55] to integrate equations (B.1) and (B.2).

The pth order equal time structure function (see fig-5.2) for the model has been defined as:

$$\Sigma_p(k_n) \equiv \left\langle \left| \mathrm{Im}\left[u_{n+1} u_n \left(u_{n+2} - \frac{1}{4} u_{n-1} \right) \right] \right|^{\frac{p}{3}} \right\rangle \sim k_n^{-\zeta_p} \tag{5.3}$$

to avoid period three oscillations[56]. The energy spectrum has been defined as: $E(k_n) = \Sigma_2(k_n)/k_n \sim k_n^{-m}$. The mean rate of dissipation of energy is, of course, $\varepsilon = \langle \sum_n \nu k_n^2 |u_n|^2 \rangle$

and flux through nth shell is calculated using the relation:

$$\Pi_n \equiv \left\langle -\frac{d}{dt}\sum_{i=1}^{n} |u_i|^2 \right\rangle \tag{5.4}$$

$$\Rightarrow \quad \Pi_n = \left\langle -\mathbf{Im}\left[k_n u_{n+1} u_n \left(u_{n+2} + \frac{1}{4}u_{n-1}\right)\right]\right\rangle \tag{5.5}$$

For studying relative structure function scaling, the ESS scaling exponents[59] are taken as $\zeta_p^* \equiv \zeta_p/\zeta_3$. m, ζ_p and ζ_p^* have all been calculated for inertial ranges only.

5.3 THE RESULTS

The results are illuminating. One can clearly see (refer to fig-5.4 and table-1), that as the rotation strength increases, the energy spectrum becomes steeper and the slope monotonically rushes from a value $\sim -5/3$ to a value of $\sim -7/3$; hence validating one of the two-dimensionalisation effect's signatures.

Table 1: Slope ($-m$) of the curves drawn for the energy spectra ($E(k) \sim k^{-m}$) as in FIG-5.3.

$\omega = 0.00, h = 0.0$ (Non-rotating case)	-1.70 ± 0.0062
$\omega = 0.01, h = 0.1$ (Rotating case)	-1.95 ± 0.0182
$\omega = 0.10, h = 0.1$ (Rotating case)	-2.14 ± 0.0232
$\omega = 1.00, h = 0.1$ (Rotating case)	-2.20 ± 0.0161
$\omega = 10.0, h = 0.1$ (Rotating case)	-2.25 ± 0.0138

As we investigate into the direction of the flux in the inertial range regime, we can find (refer to fig-5.5) that with the increase in rotation strength first the forward cascade rate starts decreasing and then instances appear when at certain shells the flux direction reverses. Again, the number of such shells increase as the rotation strength is enhanced;

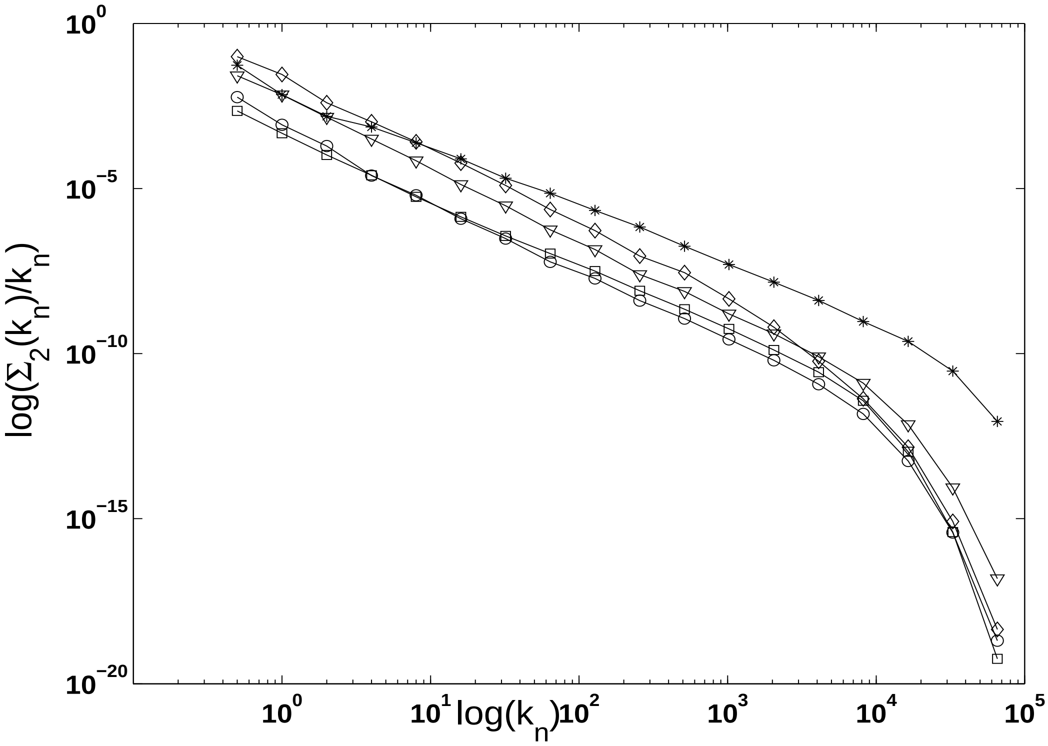

Figure 5.3: **Energy spectra**. Energy spectra $E(k_n)$ vs. k_n plotted in log-log plot. Asterisk, square, triangle, circle and diamond respectively are the markers for non-rotating, $\omega = 0.01$, $\omega = 0.1$, $\omega = 1.0$ and $\omega = 10.0$ cases. We plot for $n = 3$ to 20.

clearly suggesting that depletion in the rate of forward cascade.

Thus, yet another signature of two dimensionalisation has been upheld by the shell model. At this point, it must be appreciated how important the inclusion of term $-i(-1)^n h$ in equation (B.2) is in getting the effect of depletion in the rate of forward cascade. By setting mean level of helicity above zero, it is this very term that — in accordance with the arguments[24] given in the preceding chapter that it is the helicity that is causing this signature of two dimensionalisation effect to show up — has empowered the model with the capacity to mimic the effect. Attempts to get this very effect by setting $h = 0$ have failed miserably in our numerical experiments. The study of ESS in the shell model has been equally revealing. As it can be noted (refer to fig-5.6

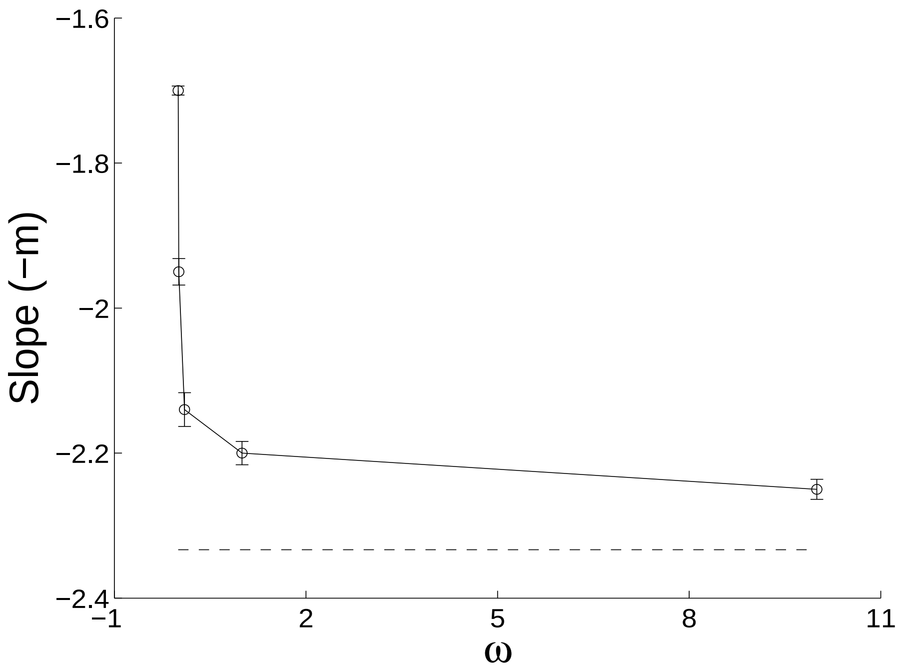

Figure 5.4: **Change in slope with "rotation".** The slopes of the energy spectra plotted againt "so-called" rotation strength. The accompanying dashed line is the value $-7/3$ of the slope that has been predicted for very rapid rotation.

and tables-2 & 3), the increase in the rotation strength is accompanied by a departure from the usual She-Leveque scaling. But, the fact that at higher p ζ_p seemingly becomes parallel to $p/2$ vs. p, is worth paying attention: This is in accordance with the direct numerical simulation (DNS) results[16] and experimental results[7]. However, most interesting observation would be that, within the statistical error, ζ_p^* obtained for the rotating system via ESS coincides with that for the non-rotating ones. Probably, this extends the ESS for 3D fluids even further by implying that rotation keeps ESS scaling intact, even though usual ζ_p changes owing to rotation. Of course, only experiments and DNS can judge if this really is true for real fluid turbulence: GOY shell, after all, is just a model that remarkably reproduces many characteristic features of turbulence by only using a fraction of computation power needed by DNS. In this context, one might be well

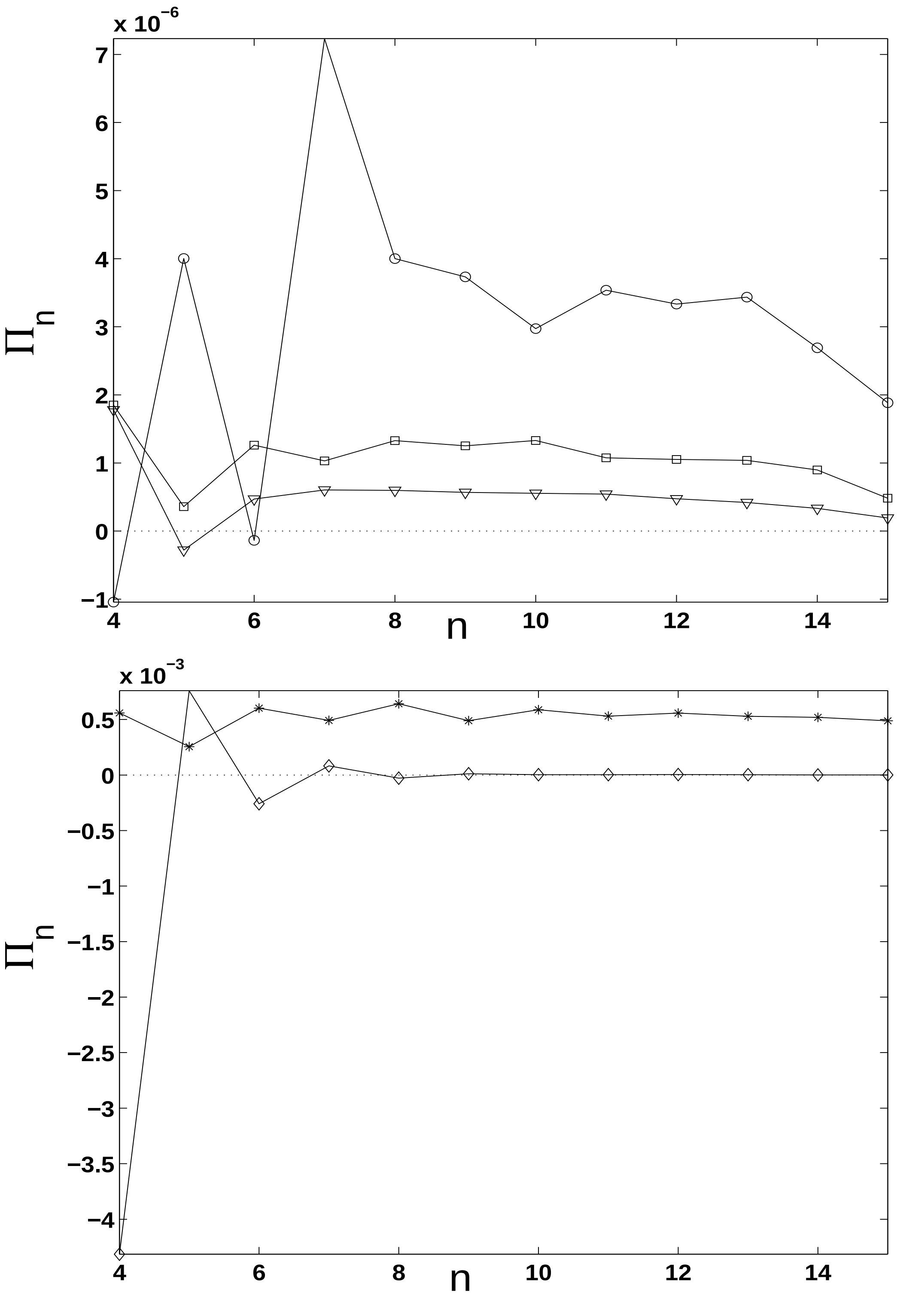

Figure 5.5: **Energy flux.** Average flux of energy through nth shell vs. shell number n. Only the inertial range ($n = 4$ to 15) has been plotted. Markers are same as that for fig-5.3.

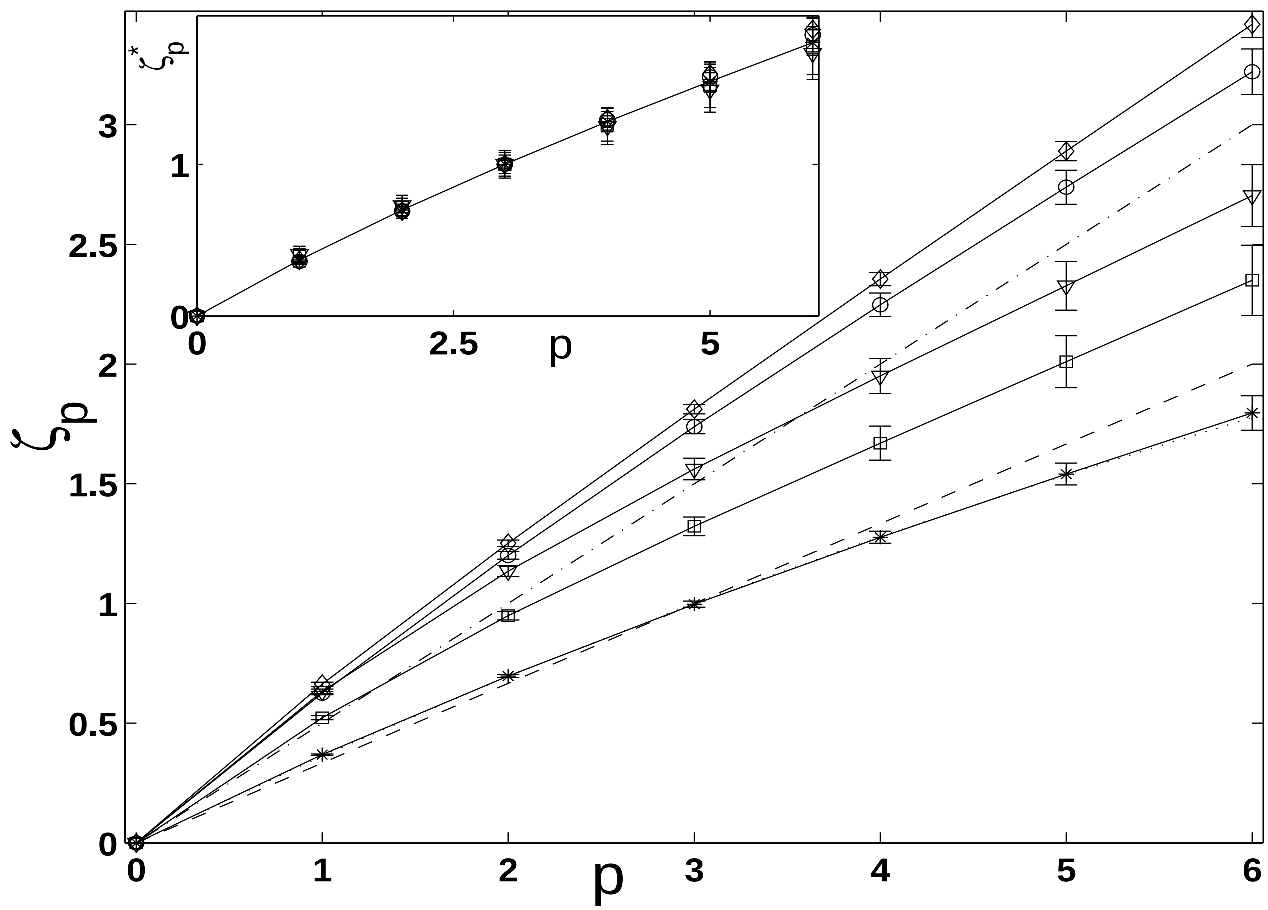

Figure 5.6: **Plots for data in the tables-2 & 3.** ζ_p vs. p plotted for the data in table-2. Markers are same as that for fig-5.3. The dashed, the chain and the dotted lines are respectively for $\zeta_p = p/3$ (K41), $\zeta_p = p/2$ and $\zeta_p = p/9 + 2[1 - (2/3)^{p/3}]$ (She-Leveque exponent[57]). The dotted curve has almost been reproduced by non-rotating GOY model, as expected. This anomalous scaling is remarkably reproduced in the model dynamical system with limited number of degrees of freedom because its chaotic evolution exhibits temporal intermittency[58]. The inset is plot for ζ_p^* vs. p plotted using the data of table-3. All the connecting lines and the fractional values of p are just aids for viewing the plots.

Table 2: ζ_p for $p = 1$ to 6 for various rotation strengths.

p	$\zeta_p(\omega = 0.00, h = 0.0)$	$\zeta_p(\omega = 0.01, h = 0.1)$	$\zeta_p(\omega = 0.10, h = 0.1)$	$\zeta_p(\omega = 1.00, h = 0.1)$	$\zeta_p(\omega = 10.0, h = 0.1)$
1	0.37 ± 0.0027	0.52 ± 0.0086	0.63 ± 0.0098	0.62 ± 0.0067	0.66 ± 0.0086
2	0.70 ± 0.0062	0.95 ± 0.0182	1.1 ± 0.0232	1.2 ± 0.0161	1.2 ± 0.0138
3	1.0 ± 0.0127	1.3 ± 0.0394	1.6 ± 0.0455	1.7 ± 0.0301	1.8 ± 0.0197
4	1.3 ± 0.0251	1.7 ± 0.0712	2.0 ± 0.0733	2.2 ± 0.0490	2.4 ± 0.0283
5	1.5 ± 0.0454	2.0 ± 0.1083	2.3 ± 0.1017	2.7 ± 0.0713	2.9 ± 0.0402
6	1.8 ± 0.0718	2.4 ± 0.1470	2.7 ± 0.1291	3.2 ± 0.0953	3.4 ± 0.0550

Table 3: $\zeta_p^* \equiv \zeta_p/\zeta_3$ for $p = 1$ to 6 for various rotation strengths.

p	$\zeta_p^*(\omega = 0.00, h = 0.0)$	$\zeta_p^*(\omega = 0.01, h = 0.1)$	$\zeta_p^*(\omega = 0.10, h = 0.1)$	$\zeta_p^*(\omega = 1.00, h = 0.1)$	$\zeta_p^*(\omega = 10.0, h = 0.1)$
1	0.37 ± 0.0153	0.40 ± 0.0480	0.39 ± 0.0553	0.36 ± 0.0368	0.37 ± 0.0283
2	0.70 ± 0.0188	0.73 ± 0.0576	0.69 ± 0.0687	0.70 ± 0.0463	0.67 ± 0.0335
3	1.0 ± 0.0253	1.0 ± 0.0789	1.0 ± 0.0910	1.0 ± 0.0603	1.0 ± 0.0393
4	1.3 ± 0.0377	1.3 ± 0.1106	1.2 ± 0.1188	1.3 ± 0.0791	1.3 ± 0.0479
5	1.5 ± 0.0580	1.5 ± 0.1477	1.4 ± 0.1472	1.6 ± 0.1014	1.6 ± 0.0598
6	1.8 ± 0.0844	1.8 ± 0.1865	1.7 ± 0.1746	1.9 ± 0.1255	1.9 ± 0.0746

aware that some modified versions of GOY model invented to model the distinguishing features of 2D turbulence have been shown to be rather useless[60]. One, thus, always has to be careful while dealing with simplified models of turbulence.

Chapter 6

DISCUSSIONS AND CONCLUSIONS

In this dissertation, the Kolmogorov-Landau approach has been invoked in 2D homogeneous isotropic unforced fluid turbulence to arrive at the various correlation functions earlier obtained using different methods. Also, some experimentally verifiable correlation functions in the dissipation range have been derived. The results derived here are 'exact' (though not rigorous) something which is a far cry in the literature on turbulence. However, we have been careless enough to assume the existence of η when $\nu \to 0$. If η doesn't exist, the one-eighth law is in jeopardy. It is really unfortunate for the law that it has been rigourously proved[61, 62] that enstrophy dissipation is not possible for any 2D Euler solutions with finite enstrophy. Thus, η may exist in the inviscid limit only when one takes rather ill-defined initial conditions for which the total initial enstrophy is infinite. In view of this, one must not be surprised at all if numerics and experiments fail to uphold the one-eighth law in many a situation. This very law of 2D turbulence, therefore, doesn't enjoy the same classic status as the Kolmogorov law of 3D turbulence.

Studies with structure functions of QG turbulence have again showcased how handy and useful the Kolmogorov-Landau approach can prove to be. The results (3.19) and

(3.39) naturally agree with what has been arrived at by Lindborg[37] earlier. Within the domain of the approximations made these results are exact, something worth getting as the literature of turbulence is comparatively barren as far as exact relations are concerned. However, the hypothesis of the equipartition of energy used in equation (3.38) is as questionable as the assumption of isotropy in the sense of Charney. This hypothesis needs to be put on more firm basis. Again, the existence of ε_q, like η, is questionable when $\nu \to 0$. However, studies in 2D and QG turbulences in the perspective of the velocity structure functions hint that finding the velocity structure functions in the rotating flows with a view to unfolding the mysteries of two-dimensionalisation effect might not be just a wild goose chase.

We emphasis on the fact that the form of two point third order structure function in a slowly rotating homogeneous 3D turbulence can strongly indicate the initiation of the effect of two-dimensionalisation of 3D turbulence. It barely needs to be mentioned that the relations are quite interesting and pertinent (at least within the approximations made in the calculations) – something which, as is being said again and again, is worth getting in the literature of turbulence since exact relations are very few therein. So any theory developed in the limit of $Ro \to 0$ and $Re \to \infty$, must satisfy the relation (4.45) derived in this dissertation in the limit of low Ω or explicitly violate the assumptions made to arrive at the result; in this sense the relation may prove to be of high importance. Moreover, true reason behind the so called two-dimensionalisation of turbulence has been figured out which accounts for the correct energy cascade direction and the correct energy spectrum found in the experiments and simulations. To settle the problem more neatly, it has been proposed that the study of passive scalars in rotating turbulence may prove to be of benefit.

Shell models have been successfully used to study statistical properties of turbulence of many authors (see ref-([63]) and ref-([64]) for details). Most of them dealt with the homogeneous isotropic turbulence. Hattori *et. al.* gave a shell model for rotating turbulence. Their result could be improved in the face of the rapid progress in the field. We, thus, have used a modified version of GOY shell model to study the two-dimensionalisation effect. Some results of the model are, no doubt, consistent with experiments and DNS. However, one can always question the effectiveness of the signatures discussed herein because (i) a scaling law for a single-component spectrum, though heavily used in literature, has poor meaning in the strongly anisotropic configuration relevant to pass from 3D-2D; different power laws can be found in terms of k_z, $k_\perp$ and k in contrast to the 3D isotropic case, and (ii) the inertial wave-turbulence theory is not consistent with an inverse cascade. Actually in weak-wave turbulence, getting rid provisionally of helicity and polarisation spectra, a two-component energy spectrum $e(k, \cos\theta)$ with $\cos\theta = k_z/\sqrt{(k_z^2 + k_\perp^2)}$ is found to be useful; if E denotes the traditional spherically averaged spectrum, the anisotropic structure is one of the best ways to quantify all intermediate states from isotropic 3D (with $e = E(k)/(4\pi k^2)$) to 2D state (with $e = E(k_\perp)/(2\pi k_\perp)\delta(k_z)$). Two-dimensional trend is therefore linked to a preferred concentration of spectral energy towards the transverse wave-plane $k_z = 0$. This concentration, however, does not necessarily yield an inverse cascade. A reasonable suggestion, in the light of this discussion, would be that in the shell model for rotating turbulence k should be interpreted as $k_\perp$. It may be concluded that this study has put the equation (5.2) as a very firm shell model for the rotating 3D turbulent flows; after all, it explains the observed signatures of the two-dimensionalisation effect so closely. Probably, this model discussed in this thesis and the model due to Hattori *et. al.* can together yield a much better shell model for the rotating turbulence.

In the absence of solutions of the Navier-Stokes equations, these various results regarding two-dimensionalisation can only be checked by data from experiments, and this endeavour seems to be difficult at present. So in the closing, we hope that validity of the results derived will be checked both numerically and experimentally in near future. After all, the structure functions traditionally provide checks for any plausible theory of turbulence.

Appendix A

ENERGY SPECTRUM

We have already discussed in the introducing chapter that there are situations in which fluid velocities seem to vary randomly in space and time. Such a state is called turbulence which we may get, *e.g.*, if we stir a cup of water with random forces at different points. In other words, turbulence is characterised by extremely irregular, disordered variation of velocity field with time at each point in space. As earlier, even in this appendix we shall denote the ensemble average of a physical quantity by angular brackets ($\langle ... \rangle$) and the fluctuating part of the physical quantity by an overhead prime ($'$). We shall restrict ourselves within the domain of the incompressible fluid ($\nabla.\mathrm{v} = 0$, where as usual v denotes the velocity field[1]). Also, unless otherwise mentioned we shall concentrate on the steady turbulence by which we mean that the mean characteristics (averaged over times that are of the order of the periods of the corresponding fluctuations but of course are small compared to the total observation time) are constant. We shall allow the vectors x and $\tilde{\mathrm{x}}$ to denote the spatial coordinates and t to denote the time.

[1]To have convenience with notations, in this appendix, we have allowed boldfaced letters to represent vectorial quantities.

Obviously, for the velocity field v in the turbulent fluid one can write:

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}' \tag{A.1}$$

Taylor in 1935 suggested that attention should be given to the velocity correlation function (or velocity correlation tensor of rank 2)[2], defined as: $\langle \mathbf{v}'(\mathbf{x}, t)\mathbf{v}'(\mathbf{x}+\tilde{\mathbf{x}}, t)\rangle$ (which technically is termed as two-point velocity correlation function); if $\tilde{\mathbf{x}} = 0$ it gives the measure of the kinetic energy as the correlation function gets reduced to $\langle \mathbf{v}'^2(\mathbf{x}, t)\rangle$ which simply measures the strength of turbulence. Now, one may also note that:

$$\lim_{\tilde{\mathbf{x}}\to\infty} \langle \mathbf{v}'(\mathbf{x}, t)\mathbf{v}'(\mathbf{x}+\tilde{\mathbf{x}}, t)\rangle = 0 \tag{A.2}$$

The range within which the correlation function is substantially non-zero is called correlation length of turbulence.

A.1 CORRELATION FUNCTION AND ENERGY SPECTRUM

Though the velocity correlation functions are hard to calculate, one can find out some properties of the velocity correlation tensors, if one considers the simplified case of isotropic and homogeneous turbulence. Let us try to toy with it.

So, what implications do the isotropicity provide us with? Obviously, it means that the mean velocity of such a turbulent fluid must be zero (i.e., $\langle v \rangle = 0$), for, isotropy forbids a net directional flow. Also, the two-point velocity correlation tensor is independent of the direction of $\tilde{\mathbf{x}}$ but is dependent on the magnitude of the $\tilde{\mathbf{x}}$. And what about homogene-

[2]Why only consider two-point velocity correlation function? Well, one does consider higher correlation functions, e.g. three-point velocity correlation function: $\langle \mathbf{v}'(\mathbf{x}, t)\mathbf{v}'(\mathbf{x}+\tilde{\mathbf{x}}, t)\mathbf{v}'(\mathbf{x}+\tilde{\tilde{\mathbf{x}}}, t)\rangle$. But just because various such correlation functions are not derivable from a fundamental theory, the turbulence remains an unsolved problems in physics.

ity? It says that the correlation function will be independent of x. Again, isotropicity and homogeneity dictates the correlation tensor C_{ij} (where i, j, k runs from 1 to 3 in three dimensional turbulent fluid) to be symmetric. So,

$$C_{ij}(\tilde{x}, t) = \langle \mathbf{v}_i(\mathbf{x}, t)\mathbf{v}_j(\mathbf{x} + \tilde{\mathbf{x}}, t) \rangle \tag{A.3}$$

The following (where Einstein's summation convention is applied) is the most general form of C_{ij}:

$$C_{ij}(\tilde{x}, t) = A_1(\tilde{x}, t)\tilde{x}_i\tilde{x}_j + A_2(\tilde{x}, t)\delta_{ij} + A_3(\tilde{x}, t)\epsilon_{ijk}\tilde{x}_k \tag{A.4}$$

where A_1, A_2, A_3 are three scalar functions, δ_{ij} is the Kronecker delta and ϵ_{ijk} is the Levi-Civita symbol which is also known as permutation symbol. Now consider the third term. It preserves isotropy, but would not be invariant under parity inversion (which is more popularly termed as "reflection"). We, for simplicity assume the turbulence to be reflectionally symmetric and omit the third term.

For analysis we split the correlation function into the longitudinal and the lateral velocity correlation functions: $C_{lon}(\tilde{x}, t)$ and $C_{lat}(\tilde{x}, t)$ respectively. The lateral component of x̃ is $\tilde{x}_{lat} = 0$ and the longitudinal component is $\tilde{x}_{lon} = \tilde{x}$ which obviously suggests following consistent definitions:

$$C_{lon}(\tilde{x}, t) \equiv A_1(\tilde{x}, t)\tilde{x}^2 + A_2(\tilde{x}, t) \tag{A.5}$$

$$C_{lat}(\tilde{x}, t) \equiv A_2(\tilde{x}, t) \tag{A.6}$$

It may be noted that we are using "lon" and "lat" subscripts to denote the longitudinal and the lateral components respectively of the concerned physical quantities. Now we define:

$$f_1(\tilde{x},t) \equiv \frac{3}{\langle v^2 \rangle} C_{lon}(\tilde{x},t) \tag{A.7}$$

$$f_2(\tilde{x},t) \equiv \frac{3}{\langle v^2 \rangle} C_{lat}(\tilde{x},t) \tag{A.8}$$

such that $f_1 = f_2 = 1$ at $\tilde{x} = 0$. Obviously, relations (A.5) to (A.8) imply that relation (A.4) becomes:

$$C_{ij}(\tilde{x},t) = \frac{\langle v^2 \rangle}{3} \left\{ \frac{f_1(\tilde{x},t) - f_2(\tilde{x},t)}{\tilde{x}^2} \tilde{x}_i \tilde{x}_j + f_2(\tilde{x},t)\delta_{ij} \right\} \tag{A.9}$$

Again, as the fluid is considered to be incompressible relation (A.3) gives:

$$\frac{\partial}{\partial \tilde{x}_j} C_{ij}(\tilde{x},t) = \langle \mathbf{v}_i(\mathbf{x},t) \frac{\partial}{\partial \tilde{x}_j} \mathbf{v}_j(\mathbf{x}+\tilde{\mathbf{x}},t) \rangle = 0 \tag{A.10}$$

which owing to the symmetry that C_{ij} possess, implies:

$$\frac{\partial}{\partial \tilde{x}_j} C_{ij}(\tilde{x},t) = \frac{\partial}{\partial \tilde{x}_i} C_{ij}(\tilde{x},t) = 0 \tag{A.11}$$

which dictates relation (A.9) to give the result:

$$f_2(\tilde{x},t) = f_1(\tilde{x},t) + \frac{\tilde{x}}{2} \frac{\partial}{\partial \tilde{x}} f_1(\tilde{x},t) \tag{A.12}$$

Here one sees that if someone can find out $f_1(\tilde{x},t)$, C_{ij} can be computed completely. Intuitively one may guess that $f_1(\tilde{x},t)$ should be a monotonically decreasing function such that $f_1(\tilde{x},t)\rho 0$ as $\tilde{x}\rho\infty$.

We shall show that an equivalent scalar function $E(k)$ exists which can completely describe $C_{ij}(\tilde{x},t)$ for a homogeneous and isotropic turbulence. For this purpose we define the Fourier transform of $C_{ij}(\tilde{x}, t)$ (From now on we shall suppress all "t" (time), assuming that it is implicitly implied that all the quantities are being calculated at the same time):

$$\widehat{C}_{ij}(k) \equiv \frac{1}{(2\pi)^3} \int C_{ij}(\tilde{x}) e^{i\mathbf{k}.\tilde{\mathbf{x}}} d^3\tilde{\mathbf{x}} \quad \text{(A.13)}$$

where we have used the fact that $\widehat{C}_{ij}(k)$ must be spherically symmetric in k-space, as $C_{ij}(\tilde{x})$ is so in the $\tilde{x}$-space. The integration is over all space. One should also not confuse the $i = \sqrt{-1}$ in the exponent of e with the subscript i which runs from 1 to 3. The inverse Fourier transform is:

$$C_{ij}(\tilde{x}) \equiv \int \widehat{C}_{ij}(k) e^{-i\mathbf{k}.\tilde{\mathbf{x}}} d^3\mathbf{k} \quad \text{(A.14)}$$

From relation (A.11) we get using the relation (A.14):

$$k_i \widehat{C}_{ij}(k) = k_j \widehat{C}_{ij}(k) = 0 \quad \text{(A.15)}$$

As earlier, homogeneity, isotropy, symmetry and parity invariance implies the general form of $\widehat{C}_{ij}(k)$ as:

$$\widehat{C}_{ij}(k) = B_1(k) k_i k_j + B_2(k) \delta_{ij} \quad \text{(A.16)}$$

From the relation (A.15), we get using the relation (A.16):

$$B_1(k) k^2 k_j + B_2(k) k_i \delta_{ij} = 0 \quad \text{(A.17)}$$

$$\Rightarrow B_2(k) = -B_1(k) k^2 \quad \text{(A.18)}$$

which lets relation (A.16) to take the form:

$$\widehat{C}_{ij}(k) = B_1(k)(k_i k_j - k^2 \delta_{ij}) \tag{A.19}$$

Now, as we had defined earlier, the strength of turbulence, we notice (using relation (A.14)):

$$\langle v^2 \rangle = C_{ii}(0) = \int \widehat{C}_{ii}(k) d^3\mathbf{k} \tag{A.20}$$

Using relation (A.19) the preceding relation becomes:

$$\langle v^2 \rangle = \int B_1(k)(k_i k_j - k^2 \delta_{ij}) 4\pi k^2 dk \tag{A.21}$$

$$\Rightarrow \langle v^2 \rangle = \int -2k^2 B_1(k) 4\pi k^2 dk \tag{A.22}$$

Let us define:

$$E(k) \equiv -4\pi B_1(k) k^4 \tag{A.23}$$

so that we get:

$$\frac{1}{2}\langle v^2 \rangle = \int_0^{\infty} E(k) dk \tag{A.24}$$

Evidently $E(k)$ is the energy spectrum of turbulence. Now the job is to show that any one of $f_1(\tilde{x})$ or $E(k)$ is enough to compute $C_{ij}(\tilde{x})$ i.e., we wish to show that $f_1(\tilde{x})$ and $E(k)$

are related. To prove this, let us define:

$$C(\tilde{x}) \equiv C_{ii}(\tilde{x}) \tag{A.25}$$

So that the relation (A.9) becomes:

$$C(\tilde{x}) = \frac{\langle v^2 \rangle}{3}\{f_1 + 2f_2\} \tag{A.26}$$

which using the equation (A.12) gives:

$$C(\tilde{x}) = \langle v^2 \rangle \left\{ f_1 + \frac{\tilde{x}}{3}\frac{df_1}{d\tilde{x}} \right\} \tag{A.27}$$

Now, the relation (A.13) yields, on setting $i = j$ and summing over all $i = 1, 2, 3$:

$$\widehat{C}_{ii}(k) = \frac{1}{(2\pi)^3}\int C_{ii}(\tilde{x}) e^{i\mathbf{k}.\tilde{\mathbf{x}}} d^3\tilde{\mathbf{x}} \tag{A.28}$$

$$\Rightarrow \frac{E(k)}{2\pi k^2} = \frac{1}{(2\pi)^3}\int C(\tilde{x}) e^{i\mathbf{k}.\tilde{\mathbf{x}}} d^3\tilde{\mathbf{x}} \tag{A.29}$$

$$\Rightarrow E(k) = \frac{k^2}{(2\pi)^2}\int_{\tilde{x}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} C(\tilde{x}) e^{ik\tilde{x}\cos\theta}\tilde{x}^2 \sin\theta d\theta d\phi d\tilde{x} \tag{A.30}$$

$$\Rightarrow E(k) = \frac{1}{\pi}\int_0^{\infty} C(\tilde{x}) k\tilde{x}\sin(k\tilde{x}) d\tilde{x} \tag{A.31}$$

where in the second step we have used the relations (A.19) to (A.25) and in the third step we have denoted the angle between k and x̃ by θ. Hence we see that $E(k)$ is related to $C(\tilde{x})$ that in turn is directly related to f_1 by the relation (A.27), proving the fact that we wished to prove.

It is any body's guess that next step should be to find a way to calculate $E(k)$ or if it is possible then at least to find the functional relationship of $E(k)$ on k.

A.2 KOLMOGOROV ENERGY SPECTRUM

The wish of finding a form for $E(k)$ takes us to the Kolmogorov's (1941) famous theory which determines the energy spectrum for a steady[3], homogeneous and isotropic turbulent system.

First of all, lets us try to comprehend the idea behind the theory. Kolmogorov imagined the turbulent velocity field to be a collection of many vortices of various sizes. He assumed that the input energy is fed into the turbulent fluid in such a manner so that production of the vortices of largest size takes place. These largest eddies then transfer energy to the smaller eddies which in turn transfer energy to still smaller eddies and so on. How exactly such a transfer of energy takes place? Well, to give an idea, in ordinary fluid dynamics, we have a theorem called the Kelvin vorticity theorem (analogous to the Alfvén's theorem) according to which for larger eddies, for which the viscous dissipation is negligible, vorticity flux through a surface is constant. Consider a vortex tube formed by the large eddies. Any two fluid elements move randomly in turbulent fluid so that the separation between them increases usually and as, in accordance with Kelvin vorticity theorem, they will carry their vorticity with them, the vortex tube will be stretched which means that the tube will be laterally squeezed due to assumed in compressibility of the fluid. This means the birth of smaller eddies which is an equivalent way of saying that energy is transferred to smaller eddies from larger ones.

As the energy is cascaded, we can have eddies of sufficiently small size, that dissipates the mean amount of energy fed at a rate, say, ε per unit mass per unit time so that equilibrium is maintained. Let the typical sizes and the typical magnitude of velocities associated with the smallest, intermediate and the largest eddies be l_s and v_s, l and v, and, l_l and v_l respectively. Also, let ν be the kinematic viscosity of the fluid. We antici-

[3]A turbulent fluid is maintained in a steady state only if it is fed with continuous energy, for, left to itself the fluid is bound to dissipate away its kinetic energy by dint of viscous dissipation.

pate:

$$R_s \equiv \frac{l_s v_s}{\nu} \sim 1 \tag{A.32}$$

$$R_l \equiv \frac{l_l v_l}{\nu} \gg 1 \tag{A.33}$$

R_s and R_l are the Reynold's number for the smallest and the largest eddies respectively. Kolmogorov postulated that ε must be expressible in terms of l and v. On dimensional grounds, this postulate states[4]:

$$\varepsilon \sim \frac{v^3}{l} \tag{A.34}$$

Which for the smallest eddies means:

$$\varepsilon \sim \frac{v_s^3}{l_s} \tag{A.35}$$

in which putting the value of v_s from the relation (A.32), we get:

$$l_s \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \tag{A.36}$$

Similarly for largest eddies, we have from the relation (A.34):

$$\varepsilon \sim \frac{v_l^3}{l_l} \tag{A.37}$$

$$\Rightarrow \frac{\varepsilon}{\nu^3} \sim \frac{1}{l_l^4}\left(\frac{v_l^3 l_l^3}{\nu^3}\right) \tag{A.38}$$

[4]The relation (A.34) can be written as $v \sim \varepsilon^{1/3}$ for a particular size ($l =$ constant). This is known as Kolmogorov scaling law.

which using the relations (A.33) and (A.36), implies:

$$\left(\frac{l_l}{l_s}\right) \sim R_l^{3/4} \tag{A.39}$$

Hence, it is the Reylonds number of the largest eddies which determines the size of the smallest eddies.

If k_l and k_s be the wave-numbers corresponding to the largest and the smallest eddies, then the range of the wave numbers from k_l to k_s is called inertial range within which energy ε per unit mass per unit time is being transferred from larger to smaller eddies. As energy spectrum $E(k)$ can most naturally depend only on ε and k, on dimensional grounds we have:

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \tag{A.40}$$

This is the celebrated -5/3 law of Kolmogorov.

Of course, this analysis can hardly be called a "derivation". But experimental results speak volumes for this theory and as a saying goes in Bengali: "*Nai mamar che kana mama bhalo.*" (It is better to have a one-eyed uncle than to have none.), there should not be any shame in absorbing the idea of Kolmogorov into the subject of turbulence which remains poor as far as a proper concrete theory is concerned.

Appendix B

TURBULENCE AND COIN TOSS

*I*n the preceding chapters we have hit upon the words 'intermittency' and 'multifractality' more than once. It has also been mentioned that one of the signatures of two-dimensionalisation of 3D turbulence is that the rotation supresses the intermittency. We believe if one can model intermittency correctly then it will be easier to find out the enigmatic way the rotation brings about a decrease in the level of intermittency in the turbulence system. Thus, how exactly this intermittency in a three-dimensional homogeneous isotropic imcompressible high Reynolds number fluid turbulence can be modelled is the issue taken up in this appendix. We shall use the theory of large deviations as a tool to model the intermittency in turbulence which shall be treated as a problem of non-equilibrium statistical physics.

B.1 THEORY OF LARGE DEVIATIONS

Large deviations play a significant role in many branches of non-equilibrium statistical physics[65, 66]. They are difficult to handle because their effects though small, are not amenable to perturbation theory. All the conventional perturbation theories in statistical physics are fashioned about a Gaussian distribution, which almost by definition, is the distribution with no large deviations. This can be seen in static critical phenomena, critical dynamics, dynamics of interfacial growth, statistics of polymer chain and myriad other problems[67]. However, the Gaussian model fails to be a starting point while discussing intermittency in fluid turbulence[30, 57, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78]. In the large deviation theory, the central role is played by the distribution associated with tossing of a coin. Our contention is: the simple coin toss is the "Gaussian model" of problems where rare events play significant role. We illustrate this by applying it to the study of intermittency in fully developed turbulence.

The high Reynolds number turbulence remains the prime age old problem dominated by rare events, which still eludes a satisfactory theoretical understanding. Before we plunge into the problem of modeling intermittency in a turbulent fluid, let us begin by briefly reviewing the large deviation theory in the context of a coin toss experiment[79]. Suppose we have a biased coin, such that for each toss the probability of obtaining "head" is 'p'. If we assign the value 1 to the outcome "head" (each outcome is denoted by X_i where $i = 1, 2, ...$) and 0 to the outcome "tail", then the mean after N trials is

$$M_N = \frac{1}{N} \sum_{i=1}^{N} X_i \tag{B.1}$$

As $N \to \infty$, it is expected that $M_N \to p$. The question is: For large N, what is the probability that M_N differs from p by at least x (where x is any pre-assigned fraction less than unity)? The meaning of large deviation is that however large N may be this probability is nonzero and if the X_i's are bounded, independent and identically distributed random variables, then Crammers theorem asserts that the tail of the probability distribution of X_i is given by

$$\left.\begin{aligned} P(M_N > x) &\approx e^{-NI(x)} \qquad \text{for } x > p \\ P(M_N < x) &\approx e^{-NI(x)} \qquad \text{for } x < p \end{aligned}\right\} \tag{B.2}$$

To apply this result in different disciplines of statistical physics, we require $P(M_N \approx x)$ and it is Varadhan's theorem that ensures that the sequence M_N itself satisfies a large deviation principle i.e. $P(M_N \approx x) \sim e^{-NI(x)}$. For the coin toss under consideration, Chernoff's formula gives the rate function $I(x)$ as follows:

$$I(x) = x \ln \frac{x}{p} + (1-x) \ln \frac{1-x}{1-p} \tag{B.3}$$

and this is the central result that we will use.

B.2 THE MODEL

Turning to turbulence, in 1941 Kolmogorov[27] invoked the concept of Richardson's cascade[80] of eddies to propose a phenomenological model (K41) for three dimensional incompressible turbulence at high Reynolds number. Even today this is the cornerstone of our understanding of turbulence. Understanding turbulence is understanding the small scale behaviour of the velocity structure function $S_q(l)$, where $S_q(l) \equiv \langle |\Delta\vec{v}.(\vec{l}/|l|)|^q \rangle$, with $\Delta\vec{v} \equiv \vec{v}(\vec{r}+\vec{l}) - \vec{v}(\vec{r})$ and 'l' is a distance which is short compared to macroscopic length scales like the system size but is large compared to molecular scale where viscous dissipation takes place. The angular bracket denotes ensemble average (i.e. average over different values of '$\vec{r}$'). The observation is that $S_q(l)$ has a scaling behaviour l^{ζ_q} where l is in the range indicated (so called inertial range). Finding ζ_q can be described as the holy grail of turbulence. K41 gives $\zeta_q = q/3$ — a result which is exact for $q = 3$ and very close to experimental findings for low value of q. There is systematic departure from $q/3$ at relatively higher values of q. This is the phenomenon of intermittency. Of particular interest is the case $q = 6$. Since $|\Delta v|^3/l$ is a measure of the local energy transfer rate (same as energy input and energy dissipation rate in K41 and thus a constant), we expect $\zeta_6 = 2$. The deviation $2 - \zeta_6$ is thus a very sensitive quantity and is often singled out for special treatment. The exponent $\mu = 2 - \zeta_6$ is formally called the intermittency exponent and the experimental measurements agree on a value 0.2 for μ. It can be viewed as the co-dimension of dissipative structures.

The model of intermittency are usually constructed on a phenomenological basis by thinking of various ways of modifying the Richardson's cascade picture. The β-model, the bifractal model and the multifractal model all belong to this class. The crucial hypothesis is that the daughter eddies produced from the mother eddies are not space filling and the active part of space is in general a multifractal. The velocity field has different scaling exponents on different fractal sets that form the multifractal structure. These scaling exponents can, in principle, yield ζ_q. This multifractality can also be defined and measured in terms of the fluctuations of the local dissipation rate rather than in terms of the fluctuations of the velocity increments Δv. The key element, that is needed to define multifractality in terms of dissipation is the local space average of energy dissipation over a ball of radius l centered around a point at $\vec{r}$: $\varepsilon_l(\vec{r}) \equiv \frac{3\nu}{8\pi l^3}\int_{|\vec{r'}-\vec{r}|<l} d^3\vec{r'}\sum_{i,j}[\partial_j v_i(\vec{r'}) + \partial_i v_j(\vec{r'})]^2$. If the dissipation is multifractal, moments of ε_l follow a power law behaviour at small l, i.e. $\langle \varepsilon_l^q \rangle \sim l^{\tau_q}$. Kolmogorov's refined similarity hypothesis relates the statistical properties of fluctuation of velocity increment to those of the space averaged dissipation and yields: $\zeta_q = \frac{q}{3} + \tau_{q/3}$. We now carry out the usual speculation that since the higher order velocity structure factors differ most strongly from K41, then the probability distribution for the velocity increments must differ most

strongly from that appropriate to K41 in the tail of the distribution. The tail of a distribution involves rare events and this is how the theory of large deviations enters the picture. Following Landau's observation on K41[29], Kolmogorov[68] and Obukhov[69] introduced fluctuations in the dissipation rate. Careful experiments revealed the existence of these fluctuations. The fluctuations, however, occur rarely and these are the rare events of turbulence. This allows us to establish a quantitative bridge between turbulence and theory of large deviations.

More than a decade ago, Stolovitzky and Sreenivasan[78], in somewhat different approach tried to validate refined similarity hypothesis by viewing turbulence as a general stochastic process (fractional Brownian motion to be precise). While this was a very significant achievement, there was a shortcoming in that the theory ruled out the existence of correlation functions like S_3. It indeed is surprising since the readers may know that the only exact non-trivial result existing in the theory of turbulence is Kolmogorov law: $S_3(l) = -\frac{4}{5}\varepsilon l$. However as we shall note, their approach allows us to make direct contact with the terms of large deviation that signify the occurrence of rare events. It can be observed that ε_l plays the role of M_N of equation (B.1) and it is the deviation from the expected mean ε that we are interested in. As $l \to \infty$, this deviation variable has a distribution according to the role of equation (B.2). We hope a simplification: The $\varepsilon_l - \varepsilon$ can range from large negative to large positive values. We bring the range between 0 to 1 by defining a variable as:

$$Z_T(\varepsilon_l) \equiv \frac{1}{2}\left[1 + \tanh\left(\frac{\varepsilon_l - \varepsilon}{\Xi}\right)\right] \tag{B.4}$$

where Ξ is a constant with dimension of ε. We now make the drastic assumption that since $\varepsilon_l - \varepsilon$ is a rare event, the distribution of Z_T can be considered similar to that for the coin-toss with a biased coin and accordingly, we can hypothesize that

$$P(Z_T) \propto e^{-NI(Z_T)} \tag{B.5}$$

Here, N is number of random variables. This simple model yields value of $\mu \approx 0.16$ which is quite close to the presently accepted value. Also, a ζ_q vs q plot has been obtained that is not only convex but also follows She-Leveque scaling[57] faithfully enough for a model as simple as this. Please refer figure-A.1 and the next section for the technical details.
Our model's inherent bias for the value 0.26 for the parameter p in order to closely mimic the realistic turbulent fluid's scaling properties would seem so natural when it is compared with a particular successful multifractal cascade model[81] based on a generalized two-scale Cantor set. In that model, as the eddies breakdown into two new ones, the flux of kinetic energy into the smaller scales is hypothesized to be dividing into non-equal fractions $p = 0.3$ (quite close to our value of $p = 0.26$!) and $1 - p = 0.7$. It could fit remarkably well the entire spectrum of generalized dimensions[82] and (equivalently) the singularity spectrum (the so-called $f - \alpha$ curve[83]) for the energy dissipation field in many a turbulent flow.
The simplicity of biased coin-toss experiments and its reasonably astonishing success in predicting μ renders the need for more complicated models redundant. We believe just by being able to find a more appropriate function Z_T, we can make big leaps in the rather complex theory of turbulence.

B.3 The method

The one dimensional velocity derivative can be use to express the global average of the full energy dissipation if local isotropy exists[84, 85]. The velocity increment is given by

$$\Delta v(l) = \int_r^{r+l} \frac{dv}{dr} dr \tag{B.6}$$

and *ergo,* the energy dissipation rate is

$$\varepsilon(l) = \frac{15\nu}{l} \int_r^{r+l} \left(\frac{dv}{dr}\right)^2 dr \tag{B.7}$$

If we define $D_i \equiv \frac{dv}{dr}\Big|_i \left[\frac{\eta\sqrt{15\varepsilon}}{(\eta\varepsilon)^{1/3}}\right]$ and $N \equiv \frac{l}{K\eta}$ (where, η is Kolmogorov scale, $(\eta\varepsilon)^{1/3}$ is Kolmogorov velocity scale and K is the number of Kolmogorov scales over which one obtains smoothness), then equation (B.7) may be rewritten, upon discretization, as:

$$\varepsilon_l - \varepsilon = \frac{1}{N}\sum_{i=1}^{N} Y_i \tag{B.8}$$

Here, $Y_i \equiv D_i^2 - \varepsilon$. In this thesis, we have assumed the relation (B.8) to be the parallel of equation (B.1). Owing to the contraction principle, the rate function for $\varepsilon_l - \varepsilon$ and $Z(\varepsilon_l)$ are same. Thus, using equations (B.3), (B.4) and (B.5), we can write:

$$\langle|\varepsilon_l - \varepsilon|^q\rangle = \left|\frac{\Xi}{2}\right|^q \left[\frac{\int_0^1 \left|\ln\left(\frac{x}{1-x}\right)\right|^q \left\{\left(\frac{p}{x}\right)^x \left(\frac{1-p}{1-x}\right)^{1-x}\right\}^N dx}{\int_0^1 \left\{\left(\frac{p}{x}\right)^x \left(\frac{1-p}{1-x}\right)^{1-x}\right\}^N dx}\right] \tag{B.9}$$

We assume that to the leading order $\langle|\varepsilon_l - \varepsilon|^q\rangle \sim l^{\tau_q}$. By trial and error, we fix the inertial range as $N = 30$ to 60 and calculate numerically $\mu(= -\tau_2) = 0.16$. Similarly, we calculate $\zeta_q(= q/3 + \tau_{q/3})$ for various q. Note that to obtain the numerical solution for the integrals in equation (B.9), we have dropped the diverging terms from the finite series that represent the integrands as they are suitably discretized for their evaluation by Simpson's one-third rule.

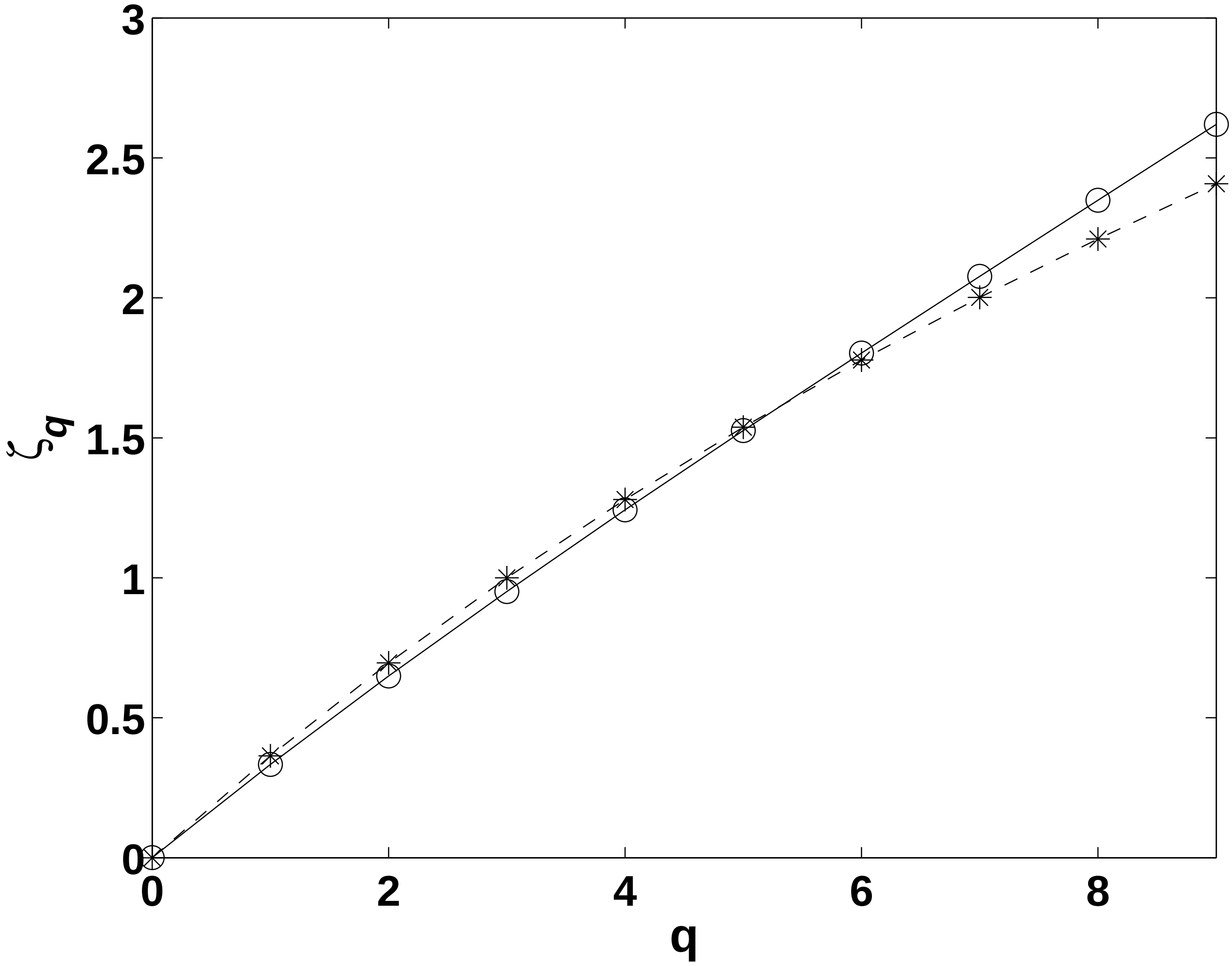

Figure B.1: ζ_q *vs.* q **curve in fully developed fluid turbulence**. The dashed line joining the asterisk is the celebrated She-Leveque scaling law.The circles joined by the solid line denote the values of ζ_p (for corresponding q) as obtained by dint of the model proposed herein. For every q, first $\langle|\varepsilon_l - \varepsilon|^q\rangle$ *vs.* N is plotted in log-log scale using the data yielded during the numerical integration of equation (B.9) and then the observation that for $N = 30$ to 60, we get a fairly straight line leads us to attempt fitting the range linearly. The process gives a value for τ_q. The relation $\zeta_q = q/3 + \tau_q$, then, tells us what is the corresponding value for ζ_q. One can see, the fit is remarkable. There is room for improvement in extending the inertial range and in getting better fit for higher ζ_q's. As mentioned earlier in this appendix, the form of Z_T is crucial.

Bibliography

[1] G. K. Batchelor, *An Introduction to Fluid Dynamics*, (Cambridge University Press), (1967).

[2] C. Cambon and L. Jacquin, J. Fluid Mech. **202**, 295 (1989).

[3] F. Waleffe, Phys. Fluids A **5**, 677 (1993).

[4] L. M. Smith, J. R. Chasnov and F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996).

[5] M. L. Waite and P. Bartello, J. Fluid Mech. **568**, 89 (2006).

[6] E. Lindborg, Geophys. Res. Lett. **32**, L010809 (2005).

[7] C. N. Baroud, B. B. Plapp, Z. S. She and H. L. Swinney, Phys. Rev. Lett. **88**, 114501 (2002).

[8] C. N. Baroud, B. B. Plapp, H. L. Swinney and Z. S. She, Phys. Fluids **15**, 2091 (2003).

[9] C. Morize, F. Moisy and M. Rabaud, Phys. Fluids **17**, 095105 (2005) .

[10] C. Morize and F. Moisy, Phys. Fluids **18**, 065107 (2006).

[11] Y. Zhou, Phys. Fluids **7**, 2092 (1995).

[12] V. M. Canuto and M. S. Dubovikov, Phys. Rev. Lett. **78**, 666 (1997).

[13] P. K. Yeung and Y. Zhou, Phys. Fluids **10**, 2895 (1998).

[14] Y. Hattori, R. Rubinstein and A. Ishizawa, Phys. Rev. E **70**, 046311 (2004).

[15] M. Reshetnyak and B. Steffen, arXiv:physics/0311001.

[16] W. C. Muller and M. Thiele, Europhys. Lett. **77**, 34003 (2007).

[17] X. Yang and J. A. Domaradzki, Phys. Fluids **16**, 4088 (2004).

[18] F. Bellet, F. S. Godeferd, J. F. Scott and C. Cambon, J. Fluid Mech. **562**, 83 (2006).

[19] S. Galtier, Phys. Rev. E **68**, R015301 (2003).

[20] C. Cambon, R. Rubinstein and F. S. Godeferd, New J. Phys. **6**, 73 (2004).

[21] S. Chakraborty, Phys. Fluids **19**, 085110 (2007).

[22] S. Chakraborty, arXiv:0806.2906 (Submitted in Phys. Fluids).

[23] S. Chakraborty, Phys. Fluids **20**, 075106 (2008).

[24] S. Chakraborty, Europhys. Lett. **79**, 14002 (2007).

[25] S. Chakraborty and J. K. Bhattacharjee, Phys. Rev. E **76**, 036304 (2007).

[26] S. Chakraborty, arXiv:0803.2273 (Submitted in Phys. Fluids).

[27] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR, **32**, 1 (1941); (English translation: Proc. R. Soc. Lond. A **434**, 15 (1991)).

[28] J. K. Bhattacharjee and S. Bhattacharyya, *Non-linear Dynamics Near and Far from Equilibrium*, (Hindustan Book Agency, New Delhi), (1995).

[29] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)*, (Reed Educational and Professional Publishing Ltd), (1987).

[30] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov*, (Cambridge University Press), (1995).

[31] K. Gawedzki, arXiv:hep-th/9710187.

[32] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).

[33] D. Bernard, Phys. Rev. E **60**, 6184 (1999).

[34] E. Lindborg, J. Fluid Mech. **388**, 259 (1999).

[35] G. L. Eyink, Phys. Rev. Lett. **74**, 3800 (1995).

[36] J. G. Charney, J. Atmos. Sci. **28**, 1087 (1971).

[37] E. Lindborg, J. Fluid Mech. **572**, 255 (2007).

[38] R. Salmon, *Lectures on Geophysical Fluid Dynamics*, (Oxford University Press, New York), (1998).

[39] A. J. Chorin, *Vorticity and Turbulence*, (Springer-Verlag New York, Inc.), (1994).

[40] C. Lim and J. Nebus, *Vorticity, Statistical Mechanics, and Monte Carlo Simulation*, (Springer), (2006).

[41] L. Biferale and I. Procaccia, Phys. Rep. **414**, 43 (2005).

[42] A. Brissaud, U. Frisch, J. Leorat, M. Lesieur and A. Mazure, *Phys. Fluids*, **16**, 1366 (1973).

[43] R. H. Kraichnan, *J. Fluid. Mech.*, **59**, 745 (1973).

[44] J. C. Andre and M. Lesieur, *J. Fluid. Mech.*, **59**, 187 (1976).

[45] W. Polifke and L. Shtilman, *Phys. Fluids A*, **1**, 2025 (1989).

[46] H. K. Moffatt, *J. Fluid Mech.*, **35**, 117 (1969).

[47] F. Waleffe, *Phys. Fluids A*, **4**(2), 350 (1992).

[48] R. Fjortoft, *Tellus*, **5**, 225 (1953).

[49] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence, Vol-***2**, (The MIT Press), (1971).

[50] D. Biskamp, *Magnetohydrodynamic Turbulence*, (Cambridge University Press), (2003).

[51] E. Gledzer, Sov. Phys. Dokl. **18**, 216 (1973).

[52] K. Ohkitani and M. Yamada, Prog. Theor. Phys. **81**, 329 (1989).

[53] R. Pandit, S. S. Ray and D. Mitra, *The Proceedings of the IUPAP Conference on Statistical Physics, Statphys 23*, Euro. Phys. Journal B, (2008).

[54] S. S. Ray, D. Mitra and R. Pandit, New Journal of Phys. **10**, 033003 (2008).

[55] D. Pisarenko, L. Biferale, D. Courvoisier, U. Frisch and M. Vergassola, Phys. Fluids A **5**, 2533 (1993).

[56] L. Kadanoff, D. Lohse, J. Wang and R. Benzi, Phys. Fluids **7**, 617 (1995).

[57] Z. S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).

[58] M. H. Jensen, G. Paladin, and A. Vulpiani, Phys. Rev. A **43**, 798 (1991).

[59] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli and S. Succi, Phys. Rev. E **48**, R29 (1993).

[60] E. Aurell, G. Boffetta, A. Crisanti, P. Frick, G. Paladin and A. Vulpiani, Phys. Rev. E **50**, 4705 (1994).

[61] G. L. Eyink, Nonlinearity **14**, 787 (2001).

[62] C. V. Tran and D. G. Dritschel, J. Fluid Mech. **559**, 107 (2006).

[63] T. Bohr, M.H. Jensen, G. Paladin and A.Vulpiani, *Dynamical Systems Approach to Turbulence*, (Cambridge University Press, Cambridge U.K.), (1998).

[64] L. Biferale, Annu. Rev. Fluid Mech. **35**, 441 (2003).

[65] Y. Oono, Prog. Theo. Phys. Supp. **99**, 165 (1989).

[66] J. Kurchan, J. Stat. Mech. **P07005**, (2007).

[67] M. Kardar, Statistical Theory of Fields, Cambridge University Press (Cambridge), (2007).

[68] A. N. Kolmogorov, J. Fluid Mechanics **13**, 82, (1962).

[69] A. M. Obukov, J. Fluid. Mech. **13**, 77 (1962).

[70] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).

[71] V. S. L'vov and I. Procaccia, Phys. Rev. E **52**, 3840 (1995).

[72] V. S. L'vov and I. Procaccia, Phys. Rev. E **54**, 6268 (1996).

[73] V. S. L'vov, E. Podivilov and I. Procaccia, Phys. Rev. E **55**, 7030 (1997).

[74] A. Das and J. K. Bhattacharjee, Europhys. Lett. **26**, 527 (1994).

[75] A. Sain, Manu and R. Pandit, Phys. Rev. Lett. **81**, 4377 (1998).

[76] D. Mitra and R. Pandit, Phys. Rev. Lett. **93**, 024501 (2004).

[77] K. R. Sreenivasan and R. A. Antonia, Ann Rev. Fluid Mech. 29, 435, (1997).

[78] G. Stolovitzky and K. R. Sreenivasan, Rev. Mod. Phys. **66**, 229 (1994).

[79] J. L. Lewis and R. Russel, An Introduction to Large deviations For teletraffic Engineeres.

[80] L. F. Richardson, Cambridge University Press (Cambridge), (1922).

[81] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).

[82] H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[83] T. C. Halsey, M. H. Jensen, L. Kadanoff, I. Procaccia and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[84] G. I. Taylor, Proc. R. Soc. London, Ser. A **151**, 421 (1935).

[85] S. Chen, G. D. Doolen, R. H. Kraichnan and Z-S. She, Phys. Fluids A **5**(2), 458 (1993).