A Study of Quantum Correlations from Different Perspectives

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY (SCIENCE)

IN

PHYSICS

BY

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UNIVERSITY OF CALCUTTA

MARCH, 2013

Abstract

This thesis covers different aspects of nonlocal feature observed in quantum correlations. Bell's theorem decisively proved that non-locality is fundamental to quantum mechanics and cannot be ascribed to any deeper local-realistic theory. Entangled states act as resource for generating nonlocal quantum correlations though the converse is not always true. Inspired by Werner's local-realistic model for certain entangled states, in second chapter, we provide the set of POVM measurements (possibly optimal) for which the singlet state statistics can be generated from pre-shared local resources (local hidden variables). Next, we discuss about Leggett's nonlocal-realistic model for entangled states which tries to assign sharp properties to constituent subsystems. Leggett's model leads to testable inequalities which are violated by quantum mechanics. However, success of Leggett's model in reproducing the correlations observed in standard Bell-CHSH tests (with co-planer observables) motivated new experiments for testing this model vis-a-vis quantum mechanics. In chapter-3, we derive two new forms of Leggett-type inequalities which, unlike the previous derived forms, puts no geometrical constraints on the relevant measurement settings. These forms may be useful for future tests of Leggett's model. In the following chapter (chapter-4) we formulate a physical situation where the no-signaling condition is applied to derive a constrained relation within quantum mechanics. After that, in chapter-5, we study quantum correlations in a generalized no-signaling framework. Though the nonlocality that can be extracted from quantum correlations respect the no-signaling principle, there are many other supra-quantum correlations which obeys it. Then a natural question is to look for physical principles that can help to distinguish between quantum and supra-quantum correlations. The principle of non violation of information causality is one such proposal. We apply this principle to study Hardy-type nonlocal correlations for two two-level systems. Finally, in the concluding chapter we point to some interesting problems for future works.

Acknowledgements

First and foremost, I would like to thank my supervisor Prof. Archan S. Majumdar, and my co-supervisors Prof. Dipankar Home and Prof. Guruprasad Kar for constant support and encouragement throughout the course of my Ph.D. work; I am fortunate to enjoy learning under their guidance. Along with their sustained effort for my academic grooming, my advisors also showed great care to my overall well being.

I also got oppertunities to learn from Prof. Sibashish Ghosh of IMSC Chennai, Prof. Prasanta Panigrahi of IISER, Kolkata and Prof. Pramod S. Joag of Pune University on many occasions, special thanks to them. I would like to thank my friend and collaborators Dr. Samir Kunkri (Samir-da), Rajjak, Manik, Subhadipa, Dr. Ramij Rahaman for numerous fruitful discussion sessions at ISI, Kolkata. Also, thanks to all kind of cooperation from my friends and colleagues Shiladitya, Anshuman, Tanumoy, Priyanka, Subhadipa, Sanjay, Sidharth, Prashant and Victor at the S. N. Bose Center.

Financial support from the DST Project No. SR/S2/PU-16/2007 and facilities provided by S. N. Bose National Center for Basic Sciences is acknowledged.

I must mention blessings and care I got from Anindita-di during this period. I would also like to thank Bikash-da who has always been very supportive and enthusiastic about my work. My dear friends Bikas, Prakash and Sonu has been always on my side whenever I needed them.

My beloved wife Nilanjana has been my most caring companion throughout the period of my research work. Apart from sharing a greater part of our common responsibilities, she even helped and assisted me in my research related work at great many occasions. Last, but definitely not the least, I would like to express my deepest gratitude to my parents and all my family members for their love and support through all this.

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List of Publications

Publications by the candidate forming part of the thesis:

- A. Rai, MD. R. Gazi, M. Banik, S. Das and S. Kunkri, '*Local simulation of singlet statistics for a restricted set of measurements*', Journal of Physics A: Mathematical and Theoretical **45**, 475302 (2012).
- A. Rai, D. Home and A. S. Majumdar, '*Leggett-type nonlocal realistic inequalities without any constraint on the geometrical alignment of measurement settings*', Physical Review A **84**, 052115 (2011).
- D. Home, A. Rai and A. S. Majumdar, '*A testable prediction of the nosignalling condition using a variant of the EPR-Bohm example*', Physics Letters A **377**, 540 (2013).
- A. Ahanj, S. Kunkri, A. Rai, R. Rahaman and P. S. Joag, '*Bound on Hardy's nonlocality from the principle of information causality*', Physical Review A **81**, 032103 (2010).
- MD. R. Gazi, A. Rai, S. Kunkri, and R. Rahaman, '*Local randomness in Hardy's correlations: implications from the information causality principle*', Journal of Physics A: Mathematical and Theoretical **43**, 452001 (2010).

Additional publications by the candidate relevant to the thesis but not forming part of it:

- S. Das, M. Banik, A. Rai, MD. R. Gazi, and S. Kunkri, '*Hardy's nonlocality argument as a witness for postquantum correlations*', Physical Review A **87**, 012112 (2013).
- M. Banik, MD. R. Gazi, S. Das, A. Rai and S. Kunkri, '*Optimal free will on one side in reproducing the singlet correlation*', Journal of Physics A: Mathematical and Theoretical **45**, 205301 (2012).
- G. Kar, MD. R. Gazi, M. Banik, S. Das, A. Rai and S. Kunkri, '*A complementary relation between classical bits and randomness in local part in the simulating singlet state*', Journal of Physics A: Mathematical and Theoretical **44**, 152002 (2011).

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Chapter 1

General Introduction

1.1 Motivation

Quantum mechanics was developed in the first quarter of the 20th century out of need to explain a number of surprising experimental observations which otherwise were not possible to comprehend from the then existing theories (classical physics). Manifestations of quantum phenomenon are diverse, each of them revealing a departure from classical concepts in its own peculiar ways. However, the present thesis is focused on studying certain aspects of an intriguing quantum feature known as *nonlocal quantum correlations*—the establishment of correlations between distant partners, through separated measurement of entangled particles, which otherwise cannot be simulated only by using some pre-shared local resources (in other words, correlations which can not have a local hidden variable description).

Historically, in the year 1935, Einstien, Podolosky and Rosen (EPR) [Einstein, Podolsky and Rosen, 1935], through an example of a nonseparable quantum state of two spatially separated particles (an EPR pair), argued that quantum mechanical description of the considered physical system is limited by not accommodating *local-realism*. According to EPR, it was plausible to assume that any complete physical theory respects the principle of *local-realism*. This assumption lead them to conclude that, though the predictions of quantum theory is an accurate description of statistical average values of observed physical quantities, at finer level (at level of a single event) the theory is short of a complete description. A paradoxical situation persisted for nearly three decades (it remained unclear, whether there can be any finer local-realistic description for EPR-type correlations or not?), until, in the year 1964, John Bell [Bell, 1964] first showed that certain quantum correlations cannot be reproduced from any deeper local-realistic description. Bell's theorem proved that correlation between results of local measurements on two spatially separated subsystems can have a local-realistic description only if these correlations satisfy certain constraints—Bell-type inequalities [Bell, 1964; Clauser and Horne, 1974; Clauser et al., 1969]. On the contrary, quantum mechanics predicted violation of these inequalities and therefore cannot embrace local-realism. Then, it remained for experiments to decide, whether in nature, Bell-type inequalities are violated or not? Many experiments, see for example [Aspect, Dalibard and Roger, 1982; Aspect, Grangier and Roger, 1982; Tittel et al., 1998], have since been performed which produce results consistent with quantum mechanics and inconsistent with local realism. This surprising feature of quantum correlations is termed as quantum non-locality. A comprehensive study on foundational issues in quantum mechanics covering different aspects of quantum nonlocality can be found in the work [Home, 1997] and [Home and Whitaker, 2007].

Quantum entanglement is the physical resource for generating non-local correlations. However, relation between concepts of entanglement and non-locality is in general too complex. Werner [Werner, 1989] first showed that, in fact, statistics generated from certain entangled states can have a local realistic description. So entanglement remains only a necessary condition for exhibition of nonlocal behavior.

The concept of entanglement still defies our understanding, it is insightful to look for alternative models to quantum correlations, based on more physical intuition. By determining whether these models are compatible or not with quantum predictions, one can identify the crucial features of entanglement and what is essential in quantum correlations. Leggett's nonlocal model [Leggett, 2003] is one such attempt which ascribes well-defined individual properties to particles in an entangled state. The resulting model and its generalizations contradicted quantum predictions which where subsequently verified in experiments [Branciard et al., 2008; Groblacher et al., 2007]. Investigations on generalized models *à la Leggett* thus demonstrate that any simulation model for quantum correlations must have fully undefined individual properties.

A key condition underpinning the 'peaceful coexistence' [Shimony, 1984] between quantum mechanics and special relativity is the no-signaling condition (NSC) which prohibits the use of quantum nonlocality for sending information in a way that can lead to causality paradoxes. An interesting line of study was initiated by Gisin showing the use of NSC as a tool to either find the limits of quantum mechanics, like constraining any conceivable nonlinear modification of the Schrödinger equation [Gisin, 1990; Gisin and Rigo, 1995], or to obtain specific bounds on quantum operations, like deriving bound on the fidelity of quantum cloning machines [Ghosh et al., 1999; Gisin, 1998]. On the other hand, Popescu and Rohrlich observed that the no-signaling principle can also allow post-quantum correlations which can exceed the optimal Bell-CHSH violation in quantum mechanics [Popescu and Rohrlich, 1994]. Indeed, they provided examples of correlations between two parties compatible with the no-signaling principle but without any quantum realization. This observation has motivated search for underlying physical principles for separating supraquantum (post-quantum) correlations from quantum ones, or ideally an optimal set of physical principles for complete characterization of the quantum set. Intensive effort has been devoted to the search for information principles characterizing the set of quantum correlations, e.g. nontrivial communication complexity [Brassard et al., 2006; van Dam, 2000], Information Causality [Pawlowski et al., 2009], and Macroscopic Locality [Navascues and Wunderlich, 2010].

Two main aspects of quantum correlations which motivates the work in this thesis are: (i) relationship between quantum entanglement and (non)local simulation models for quantum correlations, and (ii) study of quantum correlation in a nosignaling framework. Correlations considered in the present thesis are mainly those which are generated by performing local measurement on spatially separated parts of bi-partite entangled states.

1.2 A Brief Introduction to Quantum Formalism

Quantum mechanics is a mathematical model for describing phenomenon observed in the physical world. The model provides a general framework comprising of suitable mathematical objects from a complex Hilbert space which are associated with: (i) states of single/composite physical systems, (ii) measurement of physically observable quantities on the system, and (iii) time evolution of state of the system.

1.2.1 Basic Postulates

1.2.1.1 State

State space of an isolated physical system, say S, is a separable Hilbert space H_S defined over complex numbers. The state of the system is completely described by a positive operator ρ_S with trace one, acting on the state space H_S of the system. ρ_S is called the density operator of the system. If a quantum system is in the state ρ_i with probability p_i, then the density operator of the system is $\rho_S = \sum_i p_i \rho_i$.

Pure state: The state ρ_S of a system is pure if and only if it is a one dimensional projection operator acting on H_S. Then, $\rho_S = |\Psi\rangle\langle\Psi|$ for some $|\Psi\rangle \in H_S$. A pure state can be, therefore, equivalently represented by a vector (wave function) $|\Psi\rangle \in H_S$. One can now easily see that, if $|\Psi_1\rangle, |\Psi_2\rangle, ... \in H_S$ then $c_1|\Psi_1\rangle + c_2|\Psi_2\rangle + ... \in H_S$ where $c_1, c_2, ...$ are complex numbers. Thus, in quantum mechanics, any linear superposition of pure states of a physical system is again a valid pure state; this is generally known as *superposition principle.*

Mixed state: A mixed state is an ensemble (classical statistical mixture) of pure states. Suppose, a system is in state $|\Psi_i\rangle$ with probability p_i where $1 \leq i \leq n$ and $\sum_i p_i = 1$, then, it is not possible to represent the effective state of the system by a single vector (wave function) $|\Psi\rangle \in H_S$. The effective state, in the Hilbert space H_S of the system, can be then represented only by the density operator $\rho_S = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. It is important to mention here that two different ensembles $\{p_i, |\Psi_i\rangle\}$ and $\{\tilde{p}_j, |\tilde{\Psi}_j\rangle\}$ can give rise to the same effective state ρ_S.

1.2.1.2 Measurement

Quantum measurement of physical observables corresponds to a collection $\{\Pi_m\}$ of measurement operators*. These operators acts on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ_S immediately before the measurement then the probability that result m occurs is given by Born's rule, i.e., $p(m) = Tr\ (\Pi_m^\dagger \Pi_m \rho_S)$ and the state of the system after the measurement is $\frac{\Pi_m \rho_S \Pi_m^\dagger}{Tr\ (\Pi_m^\dagger \Pi_m \rho_S)}$. The measurement operators satisfy the completeness relation, $\sum_m \Pi_m^\dagger \Pi_m = I$.*

Projective measurement: A projective (von Neumann) measurement corresponds to a collection $\{P_m\}$ of mutually orthogonal projection operators satisfying the completeness relation $\sum_m P_m = I$. On preforming projective measurement on the state ρ_S, probability of obtaining the m-th outcome $p(m) = \mathrm{Tr}(P_m \rho_S)$ and the post measurement state of the system is $\frac{P_m \rho_S P_m}{\mathrm{Tr}\ (P_m \rho_S)}$. Generally, the self-adjoint operator $O = \sum_m m P_m$ is termed as an *observable* on the measured system, then, the eigenvalues and the corresponding eigenvectors of the observable O corresponds respectively to measurement results and post measurement state of the system.

POVM measurement: A collection $\{E_m\}$ of positive operators satisfying $\sum_m E_m = I$ defines a Positive Operator-Valued Measure (POVM); on performing a POVM measurement on a state ρ_S, probability of obtaining m-th outcome $p(m) = \mathrm{Tr}(E_m \rho_S)$. On comparing with the general formulation of quantum measurements, measurement operators $\{\Pi_m\}$ satisfying $E_m = \Pi_m^\dagger \Pi_m$ can have multiple solutions, which imply that a POVM measurement can be realized in many different ways. Hence, unless the measurement operators corresponding to a POVM measurement are explicitly known, there remains an ambiguity about the post measurement state after the POVM measurement. Therefore, a POVM measurement is useful for the experiments where the main objective is to collect the measurement statistics, while knowing the post measurement states is either not possible (in usual photon detection) or not relevant for the experiment.

1.2.1.3 Evolution

A closed quantum system undergoes unitary evolution with time. Suppose, initially at time t_0 the state of a quantum system is $\rho(t_0)$, which at a later time t evolves to a state $\rho(t)$, then the relationship between the initial state at time t_0 and the final state at time t is given by

$$\rho(t) = U(t_0, t)\rho(t)U^\dagger(t_0, t).$$

The unitary operator $U(t_0, t) = exp\left[\frac{-i\boldsymbol{H}(t-t_0)}{\hbar}\right]$, where $\mathbf{H}(t)$ is the Hamiltonian of the system.

1.2.2 Spin-$\frac{1}{2}$ particle

For an illustration of a simple quantum system, let us consider spin degrees of freedom of a spin-$\frac{1}{2}$ particle which defines a two level system. Then, the state space of a spin-$\frac{1}{2}$ particle is a 2-dimensional complex Hilbert space $\mathbb{C}^2$. Now, consider an orthogonal operator basis consisting of $\{I, \sigma_1, \sigma_2, \sigma_3\}$, where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the identity matrix and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, are the Pauli matrices. In the considered basis, in general, density matrix (state) of a spin-$\frac{1}{2}$ particle can be expressed as

$$\rho = \frac{1}{2}[I + \vec{n} \cdot \sigma], \quad ||\vec{n}|| \leq 1 \tag{1.1}$$

where $\vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$ (three dimensional Euclidean space) and $\vec{n} \cdot \sigma = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$. If $\vec{n}$ has unit norm the particle is in a pure state otherwise state of the particle is mixed. One can geometrically visualize these states in a solid *Bloch Sphere* defined as $S = \{(n_1, n_2, n_3) : ||\vec{n}|| \leq 1\}$. Points on the surface of the Bloch sphere S represent pure sates and interior points represent mixed states.

A projective measurement $\{P^+, P^-\}$ on state ρ of the qubit (or a spin measurement $\hat{m} \cdot \sigma \equiv (+1)P^+ + (-1)P^-$, along direction $\hat{m}$) where orthogonal projection operators $P^\pm = \frac{1}{2}[I \pm \hat{m} \cdot \sigma]$, results in any one of the two outcomes, say +1 or -1, with probability $p(\pm 1) = \mathrm{tr}(\rho P^\pm) = \frac{1}{2}[1 \pm \vec{n} \cdot \hat{m}]$. If the result is +1 (-1) the post measurement state of the qubit is $\frac{P^{+(-)}\rho P^{+(-)}}{\mathrm{tr}[\rho P^{+(-)}]}$.

1.2.3 Composite Systems and Quantum Entanglement

Let us now consider an example of two spin-$\frac{1}{2}$ particles. The state space of this system is $H_1 \otimes H_2$ where H_1 and H_2 are two dimensional complex Hilbert space $\mathbb{C}^2$ associated with particle 1 and particle 2 respectively. Let $\{|\Psi_1\rangle, |\overline{\Psi}_1\rangle\}$ be a orthonormal basis of H_1 and $\{|\Psi_2\rangle, |\overline{\Psi}_2\rangle\}$ be that of H_2. Then $\{|\Psi_1\rangle \otimes |\Psi_2\rangle, |\Psi_1\rangle \otimes |\overline{\Psi}_2\rangle, |\overline{\Psi}_1\rangle \otimes |\Psi_2\rangle, |\overline{\Psi}_1\rangle \otimes |\overline{\Psi}_2\rangle\}$ forms a natural orthonormal basis of the tensor product Hilbert space $H_1 \otimes H_2$. Now suppose state of the composite system is

$$|\Psi_{12}\rangle = a|\Psi_1\rangle \otimes |\overline{\Psi}_2\rangle - b|\overline{\Psi}_1\rangle \otimes |\Psi_2\rangle, \quad |a|^2 + |b|^2 = 1, \quad |a| \neq 0. \tag{1.2}$$

The state $|\Psi_{12}\rangle$ has a remarkable property that it cannot be factored as $|\Psi_{12}\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle$ such that $|\Phi_1\rangle \in H_1$ and $|\Phi_2\rangle \in H_2$. Such states are said to possess a property called *entanglement*. If $a = b = \frac{1}{\sqrt{2}}$ we obtain

$$|\Psi_{12}\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}}[|\Psi_1\rangle \otimes |\overline{\Psi}_2\rangle - |\overline{\Psi}_1\rangle \otimes |\Psi_2\rangle]$$

and this state is a spin singlet or a Einstein-Podolsky-Rosen (EPR) pair.

Definition of entanglement in a bipartite composite system is as follows:
If state of a bi-partite composite system ρ_{12} [1] can be expressed as $\sum_k c_k \rho_1^k \otimes \rho_2^k$ for some choice of positive integer k and density operators $\{\rho_i^k$ acting on $H_i^{d_i} : 1 \leq i \leq 2\}$, then such a state are separable. A state which is not seperable is known as *entangled* state. The spin singlet $|\Psi^-\rangle$ is an example of pure entangled state.

Density operator of composite system consisting of two spin-$\frac{1}{2}$ particles can be expressed in Hilbert-Schmidt basis (operators acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$) as follows:

$$\rho_{12} = \frac{1}{4}\left[I \otimes I + (\vec{r} \cdot \sigma) \otimes I + I \otimes (\vec{s} \cdot \sigma) + \sum_{m,n=1}^{3} t_{nm} \sigma_n \otimes \sigma_m\right] \tag{1.3}$$

where $\vec{r}, \vec{s} \in \mathbb{R}^3$, $\vec{r} \cdot \sigma = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3$, and $\vec{s} \cdot \sigma = s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3$. The coefficients $t_{nm} = \mathrm{Tr}[\rho_{12}\sigma_n \otimes \sigma_m]$ forms a 3×3 real matrix T. Here, $\sigma_1, \sigma_2, \sigma_3$ are the usual Pauli matrices. The density matrix corresponding to the singlet state $|\Psi^-\rangle$ takes the form $\rho_{\text{singlet}} = \frac{1}{4}[I \otimes I - \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3]$. A mixture of ρ_{singlet} with maximally mixed state (white noise) $\frac{I}{2} \otimes \frac{I}{2}$ in a ratio $p : 1-p$ with $0 \leq p \leq 1$ are known as Werner states, expressed as: $\rho_{\text{W}} = \frac{1}{4}[I \otimes I - p(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)]$. For $\frac{1}{3} < p < 1$, Werner states are examples of mixed entangled states.

In most general scenario, density operator of a n-partite system $\rho_{12...n} \in H_1^{d_1} \otimes H_2^{d_2} \otimes ... \otimes H_n^{d_n}$ can be entangled or separable in many different ways. The two extreme cases are *completely separable state* and *genuinely entangled state*. A n-partite state $\rho_{12...n}$ is completely separable if it can be expressed as $\rho_{12...n} = \sum_k c_k \rho_1^k \otimes \rho_2^k \otimes ... \otimes \rho_n^k$ for some positive integer k and density operators $\{\rho_i^k : 1 \leq i \leq n\}$ such that ρ_i^k acts on $H_i^{d_i}$. On the other hand if a n-partite state has the property that two components of any bipartition of the state is entangled, then the n-partite state is said to be genuinely entangled.

[1] A density operator acting on the Hilbert space $H_1^{d_1} \otimes H_2^{d_2}$ where d_1 and d_2 are the respective dimensions of subsystem's Hilbert spaces

1.2.4 Subsystem of a Composite System

Given the state of a composite quantum system it is desirable to know how to express the state of any of its components. The physical state of a part of the composite system corresponds to a *reduced density matrix* which can be obtained from the density matrix of the composite system. Suppose ρ_{12} is the density matrix of the composite system comprising of two subsystems 1 and 2. Then, reduced density matrix of the subsystem 1 is defined by

$$\rho_1 \equiv \text{tr}_2(\rho_{12})$$

where tr_2 is a map of operators known as *partial trace* over subsystem 2. The partial trace is defined by

$$\text{tr}_2(|a_1\rangle\langle b_1| \otimes |a_2\rangle\langle b_2| \equiv |a_1\rangle\langle b_1|\text{tr}(|a_2\rangle\langle b_2|) = |a_1\rangle\langle b_1|\langle a_2|b_2\rangle)$$

along with condition that the partial trace is linear in its inputs. In the definition, $|a_1\rangle$ and $|b_1\rangle$ are any two vectors in the state space of subsystem 1, and $|a_2\rangle$ and $|b_2\rangle$ are any two vectors in the state space of subsystem 2.

For example, for a state of a two qubit system given in Eq.(1.3), the reduced density matrices (state of subsystems) of qubits 1 and 2 are respectively

$$\rho_1 = \frac{1}{2}[I + \vec{r} \cdot \sigma], \quad ||\vec{r}|| \leq 1 \tag{1.4}$$

$$\rho_2 = \frac{1}{2}[I + \vec{s} \cdot \sigma], \quad ||\vec{s}|| \leq 1 \tag{1.5}$$

Therefore, for two qubit Werner class of states the two component qubits are in a maximally mixed state $\frac{I}{2}$. In particular, it is interesting to note that, for an EPR state though the state of composite system is pure (a state of maximal knowledge), the state of two subsystems are maximally mixed (a state of complete ignorance).

1.2.5 General Quantum Operations

Until now we have discussed about ideally closed quantum systems. However, in practice one has to often deal with quantum systems which are coupled to its environment. The free evolution, effect of measurements, or any other physical operations performed on such open quantum systems can be mathematically described within the general framework of quantum operations formalism.

Kraus 1st representation: Any open system under study can be combined with its environment to form a closed system. The combined state of system and environment can then be considered to evolve unitarily. The final state of the system can be obtained by taking partial trace over environment.

$$\varepsilon(\rho^f_{sys}) = \mathrm{tr}_{env}[U(\rho^i_{sys} \otimes \rho_{env})U^\dagger]. \tag{1.6}$$

Kraus 2nd representation: Though, in principle, the unitary evolution of combined system and environment can capture any possible quantum operation on the system, practically finding the global unitary operator requires constructing physical models for the environment which may be unnecessary if the interest is only in studying the principle system. Thus, it is more convenient to represent any quantum operation in terms of operators on the principle system's Hilbert space alone.

Let $|e_k\rangle$ be an orthonormal basis for the finite dimensional state space of the environment, and let $\rho_{env} = |e_0\rangle\langle e_0|$ be initial state of the environment. Without loss of generality one can assume that the environment starts in a pure state. Now one can write,

$$\begin{aligned}
\rho^f_{sys} = \varepsilon(\rho^i_{sys}) &= \mathrm{tr}_{env}[U(\rho^i_{sys} \otimes \rho_{env})U^\dagger] \\
&= \sum_k \langle e_0|U(\rho^i_{sys} \otimes |e_0\rangle\langle e_0|)U^\dagger|e_0\rangle \\
&= \sum_k E_k \rho^i_{sys} E_k^\dagger
\end{aligned}$$

where $E_k \equiv \langle e_0|U|e_k\rangle$ is an operator on the state space of the principle system. The equation

$$\rho^f_{sys} = \varepsilon(\rho^i_{sys}) = \sum_k E_k \rho^i_{sys} E_k^\dagger \tag{1.7}$$

is known as the operator-sum representation of ε. The operators $\{E_k\}$ are known as operation elements of the quantum operation ε. A quantum operation which satisfies $\mathrm{tr}(\varepsilon(\rho)) = 1$ for any ρ are trace preserving. One can easily show that operation elements $\{E_k\}$ of a trace preserving quantum operation satisfy the completeness relation $\sum_k E_k^\dagger E_k = I$.

1.3 Nonlocal Quantum Correlations

In this section we discuss about nonlocal quantum correlations which was discovered by John Bell [Bell, 1964] by following upon the EPR paradox [Einstein et al., 1935]. Bell's result state that no local realistic (classical) model exists which can explain all quantum predictions and thus points at a difference between the quantum and the classical world. Some conditions necessary for the local realistic models are given in form of Bell-CHSH inequality [Bell, 1964; Clauser et al., 1969] and Hardy's nonlocality argument [Hardy, 1992, 1993].

First we present a brief history of Bell's discovery and a few well-known versions of Bell-type arguments for bi-partite systems with two local measurement settings with binary outcomes at each site. Precisely we discuss about Bell-CHSH inequality and Hardy's nonlocality argument. We also present examples of quantum states showing these nonlocal features. Such sates are necessarily entangled, as first pointed by Werner [Werner, 1989]. To answer the question that whether the converse is also true is complex, Werner initiated this study by providing a local model for describing the statistics generated by local projective measurement on a class of entangled states. For a simple illustration, we discuss a case of the Werner's local hidden variable model for two qubit entangled states known as Werner-state.

Next, we discuss Leggett's nonlocal-realistic hidden variable model [Leggett, 2003] for quantum correlations, and finally, we discuss the relation between the no-signaling principle and quantum nonlocality.

1.3.1 EPR paradox

One of the principal features of quantum mechanics is that it is a probabilistic theory. This probability is not the expression of subjective ignorance about the pre-assigned value of a dynamical variable in a quantum state, rather it represents the probability of finding a particular value of a dynamical variable if that dynamical variable is measured. So what about the dynamical variables when the system is not subjected to any measurement; quantum mechanics remains silent in this regard. Therefore, physicists including Einstein was not at all happy by this probabilistic interpretation of quantum mechanics. In 1935, Einstein , Podolsky and Rosen, came up with their famous EPR paper [Einstein et al., 1935] where they argued that the mathematical formalism of quantum mechanics though consistent, is incomplete in its present form. Their views where founded upon the following assumptions:

1. *Necessary condition for completeness*: A necessary condition for the completeness of a theory is that "every element of the physical reality must have a counterpart in the physical theory".
2. *Sufficient condition for reality*: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".
3. *Locality principle*: "Elements of reality belonging to one system can not be affected (instantaneously) by measurements performed on another system which is spatially separated from the former".

With above three assumptions about the physical world quantum theory seems to be incomplete because according to EPR one can show the existence of elements of physical reality, whereas quantum mechanics does not embrace this concept.

In their original paper Einstein, Podolsky, and Rosen (EPR) considered quantum predictions for measurements of position and momentum. We explain their reasoning with a simpler example of two maximally entangled qubits. This approach was first presented by Bohm [Bohm, 1951]. Consider two observers, Alice and Bob, in two distant laboratories. They perform measurements on spin-$\frac{1}{2}$ particles which used to interact in the past. The quantum mechanical description of their joint state of spins in z-basis (eigen states of spin observable σ_z) reads:

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}[| \uparrow_z\rangle \otimes | \downarrow_z\rangle - | \downarrow_z\rangle \otimes | \uparrow_z\rangle] \tag{1.8}$$

A remarkable property of this state (spin singlet) is that this state is invariant under the same rotations of observables in the two labs. For instance, in another representation, the spin singlet in x-basis (eigen states of spin observable σ_x) takes the same form as in z-basis, i.e.,

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}[|\uparrow_x\rangle \otimes |\downarrow_x\rangle - |\downarrow_x\rangle \otimes |\uparrow_x\rangle] \tag{1.9}$$

This is in general true for any $\hat{n}$-basis. Thus, if Alice and Bob measure the same observable, whatever outcome of Alice, the outcome of Bob is always opposite. If Alice measures σ_z then she can predict with certainty the outcome of Bob's σ_z measurement. Thus, according to EPR there exists an element of physical reality connected with the σ_z measurement. Just as well Alice could measure σ_x and predict with certainty, without in any way disturbing the system, the outcome of a possible σ_x measurement by Bob. Again, seemingly there exists an element of reality connected with the σ_x measurement. Locality is assumed here: the physical reality at Bob's site is independent of everything that happens at Alice's site. Since quantum mechanics does not allow simultaneous knowledge of both σ_x and σ_z, it misses some concepts which are necessary for the theory to be complete.

1.3.2 Local realism and Bell's theorem

With the conclusion of the EPR paper, the natural question that appeared is whether quantum mechanics can be completed by supplementing some unknown extra parameter to the quantum state. But most of the physicists did not consider this line of thought promising as von Neumann long ago discarded this approach. Surprisingly J. S. Bell [Bell, 1964] again posed this question in a profound way. Bell asked whether any local realistic theory can reproduce all the statistical results of quantum mechanics? This new kind of approach gave birth to certain constraint (in form of an inequality) on experimentally observable correlation functions which became a touch stone to test whether physically observed correlations could be reproduced by some local realistic theory. This inequality is now famously known as Bell's inequality.

To understand the essence of Bell's inequality we consider a joint system consisting of two subsystems shared between Alice and Bob. The two observers (Alice and Bob) have access to one subsystem each, i.e. Alice and Bob perform a measurement on their respective subsystems. Imagine that Alice randomly chose to performs either measurement $\mathbf{A_0}$ or $\mathbf{A_1}$ on her subsystem. Similarly, Bob also chose to performs either measurement $\mathbf{B_0}$ or $\mathbf{B_1}$, for his subsystem. Let the corresponding measurement results be $A_0, A_1, B_0, B_1 \in \{+1, -1\}$.

Let λ be a local-realistic complete state associated with this joint system. For this state, values of every observable are fixed in a local way. i.e. the measurement results of each of the distant (space-like separated) observers (here Alice and Bob) are independent of the choice of observable of the other observer. This assumption reflects the locality concept inherent in the arguments of EPR: "The real factual situation of the Alice system is independent of what is done with Bobs system, which is spatially separated from the former". Consider now the quantity B_{CHSH} defined as:

$$\begin{aligned} B_{CHSH} &= A_0B_0 + A_0B_1 + A_1B0 - A_1B_1 \\ &= A_0(B_0 + B_1) + A_1(B_0 - B_1) \end{aligned} \tag{1.10}$$

Since A_0, A_1, B_0, B_1 takes values from $\{+1, -1\}$ it is easy to see that for any fixed λ

$$B_{CHSH}(\lambda) = \pm 2 \tag{1.11}$$

Then, the average of B_{CHSH} over some distribution $D(\lambda)$ of hidden variables λ is

$$-2 \;\leq\; \langle B_{CHSH} \rangle = \int D(\lambda)\; B_{CHSH}(\lambda)\; d\lambda \;\leq\; +2 \tag{1.12}$$

Thus we obtain the following famous Bell-CHSH inequality in terms of experimentally observable correlation functions $\langle \mathbf{A_i B_j} \rangle$, $i, j \in \{0, 1\}$

$$|\langle \mathbf{A_0B_0} \rangle + \langle \mathbf{A_0B_1} \rangle + \langle \mathbf{A_1B_0} \rangle - \langle \mathbf{A_1B_1} \rangle| \leq 2 \tag{1.13}$$

The astonishing thing about Bell's result is these constraints are not always satisfied by the predictions of quantum mechanics. The reason for the discrepancy is due to the quantum description of the two-particle correlated state. This is a so-called entangled or nonseparable state, whose correlations cannot be always explained through some local-realistic model.

Quantum mechanics violates Bell's inequality:
Suppose Alice and Bob share one particle each from an EPR pair and can only locally operate on there respective subsystem in two remote laboratories. A representation of EPR state in the z-basis denoted by $\{|0\rangle, |1\rangle\}$ can be written as:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}[\ |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\ ] \tag{1.14}$$

Let, Alice randomly choose to measure between spin observable $\mathbf{A_0} = \sigma_z$ and $\mathbf{A_1} = \sigma_x$ on her subsystem and similarly Bob on his subsystem measures either $\mathbf{B_0} = -\frac{\sigma_z+\sigma_x}{\sqrt{2}}$ or $\mathbf{B_1} = \frac{\sigma_z-\sigma_x}{\sqrt{2}}$. Then, the quantum mechanical expectation values of correlation functions $\langle \mathbf{A_i B_j} \rangle$ are:

$$\langle \mathbf{A_i B_j} \rangle_{QM} = (-1)^{ij} \frac{1}{\sqrt{2}}.$$

Substituting these values into the left hand side of the Bell-CHSH expression we get

$$|\langle \mathbf{A_0 B_0} \rangle_{QM} + \langle \mathbf{A_0 B_1} \rangle_{QM} + \langle \mathbf{A_1 B_0} \rangle_{QM} - \langle \mathbf{A_1 B_1} \rangle_{QM}| = 2\sqrt{2} > 2 \tag{1.15}$$

Thus, we see a clear violation of Bell-CHSH inequality in quantum mechanics. The experimental tests performed so far show this violation modulo some minor loopholes—these technical loopholes are gradually being closed and are now believed not to have any fundamental impact on confirmation of Bell's inequality violation. Therefore, contrary to the intuition envisaged by EPR, there can be no finer local-realistic hidden variable description for correlations from which quantum mechanical predictions can be always derived. Moreover, in this context experimental findings support the quantum mechanical predictions.

Cirel'son bound
Cirel'son [Cirel'son, 1980] asked an important question: what is the maximum value that Bell-CHSH expression can take within quantum mechanics? He answered this question by showing that this value is in fact $2\sqrt{2}$, which is same as the value we obtained for the 2-qubit EPR state for a particular choice of observable for showing Bell-CHSH violation in quantum mechanics. Below we give an outline of Cirel'son's proof:

The Bell operator corresponding to Bell-CHSH expression can be written as

$$\mathbb{B}_{CHSH} = \mathbb{A}_0 \otimes \mathbb{B}_0 + \mathbb{A}_0 \otimes \mathbb{B}_1 + \mathbb{A}_1 \otimes \mathbb{B}_0 - \mathbb{A}_1 \otimes \mathbb{B}_1 \tag{1.16}$$

Then for any pure quantum state $|\Psi_{AB}\rangle$ shared between Alice and Bob the value for the Bell-CHSH expression can be calculated as $\langle\Psi_{AB}|\mathbb{B}_{CHSH}|\Psi_{AB}\rangle$. Considering only pure sates is sufficient here as mixed states being statistical mixture of pure states must also respect the derived upper bound. In fact it is only needed to derive a bound for sup-norm $||.||_{sup}$[2] of the Bell-CHSH operator and the result easily follows. According to quantum mechanics, Alice and Bob's observables producing binary outcomes $\{+1, -1\}$ must satisfy following relations:

$$\mathbb{A}_0^2 = \mathbb{A}_1^2 = \mathbb{B}_0^2 = \mathbb{B}_1^2 = \mathbf{1} \tag{1.17}$$

$$[\mathbb{A}_0, \mathbb{B}_0] = [\mathbb{A}_0, \mathbb{B}_1] = [\mathbb{A}_1, \mathbb{B}_0] = [\mathbb{A}_1, \mathbb{B}_1] = \mathbf{0} \tag{1.18}$$

where $[\mathbb{A}_i, \mathbb{B}_j] = \mathbb{A}_i\mathbb{B}_j - \mathbb{B}_j\mathbb{A}_i$ are commutators of Alice and Bobs observables. Under these conditions one gets an identity

$$\mathbb{B}_{CHSH}^2 = 4\mathbb{I} + [\mathbb{A}_0, \mathbb{A}_1][\mathbb{B}_0, \mathbb{B}_1] \tag{1.19}$$

Also, the following inequality holds for two bounded hermitian operators $\mathbb{O}_1$ and $\mathbb{O}_2$:

$$||[\mathbb{O}_1, \mathbb{O}_2]||_{sup} \leq 2\ ||\mathbb{O}_1||_{sup}\ ||\mathbb{O}_1||_{sup} \tag{1.20}$$

Then on applying this inequality we get

$$\begin{aligned} ||\mathbb{B}_{CHSH}^2||_{sup} &\leq 8 \\ \Rightarrow ||\mathbb{B}_{CHSH}||_{sup} &\leq 2\sqrt{2} \\ \Rightarrow \langle\mathbb{B}_{CHSH}\rangle_{|\Psi\rangle_{AB}} &\leq 2\sqrt{2}\ \text{ for any state } |\Psi\rangle_{AB} \end{aligned} \tag{1.21}$$

We also saw that the Cirel'son's bound $2\sqrt{2}$ can be achieved within quantum mechanics. Moreover, Cirelson's bound teaches us that any hypothetical correlation which leads to a Bell-CHSH violation beyond the value $2\sqrt{2}$ [3] cannot be achieved from any quantum resource.

[2] Supremum norm of a bounded linear operator $\mathbb{O}$ is defined as $||\mathbb{O}||_{sup} = \text{Sup}_{|\Psi\rangle} \frac{||O|\Psi\rangle||}{|||\Psi\rangle||}$

[3] In general this is possible since the maximum algebraic value that Bell-CHSH expression can take is 4

1.3.3 Hardy's nonlocality argument

In 1989 Greenberger, Horne and Zeilinger (GHZ) [Greenberger et al., 1989], unlike Bell-CHSH inequality, provided a way to show a direct contradiction of quantum mechanics with local realism without using any statistical inequality. The state used for this demonstration by GHZ is a three qubit state $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$ shared between three spatially separated parties. Inspired by this work Lucien Hardy asked a question: whether some simple GHZ-like argument can also be provided in a bipartite setting? Hardy successfully produced one such argument which is now popular as Hardy's nonlocality argument [Hardy, 1992, 1993].

Hardy's nonlocality argument is as follows: Consider a joint system consisting of two subsystems shared between Alice and Bob. The two observers (Alice and Bob) have access to one subsystem each, i.e. Alice and Bob perform a measurement on their respective subsystems. Imagine that Alice has available two different measurement apparatuses (observables), A_0 and A_1, to measure her subsystem. Similarly, Bob also has two different measurement apparatuses, B_0 and B_1, for his subsystem. The measured result of all these apparatuses can be $+1$ or -1. Then, from the complete set of 16 joint probabilities $P(\pm 1, \pm 1|A_i, B_j)$ in any such experiment, Hardy's nonlocality argument follows by restricting a judiciously chosen set of four joint probabilities. One such set is as follows:

$$P(A_0 = +1, B_0 = +1) = q > 0 \tag{1.22}$$

$$P(A_1 = -1, B_0 = +1) = 0 \tag{1.23}$$

$$P(A_0 = +1, B_0 = -1) = 0 \tag{1.24}$$

$$P(A_1 = +1, B_1 = +1) = 0 \tag{1.25}$$

Next we show that no local realistic model can be provided for outcomes obeying above restrictions. Suppose the first condition holds, then for some local realistic hidden variable λ (such hidden variables exist since $q > 0$) $A_0^\lambda = +1$ and $B_0^\lambda = +1$. Now, for such hidden variables λ second and third condition implies that $A_1^\lambda = +1$ and $B_1^\lambda = +1$ which contradicts the fourth condition.

All bi-partite Hardy's argument (for different set of four restricted probabilities) can be compactly written as

$$P(A_0 = i, B_0 = j) = q > 0 \tag{1.26}$$
$$P(A_1 = -m, B_0 = j) = 0 \tag{1.27}$$
$$P(A_0 = i, B_0 = -n) = 0 \tag{1.28}$$
$$P(A_1 = m, B_1 = n) = 0 \tag{1.29}$$

where i, j, m, n can take any value from the set $\{+1, -1\}$. In the first discussed particular example of Hardy's set $i = j = m = n = +1$; we carry on with this particular example to show that indeed there are quantum states and observables which satisfying Hardy's condition.

Here we give the Hardy's nonlocality proof for given states of two spin-$\frac{1}{2}$ particles. Consider a system of two spin-$\frac{1}{2}$ particles in a pure nonmaximally entangled state Ψ_{AB}. Then for a proper choice of orthonormal basis $\{|u_A\rangle, |v_A\rangle\}$ and $\{|u_B\rangle, |v_B\rangle\}$ for subsystem A and B respectively, the state Ψ_{AB} can be written as:

$$\Psi_{AB} = \alpha|v_A\rangle \otimes |v_B\rangle + \beta|u_A\rangle \otimes |v_B\rangle + \gamma|v_A\rangle \otimes |u_B\rangle, \quad \text{where } \alpha\beta\gamma \neq 0 \tag{1.30}$$

Observables $A_0,\ A_1,\ B_0,\ B_1$ are chosen such that respective eigen vectors corresponding to eigen values $+1$ are, say, $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{b_0}$, $\mathbf{b_1}$, where

$$\mathbf{a_0} = \frac{\beta|v_A\rangle - \alpha|u_A\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}}, \quad \mathbf{a_1} = |u_A\rangle, \quad \mathbf{b_0} = \frac{\gamma|v_B\rangle - \alpha|u_B\rangle}{\sqrt{|\alpha|^2 + |\gamma|^2}}, \quad \mathbf{b_1} = |u_B\rangle \tag{1.31}$$

For the above choice of state and observables it follows that

$$P(A_0 = +1, B_0 = +1)_{QM} = \alpha\beta\gamma > 0 \tag{1.32}$$
$$P(A_1 = -1, B_0 = +1)_{QM} = 0 \tag{1.33}$$
$$P(A_0 = +1, B_0 = -1)_{QM} = 0 \tag{1.34}$$
$$P(A_1 = +1, B_1 = +1)_{QM} = 0 \tag{1.35}$$

Therefore, from Hardy's argument we can conclude that quantum mechanics cannot be reproduced by any local realistic model.

1.3.4 Entanglement and Nonlocality

Entanglement is necessary for showing violation of Bell-type inequalities. Whether the converse is also true, i.e., do all the entangled states violate some Bell-type inequality? Werner [Werner, 1989] first posed this question which is still an exiting area of research. For pure entangled state now it is known that every pure entangled state violate some Bell-type inequality. However, the relation between entanglement and nonlocality for mixed entangled states, is more subtle than it appears at first sight and the two concepts are not entirely equivalent. Werner [Werner, 1989] showed that for standard von Neumann measurements there exist mixed entangled states that cannot generate nonlocal correlations, i.e., cannot violate any Bell inequalities. This result was further extended by Barrett [Barrett, 2002] for any POVM measurements. On the other hand, another line of research emerged which in many novel ways show that some non-standard Bell tests can still be useful for filtering nonlocality from some (mixed) entangled state. Below we briefly discuss Werner's local hidden variable model for a class of two qubit entangled states.

Werner's local model for 2-qubit entangled state:
Earlier we discussed about 2-qubit entangled states of the form

$$\rho_{\mathrm{W}} = \frac{1}{4}[I \otimes I - p\ (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)]. \tag{1.36}$$

If parts of such a state (qubits) are distributed at two separate locations (say one qubit to Alice and the other one to Bob), joint probabilities for outcomes resulting from local projective measurement, respectively along $\hat{a}$ and $\hat{b}$, is given by:

$$P_{AB}(i,j|\hat{a},\hat{b}) = \frac{1}{4}[1-(ij)p\ \hat{a}\cdot\hat{b}] \quad \text{where} \ \ i,j \in \{+1,-1\}. \tag{1.37}$$

Though these states are entangled beyond $w = 1/3$, they violate the Bell-CHSH inequality if and only if $p > 1/\sqrt{2}$. Werner showed that a local model can be given for these states (statistics) in the range $1/3 \leq p \leq 1/2$. Werner's model goes as follows:

Suppose Alice and Bob pre-share vectors $\hat{\lambda}$ uniformly distributed over a unit sphere. Alice, for her measurement direction $\hat{a}$, declares '+1' with probability

$$P^A_{\hat{\lambda}}(+1) = \frac{1+\cos\alpha}{2} \tag{1.38}$$

where α is angle between direction $\hat{a}$ and $\hat{\lambda}$. On the other hand, for measurement along $\hat{b}$, Bob declares '+1' with probability

$$P_{\hat{\lambda}}^{B}(+1) = \frac{1}{2} - p\ \mathrm{sgn}(\cos\beta) \tag{1.39}$$

where β is the angle between the direction $\hat{b}$ and $\hat{\lambda}$, and $\mathrm{sgn}(x) = +1$ (-1) for $x \geq 0$ $(x < 0)$.

The joint probability of the outcome $(+1, +1)$, can be calculated from

$$P_{lhv}^{AB}(+1,+1) = \int \rho(\hat{\lambda})\ P_{\hat{\lambda}}^{A}(+1)\ P_{\hat{\lambda}}^{B}(+1)\ d\hat{\lambda} \tag{1.40}$$

where $\rho(\hat{\lambda})$ is the considered (uniform) distribution of the hidden variable $\hat{\lambda}$. Evaluating the above integral gives

$$P_{lhv}^{AB}(+1,+1) = \frac{1}{4}[1 - p\ \hat{a}\cdot\hat{b}] \tag{1.41}$$

which exactly matches with the quantum mechanical prediction for the outcome $(+1, +1)$. The desired quantum mechanical probabilities for the other possible outcomes also follow.

1.3.5 Nonlocal realism and Leggett's models

Demonstration of Bell-nonlocality implies that any local-realistic description for quantum correlations as envisaged by EPR is not possible: some nonlocal correlation has to be present. Still, one may try to save as much as possible of a classical world-view. This is a quite general idea but, one specific model inspired by this line of thought has been proposed by A. J. Leggett [Leggett, 2003]. Recall that one of the astonishing aspects of entanglement is the fact that a composite system can be in an overall pure state, while none of its components is. Leggett's nonlocal variable model tries to ascribe sharp properties for both the composite and the individual systems at a hidden variable level.

Consider a source emitting a photon pairs which may be entangled. Then according to Leggett's model the whole ensemble of photon pairs emitted from a source constitutes of a disjoint union of subensembles that are assumed to have the following features:

1. In any subensemble, each pair of photons is characterized by definite values of preassigned polarizations $\hat{u}$ and $\hat{v}$ so that the whole ensemble corresponds to a distribution of values of $\hat{u}$ and $\hat{v}$ denoted by, say, $D(\hat{u}, \hat{v})$.

2. For any given pair belonging to a subensemble, individual outcomes (denoted by A and B) of polarization measurements on each member of the pair along directions, say, $\hat{a}$ and $\hat{b}$ respectively are assumed to be determined by a hidden variable, say, λ whose values are distributed over the pairs comprising the given subensemble with the corresponding distribution function being denoted by $\rho_{(\hat{u},\hat{v})}(\lambda)$.

3. The outcome of polarization measurement along $\hat{a}$ ($\hat{b}$) for any individual photon in one of the two wings may be non-locally dependent on the choice of the measurement setting pertaining to its spatially separated partner in the other wing, but the statistical result for a given subensemble obtained by averaging over such effects is assumed to satisfy the Malus law. This entails that the relevant mean value depends only on the local setting.

Then, accoring to Leggett's model, mean values of outcomes of polarization measurements for the subensembles characterized by $\hat{u}$ and $\hat{v}$ pertaining to the two wings can respectively be written as

$$\begin{aligned} \overline{A}(\hat{u}) &= \int A(\hat{a}, \hat{b}, \lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda = \hat{u}\cdot\hat{a} \\ \overline{B}(\hat{v}) &= \int B(\hat{b}, \hat{a}, \lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda = \hat{v}\cdot\hat{b} \end{aligned} \tag{1.42}$$

The experimentally observable polarization correlation function for the whole ensemble is then expressible as

$$\langle AB\rangle = \iint \overline{AB}(\hat{u},\hat{v})D(\hat{u},\hat{v})d\hat{u}d\hat{v} \tag{1.43}$$

where $\overline{AB}(\hat{u},\hat{v}) = \int A(\hat{a},\hat{b},\lambda)B(\hat{b},\hat{a},\lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda$.

Leggett's model was successful in explaining all the standard experimental data (where local measurements of polarization is confined to a plane) obtained by performing the Bell-CHSH tests. However, Leggett showed that his model contradict some quantum mechanical predictions. The model thus called for new experiments to test its validity vis-a-vis quantum mechanics. A number of inequalities derived from Leggett's assumptions are violated by quantum mechanics. Experiments that have been performed to date results in agreement with the quantum mechanical prediction and thus falsify the Leggett's model.

1.3.6 Quantum correlations and the No-signaling principle

Quantum mechanics has some sort of nonlocal feature. But is it nonlocal enough to carry information with a speed greater than that of light in vacuum? According to special relativity theory, nothing can move with a speed greater than that of light in vacuum. Although the theory of relativity developed a long time before the advent of full-fledged quantum mechanics, it was still considered to transcend the then existing physical theories and was believed to hold in any future theory as it basically describes the structure of space and time in which every sort of physical phenomena take place. Quantum mechanics is such a theory which gives a nice chance to verify the principle of relativistic causality .

To check whether quantum mechanics respects relativistic causality principle, we consider two far apart systems, A and B, in a quantum state ρ_{AB}. We consider the most general trace-preserving operation acting on system A described by the Kraus operators $\{E_i\}$ where $\sum_i E_i^\dagger E_i = I$. Before performing the quantum operation on A, the density matrix for B is given by,

$$\rho_B = \mathrm{tr}_A(\rho_{AB}) \tag{1.44}$$

Let ρ'_{AB} be the transformed state of the composite system after the quantum operation on A. Then,

$$\rho'_{AB} = \sum_i (E_i \otimes I)\rho_{AB}(E_i^\dagger \otimes I) \tag{1.45}$$

For the transformed state the marginal density matrix is given by

$$\begin{aligned}
\rho_B^{'} &= \mathrm{tr}_A(\rho_{AB}^{'}) \\
&= \mathrm{tr}_A[\sum_i (E_i \otimes I)\rho_{AB}(E_i^\dagger \otimes I)] \\
&= \mathrm{tr}_A[\sum_i (E_i^\dagger E_i \otimes I)\rho_{AB}] \\
&= \mathrm{tr}_A[(\sum_i E_i^\dagger E_i \otimes I)\rho_{AB}] \\
&= \mathrm{tr}_A(\rho_{AB}) \\
&= \rho_B
\end{aligned}$$

This is true for all trace-preserving quantum operations and for all joint density matrices. Hence we see that the quantum operation respects relativistic causality.

1.4 Physical principle(s) determining the set of quantum correlations

Quantum correlations have been shown to posses nonlocality. For Bell-CHSH violation quantum nonlocality cannot supersede Cirel'son's bound. There are many other (supra-classical) quantum features like intrinsic randomness [Barnum et al., 2007; Masanes et al., 2006], incompatible measurements and uncertainty relations [Scarani et al., 2006], no-cloning, teleportation [Barnum et al., 2008] etc. On the other hand axioms of quantum physics is rather a mathematical description of its formalism. This motivates a question: is it possible to identify some set of physical principles which can uniquely characterize all quantum features (quantum mechanics). In particular, one would like to characterize the set of quantum correlations, that is correlations which can result from local measurements on quantum states. Pioneering work by Popescu and Rohrlich [Popescu and Rohrlich, 1994] showed that the no-signaling principle (that is, the impossibility of instantaneous communication) does not suffice to recover this quantum set . Indeed, they provided examples of correlations between two parties compatible with the no-signaling principle but without any quantum realization. The most paradigmatic example of these supra-quantum correlations is the so-called Popescu-Rohrlich (PR) box.

Recently a search has been started for better principles separating supraquantum correlations from quantum ones, or ideally a complete characterization of the quantum set.

An important boost to this search was due to Van Dam [van Dam, 2000], who introduced the idea that the existence of supra-quantum correlations, while not violating the no-signaling principle, could have implausible consequences from an information processing point of view. Van Dam showed that distant parties having access to PR-boxes can render communication complexity trivial and argued that this could be a reason for the non-existence of these correlations in nature [van Dam, 2000]. Since then, intensive effort has been devoted to the search for information principles characterizing the set of quantum correlations, e.g. the aforementioned non-trivial communication complexity [Brassard et al., 2006; Brunner and Skrzypczyk, 2009; Linden et al., 2007; van Dam, 2000] and Information Causality [Pawlowski et al., 2009].

1.4.1 PR correlation

The PR-box [Popescu and Rohrlich, 1994] is a specific bipartite no-signaling probability distribution with both binary input and output . Alice operating from one end can input a bit x and receives a bit a as output; and similarly Bob from a far away second end can input a bit y and receives a bit b as output. The PR-box is specified by the rule

$$P_{PR}(a,b|x,y) = \frac{1}{2}\delta_{a\oplus b=xy} \tag{1.46}$$

where the symbol $\oplus$ is addition modulo 2, and $\delta_{a\oplus b=xy} = 1$ if $a \oplus b = xy$ otherwise 0.

Correlation functions for any joint distribution $P(a,b|x,y)$ is defined by following relations

$$\langle x, y\rangle = P(0,0|x,y) + P(1,1|x,y) - P(1,0|x,y) - P(0,1|x,y). \tag{1.47}$$

Then, the standard form of Bell-CHSH inequality

$$\langle x=0, y=0\rangle + \langle x=0, y=1\rangle + \langle x=1, y=0\rangle - \langle x=1, y=1\rangle \leq 2 \tag{1.48}$$

$$\equiv S = \sum_{x,y=0}^{1} P(a\oplus b = xy|x,y) \leq 4 \tag{1.49}$$

Now, from the equivalent expression Eq.(1.49) for Bell-CHSH inequality, one can easily see that the classical limit and the quantum limit (i.e., Cirel'son's bound) for values of S is respectively 3 and $2+\sqrt{2}$. For PR-box correlation $S = 4$ which exceeds the Cirel'son's bound, therefore, PR-box correlation cannot be generated from any quantum resource. Also, note that the local distribution for Alice $P(a|x) = \frac{1}{2}$ which is independent of Bob's input, therefore, Bob cannot signal to Alice. Similarly, local distribution of Bob $P(b|y) = \frac{1}{2}$, so Alice cannot signal to Bob. Hence, PR-correlation is a no-signaling resource.

1.4.2 Generalized no-signaling framework for bi-partite correlations

In general, there are many PR-box type correlations which are supra-quantum however the no-signaling principle is insufficient to separate such correlations from quantum correlations. Then, what are the physical principle(s) which can achieve this goal? To study this problem systematically it is convenient to develop a general framework which is constrained only by the no-signaling principle.

Here we briefly describe the no-signaling framework for all bi-partite correlations. Let the inputs of Alice and Bob are written $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ respectively; the outputs (it is assumed that every input leads to the same number of possible outcomes) are written $a \in \mathbb{A}$ and $b \in \mathbb{B}$ respectively. Then the probability distribution can be expressed as:

$$\mathbf{P} = \{P(a,b|x,y) : x \in \mathbb{X}, y \in \mathbb{Y}, a \in \mathbb{A}, b \in \mathbb{B}\} \tag{1.50}$$

The no-signaling distributions are restricted by the no-signaling constraints:

$$P(a|x,y) = P(a|x) \quad \text{and} \quad P(b|x,y) = P(b|y) \quad \text{for} \quad \text{all} \quad a,b,x,y \tag{1.51}$$

Since all the constraints defining no-signaling distributions are linear the collection of all no-signaling distribution (each distribution represented as a point) defines a polytope.

1.4.3 Information causality principle

Information causality (IC) principle [Pawlowski et al., 2009; Pawlowski and Scarani, 2011] is a generalization of no-signaling condition. Relativistic causality (or the no-signaling principle) states that any party *cannot extract more* information then the amount of classical bits it receives. The principle of information causality on the other hand put an even stronger restriction by *forbidding more information to be potentially available* to the receiver than that provided by the sender.

IC principle can be formulated quantitatively through an information processing game played between two parties, say Alice and Bob. Alice receives a randomly generated N-bit string $\vec{x} = (x_0, x_1, ..., x_{N-1})$, and Bob is asked to guess Alice's i-th bit where i is randomly chosen from the set $\{0, 1, 2, ..., N-1\}$. Alice is allowed to send a M-bit message ($M < N$). Alice and Bob can pre-share no-signaling resources (correlations) which they exploit according some pre-agreed strategy while playing this game. Let Bob's answer be denoted by β_i. Then, the information that Bob can potentially acquire, about the variable x_i of Alice, is given by the Shannon mutual information $I(x_i : \beta_i)$. The statement of IC is that the total potential information [Pawlowski et al., 2009; Pawlowski and Scarani, 2011] about Alice's bit string $\vec{x}$ accessible to Bob cannot exceed the volume of message he received from Alice, i.e.,

$$\mathbb{I} = \sum_{i=1}^{N} I(x_i : \beta_i) \leq M \tag{1.52}$$

1.4.4 Quantum correlation respects Information Causality

Now let us present a simple proof given in [Pawlowski and Scarani, 2011] to show that IC holds in the classical and quantum information theory. It is sufficient to focus on quantum correlations because classical correlations form a subset of quantum correlations.

Suppose Alice and Bob share a quantum state ρ_{AB} between them. Let ρ_B be the Bobs part of the shared quantum state and $\vec{x}$ the set of all Alices variables x_i. First note that after receiving the message $\vec{m}$, which was communicated over the channel with the classical communication capacity M, from Alice all the classical and quantum information that Bob has does not contain more than M bits of information about $\vec{x}$, i.e.,

$$I(\vec{x} : \vec{m}, \rho_B) \leq M \tag{1.53}$$

Now, for the proof the chain rule for mutual information is used, $I(\vec{x} : \vec{m}, \rho_B) = I(\vec{x} : \rho_B) + I(\vec{x} : \vec{m}|\rho_B)$. Since at the beginning of the protocol Bob knows nothing about the variables of Alice $I(\vec{x} : \rho_B) = 0$, and the second term $I(\vec{x} : \vec{m}|\rho_B) = I(\vec{x}, \rho_B : \vec{m}) - I(\rho_B : \vec{m})$ is bounded by M due to the positivity of the mutual information and the fact that $\vec{m}$ is a message sent over the channel with the classical communication capacity M.

In the case of independent Alices input bits condition limits the information gain about the individual bits as well because

$$I(\vec{x} : \vec{m}, \rho_B) \geq \sum_{i=1}^{N} I(x_i : \vec{m}, \rho_B) \tag{1.54}$$

This inequality is also proved using the chain rule. Finally, one can now observe that Bobs output bit β_i is obtained at the end from $\vec{m}$ and ρ_B. Hence, the data processing inequality implies $I(x_i : \vec{m}, \rho_B) \geq I(x_i : \beta_i)$ which gives

$$\mathbb{I}_{QM} = \sum_{i=1}^{N} I(x_i : \beta_i) \leq I(\vec{x} : \vec{m}, \rho_B) \leq M \tag{1.55}$$

1.4.5 PR-correlation violates IC

Suppose, Alice is provided a random bit string $\vec{x} = (x_0, x_1)$ (bit values are 0 or 1) and Bob is asked to guess Alice's i-th bit where $i \in 0, 1$. Alice can communicate 1 cbit to Bob and both parties shares one copy of a no-signaling resource (correlation) $P(a, b|x, y)$. Say, Bob's guess is β_i for Alice's i-th bit. On following some protocol to use the available resource, let probability with which Bob correctly guesses x_i be P_i. Then, $I(x_i : \beta_i) = 1 - h(P_i)$ where $h(P_i) = -P_i \log P_i - (1 - P_i) \log(1 - P_i)$ is Shannon's binary entropy. Therefore, here the IC principle [Pawlowski et al., 2009] takes a form

$$\mathbb{I} = I(x_0 : \beta_0) + I(x_1 : \beta_1) = 2 - h(P_1) - h(P_2) \leq 1 \tag{1.56}$$

Now, suppose Alice and Bob share a single copy of PR-box correlation, $P(a, b|x, y) = \delta_{a \oplus b = xy}$ and they agree upon following protocol to play the game. Alice at her end inputs $x_0 \oplus x_1$ in the PR-box and, say, the output she obtained is A. Then she communicates $c = A \oplus x_0$ to Bob. Bob inputs i into the PR-box and obtains an output, say, B. Finally, Bob declares his guess as $\beta_i = c \oplus B$. So, Bob's guess, $\beta_i = (A \oplus x_0) \oplus B = (A \oplus B) \oplus x_0 = (x_0 \oplus x_1).i \oplus x_0 = x_i$ is always correct and in this case $P_1 = P_2 = 1$. Therefore, $\mathbb{I} = 2 > 1$ and IC principle is violated. The protocol used here is due to Wim van Dam [van Dam, 2000].

In general, when the van Dam's protocol is applied with a single copy of some arbitrary no-signaling resource $P(a, b|x, y)$, the probabilities with which Bob correctly guesses Alice's bits can be expressed as:

$$P_1 = \frac{1}{2}[P(A \oplus B = 0|0,0) + P(A \oplus B = 0|1,0)] \tag{1.57}$$

$$P_2 = \frac{1}{2}[P(A \oplus B = 0|0,1) + P(A \oplus B = 1|1,1)] \tag{1.58}$$

1.4.6 A sufficient condition for violation of IC

A generalization of the above task is—Alice is provided with $N = 2^n$ random bits, say, $\vec{x} = (x_0, x_1, ..., x_i, ...x_{2^n-1})$ and can communicate 1 cbit to Bob who aims at guessing Alice's K-th bit. If this game is played between Alice and Bob, there is a protocol (concatenation based on Van Dam's protocol) through which Bob can again guess any one of the Alice's bit with probability 1. Then, in the reference [Pawlowski et al., 2009], by replacing PR-boxes with some arbitrary no-signaling resource $P(a, b|x, y)$ a sufficient condition for violating the IC principle is obtained as:

$$E_1^2 + E_2^2 > 1, \quad \text{where} \quad E_j = 2P_j - 1 \quad \text{for } j \in \{1, 2\} \tag{1.59}$$

1.4.7 Cirel'son's bound from IC principle

Now, one can easily show that the Cirel'son's bound can be derived from the IC principle [Pawlowski et al., 2009]. First, let us recall that value of Bell-CHSH parameter $S = \sum_{x,y=0}^{1} P(a \oplus b = xy|x, y)$ corresponding to Cirel'son's bound takes the value $2 + \sqrt{2}$ (classical bound for S is 3). Also note that,

$$S = 2(P_1 + P_2). \tag{1.60}$$

Now, sufficient condition for violating the IC principle:

$$\begin{aligned} E_1^2 + E_2^2 > 1 &\equiv (E_1 + E_2)^2 > 1 + 2E_1E_2 \\ &\equiv (S-2)^2 > 1 + 2\,(2P_1 - 1)\,(2P_2 - 1) \\ &\equiv S > 1 + 2\sqrt{2}\,\sqrt{P_1P_2} \end{aligned} \tag{1.61}$$

Therefore, since $\frac{1}{2}(P_1 + P_2) \geq \sqrt{P_1P_2}$ holds, for IC violation it is sufficient that:

$$\begin{aligned} S > 1 + 2\sqrt{2}\,\frac{P_1 + P_2}{2} &\equiv S > 1 + 2\sqrt{2}\,\frac{(\frac{S}{2})}{2} \\ &\equiv S > 2 + \sqrt{2} \end{aligned} \tag{1.62}$$

Thus, applying IC principle one gets to Cirel'sons bound $S_Q = 2 + \sqrt{2}$, moreover all no-signaling correlations with $S > 2 + \sqrt{2}$ violates the IC principle.

1.5 Outline of the thesis

This thesis discusses different aspects in the study of bi-partite quantum correlations. The ordering of the chapters follow the sequence in which several related concepts have been discussed this chapter. Most of the basic ideas to be applied in the following chapters has been introduced.

Chapter-2 is motivated to explore local-realistic model(s) for certain entangled state statistics. The essence of Bells theorem is that, in general, quantum statistics cannot be reproduced by a local hidden variable (LHV) model. This impossibility is strongly manifested when statistics collected by measuring certain local observables on a singlet state, violates the Bell inequality. In Chapter-2, we search for local POVMs with binary outcomes for which an LHV model can be constructed for a singlet state. We provide various subsets of observables for which LHV model(s) can be provided for singlet statistics [Rai, Gazi, Banik, Das and Kunkri, 2012].

Chapter-3 is on Leggett's nonlocal-realistic model for entangled states. As discussed earlier, Leggett's model leads to experimentally testable inequalities which can be violated by quantum correlation. The Leggett-type nonlocal realistic inequalities that have been derived to date are all contingent upon suitable geometrical constraints to be strictly satisfied by the spatial arrangement of the relevant measurement settings. This undesirable restriction is removed in our work [Rai, Home and Majumdar, 2011] by deriving appropriate forms of nonlocal realistic inequalities, one of which involves the fewest number of settings compared to all such inequalities derived earlier. The way such inequalities would provide a logically firmer basis for a clearer testing of a Leggett-type nonlocal realistic model vis-a-vis quantum mechanics is explained.

In Chapter-4 we discusses about the fundamental significance of no-signaling condition in the study of quantum correlations. Predictive power of the no-signaling condition (NSC) is demonstrated in a testable situation involving a non-ideal SternGerlach (SG) device in one of the two wings of the EPR-Bohm entangled pairs. In this wing, for two types of measurement in the other wing, we consider the spin state of a selected set of particles that are confined to a particular half of the plane while emerging from the SG magnetic field region. Due to non-idealness of the SG setup, this spin state will have superposing components involving a relative phase for which a testable quantitative constraint is obtained by invoking NSC, thereby providing a means for precision testing of this fundamentally significant principle [Home, Rai and Majumdar, 2013].

In Chapter-5 we the study bi-partite correlations in two two-level systems in the generalized no-signaling framework. We apply the principle of non-violation of information causality (a generalization of no-signaling condition) to study the Hardy-type nonlocal correlations [Ahanj, Kunkri, Rai, Rahaman and Joag, 2010; Gazi, Rai, Kunkri and Rahaman, 2010]. First we introduce the information causality principle to explain a quantum feature: why Hardy's nonlocality cannot be observed for maximally entangled states. Next, we derive a bound on Hardy's non-locality by applying a sufficient condition for violating information causality.

In the last chapter (Chapter-6), we first summerize the main results obtained in this thesis and then discuss the future directions of our study. Here, we discuss about some important problems in the study of nonlocality involving more than two parties. Recently, some studies on few party quantum correlations has revealed an intricate structure showing limitations of bi-partite physical principle in characterizing all such correlations.

Chapter 2

Local simulation of singlet statistics for a restricted set of measurements

In this chapter we study certain entangled state statistics generated by most general two outcome POVM measurements. The essence of Bell's theorem is that, in general, quantum statistics cannot be reproduced by a local hidden variable (LHV) model. This impossibility is strongly manifested when statistics collected by measuring certain local observables on a singlet state, violates the Bell inequality. We search for local POVMs with binary outcomes for which an LHV model can be constructed for a singlet state. We provide various subsets of observables for which an LHV model can be provided for singlet statistics [Rai, Gazi, Banik, Das and Kunkri, 2012].

2.1 Introduction

A violation of the Bell-CHSH inequality [Bell, 1964; Clauser et al., 1969] by statistics generated from local measurements performed on an entangled state shared between two spatially separated parties certifies such quantum state as nonlocal. The singlet state of two qubits (an EPR state) exhibits maximum nonlocality [Cirel'son, 1980] for proper choices of local observables. Although for pure entangled states the degree of nonlocality is in direct proportion to the entanglement content of a quantum state, this is, in general, not true for mixtures of entangled states [Gisin, 1991; Gisin and Peres, 1992; Popescu, 1994; Popescu and Rohrlich, 1992]. Werner first gave the counterintuitive example of mixed entangled states (popularly known as Werner states) [Werner, 1989] whose statistics when subjected to projective measurements, can be generated by a local hidden variable (LHV) model. A similar example for tripartite entangled state which can be simulated by a local hidden variable model was first provided in a work by Toth and Acin Toth and Acin [2006]. A good review of research on hidden variable theories can be found in Genovese [2005].

Interestingly, Toner and Bacon [Toner and Bacon, 2003] in the year 2003, gave a twist to earlier studies, by providing a model for singlet simulation which requires only 1 cbit of communication supplemented with local variables. Soon after, Cerf *et al* [Cerf et al., 2005] showed that 1 *nl*-bit (single PR-Box) is also sufficient for singlet simulation. Motivated by these works, recently, another model has been provided for singlet simulation which uses (possibly) signalling resource, namely S^p correlations, which suggests a trade off relation between required communication and local randomness in measurement results [Hall, 2010*a*; Kar et al., 2011]. Degorre *et al.* [Degorre et al., 2005, 2007] could map the problem of simulating entangled states to distributed sampling problems. A more through review of simulation of entangled state statistics from communication complexity point of view can be found in [Buhrman et al., 2010]. Few other recent works [Banik, Gazi, Das, Rai and Kunkri, 2012; Barrett and Gisin, 2011; Hall, 2010*b*, 2011] show that *lack of free will* can also be considered as a resource for singlet simulation. There are also some efforts in solving the difficult problem of simulating multipartite entanglement and non-maximally bipartite entangled states either by use of communication or by nonlocal (no-signaling) resources Brunner et al. [2008, 2010]. All these various approaches have been deepening our understanding about quantum correlation and its use as a physical resource in various information processing tasks.

As of providing local variable models for class of entangled states, in a seminal work in the year 2002, Barrett Barrett [2002] generalized the work of Werner Werner [1989], by construting a LHV model for any positive-operator-valued measurements at the expense of the weight associated with singlet in Werner state. Motivated by these works we pose the problem from opposite direction i.e. rather than weakening the (singlet) state we search for the class of (weakened) dichotomic observable (POVM) for which local model can be provided. In particular, here we provide the subset of the most general two outcome measurements represented by positive operator value measure (POVM) and present local models for singlet statistics generated from them. We provide some sets of local observable which are optimal for the protocol we have suggested. First we show that, if observable on any one side is sufficiently restricted (deviates from ideal projective measurement), resulting statistics for the singlet state has a local hidden variable model. Next, we provide another model which is symmetric in a sense that observable on both the sides are put to a similar type of restriction. Finally, we identify a more general set of observable for which LHV models exists with some further restrictions. Before we derive our results, in the followings section, we give a mathematical description of a general two-outcome POVMs.

2.2 General two-outcome POVM

Generalized quantum observables are described by POVMs [Busch et al., 1995, 1996]. For finite, say n, outcome measurements on a d-dimensional state space a POVM is a collection of selfadjoint operators $\{E_i\}$ acting on a complex Hilbert space $\mathbb{C}^d$ satisfying the conditions: (i) $0 \leq E_i \leq I$ for all i, and (ii) $\sum_i E_i = I$, where $i \in \{1, 2, ..., n\}$. A measurement of such an observable $\{E_i\}$ on a quantum state ρ results in any one of the n possible outcomes; the probability of an occurrence of i-th outcome (termed as clicking of i-th *effect*) is $Tr[\rho E_i]$. A subclass of these type of general measurement has an interesting physical interpretation as unsharp spin properties, introduced by P. Busch [Busch, 1986*a*,*b*].

In this chapter, we consider general two-outcome POVMs $\{E, I - E\}$ acting on $\mathbb{C}^2$ (state space of a qubit). Effect E is characterized by some parameters, say, $a_0 \in \mathbb{R}$ (a scalar) and $\vec{a} \in \mathbb{R}^3$ (a vector). We denote norm of $\vec{a}$ by μ. Then, the selfadjoint property along with the condition $0 \leq E \leq I$ implies that E can be expressed as

$$E = \frac{1}{2}[a_0 I + \mu \hat{a} \cdot \vec{\sigma}] \tag{2.1}$$

$$0 \leq a_0 \leq 2 \tag{2.2}$$

$$0 \leq \mu \leq \min\{a_0, 2 - a_0\} \tag{2.3}$$

where $\hat{a} \cdot \vec{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$. Then, the corresponding operator $I - E$ is also selfadjoint and satisfies the requirement $0 \leq I - E \leq I$. Thus, the Eq.(2.1) along with the conditions (2.2) and (2.3), supplemented with an arbitrary direction $\hat{a}$, completely determine a two-outcome POVM $\{E, I - E\}$ acting on $\mathbb{C}^2$. The region feasible for parameters a_0 and μ for defining such an effect $E\{a_0, \mu, \hat{a}\}$ is illustrated in Fig.(2.1).

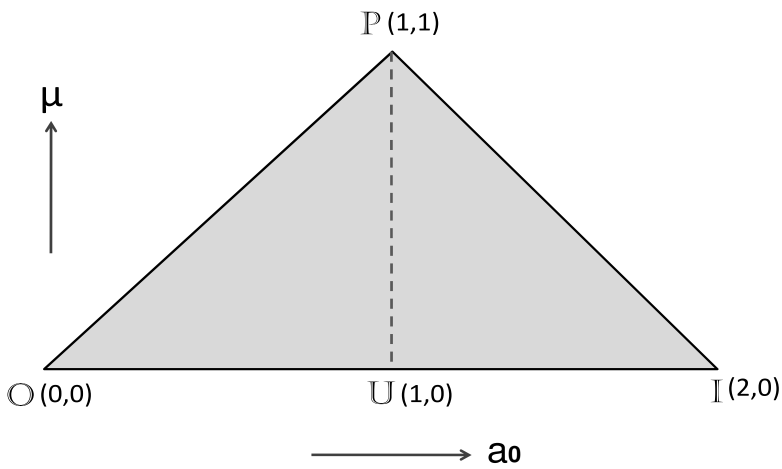

FIGURE 2.1: Parameter a_0 (μ) varies along the horizontal (vertical) axis. Any point (a_0, μ) laying in the shaded triangular region $\mathbb{POI}$ (together with an arbitrary parameter $\hat{a}$) determines a two-outcome POVM $\{E, I - E\}$. Points on the dashed line $\mathbb{PU}$ represent unsharp spin measurements. Point $\mathbb{P}(1, 1)$ represent ideal projective measurements.

An interesting application of general two-outcome measurements considered here is in the study of spin properties of spin-$\frac{1}{2}$ systems. In this context, P. Busch [Busch, 1986*a*,*b*] first showed that a subclass of general two-outcome POVMs can be interpreted as measurement of unsharp-spin property of spin-$\frac{1}{2}$ particles. Under the condition of rotation covariance, parameters $\{a_0, \mu\}$ is decoupled from $\hat{a}$ which can then be interpreted as orientation of the measuring device. Further, condition of symmetry under a rotation π of the measuring device gives $a_0 = 1$. Thus, effect operators for an unsharp spin observable is of the form $E^{\mu}_{\pm}(\hat{a}) = \frac{1}{2}[I \pm \mu\hat{a} \cdot \vec{\sigma}]$. The spectral decomposition of positive operators $E^{\mu}_{\pm}(\hat{a})$ is

$$E^{\mu}_{\pm}(\hat{a}) = (\frac{1 \pm \mu}{2})\frac{1}{2}[I + \hat{a}.\vec{\sigma}] + (\frac{1 \mp \mu}{2})\frac{1}{2}[I - \hat{a}.\vec{\sigma}]$$

where $\frac{1}{2}[I+\hat{a}.\vec{\sigma}]$ and $\frac{1}{2}[I-\hat{a}.\vec{\sigma}]$ are one dimensional spin projection operators on the Hilbert space $\mathbb{C}^2$. Now, the quantity $\frac{1+\mu}{2}$ $(\frac{1-\mu}{2})$ can be suitably interpreted as degree of reality (unsharpness) of outcomes obtained from a spin measurement along direction $\hat{a}$. From this representation it is clear that the POVM $\{E^{\mu}_{+}(\hat{a}), E^{\mu}_{-}(\hat{a})\}$ is a smeared version of the projective measurement $\{\frac{1}{2}[I+\hat{a}.\vec{\sigma}], \frac{1}{2}[I-\hat{a}.\vec{\sigma}]\}$—in case of projective measurements the unsharp parameter $\mu = 1$.

Another important property is that under suitable conditions two POVMs can be jointly measurable [Kraus, 1983]. Two POVMs of the form $\{E_1, I - E_1\}$ and $\{E_2, I - E_2\}$ are jointly measurable if there exits a four-outcome POVM $\{E_{12}, E_{\bar{1}2}, E_{1\bar{2}}, E_{\bar{1}\bar{2}}\}$ such that it can reproduce the correct marginals, i.e., $E_1 = E_{12} + E_{1\bar{2}}$ and $E_2 = E_{12} + E_{\bar{1}2}$. For unsharp spin observable it has been shown that [Busch, 1986*b*] (also see the review [Kar and Roy, 1999]) two observables parameterized by, say, $(\mu_1, \hat{a}_1)$ and $(\mu_2, \hat{a}_2)$ are jointly measurable if and only if $\|\mu_1\hat{a}_1+\mu_2\hat{a}_2\|+\|\mu_1\hat{a}_1-\mu_2\hat{a}_2\| \leq 2$. On considering unsharp parameter for both the spin observables to be same i.e., $\mu_1 = \mu_2$, along with the fact $\|\hat{a}_1+\hat{a}_2\|+\|\hat{a}_1-\hat{a}_2\| \leq 2\sqrt{2}$ for any pair of unit vectors $\hat{a}_1$ and $\hat{a}_2$, it is easy to conclude that if the unsharp parameters $\mu_1 = \mu_2 \leq \frac{1}{\sqrt{2}}$ then joint measurement of unsharp spin property can be realized for any such pair of directions.

2.3 LHV model for singlet statistics for two outcome POVMs

Suppose, two spatially separated parties Alice and Bob share one qubit each from a singlet state

$$\rho_{AB} = \frac{1}{4}[I \otimes I - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z].$$

Let Alice's (Bob's) observable be a most general two-outcome POVM $E_A[a_0, \mu_A, \hat{a}]$ $(E_B[b_0, \mu_B, \hat{b}])$, defined by Eq.(1). If the effect $E_{A(B)}$ clicks we denote the outcome by '*yes*' otherwise '*no*'. Then joint outcome probabilities are following:

$$\begin{aligned} P^{AB}(yes,yes) &= \frac{1}{4}[a_0 b_0 - \mu_A\mu_B \hat{a}\cdot\hat{b}] \\ P^{AB}(yes,no) &= \frac{1}{4}[a_0(2-b_0) + \mu_A\mu_B \hat{a}\cdot\hat{b}] \\ P^{AB}(no,yes) &= \frac{1}{4}[(2-a_0)b_0 + \mu_A\mu_B \hat{a}\cdot\hat{b}] \\ P^{AB}(no,no) &= \frac{1}{4}[(2-a_0)(2-b_0) - \mu_A\mu_B \hat{a}\cdot\hat{b}] \end{aligned} \quad (2.4)$$

2.3.1 Models for two-outcome measurements

Violation of the Bell-CHSH inequality [Bell, 1964; Clauser et al., 1969] implies that there can be no LHV model for the singlet statistics generated by projective measurements by both the parties. Therefore, the statistics of the singlet can have a LHV model only if general two outcome POVMs considered here are restricted (deviate from ideal projective measurements) in some way or the other. Following Werner's local model for some mixed entangled states [Werner, 1989], we provide two LHV models for singlet state under certain restrictions on parameters of two outcome POVMs. In both type of models vectors $\hat{\lambda} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ uniformly distributed over the unit sphere, are the local variables preshared between Alice and Bob.

2.3.1.1 A fully biased model $\mathbb{M}_{fb}$:

Let, Bob's observable $E_B(b_0, \mu_B, \hat{b})$ satisfy restriction $\mu_B \leq \frac{1}{2}\text{min}\{b_0, 2-b_0\}$ but there is no restriction on Alice's observables, see Fig(2.2).

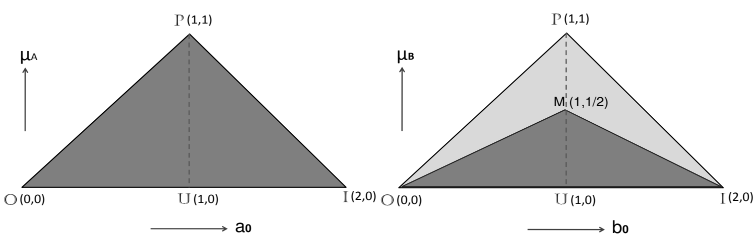

FIGURE 2.2: Alice's (Bob's) parameters can take values from the dark gray triangular region on left (right). Alice's parameters a_0 and μ_A can take any possible value, but Bob's parameters b_0 and μ_B are restricted to come from the region $\mathbb{MOI}$.

Alice, for her observable $E_A[a_0, \mu_A, \hat{a}]$, declares '*yes*' with a probability

$$P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{2}\mu_A \cos\alpha \tag{2.5}$$

where α is angle between direction $\hat{a}$ and $\hat{\lambda}$. On the other hand, for observable $E_B[b_0, \mu_B, \hat{b}]$, Bob declares '*yes*' with a probability

$$P^B_{\hat{\lambda}}(yes) = \frac{b_0}{2} - \mu_B \ \text{sgn}(\cos\beta) \tag{2.6}$$

where β is the angle between the direction $\hat{b}$ and $\hat{\lambda}$, and $\text{sgn}(x) = +1$ (-1) for $x \geq 0$ $(x < 0)$.

The joint probability of the outcome $(yes,\ yes)$, can be calculated from

$$P^{AB}_{lhv}(yes, yes) = \int \rho(\hat{\lambda})\ P^A_{\hat{\lambda}}(yes)\ P^B_{\hat{\lambda}}(yes)\ d\hat{\lambda} \tag{2.7}$$

where $\rho(\hat{\lambda})$ is the considered (uniform) distribution of the hidden variable $\hat{\lambda}$. Evaluating the above integral gives

$$P^{AB}_{lhv}(yes, yes) = \frac{1}{4}[a_0 b_0 - \mu_A \mu_B \hat{a} \cdot \hat{b}] \tag{2.8}$$

which exactly matches with the quantum mechanical prediction for the outcome $(yes,\ yes)$. The desired quantum mechanical probabilities for the other possible outcomes easily follows, for example, $P^{AB}_{lhv}(yes, no)$ is obtained simply by replacement $P^B_{\hat{\lambda}}(yes) \to P^B_{\hat{\lambda}}(no) = 1 - P^B_{\hat{\lambda}}(yes)$ in the integrand of the Eq.(2.7).

2.3.1.2 A fully symmetric model $\mathbb{M}_{fs}$:

Let Alice's and Bob's observables satisfy following restriction (see Fig.(2.3))

$$\mu_A \leq \frac{1}{\sqrt{2}} \min\{a_0, 2 - a_0\}$$
$$\mu_B \leq \frac{1}{\sqrt{2}} \min\{b_0, 2 - b_0\}$$

Alice declares '*yes*', for her observable $E_A[a_0, \mu_A, \hat{a}]$, with a probability

$$P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{\sqrt{2}}\mu_A \cos\alpha \tag{2.9}$$

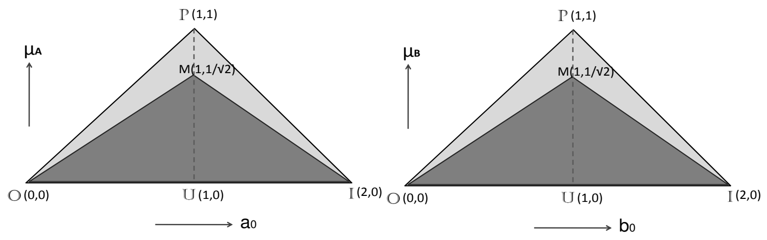

FIGURE 2.3: Alice's (Bob's) parameters can take values from the dark gray triangular region on left (right). Alice's parameters a_0 and μ_A as well as Bob's parameters b_0 and μ_B are restricted in the same way and come from the triangular region $\mathbb{MOI}$ on the left and right respectively.

where α is angle between direction $\hat{a}$ and $\hat{\lambda}$.

Where else Bob declares '*yes*', for his observable $E_B[b_0, \mu_B, \hat{b}]$, with a probability

$$P_{\hat{\lambda}}^B(yes) = \frac{b_0}{2} - \frac{1}{\sqrt{2}}\mu_B \ \mathrm{sgn}(\cos\beta) \tag{2.10}$$

where β is the angle between the direction $\hat{b}$ and $\hat{\lambda}$, and $\mathrm{sgn}(x) = +1$ (-1) for $x \geq 0$ $(x < 0)$. Like in the fully biased model $\mathbb{M}_{fb}$, we find that this model ($\mathbb{M}_{fs}$) also simulates the correct statistics for the singlet.

If we consider unsharp spin properties on both sides with uniform value of μ_A and μ_B and also assume any pair are jointly measurable **??**] on both sides, then the conditions of the model $\mathbb{M}_{fs}$ are automatically satisfied and hence this LHV model $\mathbb{M}_{fs}$ can simulate the singlet statistics for any arbitrary pair of respective directions $\hat{a}$ and $\hat{b}$ for Alice and Bob.

2.3.2 Measure of restriction on observable

By considering that observables of Alice and Bob are picked from a uniform distribution of all possible two-outcome POVMs, we can define a measure r for restriction on the observables of any of the two parties in the following way. (see Fig.(2.2) and Fig.(2.3))

$$r = \left[1 - \frac{\text{Area } (\mathbb{MOI})}{\text{Area } (\mathbb{POI})}\right] \times 100 \tag{2.11}$$

Now, one can easily calculate that in the model $\mathbb{M}_{fb}$ ($\mathbb{M}_{fs}$), there is 0% (29.3%) restriction on Alice's observables wherelse Bob's observables are restricted by 50% (29.3%).

Another interesting observation is that model $\mathbb{M}_{fb}$ and $\mathbb{M}_{fs}$ belong to a general class of LHV models, $\{\mathbb{M}_{\kappa} : \kappa \geq 0\}$.

Under the restrictions,

$$\mu_A \leq \kappa \min\{a_0, 2 - a_0\} \tag{2.12}$$

$$\mu_B \leq \frac{1}{2\kappa} \min\{b_0, 2 - b_0\} \tag{2.13}$$

Alice declares "*yes*" with a probability

$$P_{\hat{\lambda}}^{A}(yes) = \frac{a_0}{2} + \frac{1}{2\kappa}\mu_A \cos\alpha \tag{2.14}$$

Bob declares '*yes*' with a probability

$$P_{\hat{\lambda}}^{B}(yes) = \frac{b_0}{2} - \kappa\mu_B \operatorname{sgn}(\cos\beta) \tag{2.15}$$

For any nonnegative value of κ we get a LHV model—$\mathbb{M}_{fb}$ ($\mathbb{M}_{fs}$) correspond to $\kappa = 1$ ($\kappa = \frac{1}{\sqrt{2}}$). In Fig(2.4) the two curves show the % restriction on Alice's and Bob's observable for LHV models corresponding to different values of κ. The intersection point of two curves correspond to the symmetric model $\mathbb{M}_{fs}$. Observe that $\kappa = \frac{1}{2}$ correspond to another fully biased model, say $\mathbb{M}'_{fb}$, which is same as $\mathbb{M}_{fb}$ except that conditions on Alice's and Bob's observables are interchanged. In fact all the models for which $\kappa \in (0, \frac{1}{2}] \cup [1, \infty)$ are fully biased models, however one can immediately observe that within this subclass, either $\mathbb{M}_{fb}$ or $\mathbb{M}'_{fb}$ is sufficient to simulate any other fully biased model. Thus only the subclass $\{\mathbb{M}_{\kappa} : \kappa \in [1/2, 1]\}$ contains tight LHV models in a sense that they can capture any varying degree of restrictions on Alice's and Bob's observables.

2.3.3 A different class of model

In previous cases, we put restriction on the observable separately on both sides. Now we put the following restriction on the observable where one of the restrictions

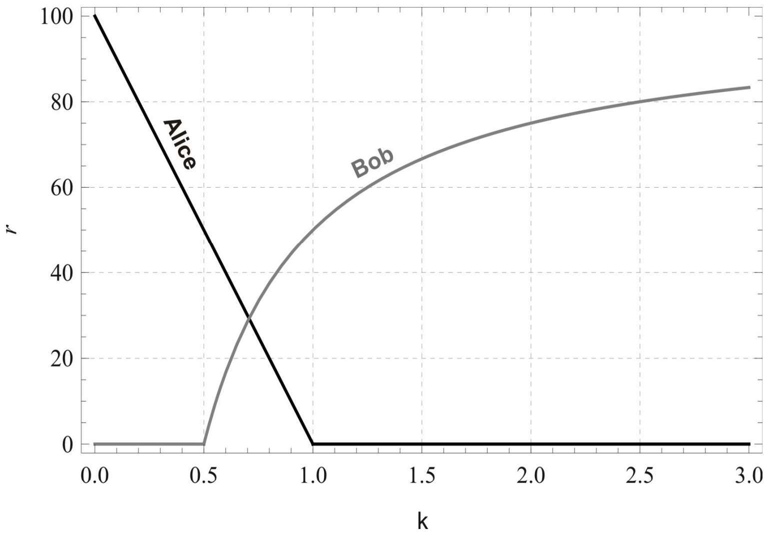

FIGURE 2.4: Black (Gray) curve show percentage restriction on Alice's (Bob's) observable for LHV models corresponding to different values of κ. The intersection point of two curves at $\kappa = \frac{1}{\sqrt{2}}$ correspond to the completely symmetric LHV model $\mathbb{M}_{fs}$. Fully biased model $\mathbb{M}_{fb}$ ($\mathbb{M}'_{fb}$) correspond to $\kappa = \frac{1}{2}$ ($\kappa = 1$). All the models for $\kappa \in (0, \frac{1}{2}] \cup [1, \infty)$ are fully biased.

involves parameters of both sides;

$$\frac{1}{\eta} \leq a_0 \leq 2 - \frac{1}{\eta} \tag{2.16}$$

$$\mu_A \mu_B \leq \frac{1}{2\eta} \min\{b_0, 2 - b_0\} \tag{2.17}$$

where $\eta \geq 1$. Alice and Bob now can simulate singlet statistics according to the following protocol

$$P^A_{\hat{\lambda}}(yes) = \frac{a_0}{2} + \frac{1}{2\eta} \cos \alpha \tag{2.18}$$

$$P^B_{\hat{\lambda}}(yes) = \frac{b_0}{2} - \eta \mu_A \mu_B \operatorname{sgn}(\cos \beta) \tag{2.19}$$

But this model is obviously non-local as Bob's output involves parameters of observable on both sides. But the model can be made local for a given η and fixed μ_A. In this case there is no restriction on direction $\hat{a}$ of Alice's POVM. It might seem that by increasing the value of η the range of a_0 can be extended but then due to the condition (2.17), the range of μ_B is also restricted accordingly. So in some sense in this model a_0 and μ_B maintain a complementary relation for a given b_0.

2.4 Conclusion

Simulation of quantum statistics for Werner state by LHV has been an interesting area for understanding the physics of entanglement [Barrett, 2002; Gisin, 1991; Gisin and Peres, 1992; Popescu, 1994; Popescu and Rohrlich, 1992; Werner, 1989]. We have studied the cases where LHV simulation is possible for singlet state. We find the optimal set of two outcomes observable for which singlet simulation by LHV is possible under the suggested protocol. It is also interesting that for uniform unsharp parameter, the joint measurability of unsharp spin property on both sides implies LHV model for singlet. It will be interesting to study whether the set can be enlarged with respect to different LHV model.

Chapter 3

Leggett-type nonlocal realistic inequalities

After discussing about local-realistic models for entangled states in the previous chapter, in the present chapter we discuss Leggett's nonlocal-realistic model for entangled states. Leggett's model leads to experimentally testable inequalities which can be violated by quantum correlation. The Leggett-type nonlocal realistic inequalities that have been derived prior to our work [Rai, Home and Majumdar, 2011] are all contingent upon suitable geometrical constraints to be strictly satisfied by the spatial arrangement of the relevant measurement settings. This undesirable restriction is removed by deriving appropriate forms of nonlocal realistic inequalities, one of which involves the fewest number of settings compared to all such inequalities derived earlier. The way such inequalities would provide a logically firmer basis for a clearer testing of a Leggett-type nonlocal realistic model vis-a-vis quantum mechanics is explained.

3.1 Introduction

Subsequent to the plethora of studies confirming experimental falsification of Bell-type inequalities [Aspect, 1999], thereby ruling out the local realist models in favor of quantum mechanics (QM), the next issue is whether the question of compatibility between QM and its plausible *nonlocal realist models* can be subjected to a deeper scrutiny. To this end, Leggett [Leggett, 2003] showed an incompatibility between QM and a testable inequality derived for a class of nonlocal realist models which we shall refer to as the Leggett-type nonlocal realist (LNR) model. This, in turn, has motivated a number of theoretical as well as experimental works from different perspectives [Aspect, 2007; Branciard et al., 2008, 2007; Colbeck and Renner, 2008; Eisaman et al., 2008; Groblacher et al., 2007; Lee et al., 2011; Leggett, 2008; Paterek et al., 2007; Paternostro and Jeong, 2010; Romero et al., 2010], including various versions of LNR inequalities. These inequalities involve correlation functions of joint polarization (spin) properties of two spatially separated photons (spin-$\frac{1}{2}$ particles), and have been largely shown to be experimentally violated for the polarization degrees of freedom of photons prepared in a maximally entangled state.

In the initial experiment by Gröblacher *et al.* [Groblacher et al., 2007], though, the form of the LNR inequality that was tested necessitated assuming the invariance of the correlation functions under simultaneous rotation (by the same angle) of the axis of each of the two polarizers. This additional assumption was, however, not required in the subsequent works [Branciard et al., 2008, 2007; Paterek et al., 2007] that showed empirical violation of the suitably derived forms of LNR inequalities. Nevertheless, an undesirable feature besets all such studies since different forms of LNR inequalities that have been derived and tested to date hold good only if certain geometrical constraints are *exactly* satisfied by the spatial arrangement of the relevant measurement settings. For example, appropriate to any such inequality, relative orientations of the planes of the relevant measurement settings need to satisfy suitable conditions such as that of orthogonality. Hence, in the experimental tests of these inequalities, even an infinitesimal error in satisfying the required restrictions would make it logically problematic to draw any firm conclusion about the falsification of the LNR model [Colbeck and Renner, 2008]. This loophole is sought to be removed in our work by deriving within the general framework of the LNR model two different forms of LNR inequalities that hold good for *any* possible geometrical alignment of the experimental setup. Further, it is important to note that the QM violation of such inequalities can be demonstrated within the experimental threshold visibility already achieved. The other significant feature is that one of our LNR inequalities involves $(3 + 3)$ number of settings which is the *least* number of settings achieved so far compared to all the LNR inequalities derived earlier.

3.2 Leggett's model

We begin by briefly recapitulating the essence of the LNR model [Leggett, 2003, 2008] which regards the whole ensemble of photon pairs emitted from a source to be a disjoint union of subensembles that are assumed to have the following features: (i) In any such subensemble, each pair of photons is characterized by definite values of preassigned polarizations $\hat{u}$ and $\hat{v}$ so that the whole ensemble corresponds to a distribution of values of $\hat{u}$ and $\hat{v}$ denoted by, say, $D(\hat{u}, \hat{v})$. (ii) For any given pair belonging to such a subensemble, individual outcomes (denoted by A and B) of polarization measurements on each member of the pair along directions, say, $\hat{a}$ and $\hat{b}$ respectively are assumed to be determined by a hidden variable, say, λ whose values are distributed over the pairs comprising the given subensemble with the corresponding distribution function being denoted by $\rho_{(\hat{u},\hat{v})}(\lambda)$. (iii) The outcome of polarization measurement along $\hat{a}$ ($\hat{b}$) for any individual photon in one of the two wings may be non-locally dependent on the choice of the measurement setting pertaining to its spatially separated partner in the other wing, but the statistical result for a given subensemble obtained by averaging over such effects is assumed to satisfy the Malus law. This entails that the relevant mean value depends only on the local setting. Thus, such mean values of outcomes of polarization measurements for the subensembles characterized by $\hat{u}$ and $\hat{v}$ pertaining to the two wings can respectively be written as $\overline{A}(\hat{u}) = \int A(\hat{a}, \hat{b}, \lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda = \hat{u}\cdot\hat{a}$, and $\overline{B}(\hat{v}) = \int B(\hat{b}, \hat{a}, \lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda = \hat{v}\cdot\hat{b}$. Then the experimentally observable polarization correlation function for the whole ensemble is expressible as $\langle AB\rangle = \iint \overline{AB}(\hat{u}, \hat{v})D(\hat{u}, \hat{v})d\hat{u}d\hat{v}$ where $\overline{AB}(\hat{u}, \hat{v}) = \int A(\hat{a}, \hat{b}, \lambda)B(\hat{b}, \hat{a}, \lambda)\rho_{(\hat{u},\hat{v})}(\lambda)d\lambda$.

3.3 Derivation of geometrical constraint-free Leggett-type inequalities

Let us consider that for a pair of emitted photons, $A = \pm 1$ and $B = \pm 1$ are the outcomes observed by two spatially separated partners Alice and Bob performing polarization measurements on each of the photons in the directions $\hat{a}$ and $\hat{b}$ respectively. An outcome $+1$ (-1) is associated with a photon getting transmitted (absorbed) through (in) the relevant polarizer. Then, one can easily verify that the algebraic identity $-1 + |A + B| = AB = 1 - |A - B|$ holds true for all the possible outcomes of Alice and Bob. Subsequently, on averaging this relation over any one of the subensembles (characterized by $(\hat{u}, \hat{v})$) mentioned earlier, one obtains $-1+\overline{|A+B|} = \overline{AB} = 1-\overline{|A-B|}$, where the bar notation denotes averaging over the hidden variables within the given subensemble. Since the average of the modulus is greater or equal to the modulus of the averages, therefore, at the level of subensembles one gets, $-1+|\overline{A}+\overline{B}| \leq \overline{AB} \leq 1-|\overline{A}-\overline{B}|$, which can be rewritten as the following inequality

$$|\overline{A} \pm \overline{B}| \leq 1 \pm \overline{AB}. \tag{3.1}$$

Next, *à* la Branciard *et al.* [Branciard et al., 2008], consider one measurement setting $\hat{a}$, with the corresponding outcome A, for Alice, and two measurement settings $\hat{b}, \hat{b}'$, with the corresponding outcomes B, B' for Bob. Applying the inequality (3.1) for the sets $\{A, B\}$ and $\{A, B'\}$ respectively, together with the use of the triangle inequality, one can obtain the following inequality $|\overline{AB} \pm \overline{AB'}| \leq 2 - |\overline{B} \mp \overline{B'}|$. Then, by invoking the Malus law on the right hand side of the preceding inequality and averaging over the distribution $D(\hat{u}, \hat{v})$, one gets

$$|\langle AB\rangle + \langle AB'\rangle| \leq 2 - \iint |\hat{v}\cdot(\hat{b}-\hat{b}')| D(\hat{u},\hat{v}) d\hat{u} d\hat{v} \tag{3.2}$$

$$|\langle AB\rangle - \langle AB'\rangle| \leq 2 - \iint |\hat{v}\cdot(\hat{b}+\hat{b}')| D(\hat{u},\hat{v}) d\hat{u} d\hat{v} \tag{3.3}$$

Now, at this stage, comes the crucial ingredient of our derivation by considering two different categories of settings that would enable us to derive the desired forms of the LNR inequalities. Note that in this derivation there is no geometrical restriction on the spatial arrangement, once any particular type of combination of settings is specified.

3.3.1 Category I settings

Category I comprising of suitable combinations of measurement settings used for deriving our first LNR inequality, pertains to the inequality (3.2). Here we consider the combinations of settings $\{(\hat{a}_i, \hat{b}_i), (\hat{a}_i, \hat{b'}_i)\}$ where $i \in \{1, 2, 3\}$ and, say, $\beta_i \in (-\pi, \pi)$ is the angle between the pair $(\hat{b}_i, \hat{b'}_i)$. Let $\hat{b}_i - \hat{b'}_i = 2\sin(\frac{\beta_i}{2})\hat{n}_i$ where the unit vectors $\hat{n}_i$'s are *linearly independent*. Then, from (3.2), after adding the corresponding inequalities for the combinations of settings $\{(\hat{a}_i, \hat{b}_i), (\hat{a}_i, \hat{b'}_i)\}$, it follows that

$$\frac{1}{3}\sum_i |\langle A_i B_i\rangle + \langle A_i B_i'\rangle| \leq$$
$$2 - \frac{2}{3}\sin(\frac{\beta_*}{2}) \iint F_n(\hat{v}) D(\hat{u},\hat{v}) d\hat{u} d\hat{v} \tag{3.4}$$

where, $\beta_* = \min\{|\beta_1|, |\beta_2|, |\beta_3|\}$ and $F_n(\hat{v}) = \sum_i |\hat{v}\cdot\hat{n}_i|$.

3.3.2 Category II settings

Category II comprising appropriate combinations of measurement settings involved in our second LNR inequality, pertains to the inequality (3.3). Here we consider the combinations of settings $\{(\hat{a}_i, \hat{b}_i), (\hat{a}_i, \hat{b}_{i\oplus 1})\}$ where $\oplus$ represents addition modulo 3, and $i \in \{1, 2, 3\}$. Let $\delta_i \in (-\pi, \pi)$ be the angle between the pair $(\hat{b}_i, \hat{b}_{i\oplus 1})$, whence $\hat{b}_i + \hat{b}_{i\oplus 1} = 2\cos(\frac{\delta_i}{2})\hat{m}_i$ where $\hat{m}_i$'s represent three *linearly independent* unit vectors. Then, from (3.3), after adding the corresponding inequalities for the combination of settings $\{(\hat{a}_i, \hat{b}_i), (\hat{a}_i, \hat{b}_{i\oplus 1})\}$, it follows that

$$\frac{1}{3}\sum_i |\langle A_i B_i\rangle - \langle A_i B_{i\oplus 1}\rangle| \leq$$
$$2 - \frac{2}{3}\cos(\frac{\delta^*}{2}) \iint F_m(\hat{v}) D(\hat{u},\hat{v}) d\hat{u} d\hat{v} \tag{3.5}$$

where $\delta^* = \max\{|\delta_1|, |\delta_2|, |\delta_3|\}$ and $F_m(\hat{v}) = \sum_i |\hat{v}\cdot\hat{m}_i|$.

3.3.3 Lower bound for the functions $F_n(\hat{v})$ and $F_m(\hat{v})$

Note that the right hand sides of the inequalities (3.4) and (3.5) still involve the unobservable supplementary variables $\hat{u}$ and $\hat{v}$. Thus, in order to recast them in experimentally verifiable forms, we need to derive the respective *lower bounds*, say L_n and L_m, for the functions $F_n(\hat{v})$ and $F_m(\hat{v})$. These lower bounds are obtained by using the following *Theorem*.

Theorem: On the Poincaré sphere, given three linearly independent unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and a variable unit vector $\hat{v}$, the minimum value, say L, of the function $F(\hat{v}) = |\hat{e}_1\cdot\hat{v}| + |\hat{e}_2\cdot\hat{v}| + |\hat{e}_3\cdot\hat{v}|$ is given by the formula $L = \frac{|\hat{e}_1\cdot(\hat{e}_2\times\hat{e}_3)|}{\max\{|\hat{e}_1\times\hat{e}_2|, |\hat{e}_2\times\hat{e}_3|, |\hat{e}_3\times\hat{e}_1|\}}$.

Proof: The minimum value of $F(\hat{v})$ would not depend on the choice of the co-ordinate axes. Thus, for convenience, let the X axis lie along $\hat{e}_1$ and the XY plane contain $\hat{e}_2$. Therefore, according to our choice, $\hat{e}_i$'s can be represented as follows: $\hat{e}_1 = (1, 0, 0)$, $\hat{e}_2 = (b_1, b_2, 0)$, $\hat{e}_3 = (c_1, c_2, c_3)$ where $b_2, c_3 \neq 0$. Further, we observe that the three great circles C_i, defined by $\hat{e}_i \cdot \hat{v} = 0$, divide the surface of the Poincaré sphere into 8 non-overlapping (except on the boundaries) regions, $R_{\xi_1\xi_2\xi_3}$, defined by the constraints $\xi_1\hat{e}_1 \cdot \hat{v} \geq 0$, $\xi_2\hat{e}_2 \cdot \hat{v} \geq 0$, and $\xi_3\hat{e}_3 \cdot \hat{v} \geq 0$, where $\xi_1, \xi_2, \xi_3 \in \{+, -\}$ [see Fig. 3.1].

Let us first minimize the function $F(\hat{v})$ in any one of the restricted regions, say, R_{+++} where it takes the form, $F(\hat{v}_{+++}) = \hat{e}_1 \cdot \hat{v} + \hat{e}_2 \cdot \hat{v} + \hat{e}_3 \cdot \hat{v}$ (here $\hat{v}_{+++}$ denote vectors belonging to the region R_{+++}). We first show that $F(\hat{v}_{+++})$ cannot attain the minimum at some interior point of R_{+++}. Note that, since $F(\hat{v}_{+++})$ is a smooth function, showing that there is no stationary point of local minimum in the interior of R_{+++} would be sufficient. For this, let us consider a function $f(\hat{v}) = \hat{e}_1 \cdot \hat{v} + \hat{e}_2 \cdot \hat{v} + \hat{e}_3 \cdot \hat{v}$ defined over all the points of the Poincaré sphere. Note that, $f(\hat{v}) = F(\hat{v})$ for any $\hat{v} \in R_{+++}$. Let $\hat{v} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with $0 \leq \theta \leq \pi$ and $-\pi < \phi \leq \pi$. Then, $f(\hat{v}) = f(\theta, \phi) = p\sin\theta\cos\phi + q\sin\theta\sin\phi + r\cos\theta$ where $p = 1 + b_1 + c_1$, $q = b_2 + c_2$, and $r = c_3$. At the stationary points of $f(\theta, \phi)$, $\partial_\phi f = \sin\theta(-p\sin\phi + q\cos\phi) = 0$ and $\partial_\theta f = \cos\theta(p\cos\phi + q\sin\phi) - r\sin\theta = 0$. However, among such stationary points, the point belonging to the interior of the region R_{+++} would satisfy $(\partial_{\phi\phi} f)(\partial_{\theta\theta} f) - \partial_{\phi\theta} f > 0$ and $\partial_{\theta\theta} f < 0$, which is the condition of maximum. Thus, $F(\hat{v}_{+++})$ can attain its minimum value only on some boundary point of the region R_{+++}.

Next, we find that the minimum value of $F(\hat{v}_{+++})$ is actually attained at any one or more vertices of the triangular region R_{+++}; these vertices are given by $\hat{v}_1 = \text{sgn}(b_2c_3)\frac{\hat{e}_2\times\hat{e}_3}{|\hat{e}_2\times\hat{e}_3|}$, $\hat{v}_2 = \text{sgn}(b_2c_3)\frac{\hat{e}_3\times\hat{e}_1}{|\hat{e}_3\times\hat{e}_1|}$, $\hat{v}_3 = \text{sgn}(c_3)\frac{\hat{e}_1\times\hat{e}_2}{|\hat{e}_1\times\hat{e}_2|}$ where $\text{sgn}(z) = +1(-1)$ for $z > 0 (z < 0)$. Here, first note that, the intersection of C_i with R_{+++} defines a side of the triangle R_{+++}. Now, if we restrict the domain of $f(\hat{v})$ on a great circle C_i, then it can be shown that, for any i, there is no stationary point of minimum of $f(\hat{v})$ in the interior of the corresponding side of the triangle R_{+++} (see Appendix A). Hence, now we can conclude that the minimum value of $F(\hat{v}_{+++})$ is attained only at some vertices of the region R_{+++}.

Note that the above proven result is true for any arbitrarily specified set of linearly independent unit vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ i.e., if one chooses, say, some other set $\{\hat{e}_1^*, \hat{e}_2^*, \hat{e}_3^*\}$ then the minimum value of the corresponding function $F^*(\hat{v}) =$

$|\hat{e}_1^* \cdot \hat{v}| + |\hat{e}_2^* \cdot \hat{v}| + |\hat{e}_3^* \cdot \hat{v}|$ in a suitably defined region R_{+++}^* is attained at one or more of its vertices.

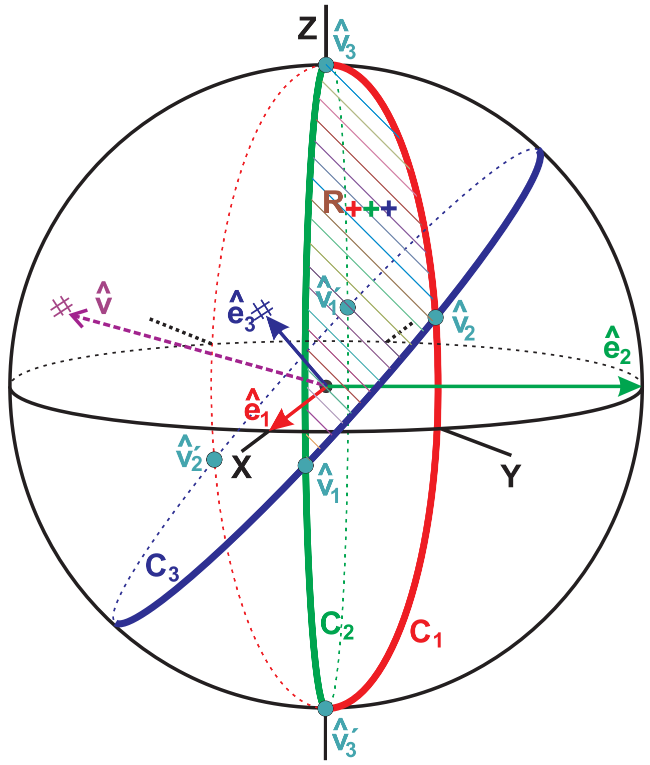

FIGURE 3.1: On the Poincaré sphere, $\hat{e}_i$'s for $i \in \{1,2,3\}$ are three linearly independent unit vectors and $\hat{v}$ is a variable unit vector. Three great circles C_i's lie in respective planes orthogonal to $\hat{e}_i$'s. The intersection points of two great circles $C_{i\oplus 1}$ and $C_{i\oplus 2}$ are denoted by $\hat{v}_i$ and $\hat{v}_{i'}$. The triangular region R_{+++} with vertices $\hat{v}_1$, $\hat{v}_2$, $\hat{v}_3$ is defined by relations $\hat{e}_1 \cdot \hat{v} \geq 0$, $\hat{e}_2 \cdot \hat{v} \geq 0$, and $\hat{e}_3 \cdot \hat{v} \geq 0$.

Finally, with the help of the above shown result and exploiting the symmetries of the function $F(\hat{v})$ we show that the desired minimum value is $\min[F(v_{+++})]$. For this, let us consider a repartitioning of the set of points on the Poincaré sphere defined by $R_{\xi_1\xi_2\xi_3}^{\chi_1\chi_2\chi_3} = \{\hat{v} : \xi_i(\chi_i\hat{e}_i) \cdot \hat{v} \geq 0,\ \forall i \in \{1,2,3\}\}$ for some fixed $\chi_1, \chi_2, \chi_3 \in \{+,-\}$ (Observe that there are 8 such ways of partitioning and in the new notation the partition represented by $R_{\xi_1\xi_2\xi_3} \equiv R_{\xi_1\xi_2\xi_3}^{+++}$). Then, we note the following two features for above type of repartition: (i) The relevant function $F^{\chi_1\chi_2\chi_3}(\hat{v}) = |(\chi_1\hat{e}_1) \cdot \hat{v}| + |(\chi_2\hat{e}_2) \cdot \hat{v}| + |(\chi_3\hat{e}_3) \cdot \hat{v}|$ remains invariant for any choice of χ_i's $\in \{+,-\}$ and (ii) $R_{\chi_1\chi_2\chi_3} \equiv R_{\chi_1\chi_2\chi_3}^{+++} \cong R_{+++}^{\chi_1\chi_2\chi_3}$. Since earlier we have shown that for any partition the minimum value of $F^*(\hat{v}_{+++})$ in the corresponding region R_{+++}^* can only be attained at one or more of its vertices, applying the property (i) and (ii) we can now conclude that minimum value of $F(\hat{v})$ in a region

$R_{\xi_1\xi_2\xi_3}$ for any ξ_1, ξ_2, $\xi_3 \in \{+,-\}$ is attained at one or more of its vertices. Thus, the required global minimum value L is attained at some point(s) belonging to the set $\{\pm\hat{v}_1, \pm\hat{v}_2, \pm\hat{v}_3\}$. Now, the use of the symmetry $F(-\hat{v}) = F(\hat{v})$ gives $L = \min\{F(\hat{v}_1), F(\hat{v}_2), F(\hat{v}_3)\} = \frac{|\hat{e}_1\cdot(\hat{e}_2\times\hat{e}_3)|}{\max\{|\hat{e}_1\times\hat{e}_2|,|\hat{e}_2\times\hat{e}_3|,|\hat{e}_3\times\hat{e}_1|\}}$.□

3.3.4 Two testable forms of LNR inequalities

By applying the above proven theorem, together with the use of the normalization relation $\int\int D(\hat{u},\hat{v})d\hat{u}d\hat{v} = 1$, to the inequalities (3.4) and (3.5), we obtain respectively the following two forms of experimentally testable LNR inequalities

$$\frac{1}{3}\sum_i |\langle A_iB_i\rangle + \langle A_iB_i'\rangle| \leq 2 - \frac{2}{3}\sin(\frac{\beta_*}{2}) \times L_n \tag{3.6}$$

$$\frac{1}{3}\sum_i |\langle A_iB_i\rangle - \langle A_iB_{i\oplus 1}\rangle| \leq 2 - \frac{2}{3}\cos(\frac{\delta^*}{2}) \times L_m \tag{3.7}$$

For these two experimentally testable forms of the LNR inequalities, respective lower bounds $L_{n,m}$ for functions $F_{n,m}(\hat{v}) = |\hat{e}_1\cdot\hat{v}| + |\hat{e}_2\cdot\hat{v}| + |\hat{e}_3\cdot\hat{v}|$ with $\hat{e}_i$'s corresponding to $\hat{n}_i$'s or $\hat{m}_i$'s for F_n or F_m respectively, can be equivalently expressed by the following convenient expression (see Appendix B for a proof)

$$L_{n,m}(\alpha_{12},\alpha_{23},\alpha_{31}) = \frac{\left(1-\sum\limits_{\substack{1\leq i\leq 3,\\ j=i\oplus 1}}\cos^2\alpha_{ij} + 2\prod\limits_{\substack{1\leq i\leq 3,\\ j=i\oplus 1}}\cos\alpha_{ij}\right)^{\frac{1}{2}}}{\max\{\sin\alpha_{12},\sin\alpha_{23},\sin\alpha_{31}\}} \tag{3.8}$$

where $\alpha_{ij} \in (0,\pi)$ denotes the angle between a pair of vectors $\{\hat{e}_i,\hat{e}_j\}$ for $i,j \in \{1,2,3\}$.

3.3.5 Discussion of details in the proof of the minimum value of $F(\hat{v}_{+++})$ in the triangular region R_{+++}

Recall that an expression for the function $f(\hat{v}) = \hat{e}_1\cdot\hat{v} + \hat{e}_2\cdot\hat{v} + \hat{e}_3\cdot\hat{v}$ defined over the points of the Poincaré sphere in terms of (θ,ϕ) coordinates ($0 \leq \theta \leq \pi$ and $-\pi < \phi \leq \pi$), where $p = 1 + b_1 + c_1$, $q = b_2 + c_2$ and $r = c_3$, is

$$f(\theta,\phi) = (p\cos\phi + q\sin\phi)\sin\theta + r\cos\theta \tag{3.9}$$

Now, analyzing this expression, we in detail show the following

3.3.6 The minimum value of $F(\hat{v}_{+++})$ is not attained at any interior point of the region R_{+++}

At the stationary points of $f(\theta,\phi)$ we have

$$\begin{aligned} \partial_\phi f &= \sin\theta(-p\sin\phi + q\cos\phi) = 0 \qquad (3.10) \\ \partial_\theta f &= \cos\theta(p\cos\phi + q\sin\phi) - r\sin\theta = 0 \qquad (3.11) \end{aligned}$$

Therefore, at some stationary point, say (θ_0, ϕ_0), in the interior of R_{+++}, since $\sin\theta_0 \neq 0$, the following equations must be satisfied

$$\begin{aligned} -p\sin\phi_0 + q\cos\phi_0 &= 0 \qquad (3.12) \\ \cos\theta_0(p\cos\phi_0 + q\sin\phi_0) &= r\sin\theta_0 \qquad (3.13) \end{aligned}$$

Then, at (θ_0, ϕ_0) we obtain

$$\begin{aligned} [(\partial_{\phi\phi} f)(\partial_{\theta\theta} f) - \partial_{\phi\theta} f]_{(\theta_0,\phi_0)} &= \{(p\cos\phi_0 + q\sin\phi_0)^2 + r^2\}\sin^2\theta_0 > 0 \quad (\because r = c_3 \neq 0) \\ (\partial_{\theta\theta} f)_{(\theta_0,\phi_0)} &= -f(\theta_0,\phi_0) < 0 \end{aligned}$$

Thus, given a stationary point (θ_0, ϕ_0), it must be a point of maximum. Consequently, the minimum cannot lie in the interior of the region R_{+++}.

3.3.7 The minimum value of $F(\hat{v}_{+++})$ is not attained at any interior point on the sides of the triangular region R_{+++}

Let us first consider the side of the triangle R_{+++} on the corresponding great circle C_1. Note that the interior points of this side is defined by

$$\hat{e}_1 \cdot \hat{v} = 0 \quad \Rightarrow \quad \sin\theta\cos\phi = 0 \tag{3.14}$$

$$\hat{e}_2 \cdot \hat{v} > 0 \quad \Rightarrow \quad (b_1\cos\phi + b_2\sin\phi)\sin\theta > 0 \tag{3.15}$$

$$\hat{e}_3 \cdot \hat{v} > 0 \quad \Rightarrow \quad (c_1\cos\phi + c_2\sin\phi)\sin\theta + c_3\cos\theta > 0 \tag{3.16}$$

Now, the inequality (3.15)$\Rightarrow \sin\theta \neq 0$. Therefore, from Eq.(3.14) one can conclude that $\cos\phi = 0 \Rightarrow \phi = \frac{\pi}{2}$ or $-\frac{\pi}{2}$. Next, substituting $\cos\phi = 0$ in the inequality (3.15) one gets $b_2\sin\theta\sin\phi > 0$. Then, it follows that $\phi = \mathrm{sgn}(b_2)\frac{\pi}{2}$ (since $\sin\theta > 0$) thereby reducing the expression for the function f at the interior points of this side of the triangle R_{+++} to

$$f(\theta, \mathrm{sgn}(b_2)\frac{\pi}{2}) = f(\theta) = \mathrm{sgn}(b_2)q\sin\theta + r\cos\theta \tag{3.17}$$

Therefore, one gets $\frac{d^2f}{d\theta^2} = -f(\theta) < 0$ which implies that a stationary point in the interior of this side cannot be a point of minimum. Thus, the minimum value of f can be attained only at some end point(s) of this side.

For the remaining two sides of the triangle R_{+++} on the respective great circles C_2 and C_3, similar analyses can be done by choosing the relevant convenient co-ordinate axes. For the side lying on C_2 (C_3), the convenient choice is the X-axis to be along $\hat{e}_2$ ($\hat{e}_3$) and the $X-Y$ plane containing $\hat{e}_3$ ($\hat{e}_1$). Then, it is again found that the minimum value of f on the remaining two sides can only occur at some end point(s) of these sides.

3.4 Maximum violation of the LNR inequality (3.6)

Let us express the LNR inequality (3.6) given in the main text in the following form

$$S_{AB}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b'}_1, \hat{b'}_2, \hat{b'}_3) \equiv \frac{1}{3}\sum_i |\langle A_i B_i\rangle + \langle A_i B'_i\rangle| + \frac{2}{3}\sin(\frac{\beta_*}{2}) \times L_n(\hat{n}_1, \hat{n}_2, \hat{n}_3) - 2 \leq 0 \tag{3.18}$$

where angles between Bob's settings $\hat{b}_i$ and $\hat{b'}_i$, $i \in \{1, 2, 3\}$, are β_i's with $\beta_* = \min\{|\beta_1|, |\beta_2|, |\beta_3|\}$ and $\hat{n}_i$'s are unit vectors along the directions of $\hat{b}_i - \hat{b'}_i$. S_{AB} represents a real valued function of settings of Alice and Bob, while the settings for which the quantum mechanically calculated value $S_{AB} > 0$ would imply a violation of the LNR inequality given by the preceding inequality (3.18).

For a pure singlet state, since the QM correlation function $\langle AB\rangle = -\hat{a} \cdot \hat{b}$, the expression (3.18) reduces to

$$S_{AB}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b'}_1, \hat{b'}_2, \hat{b'}_3) \equiv \frac{1}{3}\sum_i |\hat{a}_i \cdot (\hat{b}_i + \hat{b'}_i)| + \frac{2}{3}\sin(\frac{\beta_*}{2}) \times L_n(\hat{n}_1, \hat{n}_2, \hat{n}_3) - 2 \leq 0 \tag{3.19}$$

From the inequality (3.19) one can see that as the first step towards maximizing the function S_{AB}, Alice's settings $\hat{a}_i$'s should lie along the directions of $\hat{b}_i + \hat{b'}_i$. Then, for maximizing S_{AB}, it is sufficient to maximize the function

$$S_B \equiv \frac{1}{3}\sum_i |2\cos\frac{\beta_i}{2}| + \frac{2}{3}\sin(\frac{\beta_*}{2}) \times L_n(\hat{n}_1, \hat{n}_2, \hat{n}_3) - 2 \tag{3.20}$$

involving only Bob's settings.

Next, note that as the function $L_n(\hat{n}_1, \hat{n}_2, \hat{n}_3)$ appearing in the expression (3.20) does not depend on the values of angles β_i's, one needs to maximize L_n over the directions $\hat{n}_i$'s. Now, we proceed to show that $\max(L_n) = 1$ when the set $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ is orthonormal. For this, let us first show that the expression for L_n given in a form proved in the theorem of the main text, i.e,

$$L_n = \frac{|\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3)|}{\max\{|\hat{n}_1 \times \hat{n}_2|, |\hat{n}_2 \times \hat{n}_3|, |\hat{n}_3 \times \hat{n}_1|\}} \tag{3.21}$$

is equivalent to the expression

$$L_n(\alpha_{12}, \alpha_{23}, \alpha_{31}) = \frac{\left(1 - \sum\limits_{\substack{1 \leq i \leq 3, \\ j = i \oplus 1}} \cos^2 \alpha_{ij} + 2 \prod\limits_{\substack{1 \leq i \leq 3, \\ j = i \oplus 1}} \cos \alpha_{ij}\right)^{\frac{1}{2}}}{\max\{\sin \alpha_{12}, \sin \alpha_{23}, \sin \alpha_{31}\}} \tag{3.22}$$

which is Eq.(3.7) of the main text. The equivalence between (3.21) and (3.22) can be seen as follows. Say, $0 < \alpha_{12} < \pi$ is the angle between unit vectors $\hat{n}_1$ and $\hat{n}_2$ (α_{23} and α_{31} are defined similarly). For convenience, we use the notation α_{ij} for the angle between unit vectors $\hat{n}_i$ and $\hat{n}_j$, where $1 \leq i \leq 3$ and $j = i \oplus 1$ (here $\oplus$ denotes addition modulo 3). Then, the denominators of the expressions on the right hand side of the equations (3.21) and (3.22) are same since $|\hat{n}_i \times \hat{n}_j| = |\sin \alpha_{ij}| = \sin \alpha_{ij}$. Next, we show that the numerators of the two expressions are also equal. For this, recall that $\hat{n}_1 = (1, 0, 0)$, $\hat{n}_2 = (b_1, b_2, 0)$, $\hat{n}_3 = (c_1, c_2, c_3)$ (without any loss of generality). Then, we find that $|\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3)| = |b_2 c_3|$. Along with this, we also have the following relations $\hat{n}_1 \cdot \hat{n}_2 = b_1 = \cos \alpha_{12}$, $\hat{n}_2 \cdot \hat{n}_3 = b_1 c_1 + b_2 c_2 = \cos \alpha_{23}$, $\hat{n}_3 \cdot \hat{n}_1 = c_1 = \cos \alpha_{31}$ and the normalization relations $b_1^2 + b_2^2 = 1$, $c_1^2 + c_2^2 + c_3^2 = 1$. By using these relations we find that $|\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3)| = |b_2 c_3| = (1 - \cos^2 \alpha_{12} - \cos^2 \alpha_{23} - \cos^2 \alpha_{31} + 2 \cos \alpha_{12} \cos \alpha_{23} \cos \alpha_{31})^{\frac{1}{2}}$.

Now, we show that $L_n \leq 1$ for which note that

$$
\begin{aligned}
&- \cos^2\alpha_{23} - \cos^2\alpha_{31} + 2\cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31} \\
&= -\{(\cos\alpha_{23} - \cos\alpha_{31})^2 + 2\cos\alpha_{23}\cos\alpha_{31}(1-\cos\alpha_{12})\} \\
&= -\{(\cos\alpha_{23} + \cos\alpha_{31})^2 - 2\cos\alpha_{23}\cos\alpha_{31}(1+\cos\alpha_{12})\} \leq 0 \\
&\Rightarrow 1 - \cos^2\alpha_{12} - \cos^2\alpha_{23} - \cos^2\alpha_{31} + 2\cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31} \leq \sin^2\alpha_{12} \\
&\Rightarrow \frac{(1 - \cos^2\alpha_{12} - \cos^2\alpha_{23} - \cos^2\alpha_{31} + 2\cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{31})^{1/2}}{\sin\alpha_{12}} \leq 1 \quad (3.23)
\end{aligned}
$$

Similar other inequalities of the form (3.23) can also be obtained in which $\sin\alpha_{12}$ in the denominator of the left hand side of (3.23) is replaced by $\sin\alpha_{23}$ or $\sin\alpha_{31}$. Then, combining three such inequalities it can be easily seen that $L_n \leq 1$ where the maximum value of $L_n = 1$ occurs, for example, when $\alpha_{12} = \alpha_{23} = \alpha_{31} = \frac{\pi}{2}$.

Therefore, for maximizing S_B given by Eq.(3.20) now it sufficient to maximize the following expression

$$S_{\beta_1,\beta_2,\beta_3} \equiv \frac{2}{3}\sum_i |\cos\frac{\beta_i}{2}| + \frac{2}{3}\sin(\frac{\beta_*}{2}) - 2. \quad (3.24)$$

Now, note that from Eq.(3.24) it can be seen that for any given $\beta_1, \beta_2, \beta_3$ the value of $S_{\beta_1,\beta_2,\beta_3}$ is bounded by $S_{\beta_*,\beta_*,\beta_*}$

$$S_{\beta_1,\beta_2,\beta_3} \leq S_{\beta_*,\beta_*,\beta_*} = 2\cos\frac{\beta_*}{2} + \frac{2}{3}\sin\frac{\beta_*}{2} - 2 \quad (3.25)$$

Then, we obtain that $\max(S_{\beta_1,\beta_2,\beta_3}) \approx 0.108$ which occurs at $|\beta_1| = |\beta_2| = |\beta_3| = \beta_* \approx 36.9^o$. □

Therefore, to summarize, the above derivation implies that the settings of Alice and Bob for maximum QM violation of the LNR inequality (3.6) in the main text are as follows: (i) Alice's settings $\hat{a}_i$'s are in the directions of $\hat{b}_i - \hat{b'}_i$, and (ii) Bob's settings are such that $\hat{n}_i$'s are mutually orthogonal and $|\beta_1| = |\beta_2| = |\beta_3| \approx 36.9^o$.

3.5 Salient features of the LNR inequalities (3.6) and (3.7)

First, let us focus on the LNR inequality (3.6). Given the way this inequality has been derived by us, Alice and Bob are both free to arbitrarily choose their measurement settings, and given their choices, the *LNR bound* (the right hand side) on the combination of correlation functions (the left hand side) can be calculated with help of the formula (3.8).

Note that the inequality derived and experimentally tested by Branciard *et al.* [Branciard et al., 2008] is a special case of the inequality (3.6) by assuming a specific geometrical constraint that requires $\hat{n}_i$'s to be mutually orthogonal and $|\beta_1| = |\beta_2| = |\beta_3|$. For a photon pair prepared in a pure singlet state, one can show that settings for observing the maximum violation of the inequality (3.6) are in which Bob's choices are such that the three directions $\hat{n}_i$'s of $(\hat{b}_i - \hat{b}'_i)$ are orthogonal where $|\beta_1| = |\beta_2| = |\beta_3| \approx 36.9^o$, with $\hat{a}_i$'s chosen by Alice to be along the directions of $\hat{b}_i + \hat{b}'_i$ (see Appendix B for a proof). While the magnitude of the maximum possible violation of the inequality (3.6) corresponds to the threshold visibility of 94.3%.

Now, considering the LNR inequality (3.7), we note that for a pure singlet state, violations of (3.7) by the QM predictions can be shown, for example, by taking a class of symmetric configurations in which the angles between Bob's measurement settings satisfy $\delta_1 = \delta_2 = \delta_3 = \delta$ and Alice's measurement settings $\hat{a}_i$'s are along the directions $\hat{b}_i - \hat{b}_{i\oplus 1}$ [see Fig.3.2(a)]. For such configurations, QM violations of the inequality (3.7) occur within the domain $\delta \in [106.8^o, 116.5^o]$ where the maximum violation is obtained for $\delta \approx 112.63^o$ [see Fig.3.2(b)]. In this case, the maximum value of the ratio of the right hand bound and the corresponding QM value of the left hand side is given by 0.9836, meaning that the threshold visibility in the relevant experiment that is required to show the QM violation of the LNR inequality (3.7) is 98.36%. Now, note that already the visibility above 98.4% was achieved in an experiment by Paterek *et al.* [Paterek et al., 2007] where a LNR inequality involving 7 measurement settings for Bob and 3 for Alice was tested. A similar work [Branciard et al., 2007] was also reported using a family of LNR inequalities involving $2N$ and $4N$ $(N \geq 2)$ number of settings for Alice and Bob respectively.

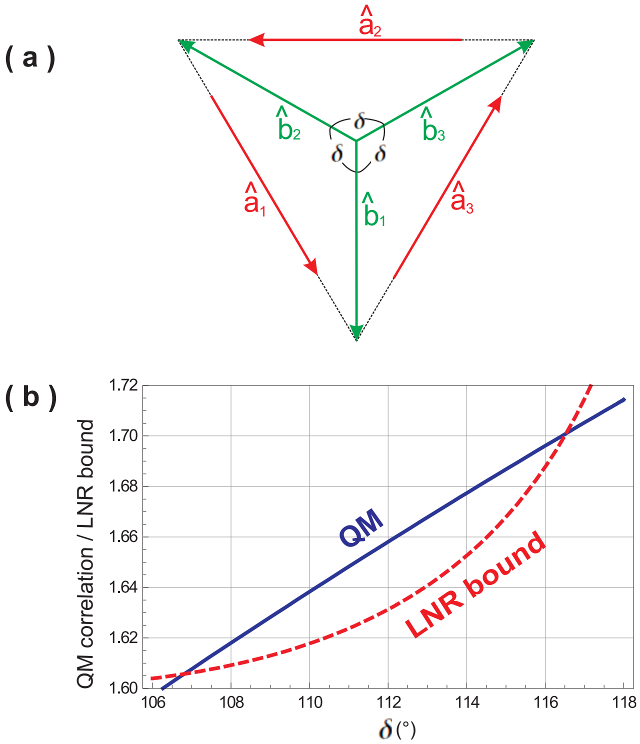

FIGURE 3.2: **(a)** As an illustrative example, a class of symmetric configurations for observing QM violation of LNR inequality (3.7) for a pure singlet state is shown. For Bob's measurement settings $\hat{b}_i$'s, where $i \in \{1, 2, 3\}$, the angle between any pair of settings is δ. Alice's measurement settings $\hat{a}_i$'s are along the directions $\hat{b}_i - \hat{b}_{i\oplus 1}$. **(b)** The dotted line shows LNR upper bounds and the bold line shows corresponding QM values of the left hand side of the LNR inequality (3.7) as δ is varied. A range of QM violations of the inequality (3.7) is obtained for $\delta \in [106.8^o, 116.5^o]$, with the maximum violation occurring at $\delta \approx 112.63^o$.

Thus, an additional significance of our LNR inequality (3.7) lies in involving *lesser* number, only 3 measurement settings for Bob and 3 for Alice such that the threshold visibility required to test this inequality is experimentally realizable. An open problem nevertheless remains whether a testable incompatibility can be shown between QM and the LNR model by using even lesser number of settings in either wing without, of course, taking recourse to any additional assumption like that of rotational invariance of correlation functions.

3.6 Concluding remarks.

A generic property of the LNR inequalities is that while the left hand side of any such inequality involves experimentally measurable quantities (correlation functions), the LNR bound (the right hand side) of such an inequality, *unlike* any Bell-type inequality, is not just a number fixed by the general assumptions used in the relevant derivation; instead, it depends on the *choice* of the geometrical configuration of measurement settings. Thus, for different configurations of settings, say, S_1, S_2, S_3... there are corresponding LNR bounds B_1, B_2, B_3.... Therefore, if due to experimental imprecision, the actual settings deviate from the required configuration within a certain domain that can be estimated by the experimenter, there will be a corresponding range of LNR bounds. As a consequence, the experimental violation of any relevant LNR inequality can be unambiguously concluded only if the supremum of such a range of LNR bounds is violated. However, an estimation of such a range of LNR bounds by taking into account all possible imprecisions that may occur in realizing the required configuration of settings is severely restricted for any of the LNR inequalities derived earlier [Branciard et al., 2008, 2007; Groblacher et al., 2007; Leggett, 2003; Paterek et al., 2007]. This is essentially because the validity of any such inequality is in itself contingent upon certain geometrical constraints being strictly satisfied by the measurement settings. Herein lies the central significance of our LNR inequalities (3.6) and (3.7) in enabling a more logically conclusive test of the LNR model vis-a-vis QM than that has been hitherto possible—this is because the forms of the LNR inequalities derived here are *free* from any constraint on the spatial alignments of the relevant measurement settings.

Chapter 4

A testable prediction of the no-signaling condition

The significance of no-signaling condition is fundamental in the study of quantum correlations. In the present chapter predictive power of the no-signaling condition (NSC) is demonstrated in a testable situation involving a non-ideal SternGerlach (SG) device in one of the two wings of the EPR-Bohm entangled pairs. In this wing, for two types of measurement in the other wing, we consider the spin state of a selected set of particles that are confined to a particular half of the plane while emerging from the SG magnetic field region. Due to non-idealness of the SG setup, this spin state will have superposing components involving a relative phase for which a testable quantitative constraint is obtained by invoking NSC, thereby providing a means for precision testing of this fundamentally significant principle [Home, Rai and Majumdar, 2013].

4.1 Introduction

A key condition underpinning the 'peaceful coexistence' [Shimony, 1984] between quantum mechanics and special relativity is the no-signaling condition (NSC) which prohibits the use of quantum nonlocality for sending information in a way that can lead to causality paradoxes. While the compatibility of NSC with quantum mechanics has been extensively analyzed with an increasing generality [Bohm, 1951; Brown and Timpson, 2006; Bussey, 1982; Clifton and Redhead, 1988; Datta et al., 1987, 1988; Finkelstein and Stapp, 1987; Ghirardi et al., 1980, 1988; Greenberger, 1998; Hall, 1987; Jordan, 1983; Scherer and Busch, 1993; Squires, 1988], an interesting line of study was initiated by Gisin showing the use of NSC as a tool to either find the limits of quantum mechanics, like constraining any conceivable nonlinear modification of the Schrödinger equation [Gisin, 1990; Gisin and Rigo, 1995], or to obtain specific bounds on quantum operations, like deriving bound on the fidelity of quantum cloning machines [Ghosh et al., 1999; Gisin, 1998]. Subsequently, NSC has been applied, for example, to obtain a tight bound on the optimal unambiguous discrimination between two nonorthogonal states [Barnett and Andersson, 2002]. A number of other studies [Feng et al., 2002; Home and Pan, 2009; Hwang, 2005; Pati and Braunstein, 2003] too have highlighted the role of NSC in limiting various quantum operations, while some features of the quantum formalism have also been derived from NSC [Svetlichny, 1998]. Interestingly, NSC has been invoked in the context of quantum cryptography as well; viz. to formulate quantum key distribution protocols which are secure against attack by any eavesdropper who is limited only by the impossibility of superluminal signaling [Acin et al., 2006; Barrett, Hardy and Kent, 2005; Kent, 1999] - this line of study is motivated by the notion [Barrett, Hardy and Kent, 2005]: "... quantum theory could fail without violating standard relativistic causality, and vice versa".

Complementing the above mentioned studies, in this chapter our goal is to derive from NSC, a *testable quantitative relation* whose empirical scrutiny would enable a dedicated precision testing of NSC in the same spirit as Born's rule has recently been tested to provide bounds on its accuracy [Sinha et al., 2010]. Such precision testing using a quantitative relation is useful in having the potential of being able to detect very small deviations that may be missed otherwise, and can provide an empirical upper bound on possible violation of NSC. To the best of our knowledge, a dedicated precision testing of NSC remains unexplored. To this end, the example analyzed here provides a constraint relation concerning the spin state of a selected set of spin-1/2 particles that are confined to a particular half of the plane while emerging from a *non-ideal* Stern-Gerlach(SG) setup. Here the particles in the other half of the plane are taken to be blocked/detected. The spin state, thus, filtered out comprises of two superposing spin components with a relative phase, the respective probability amplitudes being calculable from the solutions of the Schrödinger equation for the non-ideal SG setup. In order to evaluate the relative phase, a fully unitary treatment (albeit non-trivial) is required that would be based on appropriately modeling the post-selection process used for filtering out the spin state. The central point made in our work is that irrespective of the specifics of such a treatment, the value of the relative phase occurring in the post-selected spin state has to satisfy a testable constraint that is derivable from NSC. To show this, we proceed as follows.

4.2 Formulation of the example

For the EPR-Bohm (EPRB) entangled pairs of spin-1/2 particles in spin singlets, the corresponding wave function is given by

$$|\Psi\rangle = (1/\sqrt{2})|\psi_0\rangle_1|\psi_0\rangle_2(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2) \tag{4.1}$$

where the spatial parts $|\psi_0\rangle_1$ and $|\psi_0\rangle_2$ (assumed to be, say, represented by Gaussian wave packets) pertain to the particles 1 and 2 respectively, and the spin part represents the singlet state. Next, let a *non-ideal* SG setup be placed by Bob in one of the two wings of the EPRB pairs, say, in the wing 2 for the particles moving along the +y-axis (Fig(4.1)). After passing through an inhomogeneous magnetic field in the SG setup oriented along, say, the +z-axis, the time-evolved total wave function ($\Psi(\mathbf{x}, t)$) of any particle, in general, would involve the spatial wave functions $\psi_+(\mathbf{x}, t)$ and $\psi_-(\mathbf{x}, t)$ that are coupled with the spin-up ($|\uparrow\rangle_z$) and spin-down ($|\downarrow\rangle_z$) states respectively, given by

$$\Psi(\mathbf{x}, t) = \psi_+(\mathbf{x}, t)|\uparrow\rangle_z + \psi_-(\mathbf{x}, t)|\downarrow\rangle_z \tag{4.2}$$

Here $|\psi_+(\mathbf{x}, t)|^2$ ($|\psi_-(\mathbf{x}, t)|^2$) determines the probability of finding particles with the spin $|\uparrow\rangle_z(|\downarrow\rangle_z)$ in the upper and lower halves of the y-z plane. In the *ideal* case, $\langle\psi_+(\mathbf{x}, t)|\psi_-(\mathbf{x}, t)\rangle = 0$ with the probability of finding particles corresponding to the spin $|\uparrow\rangle_z(|\downarrow\rangle_z)$ in the lower (upper) y-z plane being negligibly small. The explicit forms of $\psi_+(\mathbf{x}, t)$ and $\psi_-(\mathbf{x}, t)$ in the general case of a *non-ideal* SG setup are available in the relevant literature (see, for example, [Home and Pan, 2009; Pan and Matzkin, 2012] and the references cited therein), where $\langle\psi_+(\mathbf{x}, t)|\psi_-(\mathbf{x}, t)\rangle \neq 0$. Note that idealness (non-idealness) of a SG setup depends on appropriately choosing the strength of the magnetic field within the SG setup, commensurate with the energy and width of the incoming wave packet.

Now, suppose two types of measurements (the types denoted by A and B respectively) are performed by, say, Alice in the wing 1 of the EPRB pairs of singlet states given by Eq.(4.1). The type A corresponds to a set of measurements of, say, the x-component of spin, while B corresponds to a set of measurements of the spin component along the z-axis which is the same as the direction of the inhomogeneous magnetic field in the non-ideal SG setup in Bob's wing. Therefore, in the cases A and B respectively, effectively mixtures of $+x$ and $-x$ spin components (with equal weighting)and that of $+z$ and $-z$ spin components (with equal weighting) are produced in Bob's wing. This can be seen from Eq. (4.1) by either invoking the collapse postulate or by taking into account the feature that the measuring apparatus states in Alice's wing are mutually orthogonal, corresponding to the distinct outcomes of the measurement of the spin. Thus, subsequently in Bob's wing, say, in the case A, for the purpose of our argument, we can consider the spin states $|\rightarrow\rangle_x$ and $|\leftarrow\rangle_x$ as pure state constituents of a mixed state. Similar is what happens in the case B where the resulting mixed state comprises of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ states. The particles in Bob's wing are then passed through a non-ideal SG setup, followed by post-selection confined to upper half of the y-z plane. Such post-selected particles are subjected to the measurement of an arbitrary component of spin (say, σ_θ) using an ideal SG setup with its inhomogeneous magnetic field oriented along a direction making an angle θ with the $+z$-axis in the x-z plane.

Our treatment hereafter will be focused on the probability of obtaining a particular outcome (say, +1) of the measurement of σ_θ on the post-selected particles in the upper half of the y-z plane in Bob's wing, calculating it in the two cases A and B corresponding to different types of measurement performed in Alice's wing. Such an observable probability needs to be the *same* in both these cases in order to ensure no-signaling between the two wings of the EPRB pairs. This requirement leads to a constraint on the relative phase occurring in the spin state of the post-selected particles on which the final spin measurement is considered.

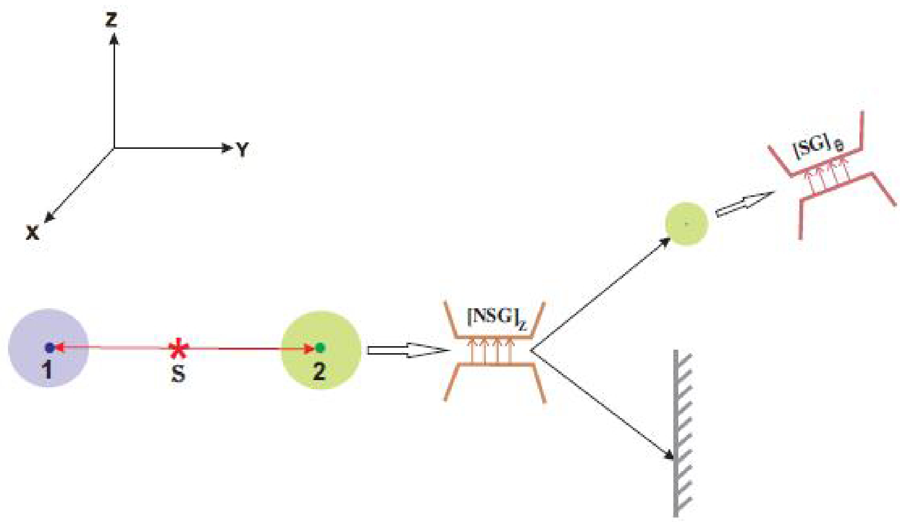

FIGURE 4.1: Spin-1/2 particles 1 and 2 are members of the EPR entangled pairs emitted from a source **S** moving along y-axis in the opposite directions. Particles 2 pass through $[NSG]_z$, a nonideal Stern-Gerlach device with its inhomogeneous magnetic field oriented along z-axis. After emerging from the $[NSG]_z$ setup, particles confined to the lower half of the y-z plane are absorbed/detected, while particles in the upper half of the y-z plane are subjected to the measurement of an arbitrary spin component, say, σ_θ by using an ideal Stern-Gerlach setup $[SG]_\theta$ where the inhomogeneous magnetic field is along a direction in the x-z plane making an angle θ with the z-axis.

4.3 Derivation of a testable consequence of the no-signaling condition

We begin by noting that post-selecting particles emerging in the upper half of the SG setup in question is ensured by absorbing/detecting particles in the lower half. Thus, this can be viewed, in principle, as a kind of approximate measurement of position of the particles lying within the lower half, for which the spatial states of the lower half particles are *coupled* with the states of the absorber/detector. Hence, this post-selection process *decoheres* the entanglement (Eq.(4.2)) between the spatial and spin states of the particles, and results in a product wave function of the spatial and spin parts for all the upper (lower) half particles that emerge from the non-ideal SG setup with its inhomogeneous magnetic field oriented along the z-axis. The spin state of any such particle will be found to be either $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$, but since it is not known a priori which particle is to be found in which of these states, the spin part of the state is to be regarded as a superposition of

$|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ spin states. The post-selection procedure that is considered may, therefore, be viewed as a state preparation for further measurements.

In what follows, the cases A and B are analyzed separately.

Case A: In this case, as mentioned earlier, due to measurements in Alice's wing, effectively a mixed state comprising of $|\rightarrow\rangle_x$ and $|\leftarrow\rangle_x$ spin components (with equal weighting) results in Bob's wing. Let us first focus on particles with the $|\rightarrow\rangle_x$ spin component in Bob's wing. Given that such particles are passed through a *non-ideal* SG setup involving an inhomogeneous magnetic field along the z-axis, the spin component of any emerging particle post-selected in the upper half of the y-z plane will be found to be either $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ with the respective probability determined by the overlap of $|\psi_+(\mathbf{x},t)|^2$ or $|\psi_-(\mathbf{x},t)|^2$ in the spatial region of the upper y-z plane, while, as mentioned earlier, one does not know a priori which particle will be found in which spin state. Since our subsequent argument is concerned solely with calculating the observational results pertaining to the spin state of the post-selected particles, we can henceforth ignore the spatial state. The normalized spin state of any particle belonging to a post-selected ensemble in the upper half will be a superposition of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ spin states with a relative phase denoted by ϕ_{+x} (the subscript +x is used for the $|\rightarrow\rangle_x$ spins passed through the non-ideal SG setup), given by

$$|\chi\rangle = \sqrt{1-E_s}|\uparrow\rangle_z + exp(i\phi_{+x})\sqrt{E_s}|\downarrow\rangle_z \tag{4.3}$$

where E_s denotes the time-saturated value of the quantity E(t) given by

$$E(t) = \int_{x\to-\infty}^{+\infty}\int_{y\to-\infty}^{+\infty}\int_{z=0}^{+\infty} |\psi_-(\mathbf{x},t)|^2 dxdydz \tag{4.4a}$$

$$= \int_{x\to-\infty}^{+\infty}\int_{y\to-\infty}^{+\infty}\int_{z\to-\infty}^{+0} |\psi_+(\mathbf{x},t)|^2 dxdydz \tag{4.4b}$$

A measure of the non-idealness of the SG setup is provided by the quantity E(t) which determines the probability of finding $|\downarrow\rangle_z$ ($|\uparrow\rangle_z$) particles in the upper (lower) y-z plane at time t. The parameter E(t) varies with time as the wave packets $|\psi_+(\mathbf{x},t)|^2$ and $|\psi_-(\mathbf{x},t)|^2$ freely propagate in opposite directions after emerging from the SG setup. E(t) finally attains a time-independent saturated value denoted by E_s, with the saturation time depending upon choices of the relevant parameters (see, for example, Home et al. [Home and Pan, 2009]). For an *ideal* SG setup, $E_s = 0$.

Here it needs to be pointed out that , although in our argument we are considering, in principle, the post-selection to be done on the whole upper half of the y-z plane, in practice, it would suffice if a representative set of a suitably large number of particles are post-selected that are confined to a finite region of the upper y-z plane such that there is sufficient non-vanishing probability for finding either $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ spin state in the half under consideration. Note that due to Gaussian nature of the wave packets $|\psi_+|^2$ and $|\psi_-|^2$, the probability for finding particles with either $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ spin state in the region corresponding to large z would be negligibly small. Of course, if one is required to take into account the finiteness of the region chosen for post-selection, the range of integration over the z-coordinate in Eqs. (4.4a), (4.4b) will vary depending upon the actual region of the upper y-z plane chosen for post-selection, thereby affecting the value of the parameter E_s in Eq. (4.3). But, it is important to stress that, whatever be the value of E_s, our subsequent argument goes through and the constraint relation concerning the relative phase we now proceed to derive would remain unaffected.

Now, given an input spin state $|\rightarrow\rangle_x = (1/\sqrt{2})(|\uparrow\rangle_z + |\downarrow\rangle_z)$ passing through a non-ideal SG setup, using Eq. (4.1), the probability for finding particles in the upper half of the y-z plane with the spin component $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ is given by $(1/2)(1-E_s)$ or $(1/2)E_s$ respectively. Then, the total probability of such particles being post-selected in the upper half of the y-z plane is given by $p_{upper} = (1/2)(1-E_s) + (1/2)E_s = 1/2$. For these post-selected particles with the normalized spin state $|\chi\rangle$ given by Eq. (4.3), using the expressions for the spin states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ in terms of the eigenstates of σ_θ with eigenvalues +1 and -1 respectively, the probability of obtaining a particular outcome, say +1, for the measurement of an arbitrary spin component σ_θ is given by

$$p_A^+ = (1/2)\left[1 + (1-2E_s)cos\theta + 2\sqrt{E_s(1-E_s)}sin\theta cos\phi_{+x}\right] \tag{4.5}$$

Next, note that due to measurements of the x-component of spin in Alice's wing, particles with the $|\rightarrow\rangle_x$ spin component are produced in Bob's wing with the probability (1/2). Hence, the total probability P_A^+ that such particles after passing through a non-ideal SG setup get post-selected in the upper half of the y-z plane and yield the outcome +1 for the measurement of σ_θ is given by $P_A^+ = (1/2)p_{upper}p_A^+$ where p_A^+ is given by Eq. (4.5) and $p_{upper} = 1/2$ as explained earlier, whence

$$P_A^+ = (1/8)\left[1 + (1-2E_s)cos\theta + 2\sqrt{E_s(1-E_s)}sin\theta cos\phi_{+x}\right] \qquad (4.6)$$

The other situation in the case A that occurs with the probability (1/2) corresponds to particles with the $|\leftarrow\rangle_x$ spin component produced in Bob's wing due to measurements of the x-component of spin in Alice's wing. The probability P_A^- of such particles to be post-selected in the upper half of the y-z plane of the *non-ideal* SG setup and yield the outcome +1 for the measurement of σ_θ is given by (obtained in a way similar to the derivation of the expression (4.6) for the quantity P_A^+)

$$P_A^- = (1/8)\left[1 + (1-2E_s)cos\theta + 2\sqrt{E_s(1-E_s)}sin\theta cos\phi_{-x}\right] \qquad (4.7)$$

where ϕ_{-x} is the relative phase occurring in the spin state (of the form Eq. (4.3)) of such post-selected particles in the upper half of the y-z plane for the $|\leftarrow\rangle_x$ spins passed through the SG setup. Therefore, in the case A, using Eqs. (4.6) and (4.7), the total probability P_A^x of obtaining the outcome +1 for the measurement of σ_θ on particles that pass through the non-ideal SG setup and get selected in the upper half of the y-z plane is given by

$$P_A^x = P_A^+ + P_A^- = (1/4)\left[1 + (1-2E_s)cos\theta + \sqrt{E_s(1-E_s)}\right.$$
$$\left. sin\theta(cos\phi_{+x} + cos\phi_{-x})\right] \qquad (4.8)$$

where the superscript x is used to denote that the quantity P_A^x is measured in Bob's wing corresponding to the total set of measurements of the x-component of spin in Alice's wing.

Case B: In this case, we consider a set of measurements of the z-component of spin in Alice's wing which is along the *same* direction as that of the inhomogeneous magnetic field in the non-ideal SG setup in Bob's wing. This results in effectively a mixed state made up of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ spin components (with equal weighting) in Bob's wing. Let us first focus on the particles with the $|\uparrow\rangle_z$ spin component occurring in Bob's wing. For a particle in the spin state $|\uparrow\rangle_z$ passing through a non-ideal SG setup involving an inhomogeneous magnetic field along the z-axis, the probability of such a particle being post-selected in the upper half of the y-z plane is given by $1 - E_s$ where, as seen from Eq. (4.4b), the quantity E_s denotes the time-saturated probability of finding $|\uparrow\rangle_z$ particles in the lower half of the y-z plane ($E_s \neq 0$ due to non-idealness of the SG setup). The post-selected normalized spin state in this case is $|\uparrow\rangle_z$.

Then, for such post-selected particles, using the expression for the spin state $|\uparrow\rangle_z$ in terms of the eigenstates of σ_θ with eigenvalues +1 and -1 respectively, the probability of obtaining a particular outcome, say +1, of measuring σ_θ is given by

$$p_B^+ = (1/2)(1 + cos\theta) \tag{4.9}$$

Now, remembering that due to measurements of the z-component of spin in Alice's wing, particles with the $|\uparrow\rangle_z$ spin component are produced in Bob's wing with the probability (1/2), the probability P_B^+ that such particles passing through the non-ideal SG setup get post-selected in the upper half of the y-z plane and yield the outcome +1 for the measurement of σ_θ is given by $P_B^+ = (1/2)(1 - E_s)p_B^+$ where p_B^+ is given by Eq. (4.9), whence

$$P_B^+ = (1/4)(1 + cos\theta)(1 - E_s) \tag{4.10}$$

Next, there is another set of particles having the $|\downarrow\rangle_z$ spin component in Bob's wing occurring with the probability 1/2 due to measurements of the z-component of spin in Alice's wing. The probability P_B^- of such particles getting post-selected in the upper half of the y-z plane and yielding the outcome +1 for the measurement of σ_θ is as follows (obtained in a way similar to the derivation of the expression (4.10) for the quantity P_B^+)

$$P_B^- = (1/4)(1 - cos\theta)E_s \tag{4.11}$$

where, as seen from Eq. (4.4a), the quantity E_s denotes the time-saturated probability of finding $|\downarrow\rangle_z$ particles in the upper half of the y-z plane.

Thus, in the case B, in Bob's wing, the total probability P_B^z of obtaining the outcome +1 for the measurement of σ_θ on particles that pass through the non-ideal SG setup and are selected in the upper half of the y-z plane is given by $P_B^z = P_B^+ + P_B^-$ where P_B^+ and P_B^- are given by Eqs. (4.10) and (4.11) respectively, whence

$$P_B^z = (1/4)\left[1 + (1 - 2E_s)cos\theta\right] \tag{4.12}$$

where the superscript z is used to denote that the quantity P_B^z is measured in Bob's wing corresponding to the total set of measurements of the z-component of spin in Alice's wing.

Now, in this example, the condition ruling out any possibility of signaling from Alice to Bob that could have occurred by comparing the cases A and B is given by $P_A^x = P_B^z$, with this equality holding good for the measurement of any arbitrary spin component σ_θ on the particles in Bob's wing that are post-selected following a non-ideal SG setup. Using Eqs. (4.8) and (4.12), the *no-signaling condition* $P_A^x = P_B^z$ reduces to

$$\begin{gathered}(1/4)\left[1 + (1 - 2E_s)cos\theta + \sqrt{E_s(1 - E_s)}sin\theta\right.\\ \left.(cos\phi_{+x} + cos\phi_{-x})\right] = (1/4)\left[1 + (1 - 2E_s)cos\theta\right]\end{gathered} \tag{4.13}$$

For $\theta = 0$, i.e., if for the post-selected particles, the spin component is measured along the z-axis (which is the direction of the inhomogeneous magnetic field in the non-ideal SG setup), Eq. (4.13) is automatically satisfied. For any value of $\theta \neq 0$, if the equality given by Eq. (4.13) is to hold good, the condition $cos\phi_{+x} + cos\phi_{-x} = 0$ needs to be satisfied, which leads to the following relation

$$\phi_{+x} \pm \phi_{-x} = \pi \tag{4.14}$$

In the above derivation of Eq. (4.14), the case A pertains to measurements of the x-component of spin in Alice's wing,while in the case B, a key feature is that measurements of the spin component in Alice's wing are considered along a direction (viz. the z-axis) which is the same as that of the inhomogeneous magnetic field in the non-ideal SG setup in Bob's wing.

Let us now discuss what happens if in the case A, in Alice's wing, an arbitrary spin component σ_ω is measured where the angle ω specifies a direction with respect to the z-axis in the x-z plane (with the proviso $\omega \neq 0, \pi$). Then, if one follows the line of calculation similar to that in the earlier case A, the total probability $P_A^\omega = P_A^+(\omega) + P_A^-(\omega)$ of obtaining the outcome +1 for the measurement of σ_θ on particles in Bob's wing that pass through a non-ideal SG setup and are selected in the upper half of the y-z plane is given by

$$P_A^\omega = (1/4)\left[1 + (1-2E_s)cos\theta + \sqrt{E_s(1-E_s)}\right.$$
$$\left. sin\omega sin\theta(cos\phi_{+\omega} + cos\phi_{-\omega})\right] \tag{4.15}$$

where the superscript ω is used to denote that here the quantity P_A^ω measured in Bob's wing corresponds to measurements of the spin component σ_ω in Alice's wing. Note that Eq. (4.15) reduces to Eq. (4.8) if the measurement of the x-component of spin is performed in Alice's wing; i.e., when $\omega = \pi/2$.

Using Eqs. (4.12) and (4.15), the *no-signaling condition* (NSC) $P_A^\omega = P_B^z$, in this general case, becomes the following equality

$$(1/4)\left[1 + (1-2E_s)cos\theta + \sqrt{E_s(1-E_s)}sin\omega sin\theta\right.$$
$$\left.(cos\phi_{+\omega} + cos\phi_{-\omega})\right] = (1/4)\left[1 + (1-2E_s)cos\theta\right] \tag{4.16}$$

For any value of $\theta \neq 0$, if Eq. (4.16) is to be valid, it is required that $cos\phi_{+\omega} + cos\phi_{-\omega} = 0$ which, in turn, implies the following relation

$$\phi_{+\omega} \pm \phi_{-\omega} = \pi \tag{4.17}$$

As a consequence of NSC, Eq. (4.17), therefore, provides a relation constraining the relative phase occurring in the spin state of the particles in Bob's wing that are post-selected following a non-ideal SG setup, confined to the upper half of the y-z plane. For $E_s = 0$, i.e., when the SG setup used in Bob's wing is *ideal*, it is seen from Eq. (4.16) that the NSC condition is automatically satisfied. Thus, *non-idealness* of the SG setup ($E_s \neq 0$) used in our example is crucial for obtaining the quantitative relation given by Eq. (4.17). The way such a relation can be subjected to an experimental test will now be discussed as follows.

4.4 Empirical testability of the constraint relation Eq. (4.17)

We begin by noting that a possible test of NSC would seem to be by measuring the statistics of the outcomes in one of the two wings of an EPRB setup while changing the measurement setting in the other wing. However, such a test would require to ensure strict space like separation between the two relevant measurements in the two wings of the EPRB pairs - a condition which is non-trivial to satisfy because of an ambiguity concerning the stage at which a measurement process can be regarded as completed (i.e., precisely when a measurement outcome can be considered to be registered); this would critically depend upon the details of how a measurement process is modeled. On the other hand, an important point to be stressed is that although the preceding treatment in our paper deriving Eq. (4.14) or (4.17) from NSC is in the context of EPRB entangled pairs, once the relation is obtained, its validity can be studied by focusing only on a single beam of spin-1/2 particles (say, neutral atoms or neutrons) passed through a non-ideal SG setup. Thus, in practice, such a test wouldn't require EPRB pairs, thereby circumventing the need to satisfy the delicate condition of space like separation.

We first consider a beam of spin-1/2 particles with their spins oriented along a direction making an angle ω with the z-axis in the x-z plane (here $\omega \neq 0, \pi$). Let this beam having the spin state $|\nearrow\rangle_\omega$ be passed through a non-ideal SG setup in which the inhomogeneous magnetic field is along the z-axis. Subsequently, attention is focused on the spin state of a set of particles confined to the upper half of the y-z plane that are selected for further measurement by blocking/detecting particles in the other half of the y-z plane. Such a spin state is of the form given by Eq. (4.3) where the relative phase is $\phi_{+\omega}$ for the initial beam of spin-polarized particles with the spin state $|\nearrow\rangle_\omega$. Similarly, if a beam of oppositely spin-polarized particles with the spin state $|\swarrow\rangle_\omega$ is passed through the non-ideal SG setup in question, the spin state of the post-selected particles given by Eq. (4.3) would involve the relative phase $\phi_{-\omega}$.

Now, note that the relative phase occurring in the spin state of the particles selected in, say, the upper half of the y-z plane cannot be fixed unless the effect of detecting particles in the other half is taken into account within a fully unitary treatment. Whatever be the details of such a treatment, the upshot of our preceding analysis is that the sum of the phase factors $\phi_{+\omega}$ and $\phi_{-\omega}$ should turn out to be π as constrained by Eq. (4.17) which is a consequence of the no-signaling condition. Therefore, an empirical test of the constraint relation given by Eq. (4.17) would provide a means for precision testing of NSC.

Next, to see explicitly how the experimental determination of $\phi_{+\omega}(\phi_{-\omega})$ required for testing Eq. (4.17) can be realized with respect to the post-selected spin state of the form given by Eq. (4.3), we write this state as follows

$$|\nu\rangle = \sqrt{1-E_s}|\uparrow\rangle_z + exp(i\phi_{\pm\omega})\sqrt{E_s}|\downarrow\rangle_z \tag{4.18}$$

Given a representative set of post-selected particles corresponding to the above state $|\nu\rangle$, if one considers the measurement of the spin variable σ_z, it is evident from Eq. (4.18) that the probability of obtaining the outcome ± 1 will yield the value of the parameter E_s. Then, if one takes another representative set of such particles and considers the measurement of any spin component, say σ_x, non-commuting with σ_z, the probability of obtaining the outcome, say $+1$, evaluated using Eq. (4.18) written in terms of the eigenstates of σ_x, will be given by $(1/2)+\sqrt{E_s(1-E_s)}cos\phi_{\pm\omega}$. This measured probability, therefore, enables to fix the phase factor $\phi_{\pm\omega}$, since the parameter E_s is known from the measurement of σ_z. Thus, in this way, by determining $\phi_{\pm\omega}$, the NSC relation given by Eq. (4.17) can be subjected to an experimental verification.

Note that the accuracy of the above experimental determination of $\phi_{\pm\omega}$ depends on the accuracy to which the *idealness* of the SG setup is ensured in measuring the relevant spin variables pertaining to the post-selected spin state, while the parameters of the *non-ideal* SG setup used before post-selection need to be chosen such that the quantity E_s has an appreciable non-zero value. As mentioned above, the measured probability from which $cos\phi_{\pm\omega}$ can be calculated involves the factor $\sqrt{E_s(1-E_s)}$ which will determine the overall precision to which NSC can be tested using the method formulated in this paper and an empirical upper bound on possible violation of NSC can be provided.

We would also like to point out that in our example, the relevant parameters of the SG setup and the region of the position space over which one introduces the projection/post-selection determine the probability amplitudes in the superposition given in Eq. (4.3) (or, Eq. (4.18)) for the post-selected spin state, through the parameter E_s fixed by time-saturated values of Eqs. (4.4a) and (4.4b). A key point of our treatment is that whatever be the value of the parameter E_s, the constraint relation Eq. (4.17) must hold good.

4.5 Summary and Conclusion

We would like to stress that our derivation of the relation (4.14) or (4.17) is based on NSC and the standard rules of quantum mechanics that include, in particular, taking the overall dynamics of the non-ideal Stern-Gerlach device to be linear and the probabilities for measurement outcomes at any given time given by the standard Born rule. If the relation (4.14) or (4.17) is found to be empirically violated, it would then seem to imply violation of NSC for the following reason. As mentioned earlier in Section I by citing Ref. [Sinha et al., 2010], Born's rule has recently been verified by a rigorous precision test. On the other hand, the linearity condition of quantum dynamics and NSC are interlinked [Gisin, 1990; Gisin and Rigo, 1995] and, in particular, the thorough analysis by Simon et al.[Simon et al., 2001] brings out the point that once NSC and the experimentally well-verified standard Born rule are taken to be valid, quantum dynamics is rather rigidly constrained to be linear. Thus, if the relation (4.14) or (4.17) turns out to be empirically invalid, it would seem plausible to infer that the violation of NSC arises from a departure from linearity in the way a non-ideal Stern-Gerlach setup acts in conjunction with the type of post-selection process that is used in our example. The tenability of such a possibility would then call for further investigation; for example, the general framework for accommodating non-linearity in quantum dynamics discussed by Weinberg [Weinberg, 1989] may be invoked in the context of our setup in order to explain any observed violation of the relation (4.14) or (4.17). Here we note that in Weinberg's paper, an indication [Weinberg, 1989] has been given of applying his general framework by analyzing an ideal Stern-Gerlach setup, while in our example, a non-ideal Stern-Gerlach device is considered, along with a suitable post-selection of particles emerging in one of the two halves.

As regards the distinctiveness of the setup used in our example, we may stress that the post-selection procedure used here following a non-ideal SG setup is *different* from the scheme of weak-measurement related studies [Aharonov et al., 1988] where the post-selection follows an ideal SG setup that is preceded by a non-ideal SG with a very weak magnetic field. In our example, the central feature is that for an incoming beam of spin-1/2 particles that are spin-polarized along any direction (other than the z-axis, the direction of the magnetic field in the non-ideal SG setup considered here), the post-selected spin state of the emerging particles in any one of the two halves is a superposition of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ states with a relative phase for which a constraint relation is obtained.

It has already been mentioned that a complete unitary treatment is required for evaluating the above mentioned relative phase, for which one would need to suitably incorporate the effect on the post-selected spin state in one of the halves, arising from the blocking/detecting of particles in the other half that emerge from the non-ideal SG setup. The extent to which the details of the modeling of such a post-selection process can affect the evaluation of the relative phase occurring in the post-selected spin state should be instructive to probe. Importantly, this type of study needs to be compatible with the constraint relation given by Eq. (4.14) or (4.17) whose validity should be independent of the specifics of the modeling of the post-selection procedure used in our example. Here it may be noted that, given the theory of approximate or generalized measurement [Busch et al., 1997, 1996] that has been developed to a considerable extent, its possible implication pertaining to the type of post-selection considered here could be worth investigating. Finally, it is hoped that the predictive power of NSC in a testable situation as illustrated by the example treated here using a non-ideal measurement setup may motivate the formulation of other such examples that could be helpful for a deeper understanding of the role of NSC in the context of non-ideal measurement situations.

Chapter 5

Information causality and Hardy's correlation

In this chapter we study bi-partite Hardy's correlations in two two-level systems in the generalized no-signaling framework. We apply the principle of non-violation of information causality (a generalization of no-signaling condition) to study the Hardy-type nonlocal correlations. First we introduce this principle to explain a quantum feature: why Hardy's nonlocality cannot be observed for maximally entangled states [Gazi, Rai, Kunkri and Rahaman, 2010]? Next, we derive bound on Hardy's non-locality and Cabello's nonlocality by applying a sufficient condition for violating information causality [Ahanj, Kunkri, Rai, Rahaman and Joag, 2010].

5.1 Introduction

Violation of the Bell-type inequalities [Bell, 1964; Clauser et al., 1969] by quantum mechanics show that nature is nonlocal. Nevertheless quantum correlations respect causality principle [Ghirardi et al., 1980]. However, there are also other non-signaling post quantum correlations [Popescu and Rohrlich, 1994] which cannot be distinguished from quantum correlation by subjecting them to the causality principle. Though post quantum correlations are not observed in experiments, but still we do not understand what underlying physical principle(s) completely distinguishes quantum correlations from nonphysical post quantum correlations.

Recent studies has shown that quantum features like violation of Bell type inequalities [Popescu and Rohrlich, 1994], intrinsic randomness, no-cloning [Barnum et al., 2007; Masanes et al., 2006], information-disturbance trade off [Scarani et al., 2006], secure cryptography [Acin et al., 2006; Barrett, Hardy and Kent, 2005; Masanes, 2009], teleportation [Barnum et al., 2008], entanglement swapping [Skrzypczyk et al., 2009] are also enjoyed by other post quantum no-signaling theories. On the other hand for no-signaling correlations some implausible features has also been noticed like: some no-signaling correlations would make certain distributed computational tasks trivial [Brassard et al., 2006; Brunner and Skrzypczyk, 2009; Linden et al., 2007; van Dam, 2000] and would have very limited dynamics [Barrett, 2007]. So the study of the nonlocal correlations in the general no-signaling framework leads us towards a deeper understanding of quantum correlations.

The principle of non-violation of information causality (IC) [Pawlowski et al., 2009] has been identified as one of the foundational principle of nature, it is compatible with experimentally observed quantum and classical correlations but rules out an unobserved class of nonlocal correlation as nonphysical. The principle states that communication of m classical bits causes information gain of at most m bits, this is a generalization of the no-signaling principle, the case $m = 0$ corresponds to no-signaling. Applying IC principle to non-local correlations, we get the Cirel'son bound [Cirel'son, 1980] and all correlations that goes beyond Cirel'son's bound violate the principle of information causality [Pawlowski et al., 2009]. In [Allcock et al., 2009] it was shown that though some part of quantum boundary can be derived from a necessary condition (given in [Pawlowski et al., 2009]) for violating IC, this condition is not sufficient for distinguishing quantum correlations from all post-quantum correlations which are below the Cirel'son's bound. So it remains interesting to see if the full power of IC (some other conditions derived from IC) can eliminate remaining post-quantum correlations below the Cirel'son's bound. Along with the research in the direction of completely distinguishing the quantum correlations from rest of the nonlocal correlations, it would also be interesting to apply the known IC condition(s) for qualitative/quantitative study of certain specific features of nonlocal correlations. In this chapter we apply IC condition to study two aspects of bipartite hardy's correlation: (i) the feature of local randomness in Hardy's correlation, and (ii) bound on success probability of Hardy's correlations constrained by the information causality principle.

First, we apply IC condition in order to study the property of local randomness for a bipartite probability distribution which exhibits Hardy's non-locality [Hardy, 1992, 1993]. Our motivation for this study came from the fact that Hardy's non-locality argument in quantum mechanics does not work for maximally entangled state [Cabello, 2000; Hardy, 1993] and at the same time for a maximally entangled state, local density matrix being completely random, both the results for a qubit are equally probable. Keeping this in mind, we asked a more general question like: for two two-level systems, how many observable and in which way, out of four entering in the Hardy's non-locality argument, can be locally random. We want to study this question in the context of probability distribution which respects an IC condition as well as in the context of quantum mechanics. We see that the applied IC condition itself imposes powerful restriction but still it does not reproduce all the restrictions imposed by quantum mechanics. In this context, it is to be mentioned that no signaling condition does not impose any such restriction. Interestingly we observed that the applied necessary condition for respecting IC allows at most two observable, one on each side, chosen in a restricted way to be completely random, and quantum mechanics allows only one of them to be completely random.

Finally, we apply the IC principle to derive an upper bound on maximum success probability of bipartite Hardy's argument. Then we extend our result to Cabello's nonlocality argument (a generalization of Hardy's argument).

5.2 Bipartite no-signaling correlations

Let us consider a bipartite black box shared between two parties: Alice and Bob. Alice and Bob input variables x and y at their end of the box, respectively, and receive outputs a and b. For a fixed input variables there can be different outcomes with certain probabilities. The behavior of a these correlation boxes is fully described by a set of joint probabilities $P(ab|xy)$. In this article, we will focus on the case of binary inputs and outputs ($a, b, x, y \in \{0, 1\}$). Then we have a set of 16 joint probabilities defining a bipartite binary input - binary output correlation box. These types of correlations can be represented by a 4×4 correlation matrix:

$$\begin{pmatrix} P(00|00) & P(01|00) & P(10|00) & P(11|00) \\ P(00|01) & P(01|01) & P(10|01) & P(11|01) \\ P(00|10) & P(01|10) & P(10|10) & P(11|10) \\ P(00|11) & P(01|11) & P(10|11) & P(11|11) \end{pmatrix}$$

We note that since $P(ab|xy)$ are probabilities, they satisfy positivity, $P(ab|xy) \geq 0 \; \forall \; a, b, x, y$, and normalization $\sum_{a,b} P(ab|xy) = 1 \; \forall \; x, y$. Since we are to study no-signaling boxes; i.e., we require that Alice cannot signal to Bob by her choice x and vice versa, the marginal probabilities $P_{a|x}$ and $P_{b|y}$ must be independent of y and x, respectively. The full set of nonsignaling boxes forms an eight-dimensional polytope [Barrett, Linden, Massar, Pironio, Popescu and Roberts, 2005] which has 24 vertices: eight extremal nonlocal boxes and 16 local deterministic boxes. The extremal nonlocal correlations have the form

$$P_{NL}^{\alpha\beta\gamma} = \begin{cases} \frac{1}{2} & \text{if} \quad a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

where $\alpha, \beta, \gamma \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2.. Similarly, the local deterministic boxes are described by

$$P_{L}^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if} \quad a = \alpha X \oplus \beta, \\ & \qquad b = \gamma Y \oplus \delta; \\ 0 & \text{otherwise}, \end{cases} \tag{5.2}$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2.
Thus we can see that any bipartite two input- two output nonsignaling correlation box can be expressed as a convex combination of the above 24 local/nonlocal vertices.

5.3 Hardy's correlations under no-signaling condition

A bipartite two input - two output Hardy's correlation puts simple restrictions on a certain choice of 4 out of 16 joint probabilities in the correlation matrix. One such choice is $P(11|11) > 0$, $P(11|01) = 0$, $P(11|10) = 0$, $P(00|00) = 0$ and it is easy to argue that these correlations are nonlocal. To show this, let us suppose that these correlations are local i.e. they can be simulated by noncommunicating observers with only shared randomness as a resource. Now consider the subset of those random variables λ shared between the two observers such that for λs belonging to this subset input $x = 1, y = 1$ give output $a = 1, b = 1$ (this subset is nonempty since $P(11|11) > 0$), now conditions $P(11|01) = 0$ and $P(11|10) = 0$ tell that within this subset input $x = 0, y = 0$ would give output $a = 0, b = 0$, this would imply that $P(00|00) > 0$), but it contradicts the condition $P(00|00) = 0$. Hence these correlations are nonlocal. If we further restrict these correlations by no-signaling condition we get Hardy's nonsignaling boxes. It is easy to check that these boxes can be written as a convex combination of 5 of the sixteen local vertices P_L^{0001}, P_L^{0011}, P_L^{0100}, P_L^{1100}, P_L^{1111} and 1 of the eight nonlocal vertex P_{NL}^{001}. Then,

$$\begin{aligned} P^{\mathcal{H}}_{ab|XY} &= c_1 P_L^{0001} + c_2 P_L^{0011} + c_3 P_L^{0100} \\ &\quad + c_4 P_L^{1100} + c_5 P_L^{1111} + c_6 P_{NL}^{001} \end{aligned} \tag{5.3}$$

where $\sum_{j=1}^{6} c_i = 1$. From here the correlation matrix for these Hardy's nonsignaling boxes can be written as

$$\begin{pmatrix} 0 & c_1 + c_2 + \frac{c_6}{2} & c_3 + c_4 + \frac{c_6}{2} & c_5 \\ c_2 & c_1 + \frac{c_6}{2} & c_3 + c_4 + c_5 + \frac{c_6}{2} & 0 \\ c_4 & c_1 + c_2 + c_5 + \frac{c_6}{2} & c_3 + \frac{c_6}{2} & 0 \\ c_2 + c_4 + c_5 + \frac{c_6}{2} & c_1 & c_3 & \frac{c_6}{2} \end{pmatrix}$$

5.4 Property of local randomness in Hardy's correlations

For a most general bipartite correlation an input x on Alice's side is locally random if the marginal probabilities of all possible outcomes on Alice's side for this input, are equal and similarly for Bob. In the case of two-input-two-output bipartite correlations: an input x on Alice's side is locally random if, $P(0|x) = P(1|x) = \frac{1}{2}$, in terms of joint probabilities this would mean that for any choice of Bob's input y, $P(00|xy) + P(01|xy) = P(10|xy) + P(11|xy) = \frac{1}{2}$. Similarly an input y on Bob's side is locally random if, $P(0|y) = P(1|y) = \frac{1}{2}$, in terms of joint probabilities this can be expressed as, for any choice of Alice's input x, $P(00|xy) + P(10|xy) =$

$P(01|xy) + P(11|xy) = \frac{1}{2}$. Let us denote the 0 and 1 inputs on Alice's (Bob's) side as $0_A(0_B)$ and $1_A(1_B)$ respectively. We would now like to see that, what choices of inputs from the set $\{0_A, 1_A, 0_B, 1_B\}$ can be locally random for a given class of Hardy's correlations.

TABLE 5.1: For the no-signaling bipartite Hardy's correlation with two dichotomic observable on either side, here each row give the conditions which coefficients c_is must satisfy for the corresponding input to be locally random.

Input	Conditions for local randomness
0_A	$c_1 + c_2 + \frac{c_6}{2} = \frac{1}{2}$; $c_3 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$
1_A	$c_1 + c_2 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$; $c_3 + \frac{c_6}{2} = \frac{1}{2}$
0_B	$c_3 + c_4 + \frac{c_6}{2} = \frac{1}{2}$; $c_1 + c_2 + c_5 + \frac{c_6}{2} = \frac{1}{2}$
1_B	$c_2 + c_3 + c_2 + c_4 + c_5 + \frac{c_6}{2} = \frac{1}{2}$; $c_1 + \frac{c_6}{2} = \frac{1}{2}$

5.4.1 Hardy's correlations respecting no-signaling

In the case of Hardy's correlations which respects no-signaling, condition of local randomness for each of the possible inputs, are given in the TABLE(5.1). Now let us see that for the Hardy's correlations respecting no-signaling, what choices of inputs can be locally random. We give the results for every case, in the TABLE(5.2). We can read from here that although in order to show the property of local randomness Hardy's correlations becomes much restricted, yet we get solutions for each case. If we get solutions for the case 1, it is obvious that there are solutions in all the remaining cases 2-15 , nevertheless we write the complete table giving the form of solutions in each case for the later reference.

TABLE 5.2: For the no-signaling bipartite Hardy's correlation with two dichotomic observable on either side, here each row gives the form of solutions for the corresponding choice of inputs to be locally random.

Cases	Locally random inputs	C_1	C_2	C_3	C_4	C_5	C_6
1.	$\{0_A, 1_A, 0_B, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	$\frac{1}{2}(1-c_6)$	0	0	c_6
2.	$\{0_A, 1_A, 0_B\}$	c_1	$\frac{1}{2}(1-c_6)-c_1$	$\frac{1}{2}(1-c_6)$	0	0	c_6
3.	$\{0_A, 1_A, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	$\frac{1}{2}(1-c_6)$	0	0	c_6
4.	$\{0_A, 0_B, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	c_3	$\frac{1}{2}(1-c_6)-c_3$	0	c_6
5.	$\{1_A, 0_B, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	$\frac{1}{2}(1-c_6)$	0	0	c_6
6.	$\{0_A, 1_A\}$	c_1	$\frac{1}{2}(1-c_6)-c_1$	$\frac{1}{2}(1-c_6)$	0	0	c_6
7.	$\{0_B, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	c_3	$\frac{1}{2}(1-c_6)-c_3$	0	c_6
8.	$\{1_A, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	$\frac{1}{2}(1-c_6)$	0	0	c_6
9.	$\{0_A, 0_B\}$	c_1	$\frac{1}{2}(1-c_6)-c_1$	c_3	$\frac{1}{2}(1-c_6)-c_3$	0	c_6
10.	$\{0_A, 1_B\}$	$\frac{1}{2}(1-c_6)$	0	c_3	c_4	$\frac{1}{2}(1-c_6)-c_3-c_4$	c_6
11.	$\{1_A, 0_B\}$	c_1	c_2	$\frac{1}{2}(1-c_6)$	0	$\frac{1}{2}(1-c_6)-c_1-c_2$	c_6
12.	$\{0_A\}$	c_1	$\frac{1}{2}(1-c_6)-c_1$	c_3	c_4	$\frac{1}{2}(1-c_6)-c_3-c_4$	c_6
13.	$\{1_A\}$	c_1	$\frac{1}{2}(1-c_6)-c_1-c_4-c_5$	$\frac{1}{2}(1-c_6)$	c_4	c_5	c_6
14.	$\{0_B\}$	c_1	$\frac{1}{2}(1-c_6)-c_1-c_5$	c_3	$\frac{1}{2}(1-c_6)-c_3$	c_5	c_6
15.	$\{1_B\}$	$\frac{1}{2}(1-c_6)$	$\frac{1}{2}(1-c_6)-c_3-c_4-c_5$	c_3	c_4	c_5	c_6

5.4.2 Hardy's correlation respecting information causality

Let us first briefly recapitulate the principle of information causality (IC) [Pawlowski et al., 2009], then we would apply it in our study of the property of local randomness for two-input-two-output Hardy's no-signaling correlations. IC principle states that for two parties Alice and Bob, who are separated in space, the information gain that Bob can reach about a previously unknown to him data set of Alice, by using all his local resources and m classical bit communicated by Alice, is at most m bits. This principle can be well formulated in terms of a generic information processing task in which Alice is provided with a N random bits $\vec{a} = (a_1, a_2, ..., a_N)$ while Bob receives a random variable $b \in \{1, 2, , ..., N\}$. Alice then sends m classical bits to Bob, who must output a single bit β with the aim of guessing the value of Alice's b-th bit a_b. Their degree of success at this task is measured by

$$I \equiv \sum_{K=1}^{N} I(a_K : \beta | b = K),$$

where $I(a_K : \beta | b = K)$ is Shannon mutual information between a_K and β. Then the principle of information causality says that physically allowed theories must have $I \leq m$. The result that both classical and quantum correlations satisfy this condition was proved in [Pawlowski et al., 2009]. It was further shown there that, if Alice and Bob share arbitrary two input-two output no-signaling correlations corresponding to conditional probabilities $P(ab|xy)$, then by applying a protocol by van Dam [van Dam, 2000], one can derive a necessary condition for respecting the IC principle. This necessary condition reads,

$$E_1^2 + E_2^2 \leq 1, \tag{5.4}$$

where $E_j = 2P_j - 1$ $(j = 1, 2)$, and P_1, P_2 are defined by,

$$\begin{aligned} P_1 &= \frac{1}{2}\left[p_{(a=b|00)} + p_{(a=b|10)}\right] \\ &= \frac{1}{2}\left[p_{00|00} + p_{11|00} + p_{00|10} + p_{11|10}\right] \\ P_2 &= \frac{1}{2}\left[p_{(a=b|01)} + p_{(a\neq b|11)}\right] \\ &= \frac{1}{2}\left[p_{00|01} + p_{11|01} + p_{01|11} + p_{10|11}\right] \end{aligned} \tag{5.5}$$

Here it is important to note that the condition (5.4) is only a necessary condition (based on the protocol give in [Pawlowski et al., 2009]) for respecting the IC principle. So a violation of (5.4)implies a violation of IC but the converse may not be true. In fact, it is shown in [Allcock et al., 2009] that there are examples where the condition (5.4) is satisfied but not the IC. We now derive some one way implications about the property of local randomness for two input - two output Hardy's nonsignaling correlations. It is easy to verify that restricting Hardy's nonsignaling correlations by condition (5.4) and interchanging the roles of Alice and Bob we get,

$$c_6^2 + 2(c_4 + c_5)c_6 + 2(c_4 + c_5)(c_4 + c_5 - 1) \leq 0 \tag{5.6}$$

$$c_6^2 + 2(c_2 + c_5)c_6 + 2(c_2 + c_5)(c_2 + c_5 - 1) \leq 0 \tag{5.7}$$

By applying these conditions for all possible choices of inputs that can be locally random for Hardy's nonsignaling correlations (TABLE(5.2)), we get that at least one of the above two conditions are violated for the cases $1 - 8$ but for the cases $9 - 15$ we can find c_is satisfying the above two conditions. Thus for the cases $1 - 8$ we can conclude that IC is violated, hence they cannot be true in quantum mechanics also. Now we shall study the cases 9-15 in the context of quantum mechanics in the following subsection.

5.4.3 Hardy's correlation in quantum mechanics

Violation of IC for cases 1-8 implies that there are no quantum solution for these cases. To resolve the remaining cases (9-15), we consider a two qubit pure quantum state. It is to be mentioned that for two qubits, Hardy's argument runs only for pure entangled state [Kar, 1997]. So without loss of any generality, we consider the following two qubit state,

$$|\Psi\rangle = \cos\beta|0\rangle_A|0\rangle_B + \exp(i\gamma)\sin\beta|1\rangle_A|1\rangle_B \tag{5.8}$$

. Then the density matrix $\rho_{AB} = |\Psi\rangle\langle\Psi|$ can be written in terms of Pauli matrices as,

$$\begin{aligned}\rho_{AB} = \frac{1}{4}[I^A \otimes I^B + (cos^2\beta - \sin^2\beta)I^A \otimes \sigma_z^B + (cos^2\beta - \sin^2\beta)\sigma_z^A \otimes I^B \\ +(2\cos\beta\sin\beta)\sigma_x^A \otimes \sigma_x^B + (2\cos\beta\sin\beta)\sigma_x^A \otimes \sigma_y^B + (2\cos\beta\sin\beta)\sigma_y^A \otimes \sigma_x^B \\ -(2\cos\beta\sin\beta)\sigma_y^A \otimes \sigma_y^B + \sigma_z^A \otimes \sigma_z^B]\end{aligned} \tag{5.9}$$

The reduced density matrices ρ_A and ρ_B are,

$$\rho_A = \frac{1}{2}[I + (cos^2\beta - \sin^2\beta)\sigma_z^A] \tag{5.10}$$

$$\rho_B = \frac{1}{2}[I + (cos^2\beta - \sin^2\beta)\sigma_z^B] \tag{5.11}$$

In general an observable on a single qubit can be written as $\hat{n}\cdot\sigma$ where, $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is any unit vector in $\mathbb{R}^3$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Then the projectors on the eigen states of these observable are,

$$P^{\pm} = \frac{1}{2}[I \pm \hat{n}\cdot\sigma] \tag{5.12}$$

For observable on Alice's side to be locally random,

$$Tr(\rho_A P^+) = Tr(\rho_A P^-) \tag{5.13}$$

similarly for observable on Bob's side to be locally random,

$$Tr(\rho_B P^+) = Tr(\rho_B P^-) \tag{5.14}$$

On simplifying this we find that, for a non-maximally entangled state an observable is locally random if and only if $\theta = \frac{\pi}{2}$ i.e. $\hat{n}$ is of the form $(\cos\phi, \sin\phi, 0)$. Here we would also like to mention that for a maximally entangled state any arbitrary observable shows the property of local randomness, but we know that Hardy's argument do not run for a maximally entangled state. This also follows from the IC principle, as for a maximally entangled state any four arbitrary observable (two on Alice's side and two on Bob's side) are locally random and we saw that if so, it violates the IC principle.

Now suppose A (0_A) and A' (1_A) are the observable on Alice's side and B (0_B) and B' $(1_{B'})$ are the observable on bob's side. Here outputs 0 and 1 will corresponds to outcomes $+1$ and -1 respectively. Then the Hardy's correlation can be written as,

$$\begin{aligned} P(A=+1, B=+1) = {} & \cos^2\beta\cos^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} + \sin^2\beta\sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2} \\ & +2\cos\beta\sin\beta\sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos\frac{\theta_A}{2}cos\frac{\theta_B}{2} \\ & \cos(\phi_A + \phi_B - \gamma) = 0 \end{aligned} \tag{5.15}$$

$$P(A=-1,B'=-1)=\cos^2\beta\sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_{B'}}{2}+\sin^2\beta cos^2\frac{\theta_A}{2}cos^2\frac{\theta_{B'}}{2}$$
$$+2\cos\beta\sin\beta\sin\frac{\theta_A}{2}\sin\frac{\theta_{B'}}{2}\cos\frac{\theta_A}{2}\cos\frac{\theta_{B'}}{2}$$
$$\cos(\phi_A+\phi_{B'}-\gamma)=0 \quad (5.16)$$

$$P(A'=-1,B=-1)=\cos^2\beta\sin^2\frac{\theta_{A'}}{2}\sin^2\frac{\theta_B}{2}+\sin^2\beta\ \cos^2\frac{\theta_{A'}}{2}\cos^2\frac{\theta_B}{2}$$
$$+2\cos\beta\sin\beta\sin\frac{\theta_{A'}}{2}\sin\frac{\theta_B}{2}\cos\frac{\theta_{A'}}{2}\cos\frac{\theta_B}{2}$$
$$\cos(\phi_{A'}+\phi_B-\gamma)=0 \quad (5.17)$$

$$P(A'=-1,B'=-1)=\cos^2\beta\sin^2\frac{\theta_{A'}}{2}\sin^2\frac{\theta_{B'}}{2}+\sin^2\beta\cos^2\frac{\theta_{A'}}{2}\cos^2\frac{\theta_{B'}}{2}$$
$$+2\cos\beta\sin\beta sin\frac{\theta_{A'}}{2}\sin\frac{\theta_{B'}}{2}cos\frac{\theta_{A'}}{2}\cos\frac{\theta_{B'}}{2}$$
$$\cos(\phi_{A'}+\phi_{B'}-\gamma)\neq 0 \quad (5.18)$$

For these Hardy's correlation if observable A and B (0_A and 0_B) are locally random, then $\theta_A=\theta_B=\frac{\pi}{2}$, then from equation (15) we get,

$$1+\sin 2\beta\cos(\phi_A+\phi_B-\gamma)=0 \quad (5.19)$$

then this equation is satisfied only if $\sin 2\beta$ takes the value $+1$ or -1, in either case corresponding state has to be a maximally entangled state, but this cannot be a case. Therefore we conclude that observable A and B cannot be locally random in quantum mechanics. Similarly we can see that local randomness of two observable in the cases, A' and B (1_A and 0_B) and A and B' (0_A and 1_B) is also not possible.

Now we consider the case of just one observable - say $A(0_A)$ from the set $\{A,A',B,B'\}$ to be locally random (and similarly for the cases A',B,B'). Then we find that there are non-maximally entangled states and choices of observable A,A',B,B' such that one of the observable is locally random. We give an example, consider the state $\beta=\frac{\pi}{6}$, and $\gamma=\pi$, choose observable A as $\theta_A=\frac{\pi}{2}$ and $\phi_A=\pi$, A'

as $\theta_{A'} = 2\tan^{-1}(\tan^2\frac{\pi}{6})$ and $\phi_{A'} = -\pi$, B as $\theta_B = \frac{2\pi}{3}$ and $\phi_B = \pi$, and B' as $\theta_{B'} = \frac{\pi}{3}$ and $\phi_{B'} = -\pi$, then it can be easily checked that for this choice of state and observable, Hardy's argument runs and the observable A is locally random. Thus by analyzing the remaining cases $(9-15)$ within quantum mechanics, we can now conclude that for a quantum mechanical state showing Hardy's nonlocality, at most one out of the four observable can be locally random.

5.5 Bound on Hardy/ Cabello correlations

5.5.1 Hardy's/Cabello-type argument for two qubits

Let us reconsider Hardy's argument and its generalization. Consider two spin-1/2 particles 1 and 2 with spin observable A, A' on particle 1 and B, B' on particle 2. These observable gives the eigenvalues ± 1. Now consider the following joint probabilities:

$$P(A=+1, B=+1) = q_1 \tag{5.20}$$

$$P(A'=-1, B=-1) = 0 \tag{5.21}$$

$$P(A=-1, B'=-1) = 0 \tag{5.22}$$

$$P(A'=-1, B'=-1) = q_4 \tag{5.23}$$

Here equation (1) tells that, if A is measured on particle 1 and B is measured on particle 2, then the probability that both get value +1 is q_1, remaining equations can also be interpreted in a similar fashion. These equations form the basis of Cabello's nonlocality argument. It can easily be seen that these equations contradict local-realism if $q_1 < q_4$. To show this, let us consider those hidden variable states λ for which $A' = -1$ and $B' = -1$. For these states, equations (2) and (3) tell that the values of A and B must be equal to +1. Thus according to local realism $P(A=+1, B=+1)$ should be at least equal to q_4. This contradicts equation (1) as $q_1 < q_4$. It should be noted here that $q_1 = 0$ reduces this argument to that of Hardy's. So by Cabello's argument, we specifically mean that the above argument runs, even with nonzero q_1.

5.5.2 Hardy and Cabello-type correlations from no-signaling polytope

Previously, we discussed that, given a set of observables $X, Y \in \{0 \;\; 1\}$ and outcomes $a, b \in \{0 \;\; 1\}$ joint probabilities $p_{ab|XY}$ form an entire correlation table with 2^4 entries, which can be regarded as a point of 2^4- dimensional vector space. Then, the positivity, normalization and non-signaling constraints lead the entire correlation table to a convex subset in the form of a polytope which is known as no-signaling polytope $\mathcal{P}$, which is eight dimensional [Barrett, Linden, Massar, Pironio, Popescu and Roberts, 2005]. There are 24 vertices of the polytope $\mathcal{P}$, 16 of which represent local correlations (called "local vertices") and 8 represent nonlocal correlations. The local vertices can then be expressed as

$$p^{\alpha\beta\gamma\delta}_{ab|XY} = \begin{cases} 1, & \text{if} \quad a = \alpha X \oplus \beta, \\ & \qquad\; b = \gamma Y \oplus \delta; \\ 0, & \text{otherwise} \end{cases} \tag{5.24}$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2.
The eight nonlocal vertices have the form:

$$p^{\alpha\beta\gamma}_{ab|XY} = \begin{cases} \frac{1}{2}, & \text{if} \quad a \oplus b = XY \oplus \alpha X \oplus \beta Y \oplus \gamma, \\ 0, & \text{otherwise} \end{cases} \tag{5.25}$$

where $\alpha, \beta, \gamma \in \{0, 1\}$.
Let us now we the correspondence $(X = 0) \leftrightarrow A, (X = 1) \leftrightarrow A'$, $(Y = 0) \leftrightarrow B, (Y = 1) \leftrightarrow B'$ and $a, b = 0(1) \leftrightarrow +1(-1)$. Then it is straight-forward to see that five of the 16 local vertices and one of the 8 nonlocal vertices satisfy Hardy's equations (1)-(4) (when $q_1 = 0$), namely those given by $p^{0001}_{ab|XY}$, $p^{0011}_{ab|XY}$, $p^{0100}_{ab|XY}$, $p^{1100}_{ab|XY}$, $p^{1111}_{ab|XY}$ and $p^{001}_{ab|XY}$. The other vertices can be covered by another set of Hardy's equations. Then the joint probabilities satisfying Hardy's conditions can be written as a convex combination of the above 6 vertices (five local vertices and one nonlocal vertex). Then

$$\begin{aligned} p^{\mathcal{H}}_{ab|XY} &= c_1 p^{0001}_{ab|XY} + c_2 p^{0011}_{ab|XY} + c_3 p^{0100}_{ab|XY} \\ &\quad + c_4 p^{1100}_{ab|XY} + c_5 p^{1111}_{ab|XY} + c_6 p^{001}_{ab|XY} \end{aligned} \tag{5.26}$$

where $\sum_{j=1}^{6} c_i = 1$.
Now if we consider $q_1 \neq 0$ (but $q_1 < q_4$), then the equations $(1) - (4)$ is known as Cabello's nonlocality conditions, which can be written as a convex combination of the above 6 vertices which satisfies Hardy's conditions along with another four local vertices $p^{0000}_{ab|XY}$, $p^{0010}_{ab|XY}$, $p^{1000}_{ab|XY}$, $p^{1010}_{ab|XY}$ and one nonlocal vertex $p^{110}_{ab|XY}$. So we get,

$$\begin{aligned} p^{\mathcal{C}}_{ab|XY} &= p^{\mathcal{H}}_{ab|XY} + c_7 p^{0000}_{ab|XY} + c_8 p^{0010}_{ab|XY} + c_9 p^{1000}_{ab|XY} \\ &\quad + c_{10} p^{1010}_{ab|XY} + c_{11} p^{110}_{ab|XY} \end{aligned} \tag{5.27}$$

where the expression $p^{\mathcal{H}}_{ab|XY}$ is given in equation (5.26) and coefficients c_i's satisfy the condition $\sum_{j=1}^{11} c_i = 1$.
One can check from equation (5.26) that the success probability for Hardy's argument is given by $p^{\mathcal{H}}_{11|11} = \frac{1}{2} c_6$. From here, one can obviously see that under the no-signaling constraint, the maximum success probability of Hardy's argument *i.e.* $(p^{\mathcal{H}}_{11|11})_{max} = \frac{1}{2}$ is achieved for $c_6 = 1$ and $c_1 = c_2 = c_3 = c_4 = c_5 = 0$. Similarly the success probability for Cabello's argument follows from equation (5.27) and can be written as, $p^{\mathcal{C}}_{11|11} - p^{\mathcal{C}}_{00|00} = (\frac{1}{2} c_6 + c_{10}) - C$, where $C = c_7 + c_8 + c_9 + c_{10} + \frac{1}{2} c_{11}$, and, here too we obtain that $(p^{\mathcal{C}}_{11|11} - p^{\mathcal{C}}_{00|00})_{max} = \frac{1}{2}$ for $c_6 = 1$ and rest of the c_i's$= 0$. This maximum success probability of Hardy's/Cabello's argument, restricted by the no-signaling condition, has also been derived in [Cereceda, 2000; Choudhary et al., 2010]. One should note that the probability set for which this maximum is achieved coincides with PR correlation for both the cases. In the following sections we will derive an upper bound on the maximum value of these success probabilities from the principle of non-violation of information causality.

5.5.3 Hardy's nonlocality and Information Causality

In this section we derive an upper bound on the maximum probability of success of Hardy's non-locality argument for a two qubit system in the context of non-violation of information causality. Let Alice and Bob share no-signaling nonlocal correlation satisfying Hardy's condition *i.e.* the joint probability $P^{\mathcal{H}}_{ab|XY}$ given in equation (5.26). Then for this nonlocal correlation we have

$$\begin{aligned} P_1 &= \frac{1}{2}(c_5 + c_4), \\ P_2 &= \frac{1}{2}(c_1 + c_2 + c_3) \end{aligned} \tag{5.28}$$

To satisfy the IC condition equation (5.28) has to satisfy the condition

$$E_1^2 + E_2^2 \leq 1$$

i.e

$$(c_5 + c_4 - 1)^2 + (c_1 + c_2 + c_3 - 1)^2 \leq 1,$$

which implies

$$c_6^2 + 2(c_4 + c_5)c_6 + 2(c_4 + c_5)(c_4 + c_5 - 1) \leq 0 \tag{5.29}$$

The above equation gives the maximum value of $c_6 = \sqrt{2} - 1$. Then an upper bound on the maximum probability of success of Hardy's non-locality is given by $P^{\mathcal{H}}_{11|11} = \frac{1}{2}c_6 \leq \frac{1}{2}(\sqrt{2} - 1) = 0.20717$.

5.5.4 Cabello's nonlocality and Information Causality

Now we try to find an upper bound on the maximum probability of success in Cabello's case in the context of non-violation of information causality. Let Alice and Bob share non-signaling nonlocal correlation satisfying Cabello's condition i.e joint probability $p^{\mathcal{C}}_{ab|XY}$ given in equation (5.27). Then for this nonlocal correlation we have:

$$\begin{aligned} P_1 &= \frac{1}{2}[C + c_5 + \frac{c_{11}}{2} + c_4 + c_7 + c_8] \\ P_2 &= \frac{1}{2}[1 + c_9 - (c_4 + c_5 + c_6 + c_{10})] \end{aligned} \tag{5.30}$$

where $C = c_7 + c_8 + c_9 + c_{10} + \frac{1}{2}c_{11}$. Then

$$\begin{aligned} E_1 &= c_7 + c_8 - c_1 - c_2 - c_3 - c_6 \\ E_2 &= c_9 - c_4 - c_5 - c_6 - c_{10} \end{aligned} \tag{5.31}$$

To satisfy the IC condition equation (5.31) has to satisfy the condition

$$E_1^2 + E_2^2 \leq 1.$$

One can easily check that

$$E_1 + E_2 = -(1 + 2x)$$

where $x = (c_{10} + \frac{1}{2}c_6) - C$. It follows that

$$E_1^2 + E_2^2 = 4x^2 + 4(1 + E_2)x + 2(1 + E_2)E_2 + 1,$$

so in order to satisfy the IC condition,

$$x^2 + (1 + E_2)x + \frac{1}{2}E_2(1 + E_2) \leq 0$$

writing E_2 in terms of P_2 we obtain,

$$x^2 + 2P_2x + P_2(2P_2 - 1) \leq 0.$$

Since we are to find x_{max} it is sufficient to consider only the equality. Then,

$$x = -P_2 + \sqrt{P_2(1 - P_2)};$$

$$0 \leq P_2 \leq \frac{1}{2}.$$

The maximum value of x we obtained from here is,

$$x_{max} = \frac{1}{2}(\sqrt{2} - 1) = 0.20717.$$

This value is same as in the Hardy's case. We conclude that on applying the IC condition, maximum probability of success of the Cabello's argument is same as that of the Hardy's argument, both achieving the same numerical value 0.20717.

5.6 Conclusions

Maximally entangled state in quantum mechanics does not reproduce Hardy's correlation whereas generalized non-signaling theory put no such restriction on the local randomness of the observable for Hardy's correlation. We study all the possibilities of local randomness in Hardy's correlation in the context of information causality condition. We observe that in term of local randomness there is gap between quantum mechanics and information causality condition.

The maximum probability of success of the Hardy's and Cabello's non-locality (for the two qubits system) in Quantum mechanics is 0.09 and 0.1078 respectively [Kunkri et al., 2006]. Interestingly, for generalized nonlocal no-signaling theories we find that this bound is 0.5 in both the cases and the probability set for which this is achieved coincides with the PR correlation. We showed that on applying the principle of Information causality this bound decreases from 0.5 to 0.20717 in both the cases, but could not reach their respective Quantum mechanical bounds. Interestingly, in quantum mechanics the maximum probability of success for the Cabello's case is not same as the Hardy's case [Kunkri et al., 2006]. Since the condition given by equation (5.4) of the present paper is a sufficient condition derived in [Pawlowski et al., 2009] for the violating IC, the probability derived here are therefore, strictly, an upper bound on the maximum probabilities allowed in an IC-respecting no-signaling theory. Restricting the no-signaling probability set by the full power of IC principle may reduce the probability to the quantum limit. However, it is curious that the same sufficient condition for violating the IC, gives the quantum bound for the CHSH expression [Pawlowski et al., 2009].

Chapter 6

Summary and Future Directions

Motivation underlying the work presented in this thesis was to highlight various fundamental aspects in the study of quantum correlations. Many of the characteristic features of quantum correlations are revealed here through study of simple bi-partite systems. The EPR-paper [Einstein, Podolsky and Rosen, 1935] and the follow up work by John Bell [Bell, 1964] demonstrated that nonlocality is intrinsic to quantum mechanics. These works renewed deep interest in the foundational studies of quantum mechanics; a comprehensive study on foundational issues in quantum mechanics can be found in the work [Home, 1997] and [Home and Whitaker, 2007]. Many crucial experiments where performed demonstrating the existence of nonlocal correlations conforming to the quantum mechanical predictions [Aspect, Dalibard and Roger, 1982; Aspect, Grangier and Roger, 1982; Tittel et al., 1998]. On the other hand, people started harnessing quantum mechanics for various information processing tasks like superdense coding [Bennett and Wiesner, 1992], quantum teleportation [Bennett et al., 1993], quantum cryptography [Bennett, 1992] and quantum computing [Steane, 1998] which lead to the rise of quantum information science [Nielsen and Chuang, 2002].

Quantum entanglement which is necessary for generating nonlocality is also the key resource in quantum information processing tasks. However, relationship between quantum nonlocality and quantum entanglement is complex—the two concepts are not always proportional; the study along this line was first initiated by Werner by showing that there can be local-realistic model for certain entangled states [Werner, 1989]. A. J. Leggett gave a new twist to the study of quantum nonlocality

[Leggett, 2003] by asking a question that whether there can be some nonlocal-realistic model for entangled states? Contrary to the quantum mechanical feature that subsystems of a pure entangled states are in mixed state (with no definite properties), the class of Leggett's nonlocal-realistic modal tries to introduce sharp properties to the subsystems at a hidden variable level.

Some other important questions arose in the study of nonlocal quantum correlations in the context of causality principle. The no-signaling condition was shown to be respected by quantum correlations. An interesting line of study was initiated by Gisin showing the use of no-signaling condition as a tool to either find the limits of quantum mechanics, like constraining any conceivable non-linear modification of the Schrdinger equation [Gisin, 1990; Gisin and Rigo, 1995], or to obtain specific bounds on quantum operations, like deriving bound on the fidelity of quantum cloning machines [Ghosh et al., 1999; Gisin, 1998]. On the other hand, Popescu and Rohrlich [Popescu and Rohrlich, 1994] asked an important question: why nonlocality in quantum mechanics is limited by Cirel'son's bound? They showed the existence of supra-quantum no-signaling correlation, like a PR-correlation. A new line of research emerged which tries to distinguish quantum correlation from supra-quantum correlations by proposing physical principles like non-trivial communication complexity [Brassard et al., 2006; van Dam, 2000], Information Causality [Pawlowski et al., 2009], and Macroscopic Locality [Navascues and Wunderlich, 2010].

We briefly summarize the results presented in this thesis:

Chapter-2: Simulation of quantum statistics for Werner state by LHV has been an interesting area for understanding the physics of entanglement [Barrett, 2002; Gisin, 1991; Gisin and Peres, 1992; Popescu, 1994; Popescu and Rohrlich, 1992; Werner, 1989]. In chapter-2, we posed the problem from opposite direction i.e. rather than weakening the (singlet) state we search for the class of (weakened) dichotomic observable (POVM) for which local model can be provided [Rai, Gazi, Banik, Das and Kunkri, 2012]. We provide the subset of the most general two outcome measurements represented by positive operator value measure (POVM) and presented local models for singlet statistics generated from them. It will be interesting to study whether the set can be enlarged with respect to different LHV model.

Chapter-3: Next, we discussed about Leggett's nonlocal-realistic model for entangled states which tries to assign sharp properties to constituent subsystems. Leggett's model lead to testable inequalities which are violated by quantum mechanics. However, success of Leggett's model in reproducing the correlations observed in standard Bell-CHSH tests (with co-planer observables) motivated new experiments for testing this model vis-a-vis quantum mechanics. In chapter-3, we derive two new forms [Rai, Home and Majumdar, 2011] of Leggett-type inequalities which, unlike the previous derived forms [Branciard et al., 2008, 2007; Groblacher et al., 2007; Leggett, 2003; Paterek et al., 2007], puts no geometrical constraints on the relevant measurement settings. These forms are believed to be more convenient for performing future tests of Leggett's model.

Chapter-4: In chapter-4, predictive power of the no-signaling condition is demonstrated [Home, Rai and Majumdar, 2013] in a *testable* situation involving a *non-ideal* measurement setup. To this end, an example is formulated using a non-ideal Stern-Gerlach (SG) device in one of the two wings of the EPR-Bohm entangled pairs. In this wing, for two types of measurement in the other wing, we consider the spin state of a set of particles selected such that they are confined to a particular half of the plane after they emerge from the SG magnetic field region. Due to non-idealness of the SternGerlach setup, this spin state will have superposing components involving a relative phase for which a testable *quantitative constraint* is obtained by invoking the no-signaling condition, thereby providing a means for precision testing of this fundamentally significant principle.

The derivation of our main result here is based on no-signaling condition and the standard rules of quantum mechanics that include, in particular, taking the overall dynamics of the non-ideal Stern-Gerlach device to be linear and the probabilities for measurement outcomes at any given time given by the standard Born rule. The linearity condition of quantum dynamics and the no-signaling condition are interlinked [Gisin, 1990; Gisin and Rigo, 1995] and, in particular, the thorough analysis by Simon et al.[Simon et al., 2001] brings out the point that once the no-signaling condition and the experimentally well-verified standard Born rule are taken to be valid, quantum dynamics is rather rigidly constrained to be linear. Thus, our result turns out to be empirically invalid, it would seem plausible to infer that the violation of no-signaling condition arises from a departure from linearity in the way a non-ideal Stern-Gerlach setup acts in conjunction with the type of post-selection process that is used in our example. The tenability of such a possibility would then call for further investigation; for example, the general framework for accommodating non-linearity in quantum dynamics discussed by Weinberg [Weinberg, 1989] may be invoked in the context of our setup. It is hoped that the predictive power of the no-signaling condition in a testable situation as illustrated by the example treated here using a non-ideal measurement setup may motivate the formulation of other such examples that could be helpful for a deeper understanding of the role of NSC in the context of non-ideal measurement situations.

Chapter-5: In this chapter, in a generalized no-signaling framework, we study applications of information causality principle [Pawlowski et al., 2009] to two two-level systems showing Hardy-type nonlocality. First, by noting that maximally entangled state in quantum mechanics does not reproduce Hardy's correlation whereas generalized non-signaling theory put no such restriction on the local randomness of the observable for Hardy's correlation, we study [Gazi, Rai, Kunkri and Rahaman, 2010] all the possibilities of local randomness in Hardy's correlation in the context of information causality condition. We observe that in term of local randomness there is gap between quantum mechanics and information causality condition.

Next, we derive an upper bound on maximum success probability of Hardy-type arguments under the information causality principle [Ahanj, Kunkri, Rai, Rahaman and Joag, 2010]. The maximum probability of success of the Hardy's and Cabello's non-locality (for the two qubits system) in Quantum mechanics is 0.09 and 0.1078 respectively [Kunkri et al., 2006]. Interestingly, for generalized nonlocal no-signaling theories this bound is 0.5 in both the cases and the probability set for which this is achieved coincides with the PR correlation. We showed that on applying the principle of information causality this bound decreases from 0.5 to 0.20717 in both the cases, but could not reach their respective Quantum mechanical bounds. Interestingly, in quantum mechanics the maximum probability of success for the Cabello's case is not same as the Hardy's case [Kunkri et al., 2006]. Since the condition applied here is only a sufficient condition derived in [Pawlowski et al., 2009] for the violating IC, the probability derived here are therefore, strictly, an upper bound on the maximum probabilities allowed in an IC-respecting no-signaling theory. Restricting the no-signaling probability set by the full power of IC principle may reduce the probability to the quantum limit.

6.1 Future directions

Below some open problems are listed which are going to be considered in the near future (some of them are more general, others are particularly linked with this thesis):

- The nonlocal feature of quantum mechanics has given rise to an area where this non-classical feature can be studied independently of any particular physical theory. It has been shown that there are nonlocal correlations [Popescu and Rohrlich, 1994] which is post quantum but still respects no signaling condition. So the natural question arises what are the other conditions that restrict the physical world to obey quantum mechanics? There are various suggestions and one of the surprising discoveries in this direction is the Information Causality principle [Pawlowski et al., 2009]. This condition exactly reproduces the optimal Bell-CHSH inequality violation by quantum mechanics. In this sense it rejects all post quantum correlation whose Bell violation goes beyond Cirel'son bound. But still it does not detect all post quantum correlation as for example it does not reproduce the optimal success probability for Hardy's nonlocality [Ahanj, Kunkri, Rai, Rahaman and Joag, 2010] argument in quantum mechanics. So it is an active area of research to suggest further conditions which could reproduce some quantum features if not the whole quantum mechanics.

- Recently, there are some results which show that optimal quantum Bell violation can also be reproduced by uncertainty principle [Oppenheim and Wehner, 2010] and complementarity principle [Banik, Gazi, Ghosh and Kar, 2012]. This is surprising because uncertainty relation and complementarity principle can restrict the correlation that can be achieved in physical world. So these results provide a new direction where various quantum features like uncertainty, complementarity, nonlocality, steering are intrinsically related with each other and all these features can be also explored in general non-signaling probabilistic theories—see for example some recent works [Dey et al., 2013; Mal et al., 2013; Pramanik et al., 2013].

- In this thesis we have discussed only bi-partite correlations. The multipartite correlations in quantum mechanics are more complex and these correlations have no unique measure. So reproduction of multi-partite quantum correlations by some general physical principles is itself a difficult problem and still there is no such solution. Recently it has been shown [Das et al., 2012, 2013; Gallego et al., 2011] that there are post quantum tri-partite correlations which satisfy all discovered or yet to be discovered bi-partite principles. This result immediately suggests that there must be some multipartite principles along with bi-partite candidates to detect post quantum correlations. There has been little development in this direction. Only there is an inequality which arises in the context of a game known as 'Guess your neighbor's input' (GYNI game) [Almeida et al., 2010]. There are post quantum multipartite correlations that violate the GYNI inequality but there are also post quantum multipartite correlations that satisfy this (GYNI) inequality. It will be interesting to search for more physically motivated information theoretic multipartite principle which would draw line between quantum and post quantum multi-partite correlations.

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