Chern Simons Theory in the Context of 2+1 and 3+1 Quantum Gravity

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TO
MY BEST FRIEND

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Chapter 0

Introduction

The Chern Simons form as a topological invariant of three-manifolds first appeared in [1]. Its use as a Lagrangian of a quantum field theory was done way back in '82 [2] in the context of generating topological mass in gauge theories. However it was Witten [3], who first related its quantum field theoretic properties with knot polynomials, thus establishing 'Chern-Simons (to be referred as CS from now on) theory' as a viable non-trivial theory and hence appealing in its own right to both mathematicians and mathematical physicists.

It is a first order topological gauge theory and has the following physical properties.

- It does not have local (or) propagating degrees of freedom. This means that there are exactly that many constraints (in Dirac's classification) coming from gauge invariance, which suffice to 'kill' the superficial local field degrees of freedom. This essentially boils down to the fact that the physical or gauge invariant phase space is at most finite dimensional as is the case for particle mechanics with finite degrees of freedom. Put differently, there is no gauge mediator here, unlike any other gauge theory describing particle interactions.

- Secondly, from the physicists point of view, it is striking to observe that the CS Lagrangian 'does not need' a 'space-time' metric unlike other field theories. Hence it can consistently be defined on a manifold without (pseudo) Riemannian structures. However in order to connect with the physical world we assume manifolds which can be split locally as $X = [0, 1] \times \Sigma$, where Σ is any orientable 2-surface. This assumption (by introducing by hand a 'time' interval) helps us performing the canonical analysis of the dynamics.

- Independence of any background metric enables it with the whole diffeomorphism group as its (gauge) invariance beside the usual 'gauge group' relevant to any gauge theory.

- Time re-parametrization is among the set of diffeomorphism invariances (as is also the case of any generally covariant theory). This means that the local Hamiltonian function vanishes as a first class constraint (in Dirac's terminology).

- The Lagrangian contains only first derivative of the basic dynamical variable, the connection. Hence equation of motion also involves first derivative unlike other field theories.

0.1 Defining the classical CS

Before proceeding further with properties of CS theory which have been chronologically unveiled in literature, we digress a little bit on the basic framework needed for the definition of a classical CS theory. As a set-up, one needs the following ingredients: a differential 3 manifold X, a Lie group G (preferably connected, simply connected and having a semi-simple Lie algebra $\mathfrak{g}$) and a chosen finite dimensional representation R of $\mathfrak{g}$. We should then construct a principal G bundle $\pi : P \to X$ and its associated vector bundle $E \to X$ with structure group G. The connections on P (or rather its pull-back to X) are the dynamical variables of CS theory (like any other gauge theory). Now, these connections take value in the Lie algebra $\mathfrak{g}$. On the other hand, for obvious reasons, the physical action functional maps connections to $\mathbb{R}$ (at most $\mathbb{C}$ in certain physically allowed cases). It is then intuitively clear to us, that in order to construct an action we need a symmetric, bilinear, Ad-invariant form $\mathrm{tr}_R : \langle \cdot, \cdot \rangle \to \mathbb{R}$ on $\mathfrak{g}$ for a particular representation R. Consider now the vector bundle E. From Chern-Weil theory, we know the de-Rahm cohomology of the Chern characteristic class (they are closed). The second Chern class is a four form whose anti-derivative in the local form is the Chern-Simons 3-form.

One should remember, at this point, that the CS form just described is defined on P. However for all physical purposes we need its local version back on X. In this situation we have to choose smooth local sections of $p : X \to P$. The next step is to pull back the global CS form on P through the map p^* to X. If the gauge group G is connected and simply connected, it can be shown that these principal bundles can be made trivial and that they allow smooth global sections [4]. Therefore one can write the action functional as an integral of the pulled back CS form from P to X:

$$S[A] = \frac{k}{4\pi}\int_{\mathcal{X}} \mathrm{tr}_R\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \tag{0.1}$$

where $A : X \to \mathfrak{g}$ is the local form of the connection on X. In general, $k \in \mathbb{R}$, which however takes integral values in special cases. Obviously now, maps between sections of P induce transformation in the local (section dependent) connection A. These are the standard gauge transformations: $A \to g^{-1}Ag + g^{-1}dg$, where $g : X \to G$. If $\partial X = \varnothing$, then (0.1) is gauge invariant, upto integral of the the Wess-Zumino-Witten term

$$\sim \mathrm{tr}_R(g^{-1}d\,g\wedge g^{-1}d\,g\wedge g^{-1}d\,g) \in H^3(G,\mathbb{R}).$$

However in the other case, where X does have a non-empty boundary, the action depends upon the choice of section (and hence choice of gauge) on ∂X through $\mathrm{tr}_R(g^{-1}A\wedge dg)$. The analysis of functional differentiability of (0.1) with respect to A is somewhat involved and we prefer not to delve into that right now. However we take note of the fact that the answer to the questions regarding this is affirmative and the equation of motion that comes out of the ensuing variational calculation is summarized as:

$$dA + A\wedge A = 0; \tag{0.2}$$

ie, the connections should be 'flat'. We will come back to the classical properties of the theory later, with phase space analysis.

0.2 Relationships with conformal field theory and knot theory

Despite being a theory of flat connections, which is devoid of propagating degrees of freedom, quantum CS theory have been attracting much interest of mathematical physicists for a long period. It was Atiyah, who suggested that a connection should exist between knot theory (Jones' braid group) [5] representations and mapping class group representations coming from conformal field theories [1]. He moreover stressed that these apparently disconnected topics be merged by a gauge theory in 3 or 4 dimensions. This is because 2-D conformal field theory on its own does not provide any explanation for this seemingly weird relation with invariants of 3-manifolds. Now, the later representations were extensively studied by physicists [7–9]. It was however Schwarz's, who pinned down that such a link may exist through CS theory [10]. Finally, Edward Witten's exposition [3] made this relation explicit.

The primary question that crops up in this milieu is regarding the entente between CS theory and conformal field theories. [2] The structure of the Hilbert space (of quantum states) of quantum CS theory is again same as the space of conformal blocks that appear in conformal field theory of current algebra for CFTs with Lie algebra symmetry.

On the other hand, the naturally gauge invariant objects of a gauge theory are Wilson loops. For a closed loop $\mathcal{C}$ and for a particular representation R of the Lie algebra, one can construct a Wilson loop as :

$$W_R^{\mathcal{C}} = \mathrm{tr}_R \left(P \exp \left(\int_{\mathcal{C}} A \right) \right) \tag{0.3}$$

with P denoting a path ordering prescription. Now, consider a link

$$L = \bigcup_{i=1\ldots n} C_i$$

with C_i being oriented knots. For example, the vacuum expectation value

$$\langle W^L_{j=\frac{1}{2}} \rangle = Z^{-1} \int [DA]/(\text{gauge}) W^L_{j=\frac{1}{2}} \exp(iS)$$

[1]However, one should not forget to mention other (than 2 dimensional conformal field theories) affairs knot polynomials share with physical systems in 2 dimensions. These include essentially integrable systems [6].

[2]Along with the development of the quantization procedure of CS later in this thesis, it will become clear, that for three manifolds, (which can be split as $\mathbb{R} \times \Sigma$) the topology of Σ becomes one of the most important factors. The moduli space of connections, which serves as the symplectic manifold of the theory depends upon Σ both in terms of its topology and complex (Kähler) structure. The Hilbert space of physical states (in terms of theta functions) constructed upon the moduli space is in general finite dimensional (considering the moduli space to be compact and connected).

of the Wilson loops over the link L defined for the representation $j = \frac{1}{2}$ of gauge group $SU(2)$ gives the Jones polynomial (in the variable $q = \exp(\frac{2\pi i}{k+2})$, k being the CS level) invariant associated with the link. One has other invariant polynomials for other representations and other groups. We refer to [11] and references therein for an elaborate account of knot invariants coming from different compact gauge groups.

However when one does not consider links in a manifold, CS theory still offers topological invariants. For example, perturbative evaluation of the path integral at one loop [12,13] gives the Ray Singer analytic torsion, which are well-known 3-manifold invariants.

0.3 Entrance of quantum gravity

Einstein's general relativity (GR), is till date the most successful candidate for describing gravitational interaction. It has enormous precision, compared to the ones enjoyed by electro-weak theory in predicting celestial events, be it bending of light or mercurial precession. However like all other physical theories, general relativity has its limitations and self-consistency problems, which force one to look beyond classical version of the theory. Any plausible theory of quantum gravity should address some fundamental problems where classical GR gets stuck. These include the problem of singularities (big bang and the physical singularities covered by the black-hole horizon), the *essentially non-classical* and statistical mechanical origin of black-hole thermodynamics, information loss paradox and re-establishing semi-classical physics starting from the Plank-scale arena among others.

We must spell out here that as a theoretical problem, quantizing gravitational interactions, is one of the most notorious ones faced by physicists till now. This is primarily because of non-renormalizability of 4 dimensional general relativity, unlike the viable models of particle physics. The way-out from this obstacle may include looking beyond standard wisdom of perturbative renormalization program, the holy grail of high energy physics. The most promising candidate in this direction is string theory where one adopts a framework, which uses manifestly finite amplitudes. String theory has shown its robustness and stood upto many of the expectations, as far as quantum gravity is concerned. But string theory has its own problem regarding background independence, which any perturbative approach towards quantum gravity is plagued with. In addition to that, unitarity of graviton scattering S matrix and the associated information-loss problem are among some questions, which the theory has still fallen short of answering.

The immediately next (in terms of effort put into) avenue is loop quantum gravity (LQG), which starts with the goal of considering gravity alone and does not attempt to develop a quantum theory of all other matter and interaction at a same go. This framework in manifestly non-perturbative, background independent and retains general covariance at each step. Number of successful resolution of physical problems in LQG is somewhat satisfactory. However, it suffers with ailments like non-uniqueness of the regularization of the diffeomorphism, Hamiltonian and volume operator.

With this extremely brief outline of the treatment of quantum gravity at hand, we wish to refuge to the original discussion: some of the uses of CS theory in quantum gravity and associated literature survey.

0.3.1 ABJM and Bagger-Lambert theory

Let us briefly take note of another very interesting recent advancement in CS theory in context of string/ M theory. To be precise, this is achieved through M theory duality [14] and named ABJM theory. Unlike our study in pure gravity, in this case one needs to consider a super-symmetric ($\mathcal{N} = 6$) version of it, in order to have conformal invariance. This is done in view of duality (AdS_4/ CFT_3) between bulk M theory on $AdS_4 \times S^7/\mathbb{Z}_k$ and superconformal CS theory. Actually a $U(N) \times U(N)$ Chern Simons theory with k and $-k$ as levels, sufficiently supplemented with matter fields are employed to describe N M2 branes. At large k 't Hooft limit however (one dimension for the M theory description getting compactified) the duality is precisely expressed in terms of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. At this limit this duality is considered as the second most important example of AdS/CFT duality.

Number of plausible conformal field theories in dimensions greater than 3, is limited. On the other hand world-volume theory of eleven dimensional M theory is supposed to be conformal. Highly super-symmetric CS theories are conformal and therefore are good candidates for describing world-volume theories of M theory (M2 brane) [15]. This is another interesting way, in which, CS is being used in M theory context.

0.3.2 3 dimensional quantum gravity and Chern Simons theory

Resorting to dimensions less than 4 may be uncomfortable in general consensus [3]. We assert that results from 3 dimensional quantum gravity do not directly yield physical results for the real 4 dimensional case. Moreover classical gravity in three space-time dimensions does not possess local degrees of freedom, which may render it apparently trivial as a physical theory. This feature however makes sure that 3D quantum gravity can be exactly solved [16] in certain cases. Despite this superficial simplicity, 3D quantum gravity has its usefulness in providing interesting information [17] about what to expect from some of the major issues in the 4 dimensional counterpart. Studies in 3D gravity captured attention mainly after one of the most important discoveries in the field: finding the BTZ black-hole solution [18]. Another result, which turns out to be of great significance in the light of relatively recent studies in holographic duality was given way back in '86 [19]. This work showed firstly, the existence of the conformal algebra as the asymptotic symmetry of asymptotically AdS_3 spaces and secondly appearance of central extensions in its canonical realization.

One may now ask: where does CS theory enter in this set up? Answer again lies within Witten's work [16]. We take this chance to remind ourselves that although Einstein's metric formulation of a generally covariant theory of gravity is the most standard one, a number of alternative descriptions have arisen with time. The most prominent one is the first order formulation with frames and connections. Once formulated in these variables, classical 3D gravity theory essentially becomes a Chern Simons theory with the gauge group depending upon the cosmological constant and the equivalence becomes exact as far as classical analysis is concerned [4]. At the face value one can appreciate this connection, since both 3D gravity and Chern Simons theory are devoid of propagating degrees of freedom. But, there are ways to incorporate local physical modes without incorporating matter degrees of freedom, which became known as topologically massive

[3]Dimensions greater than 4 are of somewhat ease, thanks to string/M theory. That is, decent to 4 dimensions by shrinking compact dimensions is a fairly understood mechanism.

[4]We will get back to the credibility of this equivalence at the quantum level later [20]

gravity [2]. Relatively recently, newer excitement in this field has been observed, due to the connection of higher spin theories in 3d gravity, its Chern Simons avatar and holographic dualities (see [21], [22] for example). As we progress through this thesis, we will present more commentaries on the development and features as they appeared chronologically in literature.

0.3.3 4 dimensional quantum gravity and Chern Simons theory

The appearance of Chern Simons theory in 4 dimensional gravity is more dramatic than the 3 dimensional counterpart and the results following from it are even more interesting. In order to present a brief description, we first consider a null three dimensional submanifold (which is a boundary)*in* a 4 dimensional space-time (arbitrary), equipped with a set of geometric properties. However no a-priori background is fixed. The minimal geometric structures render the 3-manifold with the properties of a local *horizon* [23], giving it a name isolated horizon. One of the advantages of it having a local description is that arbitrary matter and radiation are allowed to exist outside the horizon. This construction is purely kinematical. One then ventures into the quantum theory. As a first step towards that is to study gravitational dynamics on and in presence of isolated horizons. The favoured formulation for gravitational dynamics for this purpose is the first order one. In this framework, it turns out that the degrees of freedom on the horizon are governed effectively by a CS theory [24]. It was demonstrated that this has $U(1)$ [5] gauge invariance.

Once one has CS (level of which is proportional to the area of the compact slicing of the isolated horizon), quantum modes residing on the horizon itself become known. But as one progresses in LQG, such a quantization has to be done consistently with the quantization of the bulk degrees of freedom. A careful calculation about the boundary quantum states consistent with the bulk states gives us a handle to estimate a microstate counting with a fixed horizon area (and hence CS level). This results in reproducing the Bekenstein-Hawking black hole entropy. Using a finer counting later, true quantum corrections to this entropy was calculated [25].

[5] However it was later contested by many calculations from other groups who advocated for an $SU(2)$ theory. We will comment in detail later in the thesis on this topic.

0.4 Plan of the thesis

Clearly the thesis can be viewed as a union of two 'almost' mutually disjoint topics, ie quantum gravity in 3 and 4 dimensions, unified by the common theme of Chern Simons theory. In this view we partition the work broadly in these two sectors.

Part-1: 3D Gravity

In chapter 2, the first one in the first sector, we present a classical kinematical and dynamical analysis of 2+1 gravity. We discuss the equivalence of it with CS theory, including the variation of the gauge group in different cases. Moreover classical dynamics is studied in both canonical and covariant formalism. As an improvement over the conventional theory we introduce a dimensionless parameter γ in the theory which would prove to be useful in developing a quantum theory later on.

With this basic set-up at hand, we delve into building a model quantum theory of Lorentzian 3d gravity with negative cosmological constant in chapter 3. As a choice of space-time topology we took spatial slices as genus-1 Riemann surfaces. We carry out the quantization in constrain first approach, which deals with first finding out the physical phase space (moduli space of flat connections) and then quantizing it (geometric quantization is chosen for this purpose, owing to the non-trivial topological nature of the phase space). We construct a finite dimensional Hilbert space and stress the importance of γ in light of this Hilbert space.

Topic of the next chapter (ch. 4) however roams mostly in the classical and semi-classical arena. We consider asymptotically AdS_3 spaces which also incorporate isolated horizons. We derive dynamics of the horizon, zeroth and the first law of black hole mechanics and observe the influence of γ. Moreover, we show the canonical implementation of asymptotic Witt algebra in symplectic geometry framework and recover the asymptotic Virasoro algebra, from which semi-classical entropy of BTZ black hole (as a function of γ) is calculated.

Our next project, which is the subject matter of chapter 5 is in the regime of Euclidean signature and positive cosmological constant. We calculate partition function of 3d gravity starting from CS framework on prototype Lens space topologies and sum over all possible topologies. Once again the might of γ was revealed, in taming the divergences.

Part-2: 4D Gravity

We then shift gear and move over to problems in more realistic world of 4 dimensions. Our work in 4 dimensional context revolves around generalized (local) versions of black-hole event horizon, namely isolated or non-expanding horizon. We take-up this framework in order to facilitate the study of quantum excitations on black hole horizons in the regime of Loop Quantum Gravity (LQG). In doing so we would focus on the utility of CS theory in studying horizon mechanics.

In chapter 6, we would like to make the stage for later quantum calculations. This involves studying the classical kinematical and dynamical issues of gravitational interaction on non-expanding horizon. In particular we analyse the reduced local symmetry on the horizon.

The next and the penultimate chapter involves the details of black hole entropy calculation upto logarithmic correction using $SU(2)$ Chern Simons theory.

We then come to an end by summarizing the works presented in the thesis and citing some possible future directions.

3D Gravity

Chapter 1

Kinematics and Classical Dynamics

General theory of relativity is usually formulated as a theory of metrics. But it can also be recast as a theory of connections (on a principle G-bundle). Even Einstein, the master himself and Schroedinger initiated a project to formulate general relativity in terms of connections. It became complicated because they chose to use Levi-Civita connection. Things become more systematic and simpler if one wishes to use spin-connections. In connection variables general relativity apparently a close relative of gauge theories. In fact, it is well known (and we will demonstrate below) that in three dimensions, gravity exactly becomes a gauge theory in connection (first order) formulation. Despite this alluring similarity, as a cautionary remark, we must also mention that unlike the other gauge theories encountered in the arena of particle physics, gravity as a gauge theory should be independent of any background geometrical structure. In this chapter our goal is to set up the classical framework, on which most of the first part of this thesis is based.

To start with, we consider a three manifold M, preferably without boundary. Consider a frame bundle $\pi : F \to M$ on it with fibres as 3-D vector spaces. We also have the canonical bundle $\pi : T \to M$, the tangent bundle on it, whose fibres are also 3-D vector spaces. We now define define isomorphism between these fibres point wise on M:

$$e_{(p)} : F_{(p)} \to T_{(p)} \qquad \forall p \in M.$$

For $\dim(M) = 3$ these invertible maps are called triads (dreibeins). In particular its action on a vector $V_I \in F_{(p)}$ to the vector $V_\mu \in T_{(p)}$ is expressed as: $e^I_\mu(V_I) = V_\mu$ and its inverse appears trivially. This action can thus be generalized on contravariant vectors and arbitrary higher rank tensors.

Keeping in mind the pseudo-Riemannian structure of gravity, we associate a metric tensor (degenerate, as it should be) on F_p for each p. In suitable basis, this metric can always be viewed as $\eta = \text{diag}(-1, 1, 1)$. Automatically we have $SO(2,1)$ or $SU(1,1)$ as the structure group of F.

From now on we will use the 1-form $e^I = e^I_\mu dx^\mu$ as one of the dynamical variables and action of the structure group in the defining representation on it will be $e^I \to \Lambda^I{}_J e^J$.

The other dynamical variable we will use is the connection ω on F. In general it is an antisymmetric tensor in fibres of F and a 1-form in fibres of T. Since M is 3 dimensional, using the total anti symmetric form of $F_{(p)}$ we can use ω (rather its local section form) as the 1-form $\omega^I = \omega^I{}_\mu dx^\mu$.

We will be using this pair e, ω throughout the first part (3D gravity part of this thesis) as the set of dynamical variables. We would start with the classical action of 3D gravity, then construct its classical phase space and perform the canonical analysis. Results of this chapter, which are well known in 3-d gravity literature will be recalled frequently in the first part of the thesis. Also note that these analyses broadly follow those presented in our paper [26].

1.0.1 Formulating 2+1 Gravity as a Chern Simons theory

Action for 2+1 gravity with negative cosmological constant $\Lambda = -\frac{1}{l^2}$ on a space time manifold M in first order formalism is

$$I_{GR} = 2\int_M e^I \wedge \left(2d\omega_I + \epsilon_I{}^{JK}\omega_J \wedge \omega_K + \frac{1}{3l^2}\epsilon_I{}^{JK} e_J \wedge e_K\right) \tag{1.1}$$

We have chosen units such that $16\pi G = 1 = c$ and will continue using this unit unless mentioned at special instances, where explicit presence of G is crucial in terms of physical plausibility of a result. Moreover the sign of cosmological constant can be chosen to be positive and we will point out the changes due to that when needed.

The above action is well defined and functionally differentiable in absence of boundaries. In presence of boundary (internal and/or asymptotic) [27] one has to add suitable boundary terms to the action in order to have a finite action with well defined (differentiable) variation.

The equations of motion that follow from the action (1.1) are:

$$F_I := 2d\omega_I + \epsilon_I{}^{JK}\omega_J \wedge \omega_K = -\frac{1}{l^2}\epsilon_I{}^{JK} e_J \wedge e_K \quad (1.2)$$

$$T_I := de_I + \epsilon_{IJK} e^J \wedge \omega^K = 0 \quad (1.3)$$

As expected, these equations of motion, combined are equivalent to the Einstein's equation. To be more explicit, one first makes the identification with pseudo Riemann metric: $e^I_\mu e^J_\nu \eta_{IJ} = g_{\mu\nu}$. The next step is to solve ω from (1.3) in favour of e and substitute back in (1.2). This then gives Einstein's equation.

A more general model for 2+1 gravity with negative cosmological constant was introduced by Mielke *et al* [28, 29] and later studied extensively in [30–32], which without matter fields read:

$$I = aI_1 + bI_2 + \alpha_3 I_3 + \alpha_4 I_4 \quad (1.4)$$

where

$$I_1 = \int_M e^I \wedge \left(2d\omega_I + \epsilon_I{}^{JK}\omega_J \wedge \omega_K\right)$$

$$I_2 = \int_M \epsilon^{IJK} e_I \wedge e_J \wedge e_K$$

$$I_3 = \int_M \omega^I \wedge d\omega_I + \frac{1}{3}\epsilon_{IJK}\omega^I \wedge \omega^J \wedge \omega^K$$

$$I_4 = \int_M e^I \wedge de_I + \epsilon_{IJK}\omega^I \wedge e^J \wedge e^K$$

However this model does not reproduce the equations of motion (1.2) and (1.3) for arbitrary values of the parameters a, b, α_3, α_4. We choose, as a special case of the above model, those values of these parameters which gives the expected equations of motion as in [16, 20, 30, 33–41]:

$$a = 2 \quad b = \frac{2}{3l^2} \quad \alpha_3 = \frac{2}{\gamma} \quad \alpha_4 = \frac{2}{\gamma l} \quad (1.5)$$

γ is introduced as new dimensionless parameter from 2+1 gravity perspective. Effectively (1.5) is the equation of a 3 dimensional hypersurface parametrized by G, l, γ in the 4-d parameter space of a, b, α_3, α_4. It may now be tempting to argue that the term added to the 'original' action (1.1) is a total derivative for the choice (1.5). However the reality is that, the new term on its own also produces the same equations of motion. This will become clearer as we go through the CS counterpart of the story in the following.

We take this opportunity for a little digression for the Chern Simons formulation. Following [16, 30] one introduces the $SO(2,1)$ or equivalently $SL(2,\mathbb{R})$ or $SU(1,1)$ connections for a principal bundle over the same base space of the frame bundle:

$$A^{(\pm)I} := \omega^I \pm \frac{e^I}{l} \ .$$

It is easily verifiable that the action

$$\tilde{I} = l\left(I^{(+)} - I^{(-)}\right) \tag{1.6}$$

is same as (1.1) in absence of boundaries. Where

$$I^{(\pm)} = \int_M \left(A^{(\pm)I} \wedge dA^{(\pm)}_I + \frac{1}{3}\epsilon_{IJK} A^{(\pm)I} \wedge A^{(\pm)J} \wedge A^{(\pm)K} \right) \tag{1.7}$$

are two Chern Simons actions with gauge group $SO(2,1)$, the lie algebras being given by

$$\left[J^{(+)}_I, J^{(+)}_J\right] = \epsilon_{IJK} J^{(+)K} \qquad \left[J^{(-)}_I, J^{(-)}_J\right] = \epsilon_{IJK} J^{(-)K}$$
$$\left[J^{(+)}_I, J^{(-)}_J\right] = 0. \tag{1.8}$$

The metric on the Lie algebra is chosen to be

$$\langle J^{(\pm)I}, J^{(\pm)J} \rangle = \frac{1}{2}\eta^{IJ}$$

where $J^{(\pm)I}$ span the $\mathfrak{so}(2,1)$ (or $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(1,1)$) Lie algebras for the two theories.

One striking feature of this formulation is that the last two terms of (1.4) can also be incorporated in terms of $A^{(\pm)}$, for ($\alpha_3 = l^2\alpha_4$) as:

$$\begin{aligned} I^{(+)} &+ I^{(-)} \\ &= 2\int_M \left(\omega^I \wedge d\omega_I + \frac{1}{l^2} e^I \wedge de_I + \frac{1}{3}\epsilon_{IJK}\omega^I \wedge \omega^J \wedge \omega^K + \frac{1}{l^2}\epsilon_{IJK}\omega^I \wedge e^J \wedge e^K \right) \end{aligned} \tag{1.9}$$

and the same equations of motion (1.2) and (1.3) are also found from varying this action.

We thus propose the action

$$\begin{aligned} I &= l\left(I^{(+)} - I^{(-)}\right) + \frac{l}{\gamma}\left(I^{(+)} + I^{(-)}\right) \\ &= l\left[(1/\gamma + 1)\, I^{(+)} + (1/\gamma - 1)\, I^{(-)}\right] \end{aligned} \tag{1.10}$$

with a dimensionless non-zero coupling γ. This action (1.10) upon variations with respect to $A^{(+)}$ and $A^{(-)}$ give equations of motion as expected from Chern Simons theories. This imply that the connections $A^{(\pm)}$ are flat:

$$\mathcal{F}_I^{(\pm)} := dA_I^{(\pm)} + \epsilon_{IJK} A^{(\pm)J} \wedge A^{(\pm)K} = 0. \tag{1.11}$$

It is also easy to check that the above flatness conditions of these $SO(2,1)$ bundles (1.11) are equivalent to the equations of motion of general relativity (1.2), (1.3).

So far we have been talking about a case, where cosmological constant Λ is taken to be negative definite. We also considered the frame space to have an internal Lorentzian metric. However, there are cases, especially for path integral quantization, one has to take recourse to a Euclideanized version of the theory. Depending upon the choice of signature and the sign of the cosmological constant Λ, the CS gauge groups vary. The possible gauge groups in all the possible scenarios are tabulated below. We also list the homogeneous solutions (of the Einstein's equation) for various values of Λ

Λ	Homogeneous solution	CS gauge group	
		Lorentzian	Euclidean
$=0$	Minkowski	$ISO(2,1)$	$ISO(3)$
>0	dS_3	$SO(3,1)$	$SO(4) \simeq SU(2) \times SU(2)$
<0	AdS_3	$SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3,1)$

This is a good point to stop and probe into the physical relevance of this new parameter comparing with 3+1 dimensional gravity. In order to proceed we notice that the new action is in spirit very much like the Holst action [42] used in 3+1 gravity. In our case the parameter γ can superficially be thought of being the 2+1 dimensional counterpart of the original Barbero-Immirzi parameter. Moreover the part $I^{(+)} + I^{(-)}$ of the action in this light qualifies to be at par with the topological (non-dynamical) term one adds with the usual Hilbert-Palatini action in 3+1 dimensions, since this term we added (being equal to a Chern Simons action for space-times we consider) is also non-dynamical. But more importantly the contrast is in the fact that the original action, which is dynamical in the 3+1 case is also non-dynamical here, when one considers local degrees of freedom only.

Another striking contrast between the original B-I parameter and the present one lies in the fact that in the 3+1 scenario γ parameterizes canonical transformations in the phase space of general relativity. From the canonical pair of the $SU(2)$ triad (time gauge fixed and on a spatial slice) and spin-connection one goes on finding an infinitely large set of pairs parameterized by γ. The connection is actually affected by this canonical transformation, and this whole set of parameterized connections is popularly known as the Barbero-Immirzi connection. The fact that this parameter induces canonical transformation can be checked by seeing that the symplectic structure remains invariant under the transformation on-shell. On the other hand for the case at hand, ie 2+1 gravity, as we will see in the following sub-section that inclusion of finite γ is not a canonical transformation and it does not keep the symplectic structure invariant.

1.0.2 Symplectic Structure on the Covariant Phase Space

Consider a globally hyperbolic space-time manifold endowed neither with an internal nor an asymptotic boundary and let it allow foliations [1] $M \equiv \Sigma \times \mathbb{R}$, with Σ being compact and $\partial\Sigma = 0$.

In view of [43, 44] the covariant phase space, ie the space of solutions of the equations of motion the theory, is $\mathcal{V}_F^{(+)} \times \mathcal{V}_F^{(-)}$, product of spaces of flat $SO(2,1)$ connections as discussed in the last section. We now intend to find the pre-symplectic structure [2]. For

[1] On-shell (1.3) implies the space-time can be given (pseudo) Riemannian structure. With respect to the associated metric (0,2)-tensor and a time like vector field t^a the manifold is assumed to be Cauchy-foliated.

[2] It is being called the pre-symplectic structure since as we will point out later that only on the constraint surfaces this has the property to be gauge invariant. When we have a phase space parameterized by gauge invariant variables, this pre-symplectic structure will induce a symplectic structure on that.

that purpose, we start with the Lagrangian 3-form that gives the action (1.10):

$$
\begin{aligned}
L &= l\,(1/\gamma+1)\left(A^{(+)I}\wedge dA_I^{(+)}+\frac{1}{3}\epsilon_{IJK}A^{(+)I}\wedge A^{(+)J}\wedge A^{(+)K}\right)\\
&+ l\,(1/\gamma-1)\left(A^{(-)I}\wedge dA_I^{(-)}+\frac{1}{3}\epsilon_{IJK}A^{(-)I}\wedge A^{(-)J}\wedge A^{(-)K}\right)
\end{aligned}
\tag{1.12}
$$

The standard variation gives on-shell:

$$\delta L =: d\Theta(\delta) = d\Theta^{(+)}(\delta) + d\Theta^{(-)}(\delta)$$

where

$$(16\pi G/l)\,\Theta^{(\pm)}(\delta) = (1/\gamma \pm 1)\,\delta A^{(\pm)I}\wedge A_I^{(\pm)}. \tag{1.13}$$

The procedure of second variations [43, 44] then gives the pre-symplectic current

$$
\begin{aligned}
J(\delta_1,\delta_2) &= J^{(+)}(\delta_1,\delta_2) + J^{(-)}(\delta_1,\delta_2)\\
\text{where} &\qquad J^{(\pm)}(\delta_1,\delta_2) = 2\delta_{[1}\Theta^{(\pm)}(\delta_{2]})
\end{aligned}
$$

which is a closed 2-form ($dJ(\delta_1,\delta_2) = 0$) on-shell. The closure of J and the fact that we are considering space-time manifolds which allow closed Cauchy foliations imply that the integral $\int_\Sigma J(\delta_1,\delta_2)$ is actually foliation independent, ie independent of choice of Σ. Hence the expression $\int_\Sigma J(\delta_1,\delta_2)$ is manifestly covariant and qualifies as the pre-symplectic structure on $\mathcal{V}_F^{(+)}\times\mathcal{V}_F^{(-)}$. We thus define the pre-symplectic structure:

$$\Omega = \Omega^{(+)} + \Omega^{(-)} \tag{1.14}$$

where

$$
\begin{aligned}
\Omega^{(\pm)}(\delta_1,\delta_2) &= \int_\Sigma J^{(\pm)}(\delta_1,\delta_2)\\
&= \frac{k_{(\pm)}}{\pi}\int_\Sigma \delta_1 A^{(\pm)I}\wedge\delta_2 A_I^{(\pm)}
\end{aligned}
\tag{1.15}
$$

where the CS levels, $k_{(\pm)} = \frac{l(1/\gamma\pm1)}{8G}$, restoring G.

At this point we would like to note two important features of this symplectic structure, in light of the comparison we have been doing with the Barbero-Immirzi modified phase space of 3+1 gravity:

- In the 3+1 case the extra contribution of the Holst term (with coeffecient $1/\gamma$)in the symplectic structure can be shown to vanish on-shell. Hence in that context, it is clear that in the covariant phase space γ has the role of inducing canonical transformations. On the other hand, in the present case, it is very much clear from the above expression, that the γ dependent term cannot vanish, as suggested in the previous subsection. So, what we have at hand are infinite inequivalent theories for 2+1 gravity each having different canonical structure and parameterized by different values of γ at the classical level itself.

- The other point worth noticing is that Ω is indeed gauge invariant and it can be checked by choosing one of the two δ s to produce infinitesimal $SO(2,1)$ gauge transformations or infinitesimal diffeomorphisms and keeping the other arbitrary. In both these cases $\Omega\left(\delta_{SO(2,1)},\delta\right)$ and $\Omega\left(\delta_{\text{diffeo}},\delta\right)$ vanish on the constraint surface, recognizing these two classes of vectors in the covariant phase space as the 'gauge' directions.

1.0.3 Canonical Phase Space

From the above covariant symplectic structure one can instantly read off the following canonical equal-time (functions designating the foliations as level surfaces) Poisson brackets:

$$\{A_i^{(\pm)I}(x,t), A_j^{(\pm)J}(y,t)\} = \frac{1}{2l(1/\gamma \pm 1)}\varepsilon_{ij}\eta^{IJ}\delta^2\left(x,y\right) \tag{1.16}$$

where ε_{ij} is the usual alternating symbol on Σ.

It is worthwhile to see the Poisson bracket structure in terms of Palatini variables:

$$\begin{aligned}
\{\omega_i^I(x,t), e_j^J(y,t)\} &= \frac{1}{4}\frac{\gamma^2}{\gamma^2-1}\varepsilon_{ij}\eta^{IJ}\delta^2\left(x,y\right) \\
\{\omega_i^I(x,t), \omega_j^J(y,t)\} &= -\frac{1}{4}\frac{\gamma/l}{\gamma^2-1}\varepsilon_{ij}\eta^{IJ}\delta^2\left(x,y\right) \\
\{e_i^I(x,t), e_j^J(y,t)\} &= -\frac{1}{4}\frac{\gamma l}{\gamma^2-1}\varepsilon_{ij}\eta^{IJ}\delta^2\left(x,y\right)
\end{aligned} \tag{1.17}$$

As expected in the limit $\gamma \to \infty$ the Poisson brackets reduce to those of usual Palatini theory:

$$\begin{aligned}\{\omega_i^I(x,t), e_j^J(y,t)\} &= \frac{1}{4}\varepsilon_{ij}\eta^{IJ}\delta^2(x,y)\\ \{\omega_i^I(x,t), \omega_j^J(y,t)\} &= 0 \\ \{e_i^I(x,t), e_j^J(y,t)\} &= 0\end{aligned} \tag{1.18}$$

We here wish to concentrate on the Hamiltonian and the constraint structure of the theory. In terms of the Chern Simons gauge fields these are the $SO(2,1)$ Gauss law constraints as illustrated below. The Legendre transformation is done by space-time splitting of the action I given by(1.10)

$$\begin{aligned}I = l(1/\gamma+1)\int_{\mathbb{R}} dt \int_{\Sigma} d^2x\varepsilon^{ij}\left(-A_i^{(+)I}\partial_0 A_{jI}^{(+)} + 2A_0^{(+)I}\partial_i A_{jI}^{(+)} + \epsilon^{IJK}A_{0I}^{(+)}A_{iJ}^{(+)}A_{jK}^{(+)}\right)\\ +l(1/\gamma-1)\int_{\mathbb{R}} dt \int_{\Sigma} d^2x\varepsilon^{ij}\left(-A_i^{(-)I}\partial_0 A_{jI}^{(-)} + 2A_0^{(-)I}\partial_i A_{jI}^{(-)} + \epsilon^{IJK}A_{0I}^{(-)}A_{iJ}^{(-)}A_{jK}^{(-)}\right)\end{aligned} \tag{1.19}$$

First terms in the integrands are kinetic terms and from them one can again extract (1.16). The Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}^{(+)} + \mathcal{H}^{(-)}$$

where

$$\mathcal{H}^{(\pm)} = l\left(1/\gamma \pm 1\right)\varepsilon^{ij}\left(2A_0^{(\pm)I}\partial_i A_{jI}^{(\pm)} + \epsilon^{IJK}A_{0I}^{(\pm)}A_{iJ}^{(\pm)}A_{jK}^{(\pm)}\right)$$

The fields $A_{0I}^{(\pm)}$ are the Lagrange multipliers and we immediately have the primary constraints

$$\mathcal{G}_I^{(\pm)} = l\left(1/\gamma \pm 1\right)\varepsilon^{ij}\left(\partial_i A_{jI}^{(\pm)} + \frac{1}{2}\epsilon_I{}^{JK}A_{iJ}^{(\pm)}A_{jK}^{(\pm)}\right) \approx 0 \tag{1.20}$$

Since $\mathcal{H}^{(\pm)} = A_0^{(\pm)I}\mathcal{G}_I^{(\pm)} \approx 0$ the Hamiltonian is therefore weakly zero. Again the primary constraint being proportional to the Hamiltonian, there are no more secondary constraints in the theory, in Dirac's terminology. Now consider the smeared constraint

$$\mathcal{G}^{(\pm)}(\lambda) = \int_{\Sigma} d^2x\lambda^I\mathcal{G}_I^{(\pm)}$$

for some $\lambda = \lambda^I J_I \in \mathfrak{so}(2,1)$ in the internal space. It now follows that these smeared constraints close among themselves:

$$\{\mathcal{G}^{(\pm)}(\lambda), \mathcal{G}^{(\pm)}(\lambda')\} = \mathcal{G}^{(\pm)}([\lambda,\lambda']) \tag{1.21}$$

and the $SO(2,1)$ Lie algebra is exactly implemented on the canonical phase space. Hence clearly these are the 'Gauss' constraints generating $SO(2,1)$ gauge transformations separately for the $(+)$ type and the $(-)$ type gauge fields. The closure of these constraints on the other hand means that these are first class and there are no second class constraints. A close look on (1.20) reveals that this constraint is nothing but vanishing of the gauge field curvature (1.11) when pulled back to Σ. The temporal component A_{0I} is non-dynamical, being just a Lagrange multiplier. Hence all the dynamics of the theory determined by (1.11) is constrained as (1.20). It follows immediately there is no local physical degree of freedom in the theory. This is related to the justified recognition of Chern Simons theories as 'topological'. We now wish to probe in to the implications of this constraint structure in the gravity side. These issues were discussed by various authors, e.g. [16] for the theory which corresponds to the limit $\gamma \to \infty$ of our system.

We now carry out the Legendre transformation through the space-time split action (1.19) in terms of the variables pertaining relevance to the gravity counterpart of the theory as

$$\begin{aligned} I &= -2l \int_{\mathbb{R}} dt \int_{\Sigma} d^2x \varepsilon^{ij} \underbrace{\left[1/\gamma \left(\omega_i^I \partial_0 \omega_{jI} + \frac{1}{l^2} e_i^I \partial_0 e_{jI} \right) + \frac{2}{l} \omega_i^I \partial_0 e_{jI} \right]}_{\text{kinetic terms}} \\ &\quad +4l \int_{\mathbb{R}} dt \int_{\Sigma} d^2x \varepsilon^{ij} \left[\frac{1}{l} \left(\omega_0^I + \frac{1}{\gamma l} e_0^I \right) \left(\partial_i e_{jI} + \epsilon_I{}^{JK} \omega_{iJ} e_{jK} \right) \right. \\ &\quad \left. + \left(\frac{1}{\gamma} \omega_0^I + \frac{1}{l} e_0^I \right) \left(\partial_i \omega_{jI} + \frac{1}{2} \epsilon_I{}^{JK} \left(\omega_{iJ} \omega_{jK} + \frac{1}{l^2} e_{iJ} e_{jK} \right) \right) \right] \end{aligned} \tag{1.22}$$

One then envisages the part save the kinetic part as the Hamiltonian with $\left(\omega_0^I + \frac{1}{\gamma l} e_0^I\right)$, $\left(\frac{1}{\gamma}\omega_0^I + \frac{1}{l} e_0^I\right)$ as the Lagrange multipliers with the following as the constraints, after suitable rescaling[3]:

$$\begin{aligned} P^I &:= 2\frac{1-\gamma^2}{\gamma^2} \varepsilon^{ij} \left(\partial_i e_j^I + \epsilon^I{}_{JK} \omega_i^J e_j^K \right) \approx 0 \\ S^I &:= 2\frac{l(1-\gamma^2)}{\gamma^2} \varepsilon^{ij} \left(\partial_i \omega_j^I + \frac{1}{2} \epsilon^I{}_{JK} \left(\omega_i^J \omega_j^K + \frac{1}{l^2} e_i^J e_j^K \right) \right) \approx 0 \end{aligned} \tag{1.23}$$

[3]A rescaling with the factor $\frac{1-\gamma^2}{\gamma^2}$ is done in order to avoid apparent divergences in the constraint algebra at the points $\gamma \to \pm 1$

Let us define their smeared versions as

$$P(\lambda) := \int_\Sigma d^2x \lambda^I P_I \text{ and } S(\lambda) := \int_\Sigma d^2x \lambda^I S_I$$

for $\lambda \in \mathfrak{so}(2,1)$. One can also check the expected closure of the constraint algebra of S and P which guarantees their first class nature:

$$\begin{aligned} \{S(\lambda), S(\lambda')\} &= \gamma^{-1} S([\lambda, \lambda']) - P([\lambda, \lambda']) \\ \{S(\lambda), P(\lambda')\} &= -S([\lambda, \lambda']) + \gamma^{-1} P([\lambda, \lambda']) \\ \{P(\lambda), P(\lambda')\} &= \gamma^{-1} S([\lambda, \lambda']) - P([\lambda, \lambda']) \end{aligned} \tag{1.24}$$

Since linear combinations of ω_0^I and e_0^I are Lagrange multipliers, these fields themselves are non dynamical. We thus infer that all the dynamical informations through the equations of motion (1.2), (1.3) are encoded in the constraints (1.23). In the limit $\gamma \to \infty$, as e.g. in [16] P generate local Lorentz ie, $SO(2,1)$ Lorentz transformations and S generate diffeomorphisms for the frame variables. Since in finite γ case too these are first class, one should expect them to generate some gauge transformation. To see changes brought in by the modified symplectic structure we first compute the transformations induced by these constraints:

$$\{e_i^I(x,t), P(\lambda)\} = -\frac{l}{2}\left[\gamma^{-1} \underbrace{\left(\partial_i \lambda^I + \epsilon^{IJK} \omega_{iJ} \lambda_K\right)}_{D_i \lambda^I} + \frac{1}{l} \epsilon^{IJK} \lambda_J e_{iK}\right] \tag{1.25}$$

$$\{\omega_i^I(x,t), P(\lambda)\} = \frac{1}{2}\left[D_i \lambda^I + \frac{1}{l\gamma} \epsilon^{IJ}{}_K \lambda_J e_i^K\right], \tag{1.26}$$

where D_i defined above in the frame bundle covariant derivative induced by ω_i^I. The infinitesimal local $SO(2,1)$ Lorentz transformations, ie $e \to e + \lambda \times e, \quad \omega \to \omega + d\lambda + \lambda \times \omega$ are seen to be successfully generated by $P(\lambda)$ in the limit $\gamma \to \infty$. But for finite γ, the transformations are deformed in a sense that infinitesimal diffeomorphisms are also generated along with Lorentz transformations. Similarly the Lie transports generated by the diffeomorphism generator S are also deformed due to the modified symplectic structure as:

$$\{e_i^I(x,t), S(\lambda)\} = \frac{l}{2}\left[D_i \lambda^I + \frac{1}{l\gamma} \epsilon^{IJ}{}_K e_{Ji} \lambda^K\right] \tag{1.27}$$

$$\{\omega_i^I(x,t), S(\lambda)\} = -\frac{1}{2}\left[\gamma^{-1} D_i \lambda^I + \frac{1}{l} \epsilon^{IJ}{}_K \lambda_J e_i^K\right] \tag{1.28}$$

In this case we also notice that the usual diffeomorphism generator is generating local Lorentz transformations for finite γ. We can find suitable linear combinations of these two generators which separately generate local Lorentz transformation and diffeomorphisms.

The striking consequence of the γ-deformed Poisson algebra (1.17) can again be envisaged in terms of the usual ADM canonical pairs: the spatial metric h_{ij} and the dual momentum $\pi^{ij} = \sqrt{h}\,(K^{ij} - h^{ij}K)$, where K^{ij} is the extrinsic curvature and K is its trace. Using $h_{ij} = g_{ij} = e_i^I e_{jI}$, we have:

$$\{h_{ij}(x,t), h_{kl}(y,t)\} = -\frac{1}{4}\frac{\gamma}{\gamma^2-1}\left(\varepsilon_{ik}h_{jl} + \varepsilon_{il}h_{jk} + \varepsilon_{jk}h_{il} + \varepsilon_{jl}h_{ik}\right)\delta^2(x,y).$$

Similar Poisson brackets involving π^{ij} can also be calculated, which are more cumbersome and we omit their explicit forms. The take-home message from this analysis is that, while the spatial metric Poisson commute with itself in the limit $\gamma \to \infty$, it fails to do so for finite γ. In contrast to the 3+1 dimensional case, therefore, γ does not induce canonical transformation in the ADM phase space.

1.0.4 The Singularity and its Resolution at $\gamma \to 1$

As it is apparent from (1.16), (1.17) the canonical structure blows up at the point $\gamma \to \pm 1$. This is due to the fact that the Lagrangian (1.12) and the action functional (1.10) become independent of either of the 1-form fields $A^{(\pm)}$ for $\gamma \to \pm 1$. As a result the symplectic structure we have constructed (1.14), becomes degenerate on the space $\mathcal{V}_F^{(+)} \times \mathcal{V}_F^{(-)}$ (leaving the gauge degeneracies apart), resulting it to be non-invertible. This is clearly the reason for blowing up of the equal time Poisson brackets (1.16).

In order to avoid this singularity we restrict our theory to $\gamma \in \{\mathbb{R}^+ - \{1\}\}$ and propose the theory (1.10) for gravity in 2+1 dimensions. We will see further restriction on the values of γ allowed by the quantum theory. The borderline case $\gamma = 1$ can however be dealt as follows. At the point $\gamma = 1$ the effective theory of 2+1 gravity, as recovered from (1.10) easily, is described by the single gauge 1-form $A_I^{(+)}$ and we consider the phase space to be only coordinatized by flat connections $A_I^{(+)}$, ie $\mathcal{V}_F^{(+)}$ with the action functional:

$$I = l\int_M \left(A^{(+)I} \wedge dA_I^{(+)} + \frac{1}{3}\epsilon_{IJK}A^{(+)I} \wedge A^{(+)J} \wedge A^{(+)K}\right) \tag{1.29}$$

On the space $\mathcal{V}_F^{(+)}$ we now have the symplectic structure

$$\Omega\left(\delta_1, \delta_2\right) = 2l \int_\Sigma \delta_{[1} A^{(+)I} \wedge \delta_{2]} A_I^{(+)} \tag{1.30}$$

This gives the non-singular Poisson bracket:

$$\{A_i^{(+)I}(x,t), A_j^{(+)J}(y,t)\} = \frac{1}{2l}\varepsilon_{ij}\eta^{IJ}\delta^2\left(x,y\right) \tag{1.31}$$

In a more generalized theory, such as cosmological topologically massive gravity (CTMG) dealt in the first order formalism [45–47] one deals with the action

$$\begin{aligned} I_{\text{CTMG}} &= l\left[\left(1/\gamma+1\right)I^{(+)} + \left(1/\gamma-1\right)I^{(-)} + \varrho^I \wedge \left(de_I + \epsilon_{IJK}e^J \wedge \omega^K\right)\right] \\ &= l\Bigg[\left(1/\gamma+1\right)I^{(+)} + \left(1/\gamma-1\right)I^{(-)} + \frac{1}{4l}\int_M \varrho^I \wedge \left(dA_I^{(+)} + \epsilon_{IJK}A^{(+)J} \wedge A^{(+)K}\right) \\ &\quad -\frac{1}{4l}\int_M \varrho^I \wedge \left(dA_I^{(-)} + \epsilon_{IJK}A^{(-)J} \wedge A^{(-)K}\right)\Bigg] \end{aligned} \tag{1.32}$$

where ϱ is a new 1-form field which enhances the covariant phase space and emerges as a Lagrange multiplier. The corresponding symplectic structure is

$$\begin{aligned} \Omega_{\text{CTMG}}\left(\delta_1,\delta_2\right) &= 2l\Bigg[\left(1/\gamma+1\right)\int_\Sigma \delta_1 A^{(+)I} \wedge \delta_2 A_I^{(+)} \\ &+ \left(1/\gamma-1\right)\int_\Sigma \delta_1 A^{(-)I} \wedge \delta_2 A_I^{(-)} - \frac{1}{2l}\int_\Sigma \left(\delta_{[1}\rho^I \wedge \delta_{2]}A^{(+)}\right) \\ &+ \frac{1}{2l}\int_\Sigma \left(\delta_{[1}\rho^I \wedge \delta_{2]}A^{(-)}\right)\Bigg] \end{aligned} \tag{1.33}$$

In contrast to the theory we have considered, this theory does not become independent of any of the dynamical variables $\left(A^{(-)}, A^{(+)}, \varrho\right)$ as $\gamma \to 1$:

$$\begin{aligned} I_{\text{CTMG}}\Big|_{\gamma=1} &= l\Bigg[2I^{(+)} + \frac{1}{4l}\int_M \varrho^I \wedge \left(dA_I^{(+)} + \epsilon_{IJK}A^{(+)J} \wedge A^{(+)K}\right) \\ &\quad -\frac{1}{4l}\int_M \varrho^I \wedge \left(dA_I^{(-)} + \epsilon_{IJK}A^{(-)J} \wedge A^{(-)K}\right)\Bigg] \end{aligned} \tag{1.34}$$

and in this limit $\gamma \to 1$ the symplectic structure (1.33) remains non-degenerate:

$$\begin{aligned} \Omega_{\text{CTMG}}\Big|_{\gamma=1}\left(\delta_1,\delta_2\right) &= 2l\Bigg[2\int_\Sigma \delta_1 A^{(+)I} \wedge \delta_2 A_I^{(+)} - \frac{1}{2l}\int_\Sigma \left(\delta_{[1}\rho^I \wedge \delta_{2]}A^{(+)}\right) \\ &+ \frac{1}{2l}\int_\Sigma \left(\delta_{[1}\rho^I \wedge \delta_{2]}A^{(-)}\right)\Bigg] \end{aligned} \tag{1.35}$$

On the other hand there is a price one has to pay for considering TMGs [2, 48, 49] in general. The theory develops a local propagating degree of freedom (graviton) and the complete non-perturbative quantization (as we present in the next section) seems far from being a plausible aim. Progress in perturbative quantization about linearized modes in TMG although have been made in [50]. Also the relevance of the limit $\gamma \to 1$ in the context of chiral gravity was made clear there.

Topologically massive gravity has so far been talked about in the first order framework only. It is however interesting to learn about the second order version of it. To see this one first solves the torsionless part of the equation of motion (1.2) for $e(\omega)$. The next part is to substitute it in the action (1.10) and use $e^I_\mu e_{I\nu} = g_{\mu\nu}$. Then the new action becomes original Einstein-Hilbert plus a 'gravitational Chern Simons' term. Surprisingly, now this total action is no more topological and has propagating degrees of freedom. We will be talking about other features of TMG in the quantum perspective later.

Chapter 2

2+1 Quantum Gravity on Toric Spatial Foliation

2.1 Introduction

As promised in 0.3.2, we here take the opportunity to discuss the relationship between Chern Simons theory and 3D gravity at the level of quantum dynamics, in greater detail. Quantum gravity in 2+1 dimensions have been an object of serious research for quite some time. From our previous discussion, we understand that being equivalent to a Chern Simons topological gauge theory, there is no propagating mode in this theory [51]. However it admits of a CFT at the boundary when the theory is considered in an asymptotically AdS space-time [19]. Gravity in 2+1 space-time even without graviton modes took an intresting turn after existence of black hole solutions was ensured [18]. Subsequent important works in the context of AdS/CFT correspondence [52, 53] warrants the importance of this model.

Even if one restricts oneself with 2+1 gravity models without propagating degrees of freedom, quantization of the theory poses a non-trivial problem in its own right in the sense that one has to study this problem keeping in mind that topology of the space-time would play an important role. We would be working with non-perturbative canonical quantization. In that case, if the phase space is finite dimensional one can do away with problems regarding renormalizability even in the non-perturbative regime [16, 20]. In this work we would deal with 2+1 gravity on a (pseudo)Riemann-Cartan manifold which is not asymptotically AdS and aim to compare results with asymptotically AdS calculations already available in literature.

As discussed in the chapter 1, in the case of negative cosmological constant the Chern-Simons action corresponding to 2+1 gravity (hereafter referred as CSG) can be written as an $SO(2,1) \times SO(2,1)$ gauge theory [16], which is purely topological as opposed to TMGs. For the specific type of topology of space-time, that we will be choosing, the topology of the physical phase space of the theory will become nontrivial and one has to take recourse to geometric quantization [54]. This approach of quantization in the 'constrain first' line was studied for an SL(2,R) Chern-Simons theory with rational charges in [55] where a finite dimensional Hilbert space was constructed on the almost torus part of the physical phase space and it was argued that the Hilbert space on the total phase space would be finite dimensional. Spatial slice in this case was chosen to be a torus. We must admit, the choice of the spatial slice being a compact Riemann surface stems from a desire to construct a finite dimensional Hilbert space of states, which was still unexplored in literature. Our work presented in this chapter is particularly motivated from [55]. Since we are dealing with a theory with non-compact gauge group, we fail to observe the 'shift' in the CS level [56]. We now note the connection of CS theory with Wess-Zumino-Witten(WZW) conformal field theory [57]. In this view, the central charge of the current algebra of the WZW also should miss the corresponding shift. Inspite of that, as explained in [55] the above difficulty is overcome as the Hilbert space of quantum states of we will be having, will enjoy exactly the same unitary structure of the vector space of the current blocks of the WZW theory. One can however consider physically more interesting topologies giving rise to asymptotically AdS ones. A non-perturbative quantization in this regard can be proposed following [56], based on loop-group moduli space formulation. However in a more modern light of holographic interpretation, the required states can be more easily read out from the asymptotic CFT [20].

As warned earlier, the equivalence of gravity with Chern Simons has to be taken with a grain of salt, particularly following the caveat presented relatively recently by Witten [20]. For example, in the Chern Simons side, one has flat connections which are trivially zero. But once one makes contact with gravity variables, these give triads (as well as frame connections) trivially zero and hence non-invertible (they do not give rise to physically meaningful metrics). Naturally the question around viability of a non-perturbative quantum theory, which should include all smooth classical solutions from

Chern Simons as well as frame variables, is not quite answered.

We also wish to point out that in a later work [58], an explicit parameterization of the physical phase space for CS gravity on toric spatial foliation and negative cosmological constant was done. There, in contrast to geometric quantization, the phase space was modified to a suitable cotangent bundle by a surgery of the non-trivial phase space and trivializing its topology. Conventional procedure of canonical quantization was carried out in that modified phase space. Also no comment on the dimensionality of the Hilbert space was made.

With most of the classical dynamical analysis already done in chapter 1, we directly go into constructing the physical phase space of theory in section 2.2. While doing so, we keep in mind that the physical (gauge moduli) space actually depends on the topology of the spatial slice of the 3D manifold M initially chosen. In our case, as mentioned earlier, it is genus-1 Riemann surface initially with no other additional structures. The physical phase space is the moduli space of flat gauge connections, modulo gauge transformations on our choice of spatial foliation. The topology of the moduli space turns out to be a torus punctured at a point (may be chosen to be origin) with a plane also punctured at a point (also chosen to be origin of the plane) and glued to the torus through a closed curve (S^1) around the origin (common puncture) the plane being Z_2 folded through the origin .

In section 2.3 we discuss the geometric quantization of the phase space [54]. A complete basis for the physical Hilbert space is constructed in terms of theta functions. Note that due to introduction of this new parameter, both the Chern-Simons levels can be adjusted to be positive and rational. During quantization this becomes important since the dimensionality of the Hilbert space of the quantized theory is directly related with these levels. The corresponding charge in the CSG is no longer an integer owing to the fact that the Weil's integrality condition on the Chern-Simons charge disappears as a consequence of the non compactness of the gauge group which in our case is $SO(2,1)$ [3], [56]. A discussion on the restrictions on physical parameters coming from the quantization is also presented and compared with those from [20].

In this chapter we will be discussing the analysis presented in the paper [26]. However unit conventions used in this thesis is different from the paper.

2.2 The Physical Phase space for $\Sigma = T^2$

The classical phase space, with all redundant (gauge) degrees of freedom was described in chapter 1. While constructing the physical phase space, we closely follow the analysis of [55] and present the internal details of the construction.

2.2.1 The physical phase space

The route we choose for quantization of the system involves eliminating the gauge redundancy inherent in the theory, ie, finding the solution space modulo gauge transformations, in the classical level itself. For the present purpose this approach is useful in contrast to the other one which involves quantizing all degrees of freedom and then singling out the physical state space as the solution of the equation:

$$\hat{\mathbf{M}}|\mathbf{\Psi}\rangle = 0$$

ie, the kernel of the quantum version of the constraints. For illustrations of this later path one may look up the context of quantization of diffeomorphism invariant theories of connections [59], eg, loop quantum gravity in 3+1 dimensions [60].

The advantage of the first approach, ie, the reduced phase space (constrained first) one is that the phase space is completely coordinatized by gauge invariant objects; another manifestation being its finite dimensionality. Quantization of a finite dimensional phase space may acquire non-triviality only through the topology of it, as will be illustrated in the case at hand.

Now, the physical phase space is clearly

$$\left(\mathcal{V}_F^{(+)}/\sim\right) \times \left(\mathcal{V}_F^{(-)}/\sim\right),$$

where $\sim$ means equivalence of two flat connections which are gauge related. It is thus understood [16] that at least for the case when Σ is compact, each of the $\mathcal{V}_F^{(\pm)}/\sim$ spaces is topologically isomorphic to the space $(\text{hom} : \pi_1(\Sigma) \to SO(2,1))/\sim)$ of homomorphisms from the first homotopy group of Σ to the gauge group modulo gauge transformations. This isomorphism is realized (parameterized) by the holonomies of the flat connections around non-contractible loops on Σ which serve as the homomorphism maps.

For the choice of the topology of compact Σ, one may start by choosing a general g-genus Riemann surface. The case $g = 0$ is trivial, and the moduli space consists of two points. For $g \geq 2$, parametrization of the phase space is highly non-trivial and topology of it is still not clear in literature, although construction of canonical structure on those moduli spaces have been constructed [61]. As the first non-trivial case we therefore choose the case when Σ is a genus 1 Riemann surface T^2. For this torus, we know that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ i.e. this group is freely generated by two abelian generators α and β with the relation

$$\alpha \circ \beta = \beta \circ \alpha. \tag{2.1}$$

Since the connections at hand are flat, their holonomies depend only upon the homotopy class of the curve over which the holonomy is defined. For this reason, as parameterizations of the $\mathcal{V}_F^{(\pm)}$ we choose the holonomies

$$h^{(\pm)}[\alpha] := \mathcal{P}\exp\left(\int_\alpha A^{(\pm)}\right) \quad \text{and} \quad h^{(\pm)}[\beta] := \mathcal{P}\exp\left(\int_\beta A^{(\pm)}\right)$$

[1]with (2.1) being implemented on these $SO(2,1)$ group valued holonomies as

$$h^{(\pm)}[\alpha]h^{(\pm)}[\beta] \;\; = \;\; h^{(\pm)}[\beta]h^{(\pm)}[\alpha]. \tag{2.2}$$

As is well-known these are gauge covariant objects although their traces, the Wilson loops are gauge invariant. Although the classical Poisson bracket algebra of Wilson loops for arbitrary genus were exhaustively studied in [61], the phase space these loops constitute is absent. On the other hand there is another simple way of finding the gauge invariant space especially for the case of genus 1, as outlined in [55, 58]. We will for completeness briefly give the arguments reaching the construction.

Under the gauge transformations $A^{(\pm)} \to \tilde{A^{(\pm)}} = g^{-1}\left(A^{(\pm)} + d\right)g$ the holonomies transform as $h^{(\pm)}[c] \to \tilde{h}^{(\pm)}[c] = \chi^{-1}h^{(\pm)}[c]\chi$ for any closed curve c and some element $\chi \in SL(2,\mathbb{R})$.

Again from (2.25) we know that any $SO(2,1)$ element is conjugate to elements in any of the abelian subgroups: f_ϕ or g_ξ or h_η. Out of the three cases, for illustrative purpose we present the elliptic case.

[1]here the path ordering $\mathcal{P}$ means ordering fields with smaller parameter of the path to the left

Let $h^{(\pm)}[\alpha]$ is conjugate to an element in the elliptic class. Up to proper conjugation we can write

$$h^{(\pm)}[\alpha] = e^{-\lambda_0 \rho_{(\pm)}}$$

and from the discussion of 2.4 with (2.2) we must have that

$$h^{(\pm)}[\beta] = e^{-\lambda_0 \sigma_{(\pm)}}.$$

Hence we have $\rho_{(\pm)}$ and $\sigma_{(\pm)}$ with range $(0, 2\pi)$ parameterizing a sector of the gauge invariant phase space with topology of a torus : $S^1 \times S^1 \simeq T^2$.

Similarly structures of the other two sectors can also be found out. One is $(\mathbb{R}^2 \backslash \{0,0\}) / \mathbb{Z}^2$, containing an orbifold singularity and another is S^1 topologically. The total phase space is therefore product of two identical copies of $T^2 \cup (\mathbb{R}^2 \backslash \{0,0\}) / \mathbb{Z}^2 \cup S^1$. To be more precise the total phase space can be thought of as a union of a punctured torus $\tilde{T}$, the punctured orbifold $(\mathbb{R}^2 \backslash \{0,0\}) / \mathbb{Z}^2$ named as $\tilde{P}$ glued together at the repsective punctures through the circle S^1, identifying the S^1 as a point.

2.2.2 Symplectic structure on the phase space

If one considers periodic coordinates x, y on $\Sigma \simeq T^2$ with period 1, then it follows immediately that the connections

$$A^{(\pm)} = \lambda_0(\rho_{(\pm)} dx + \sigma_{(\pm)} dy) \tag{2.3}$$

give the above written holonomies parameterizing the $\tilde{T}$ sector.

Now using (1.14) and (4.4) we have the symplectic structure ω, whose pull back to the pre-symplectic manifold is Ω (1.14) on this $\tilde{T}$ sector of the phase space is given by (we have restored G at this point):

$$\omega\left(\delta_1, \delta_2\right) = \frac{l}{16\pi G}\left[\left(1/\gamma + 1\right)\delta_{[1}\rho_{(+)}\delta_{2]}\sigma_{(+)} + \left(1/\gamma - 1\right)\delta_{[1}\rho_{(-)}\delta_{2]}\sigma_{(-)}\right] \tag{2.4}$$

or,

$$\begin{aligned} \omega &= \frac{l}{16\pi G}\left[\left(1/\gamma + 1\right)\mathbf{d}\rho_{(+)} \wedge \mathbf{d}\sigma_{(+)} + \left(1/\gamma - 1\right)\mathbf{d}\rho_{(-)} \wedge \mathbf{d}\sigma_{(-)}\right] \\ &= \frac{k_{(+)}}{2\pi}\mathbf{d}\rho_{(+)} \wedge \mathbf{d}\sigma_{(+)} + \frac{k_{(-)}}{2\pi}\mathbf{d}\rho_{(-)} \wedge \mathbf{d}\sigma_{(-)} \end{aligned} \tag{2.5}$$

where $k_{(\pm)} = \frac{l(1/\gamma \pm 1)}{8G}$ and the $\mathbf{d}$ are exterior differentials on the phase and the $\wedge$ is also on this manifold, not on space time. Here we introduce holomorphic coordinates on $\tilde{T}$ corresponding to a complex structure τ on the two dimensional space manifold Σ as

$$z_{(\pm)} = \frac{1}{\pi}\left(\rho_{(\pm)} + \tau\sigma_{(\pm)}\right).$$

Then the symplectic structure in (2.5) takes the form

$$\omega = \frac{ik_{(+)}\pi}{4\tau_2}\mathbf{d}z_{(+)} \wedge \mathbf{d}\bar{z}_{(+)} + \frac{ik_{(-)}\pi}{4\tau_2}\mathbf{d}z_{(-)} \wedge \mathbf{d}\bar{z}_{(-)} \tag{2.6}$$

In a similar fashion the symplectic structure on $\tilde{P}$ is given by:

$$\omega = \frac{ik_{(+)}\pi}{4\tau_2}\mathbf{d}z_{(+)} \wedge \mathbf{d}\bar{z}_{(+)} + \frac{ik_{(-)}\pi}{4\tau_2}\mathbf{d}z_{(-)} \wedge \mathbf{d}\bar{z}_{(-)} \tag{2.7}$$

where $z_{(\pm)} = \frac{1}{\pi}\left(x_{(\pm)} + \tau y_{(\pm)}\right)$, x, y being the coordinates on $\tilde{P}$.

2.3 Geometric quantization of the phase space

As explained in 2.2.1 the total phase space is product of two identical copies of $\tilde{T} \cup \tilde{P}$, $\tilde{T}$ and $\tilde{P}$ being glued through a circle S^1 around the puncture at $(0, 0)$. Variables relevant to each factor of this product has been distinguished until now by $\pm$ suffices. From now on, we will remove this distinction for notational convenience and will restore when it is necessary.

Upon quantization the total wave functions (holomorphic sections of the line bundle over $\tilde{T} \cup \tilde{P}$) should be such that the wave function (holomorphic sections of the line bundle over $\tilde{T} \cup \tilde{P}$) on $\tilde{T}$, say $\psi(z)$ and the wave function on $\tilde{P}$, say $\chi(z)$ should 'match' on the circle. The plan of quantization is therefore simple. We will first carry out the quantization on $\tilde{T}$. Then we will consider those functions on $\tilde{P}$ which can be found by continuation in some sense of the wave functions on $\tilde{T}$.

Note that as we will be developing the quantum theory in this chapter, the Planck constant would evidently arise and we will be using the natural units, in which $\hbar = 1$. The first appearance of it will be kept note of.

2.3.1 Quantization on $\tilde{T}$

While performing quantization on $\tilde{T}$ with the symplectic structure

$$\omega = \frac{k}{2\pi}\mathbf{d}\rho \wedge \mathbf{d}\sigma = \frac{ik\pi}{4\tau_2}\mathbf{d}z \wedge \mathbf{d}\bar{z}$$

one must keep in mind the fact that $\tilde{T}$ is in fact punctured as opposed to being compact. [2] The distinction occurs from the non triviality of the algebra of the generators of the homotopy group. The three generators of $\pi_1(\tilde{T})$, denoted as $a, b, \&\Delta$ respectively correspond to the usual cycles of the compact torus and the cycle winding around the puncture. They should satisfy the following relations:

$$aba^{-1}b^{-1} = \Delta \qquad a\Delta a^{-1}\Delta^{-1} = 1 \qquad b\Delta b^{-1}\Delta^{-1} = 1$$

As explained in [55, 62] $q \in \mathbb{Z}$ dimensional unitary representation of these relations are given as follows. The unitary finite dimensional non-trivial representations of this algebra must have the commuting generator δ proportional to identity. Hence we have the for some q dimensional representation

$$\Delta_{\alpha,\beta} = \mathrm{e}^{2\pi ip/q}\delta_{\alpha,\beta}$$

where p, q are positive integers, co prime to each other. Reason behind choosing rational phase will become clear shortly when we complete the quantization.

Again, up to arbitrary $U(1)$ phase factor a, b are represented as

$$a_{\alpha,\beta} = \mathrm{e}^{-2\pi i\frac{p}{q}\alpha}\delta_{\alpha,\beta} \qquad b_{\alpha,\beta} = \delta_{\alpha,\beta+1}$$

with $\alpha, \beta \in \mathbb{Z}_q$. It is also being expected that the space of holomorphic sections should also carry the q representation of this homotopy group.

Let us now consider quantization on $\mathbb{R}^2$ endowed with complex structure τ and the above symplectic structure. The fact that the actual phase space we wish to quantize is a punctured torus will be taken into account by action of the discretized Heisenberg

[2] Had the symplectic manifold $\tilde{T} \cup \tilde{P}$ been compact, Weil's integrality criterion would require the Chern-Simons level k to be integer valued. At this point we keep open the possibility of k being any real number.

group operators on the Hilbert space of parallel sections of the line bundle over $\mathbb{R}^2$. A very similar quantization scheme for a different situation may be found in [63–65].

We start from the symplectic structure on $\mathbb{R}^2$ instead of the the punctured torus $\tilde{T}$ and coordinatize it by ρ and σ and endowed with complex structure τ, such that holomorphic anti holomorphic coordinates are chosen as before:

$$\omega = \frac{k}{2\pi}\mathbf{d}\rho \wedge \mathbf{d}\sigma.$$

With definition of holomorphic coordinate $z = \frac{1}{\pi}(\rho + \tau\sigma)$ defined through arbitrary complex structure τ this becomes

$$\omega = \frac{ik\pi}{4\tau_2}\mathbf{d}z \wedge \mathbf{d}\bar{z},$$

where $\tau_2 = \Im\tau$. It is easy to check that the symplectic potential

$$\Theta = \frac{ik\pi}{8\tau_2}\left[-\left(\bar{z} - 2z\right)\mathbf{d}z + \left(z + \xi(\bar{z})\right)\mathbf{d}\bar{z}\right]$$

gives the above symplectic structure, for arbitrary anti-holomorphic function $\xi(\bar{z})$. Let us now consider the hamiltonian vector fields corresponding to the variables ρ and σ

$$\zeta_\rho = \frac{2\pi}{k}\partial_\sigma \tag{2.8}$$

$$\zeta_\sigma = -\frac{2\pi}{k}\partial_\rho \tag{2.9}$$

The corresponding pre-quantum operators to these variables are therefore

$$\begin{aligned}\hat{\rho} &= -i\zeta_\rho - \Theta(\zeta_\rho) + \rho \\ &= -\frac{2i}{k}\left(\tau\partial_z + \bar{\tau}\partial_{\bar{z}}\right) + \frac{i\pi}{4\tau_2}\left(\bar{\tau}z - \tau\bar{z} - 2\tau z - \bar{\tau}\xi(\bar{z})\right) \qquad (2.10)\\ \hat{\sigma} &= -i\zeta_\sigma - \Theta(\zeta_\sigma) + \sigma \\ &= -\frac{2i}{k}\left(\partial_z + \partial_{\bar{z}}\right) + \frac{i\pi}{4\tau_2}\left(z + \bar{z} + \xi(\bar{z})\right) \qquad (2.11)\end{aligned}$$

Now, parallel (holomorphic) sections of the line bundle $\pi : L\tilde{T} \to \tilde{T}$ over the symplectic manifold $\tilde{T}$ are classified through the kernel of the Cauchy-Riemann operator defined via the connection $\nabla = \mathbf{d} - i\Theta$ on $L\tilde{T}$ as (in units of $\hbar$=1)

$$\nabla_{\partial_{\bar{z}}}\Psi(z, \bar{z}) = 0. \tag{2.12}$$

Ansätz for Ψ can be chosen as:

$$\Psi(z,\bar{z}) = \mathrm{e}^{-\frac{k\pi}{8\tau_2}(z\bar{z}+\Xi(\bar{z}))}\psi(z) \tag{2.13}$$

with $\Xi(\bar{z})$ being the primitive of $\xi(\bar{z})$ with respect to $\bar{z}$ and $\psi(z)$ is any holomorphic function. This is how the holomorphic factor $\psi(z)$ of the function $\Psi(z)$ is being singled out by the $\nabla_{\partial_{\bar{z}}}$. To find the representations of the operators corresponding to $\hat{\sigma}$ and $\hat{\rho}$ on the space of the holomorphic functions, we see the actions:

$$\begin{aligned} \hat{\rho}\Psi(z,\bar{z}) &= \mathrm{e}^{-\frac{k\pi}{8\tau_2}(z\bar{z}+\Xi(\bar{z}))}\left[-\frac{2i}{k}\tau\partial_z + \pi z\right]\psi(z) \\ &=: \mathrm{e}^{-\frac{k\pi}{8\tau_2}(z\bar{z}+\Xi(\bar{z}))}\hat{\rho}'\psi(z) \qquad (2.14) \\ \hat{\sigma}\Psi(z,\bar{z}) &= \mathrm{e}^{-\frac{k\pi}{8\tau_2}(z\bar{z}+\Xi(\bar{z}))}\left[\frac{2i}{k}\partial_z\right]\psi(z) \\ &=: \mathrm{e}^{-\frac{k\pi}{8\tau_2}(z\bar{z}+\Xi(\bar{z}))}\hat{\sigma}'\psi(z). \qquad (2.15) \end{aligned}$$

These give the representations for σ and ρ on the space of holomorphic sections in terms of $\hat{\rho}'$ and $\hat{\sigma}'$.

At this point it is necessary to notice that we aim to quantize the punctured torus instead of $\mathbb{R}^2$. This is done by imposing periodicity conditions (for being defined on torus) through action of the Heisenberg group and the homotopy group (accounting for the puncture) on the space of holomorphic sections. Let us therefore define homotopy matrix-valued Heisenberg operators:

$$U(m) := b^m \mathrm{e}^{ikm\hat{\rho}'} \tag{2.16}$$

$$V(m) := a^m \mathrm{e}^{-ikm\hat{\sigma}'} \tag{2.17}$$

The periodicity condition that,

$$U(m)V(n)\psi(z) = \psi(z)$$

for $m, n \in \mathbb{Z}$ therefore reduces to

$$\begin{aligned} \psi(z+2m+2n\tau) &= \mathrm{e}^{-ikn^2\pi\tau - ikn\pi z}a^{-m}b^{-n}\psi(z) \qquad \text{or} \\ \psi_\alpha(z+2m+2n\tau) &= \mathrm{e}^{-ikn^2\pi\tau - ikn\pi z + 2\pi i(p/q)m\alpha}\psi_{\alpha+n}(z). \qquad (2.18) \end{aligned}$$

in terms of components.

Let us now as a digression concentrate upon level I, J $SU(2)$ theta functions

$$\vartheta_{I,J}(\tau, z) := \sum_{j\in\mathbb{Z}} \exp\left[2\pi i J\tau\left(j+\frac{I}{2J}\right)^2 + 2\pi i Jz\left(j+\frac{I}{2J}\right)\right]$$

and define

$$\tilde{\vartheta}_{\alpha,N}(\tau, z) := \vartheta_{qN+p\alpha,pq/2}(\tau, z/q)$$

for pq even [55]. After some manipulations, it is easy to check that

$$\tilde{\vartheta}_{\alpha,N}(\tau, z+2m+2n\tau) = \mathrm{e}^{-\pi i(p/q)n^2\tau-\pi i(p/q)nz}\mathrm{e}^{2\pi i(p/q)m\alpha}\tilde{\vartheta}_{\alpha+n,N}(\tau, z) \qquad (2.19)$$

the indices $\alpha \in \{0, 1, \ldots, q-1\}$ and $N \in \{0, 1, \ldots, p-1\}$. These theta functions are known to form a complete p dimensional set over the field of complex numbers [66].

Again comparing the transformations (2.18) and (2.19) we infer that for the value $k = p/q$, [3] a positive rational, we have a finite p dimensional vector space of physical states spanned by q component wave-functions, represented by theta functions depicted as above. For instance the N th wave function is

$$\psi^N(z) = \begin{pmatrix} \tilde{\vartheta}_{0,N}(\tau, z) \\ \vdots \\ \vdots \\ \tilde{\vartheta}_{q-1,N}(\tau, z) \end{pmatrix}.$$

Here we have only considered the case pq even. In spirit the case pq odd [55]can also be dealt at par. Distinction of that case from the present one occurs as identification of the wave functions satisfying (2.18) has to be made with a theta functions with different levels.

We are considering $k = p/q$, a positive rational. From our earlier discussions (2.5), we had $k_{(\pm)} = \frac{l(1/\gamma\pm1)}{8G}$ (in the units of $\hbar = 1 = c$, and in 2+1 space time dimensions G is of

[3]From another point of view it can be seen that the monodromy of wave functions about the puncture satisfying the above relation is measured to be $\mathrm{e}^{2\pi ik}$. When this is related related with the measure of non-commutativity $\mathrm{e}^{2\pi ip/q}$ of the homotopy generators due to the puncture [55] we have the relation:

$$k = p/q$$

up to additive integers.

dimension that of length, hence making k dimensionless) in terms of the parameters of the classical theory. From the point of view of quantization, we are restricting only those values of classical parameters for which the combinations $k_{(\pm)}$ are positive rational.

2.3.2 Continuation to $\tilde{P}$

The wave function $\chi(z)$ on $\tilde{P}$ must be of the form

$$\chi(z) = z^{\kappa}\phi(z)$$

where $\phi(z)$ is holomorphic and κ is a positive rational. The factor z^{κ} in the wave function is necessary since it must be allowed to pick up a non-trivial phase in going around the orbifold singularity.

Also the wave function on the entire phase space should be such that the two functions ψ and χ agree on the intersection and the wave function $\psi(z)$ on $\tilde{T}$ should uniquely determine that on $\tilde{P}$ in a neighbourhood of the intersection. Hence $\chi(z)$ must take the following form around the origin.

$$\chi_N^{\alpha}(z) = \mathrm{e}^{\frac{2\pi i\alpha r}{q}} z^{\frac{r}{q}} \phi_N^{\alpha}(z) \tag{2.20}$$

In the above equation we have chosen $\kappa = \frac{r}{q}$ keeping in mind that $\chi_N^{\alpha}(z)$ should have exactly q number of branches. This is necessary for agreement of ψ and χ around the puncture. $z^{\frac{r}{q}}$ in (2.20) is the principal branch of z^{κ}.

Again since $\tilde{P} \equiv (\mathbb{R}^2\backslash 0, 0)/\mathbb{Z}_2$, the wave functions defined on it must have definite 'parity' since this results into a constant phase factor in the wave function. As a result $\phi_N^{\alpha}(z)$ must be even or odd. This property must hold for the wave functions on $\tilde{T}$ in order that the wave functions agree on a circle around the origin. For example in the case pq even [55] we construct from (2.19) wave functions with definite parity through the combination:

$$\psi_N^{\alpha(\pm)}(\tau, z) = \tilde{\vartheta}_{\alpha,N}(\tau, z) \pm \tilde{\vartheta}_{-\alpha,-N}(\tau, z) \tag{2.21}$$

Now in order to match the wave-functions, we have to do Laurent expansion around the origin. Laurent expansion about a point say P_+ on a torus was studied in [67]. It was shown that there exists a basis on C_η which is anlogous to the basis z^n on a circle, where C_η parametrizes a compact Riemann surface in same way as a circle can parametrizes the extended complex plane, η being a well defined global parameter that labels the curve $C_\eta = \{Q : \Re[p(Q)]\} = \eta$, $p(Q) = \int_{Q_0}^{Q} dp$, dp on the other hand is a differential of third kind on the Riemann surface with poles of the first order at the points $P_\pm$ with residues ± 1. In the case of a torus an exact basis $A_n(z)$ on C_η is given in [67]. These are the Laurent basis for curves on the torus on a special system of contours C_η . As $\eta \to \pm\infty$, C_η are small circles enveloping the point , $P_\mp$. We have to match the wave function on the the torus around a small circle about P_+ with that on $\tilde{P}$. We could have expanded of $\psi(z)$ in terms of the basis $A_n(z)$ while for the latter we can expand $\chi(z)$ in terms of z^n. The two expressions must be equal, when an expansion of the basis $A_n(z)$ is performed in terms of z^n. Since $\psi_N^{\alpha(\pm)}(\tau, z)$ is holomorphic we have the Laurent expansion for $\phi_N^{\alpha(\pm)}(\tau, z)$, (which is related to $\chi_N^{\alpha(\pm)}(\tau, z)$ through (2.20)) around the origin as follows:

$$\begin{aligned}
\phi_N^{\alpha(\pm)}(\tau, z) = 1 + \left[\frac{u^2}{2!}\left(\pi ipq\right)^{-1}\partial_\tau + \frac{u^4}{4!}\left(\pi ipq\right)^{-2}\partial_\tau^2 + \cdots\right] \times \sum_j \left(\mathrm{e}^{\pi ipq\tau x_j^2} \pm \mathrm{e}^{\pi ipq\tau \tilde{x}_j^2}\right) \\
+ \left[u + \frac{u^3}{3!}\left(\pi ipq\right)^{-1}\partial_\tau + \frac{u^5}{5!}\left(\pi ipq\right)^{-2}\partial_\tau^2 + \cdots\right] \times \sum_j \left(x_j \mathrm{e}^{\pi ipq\tau x_j^2} \pm \tilde{x}_j \mathrm{e}^{\pi ipq\tau \tilde{x}_j^2}\right) \qquad (2.22)
\end{aligned}$$

with $u = i\pi pz$ and $x_j = j + \frac{qN+p\alpha}{pq}$ and $\tilde{x}_j = j - \frac{qN+p\alpha}{pq}$. ϕ_N^α in the wave function (2.20) should have the same form as above (2.22). This does not determine the exact form of the above function on the entire $\tilde{P}$. But this asymptotic form on $\tilde{P}$ ensures the finite number p of the wave functions each with q components.

Hence we have at hand the full Hilbert space of the quantized theory. Dimension of the Hilbert space is $p_{(+)}p_{(-)}$. The extensions determined by the above asymptotic form should also be 'square integrable' with respect to some well-defined measure $\mathbf{d}\mu_{\tilde{P}}$. The unitarily invariant, polarization independent inner product associated with this Hilbert space of wave functions (to be more precise 'half-densities') is given in temrs of the Kähler potential on $\tilde{T}$ corresponding to (2.6) or (2.13) and measure $\mathbf{d}\mu_{\tilde{P}}$ on $\tilde{P}$ is given as:

$$\langle \Psi, \Psi' \rangle = \int_{\tilde{T}} \sum_\alpha \mathbf{d}z\mathbf{d}\bar{z}\tau_2^{-1/2}\mathrm{e}^{-\frac{k\pi}{8\tau_2}(2z\bar{z}+\Xi(z)+\Xi(\bar{z}))}\psi'_\alpha(z)\psi_\alpha(\bar{z}) + \int_{\tilde{P}} \sum_\alpha \mathbf{d}\mu_{\tilde{P}}\chi'_\alpha(z)\chi_\alpha(\bar{z})$$

2.3.3 $\gamma \to 1$ limit in quantum theory

Brown and Henneaux [19], way back in '86 showed that asymptotic symmetries asymptotically AdS manifolds, which are of course solutions of $2+1$ gravity with negative cosmological constant (not necessarily the BTZ solution), is given by a pair of Witt algebras- the deformation algebra of S^1 instead of the expected isometry $SO(2,2)$ of AdS$_3$. Canonical phase space realization of these asymptotic symmetries however are given by a pair of Virasoro algebras, which are centrally extended versions of the symmetry algebra. Later various authors [32] for example, reproduced the result with equivalent theories of (1.4) or toplogically massive gravity (TMG) [46, 68, 69] confirming an AdS(3)/CFT(2) correspondence, although with unequal central charges. In the theory we are dealing with, these central charges come out to be $\left(c_{(+)}, c_{(-)}\right) = \frac{3l}{2G}\left(\left(1+\frac{1}{\gamma}\right), \left(1-\frac{1}{\gamma}\right)\right)$ in our conventions and notations.

The chiral limit ie $\gamma \to 1$ in this direction has gained importance in recent literature for various reasons. In view of results from [50], where second order TMG was studied on an asymptotically AdS spacetime, we see that in order to make sense of all the graviton modes γ should be restricted to 1. At this limit the theory becomes chiral with $\left(c_{(+)}, c_{(-)}\right) = \left(\frac{3l}{G}, 0\right)$. Another interesting result by Grumiller et al [70] reveals that at the quantum level chiral limit of TMG is good candidate as a dual to a logarithmic CFT (LCFT) with central charges $\left(c_{(+)}, c_{(-)}\right) = \left(\frac{3l}{G}, 0\right)$. More recent works with some of the interesting ramifications of TMG 'new massive gravity' [48] shows similar progress [71]. These results were worked out on an asymptotically AdS space-time. In the present case however, we have considered spatial slice to be a genus 1 compact Riemann surface, without boundary. Hence chance of a CFT living at the boundary doesn't arise. Even if we had worked on a asymptotically AdS manifold, the theory would not be dual to an LCFT, because for that a propagating degree of freedom is necessary, which is absent in our case.

However there are some interesting issues in the present discussion for the limit $\gamma \to 1$: We have inferred in 2.3.1 from (2.5),(2.18) and (2.19) $k_{(\pm)} = \frac{l}{8G}(1/\gamma \pm 1)$, which are related to above discussed central charges through $k_{(\pm)} = \pm\frac{1}{12}c_{(\pm)}$ must be positive rationals. As a result, if the ratio of the AdS radius l and and Planck length G (in units of $\hbar = 1 = c$) is positive, we must restrict $0 < \gamma < 1$. This is in apparent contradiction to the restriction

$\gamma \geq 1$ [72] put by the CFT (living in the boundar, in the case of asymptotically AdS formulation). But this may well be resolved from the point of view that our analysis is completely performed on spacetime topology (as seen clearly in the construction of the physical phase space) whose spatial foliations are compact tori and relevant ranges of γ should depend non-trivially on the topology of spacetime and in our case restrictions coming from suitable CFT is not clear as explained in next paragraph.

As argued in 1.0.4 at the point $\gamma = 1$, we describe 2+1 gravity with negative cosmological constant through a single $SO(2,1)$ Chern Simons action (1.29). On the other hand, for a rational $SO(2,1)$ (or any of its covers) Chern Simons theories on genus-1 spatial foliation, existence of a dual CFT too is still not very clear, as argued in [55]. The modular transformation ($SL(2,\mathbb{Z})$) representations acting on the physical hilbert space (as found in 2.3.1, 2.3.2) appaear to be one of the two factors in to which modular representations of the conformal minimal models factorize. This observation points that a 2-D dual theory may not be conformal, although one may identify conformal blocks (of a CFT, if it exists) labelling our wavefunctions [55].

2.3.4 Results on the quantization of parameters

We have explained in section 2.3.1 that $k_{(\pm)} = \frac{p_{(\pm)}}{q_{(\pm)}}$ are positive rationals. In [20] it has been shown that for the gauge group being an n-fold diagonal cover of $SO(2,1)\times SO(2,1)$, one requires the couplings

$$\begin{aligned} k_{(+)} &\in 8n^{-1}\mathbb{Z} \quad \text{for } n \text{ odd} \\ k_{(+)} &\in 4n^{-1}\mathbb{Z} \quad \text{for } n \text{ even and} \\ k_{(+)} + k_{(-)} &\in 8\mathbb{Z} \end{aligned} \tag{2.23}$$

in our notation and convention. This is in agreement with our finding that the consistent quantization procedure reveals $k_{(\pm)} = \frac{p_{(\pm)}}{q_{(\pm)}} \in \mathbb{Q}^+$ and we are considering $q_{(\pm)}$ covers of the phase space (see section 2.3.2) which is constructed from the gauge group. In terms of physical parameters we have

$$\begin{aligned} \frac{l}{G} \in \mathbb{Q}^+ \quad \text{and} \\ \frac{l}{G\gamma} \in \mathbb{Q}^+ \\ \Rightarrow \gamma \in \mathbb{Q}^+ \end{aligned} \tag{2.24}$$

which are slightly less restrictive than the results of the analysis done in [20] $\frac{l}{G} \in \mathbb{Q}^+$ and $\frac{l}{G\gamma} \in \mathbb{N} \subset \mathbb{Q}^+$.

2.4 Conclusion

The features which come out of our analysis can be summarized as follows.

Classically it is observed that γ fails to induce canonical transformations on the canonical variables although equations of motion do not involve γ. The role of γ is best viewed in the constraint strucure of the theory which is also studied in detail. On the other hand the 'chiral' limit relevant in our case is $\gamma \to 1+$ as opposed to the TMGs on asymptotically AdS space times, where it is $\gamma \to 1-$. In the canonical structure the apparent singularity can also be removed as discussed in 1.0.4.

Natuarally different values of γ results in inequivalent quantizations of the theory. Dimensionless γ and the cosmological constant $-\frac{1}{l^2}$ give the dimensionality of the physical state space in a subtle manner. Note that we had $k_{(+)}k_{(-)} = l^2 \frac{1/\gamma^2 - 1}{64G^2}$, $k_{(\pm)} = \frac{p_{(\pm)}}{q_{(\pm)}}$, $p_{(\pm)}$ and $q_{(\pm)}$ being both positive integers and prime to each other. Dimension of the Hilbert space turns out to be $p_{(+)}p_{(-)}$ which must be a positive integer .This requirement, provides allowed values of γ, for a given $\frac{l}{8G}$ such that $\frac{l}{8G} \in \mathbb{Q}^+$ and $\frac{l}{8G\gamma} \in \mathbb{Q}^+$.

Appendix : Conjugacy classes of $SL(2, \mathbb{R})$

Any $SL(2, \mathbb{R})$ (which is the double cover of $SO(2, 1)$[4]) element G can be written in its defining representation as the product of three matrices by the Iwasawa decomposition

[4]Since from gravity action we got a gauge theory with a lie algebra shared commonly by $SO(2, 1)$, $SL(2, \mathbb{R})$, $SU(1, 1)$ or any covering of them, the actual group used is quite irrelevant unless one is considering transformations between disconnected components of the group manifold.

uniquely

$$G = \underbrace{\begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}}_{f_\phi} \underbrace{\begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{\xi/2} \end{pmatrix}}_{g_\xi} \underbrace{\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}}_{h_\eta} \tag{2.25}$$

with the range of ϕ being compact $(-2\pi, 2\pi)$ and those of ξ and η noncompact. Note that these three matrices fall in respectively the elliptic, hyperbolic and the null or parabolic conjugacy class of $SL(2,\mathbb{R})$, in addition to forming three abelian subgroups themselves. Also note that

$$\begin{aligned} f_\phi &= \exp\left(i\sigma_2\phi/2\right) = e^{-\lambda_0\phi} \\ g_\xi &= \exp\left(\sigma_3\xi/2\right) = e^{\lambda_2\xi} \\ h_\eta &= \exp\left[(i\sigma_2+\sigma_1)\eta/2\right] = e^{(-\lambda_0-\lambda_1)\eta} \end{aligned}$$

where $\lambda_I \in \mathfrak{sl}(2,\mathbb{R})$ with $[\lambda_I, \lambda_J] = \epsilon_{IJK}\lambda^K$.

We now state an important result which is used in the text. Let $g = \exp\left(\kappa_I\lambda^I\right)$ and $g' = \exp\left(\kappa'_I\lambda^I\right)$ be two $SL(2,\mathbb{R})$ elements. Then the necessary and sufficient condition for $g_1g_2 = g_2g_1$ to hold is $\kappa^I = c\kappa'^I$ for $I = 0,1,2$ and any $c \in \mathbb{R}$. This can be seen by using the Baker Campbell Hausdorff formula.

Chapter 3

Isolated Horizons and Asymptotic Symmetries in 2+1 dimensions

3.1 Introduction

In chapter 2, we studied 3d quantum gravity on space-times with toric spatial foliation in presence of negative cosmological constant. As we have already discussed, all classical solutions of this theory are locally AdS_3. However most important of these solutions are those which are asymptotically AdS_3. It came into prominence mainly due to two results. One of them was about asymptotic symmetry, which takes the form of the Virasoro algebra. This was discussed briefly in section 2.3.3. The other factor which triggered studies in asymptotically AdS spaces, was the existence of the black hole solution. The theory has been shown to admit the BTZ black hole as an excited state and the AdS_3 solution as it's vacuum [18] [1]. As discussed multiple times earlier, the theory of 3d gravity we have been dealing with (with a the parameter γ), shares the same solution space as the ordinary theory of gravity. It naively follows that we have potential scope of studying asymptotically AdS_3 solutions and their quantum theory, in this γ deformed phase space.

We have also been comparing the outcomes of our theory with similar studies in topologically massive gravity (TMG) at each stage of development. In that spirit it should be apt to look at what TMG offers for space-times which are asymptotically AdS_3. The theory of TMG has some peculiarities - the massive excitations carry negative energy for a positive coupling constant (in this case, it is the G) [2]. In case of negative cosmological

[1] At this point, we must take note of the fact that there are a more general class of black holes in 2+1 topological gravity theories [73], of which BTZ is a special one.

constant TMG, the situation is drastic. Change in sign of the coupling constant gives excitations with positive energy but gives negative mass BTZ black hole solutions [74].

Studies on 3d black holes (see [74–77] and references therein) and their entropy calculation in topological 3d gravity as well as TMG have formed a major body of research in the field of 2+1 quantum gravity. For example, particularly entropy has been studied in great detail in [68, 72, 78, 79] for a large class of interactions governed by Chern Simons theory. These computations are majorly based upon two different routes. Most popular is the one which follows [52]. A simple use of Cardy formula for the central extensions and the Hamiltonian modes in the resulting Virasoro algebras gives the entropy (see [80] for discussions). This is again based on the results of the seminal paper by Brown and Henneaux [19], briefly described in 2.3.3. On the other hand, there is another path, (eg [68, 79]) which uses covariant phase space framework, following Wald [81, 82]. Unfortunately this approach heavily relies on the (bifurcate) Killing horizon structures and they have their own problems including restriction to non-extremal horizons only.

Dynamical issues, conserved charges in similar class of theories (for asymptotically AdS cases and with or without inner horizon), including the canonical realization of asymptotic symmetries have been studied in [31, 32, 46, 47, 78, 83–89]. Entropy of the BTZ black hole in these modified topological theories and the TMG were also presented in these papers. Contrary to the Bekenstein-Hawking expectation, the entropy turns out not only to be proportional to the black hole area, but also to some extra terms, involving even the horizon angular momentum [46, 78]. In this chapter, we shall investigate related issues for a general class of theories in a covariant manner and exhibit reasons why such results are expected. Moreover, we shall establish that our method is equally applicable to extremal and non-extremal black holes since it does not rely on the existence of bifurcation spheres.

Our aim will be to establish the laws of black hole mechanics in this theory and to determine entropy in fully covariant framework. We stress here, that our analysis of the horizon dynamics applies for an infinite class of space-time manifolds which have an inner boundary (with some specific boundary conditions) and are asymptotically AdS which may allow arbitrary matter or radiation outside the horizon. [2]

[2]But one may argue that since 3d pure gravity is locally non-dynamical, solution space may be essentially finite dimensional and BTZ solution specifies that uniquely for such given asymptotic conditions.

In this respect, it becomes important to introduce conserved charges like the angular momentum and mass. We shall use the formalism of isolated horizons to address these issues. The set-up is robust and conceptually straightforward, resulting in surprising simplicity in calculations. The detailed definition of isolated horizons, their classical dynamics including application to black-hole dynamics were developed in a series of papers [23,90–96]. For 2+1 dimensions the analysis needed some attentions, which were addressed in [27]. We shall however use a weaker set of boundary conditions than [27], extend to more general theories, study the asymptotic symmetries (apart from working out the horizon dynamics) and eventually determine the entropy of horizons.

The basic idea behind the quasi-local description of isolated horizons is following: a horizon (black hole or cosmological) is a null hyper-surface which can be described locally, by providing the geometric description of that surface only. Black hole horizon (we primarily are interested in these horizons here) is described in this formalism as an internal boundary of space time which is expansion free and on which the field equations hold[3]. Unlike in the case of Killing horizons/event horizons, we need not look in to the bulk near-horizon or asymptotics of the space time to define isolated horizons; only horizon properties are enough. It is because of this generality that isolated horizon is useful to describe even solutions where the asymptotic structure is still not well-defined or has not fully developed. As it happens (and we shall show this below), the boundary conditions on the horizon itself enable us to prove the zeroth law of black hole mechanics directly. The first law of black hole mechanics and construction of conserved charges is not difficult in this formalism. We will employ the formalsim of covariant phase space, already discussed in 1.0.2, in order to study those charges. This has already been applied successfully to study dynamics of space times with isolated horizon as an internal boundary. The conserved charges (like angular momentum) are precisely the *Hamiltonian functions* corresponding to the vector field generating canonical transformations or the so called

As we will be exploring through the chapter, it will be revealed that indeed there are infinite number of dynamically independent degrees of freedom coming from boundary excitations (both inner and asymptotic). Moreover solutions of more general class containing black holes are known [73].

[3]This construction is more general than that of Killing horizon. Laws of black hole mechanics were proved for this quasi-local definition too [81, 82, 97]. But, as mentioned earlier this formalism does not seem useful to address extremal horizons. Improvements by introducing extremal horizons in the same space of the non-extremal horizons in the isolated horizons framework were made in [98, 99].

Hamiltonian vector fields (which in this case is related to rotational Killing vector field on). The first law, in this description, comes out as a result of imposing the condition that the null generator of the horizon should give rise to Hamiltonian flow on the phase space. These features, as we shall show below, can be established very easily in the regime of symplectic geometry.

Now let us focus on the dynamical implications of kinematical asymptotic symmetries, in the above-mentioned framework. We note that there is a precise definition of the asymptotic symmetry group, if we know the fall-off behaviour of the geometry asymptotically. A natural question to ask is whether the action of this group on the pre-symplectic manifold, defined through the degenerate symplectic structure, is a Hamiltonian. As have been the expectation through canonical analyses made earlier, the answer is not in the affirmative, rather the algebra of symmetry generators get centrally extended. This leads to finding the black hole entropy using the Cardy formula (just as for the TMG case). In a sense it is similar to TMG where the parameters are the topological mass and the cosmological constant. (For TMG, this implies that the massive graviton introduced through the extra couplings have no effect on the entropy.)

In this chapter our plan is to first 3.2, recall the definition of Weak Isolated Horizons (WIH) and prove the zeroth law of black hole mechanics [98, 99]. The proof of zeroth law is purely kinematical and does not require any dynamical information. Since we shall be interested in manifolds with inner and outer boundaries, an extension of the dynamical questions addressed in the first chapter 1 is in order. That will be taken care of next 3.3.1. A brief description of the BTZ solution as an example of a black hole solution in this theory will also be presented. In sections 3.3.2 and 3.3.3, we shall establish that indeed the action principle is well defined even when the inner boundary is a WIH. In section 3.3.4, we construct the space of solutions and pre-symplectic structure for space-time manifolds equipped with a WIH. The phase-space contains all solutions, (extremal as well as non-extremal black hole solutions) which satisfy the boundary conditions of WIH for the inner boundary and are asymptotically AdS at infinity. In section 3.3.6, we shall show how the angular momentum can be extracted from the symplectic structure. The angular momentum will naturally arise as a Hamiltonian function (on the phase-space) corresponding to the Hamiltonian vector field associated with rotational Killing vector field on the space-time. When the definition is applied to the BTZ solution, it will naturally arise that the angular momentum depends on the parameters J and M of the solution. Next (3.4), we will construct the vector fields which generate diffeomorphisms preserving the asymptotic conditions. We shall construct Hamiltonians functions corresponding to these vector fields and show that in presence of a WIH inner boundary, the Hamiltonian charges do not realize the algebra of vector fields. The difference is a central extension which gives rise to the entropy for black holes in these theories. We shall also observe that the parameter γ shows up in all stages. We shall discuss these issues in the section (3.5).

The work presented in this chapter follows those presented in our paper [100].

3.2 Weak Isolated Horizon: Kinematics

We now give a very brief introduction to weak isolated horizons [98]. Let $\mathcal{M}$ be a three-manifold equipped with a metric g_{ab} of signature $(-,+,+)$. Consider a null hypersurface Δ in $\mathcal{M}$ of which ℓ^a is a future directed null normal. However, if ℓ^a is a future directed null normal, so is $\xi\ell^a$, where ξ is any arbitrary positive function on Δ. Thus, Δ naturally admits an equivalence class of null normals $[\,\xi\ell^a\,]$. The hypersurface Δ being null, the metric induced on it by the space-time metric g_{ab} will be degenerate. We shall denote this degenerate metric by $q_{ab} \triangleq g_{\underleftarrow{ab}}$ (since we are using abstract indices, we shall distinguish intrinsic indices on Δ by pull-back and $\triangleq$ will mean that the equality holds *only on* Δ). The *inverse* of q_{ab} will be defined by q^{ab} such that $q^{ab}q_{ac}q_{bd} \triangleq q_{cd}$. The expansion $\theta_{(\ell\,)}$ of the null normal ℓ^a is then defined by $\theta_{(\ell\,)} = q^{ab}\nabla_a\ell_b$, where ∇_a is the covariant derivative compatible with g_{ab}. Null surfaces are naturally equipped with many interesting properties. Firstly, the null normal is hyper-surface orthogonal and hence is twist-free. Secondly, the ℓ^a is also tangent to the surface. It is tangent to the geodesics generating Δ. Thus, any ℓ^a in the class $[\xi\ell^a]$ satisfies the geodesic equation:

$$\ell^a\nabla_{\underleftarrow{a}}\,\ell^b \triangleq \kappa_{(\ell)}\,\ell^b. \tag{3.1}$$

We shall interpret the acceleration $\kappa_{(\ell)}$ as the surface gravity. If the null normal to Δ is such that κ vanishes, we shall call it to be extremal surface. Otherwise, the surface will be called non-extremal. The variation of κ in the null class [4] $[\xi\ell]$ being as $\kappa_{(\xi\ell)} = \xi\kappa_{(\ell)} + \pounds_{\ell}\xi$.

In what follows, we shall use the Newmann-Penrose (NP) basis for our calculations. In three dimensions, this will consist of two null vectors ℓ^a and n^a and, one space-like vector m^a They satisfy the condition $\ell.n = -1 = -m.m$ while other scalar products vanish. This basis is particularly useful for our set-up because the normal to Δ, denoted by ℓ^a can be chosen to be the ℓ^a of NP basis. The space-like m^a will be taken to be tangent to Δ. In this basis, the space-time metric will be given by $g_{ab} = -2\,\ell_{(a}n_{b)} + 2\,m_{(a}m_{b)}$ whereas the pull-back metric q_{ab} will be simply, $q_{ab} \triangleq m_a m_b$.

3.2.1 Weak Isolated Horizon and the Zeroth Law

The null surface Δ introduced above is an arbitrary null surface equipped with an equivalence class of null normals $[\xi\ell^a]$. The conditions on Δ are too general to make it resemble a black hole horizon. To enrich Δ with useful and interesting information, we need to impose some additional structures (the imposed conditions will be weaker than that in [27] in the sense that our equivalence class of null normals will be related by functions on Δ rather than constants). As we shall see, the zeroth law and the first law of black hole mechanics will naturally follow from these conditions. These definitions will be local and only provides a construction of black hole horizon and do not define a black hole spacetime which is a global object. However, if there is a global solution, like the BTZ one, then these conditions will be satisfied.

The null surface Δ, equipped with an equivalence class of null normals $[\xi\ell^a]$, will be called a *weak isolated horizon* (WIH) if the following conditions hold:

1. Δ is topologically $S^1 \times \mathbb{R}$.

2. The expansion $\theta_{(\xi\ell)} \triangleq 0$ for any $\xi\ell^a$ in the equivalence class.

3. The equations of motion and energy conditions hold on the surface Δ and the vector field $-T^a_b\xi\ell^b$ is future directed and causal.

[4]Without the positive function ξ, κ cannot be set to zero unless the null hyper-surface is part of a neigbourhood of family of null hyper-surfaces. This follows from the fact that not only the hyper-surface normal be null but also that it's derivative off the surface vanishes (see sec 3.1 of [101])

4. The above conditions imply existence of a 1-form ϖ such that

$$\nabla_{\underleftarrow{a}}\ell^b \triangleq \varpi_a^{(\ell)}\ell^b.$$

and it is Lie-dragged along the horizon Δ,

$$\pounds_{\xi\ell}\,\varpi^{(\xi\ell)} \triangleq 0 \tag{3.2}$$

In the literature, Δ is called a *non-expanding horizon* (NEH) if it satisfies only the first three conditions. It is clear that that the boundary conditions for a NEH hold good for the entire class of null normals $[\xi\ell^a]$ if it is valid for one null normal in that class. The Raychaudhuri equation imply that NEHs are also shear free. Thus, NEHs are twist-free, expansion-free and shear-free and this implies that the covariant derivative of ℓ^a on Δ is much simple. This is the reason for existence of the 1-form ϖ in the third point above (see appendix 3.5 for a Newman-Penrose type discussion), such that

The one form $\varpi_a^{(\ell)}$ varies in the equivalence class $[\xi\ell^a]$ as

$$\varpi^{(\xi\ell)} \triangleq \varpi^{(\ell)} + d\ln\,\xi \tag{3.3}$$

A few other conclusions also follow. Firstly, from equations (3.1) and (3.3), it follows that $\kappa_{(\xi\ell)} \triangleq \xi\ell.\varpi^{(\xi\ell)}$. Secondly, that the null normals in the equivalence class are Killing vectors on NEH $\pounds_\ell\, q_{ab} \triangleq 2\underleftarrow{\nabla_{(a}\ell_{b)}} \triangleq 0$. Thirdly, the volume form on Δ, is Lie-dragged by the null normal in the equivalent class, ie, $\pounds_{\xi\ell} m \triangleq 0$.

At this point one should note that the acceleration $\kappa_{(\xi\ell)}$ is in general a function on Δ. Now we would want NEH to obey the zeroth law of black hole mechanics, which requires constancy of the acceleration of the null normal on Δ. This puts further restriction on it. This is done by demanding the fourth condition in the list of boundary conditions, equation (3.2). Although this is not a single condition, (*i.e.* unlike the other three conditions, it is not guaranteed that if this condition holds for a single vector field ℓ^a, it will hold for *all* the others in the class $[\xi\ell^a]$ for any arbitrary ξ), one can always choose a class of functions ξ on Δ [98,99], for which this reduces to a single condition. For example, if the class of function is, $\xi = F\exp(-\kappa_{(\ell)}v) + \kappa_{(\xi\ell)}/\kappa_{(\ell)}$, where $\ell^a = (\partial/\partial v)^a$ and F is a function such that $\pounds_\ell F \triangleq 0$, the condition holds for the entire equivalence class.[5]. Also note that from (3.3) that $d\varpi^{(\xi\ell)}$, which is proportional to the Weyl tensor, is independent of variation of ξ. Since the Weyl tensor vanishes identically in three dimensions, we have $d\varpi^{(\xi\ell)} \triangleq 0$. The equation (3.2) then gives the *zeroth law*: $d\kappa_{(\xi\ell)} \triangleq 0$.

3.3 Weak Isolated Horizon: Dynamics

In this section, we would derive the first law of black hole mechanics. The required ingredients, namely the dynamics of the theory and the boundary conditions are already present in our hands. In the following subsections we would set the stage for demonstrating the first law. These computations do work for any black hole solution which obeys the very week boundary conditions spelt above. However for the purpose of comparison we will be referring to results pertaining to the well known BTZ solution at each step. As have been the theme of this thesis, we shall use the first order connection formulation. This formulation is tailor-made for our set-up and the calculational simplicity will be enormous. In particular, the construction of the covariant phase-space and it's associated symplectic structure is a straightforward application of the notions used in higher dimensions [98,99]. The use of forms also simplifies the calculation of first law and the conserved charges.

Our 3-manifold $\mathcal{M}$ will be taken to be topologically $M\times\mathbb{R}$ with boundaries. The inner null boundary will be denoted by Δ which is taken to be topologically $S^1\times\mathbb{R}$. The initial and final space-like boundaries are denoted by M_- and M_+ respectively. The boundary at infinity will be denoted by i_0. In what follows, the inner boundary will be taken to be a WIH. In particular, this implies that the surface Δ is equipped with an equivalent class of null-normals $[\xi\ell^a]$ and follows eqn. (3.2).

3.3.1 The BTZ solution

Just like in $2+1$ gravity with a negative cosmological constant, the 2 parameter family of BTZ black holes is a solution of our theory (1.10). In the standard coordinates, the

[5] This is a virtue in disguise in the sense that we can interpolate between extremal horizons, with $\kappa \triangleq 0$ to non-extremal horizons with $\kappa \neq 0$ using this ξ. In other words, we can use this formalism to accommodate extremal as non-extremal horizons in the same phase space.

solution is given by:

$$ds^2 = -N^2\, dt^2 + N^{-2}\, dr^2 + r^2(N^\phi\, dt + d\phi)^2, \tag{3.4}$$

where the lapse and the shift variables contain the two parameters M and J and are defined by:

$$N^2 = (-\frac{M}{\pi} + \frac{r^2}{l^2} + \frac{J^2}{4\pi r^2}) \qquad \text{and} \qquad N^\phi = -\frac{J}{2\pi r^2} \tag{3.5}$$

More suitable it is for our purpose, when expressed in terms of the first order variables. In a particular frame (being commensurate with (3.4)), these read as:

$$\begin{aligned} e^0 = N\, dt, \quad e^1 = N^{-1}\, dr \quad \text{and} \quad e^2 = r\,(\, d\phi + N_\phi\, dt\,) \\ \omega^0 = -N\, d\phi, \quad \omega^1 = N^{-1}\, N_\phi\, dr \quad \text{and} \quad \omega^2 = -\frac{r}{l^2}\, dt - r\, N_\phi\, d\phi. \end{aligned} \tag{3.6}$$

The horizon is defined through the zeros of the lapse function N which gives the position of the horizon to be:

$$r_\mp = l\left[\frac{M}{2\pi}\left\{1 \mp (1 - (J/Ml)^2)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}} \tag{3.7}$$

It is not difficult to see that the outer horizon (at r_+) satisfies the conditions of WIH Δ. It is a null surface with null normal $\ell^a = (\partial/\partial v)^a + N^\phi(r_+)\,(\partial/\partial\phi)^a$. A simple calculation also shows that $\theta_{(\ell)} \triangleq 0$. In what follows, we shall always refer back to this solution to check if our definitions for conserved charges are consistent.

3.3.2 Computing tetrads and connection on Δ

Before proceeding with the variation of the action and determining the equations of motion, it will be useful to have the values of the tetrad and connection on the null surface Δ. The usefulness of such calculation will be apparent soon. We shall assume that it is possible to fix an internal null triad (ℓ^I, n^I, m^I) such that $\ell^I n_I = -1 = -m^I m_I$ and all others zero. The internal indices will be raised and lowered with η_{IJ}. Given the internal triad basis (ℓ^I, n^I, m^I) and e^I_a, the spacetime null basis (ℓ^a, n_a, m^a) can be constructed. We shall further assume that the internal basis is annihilated by the partial derivative operator, $\partial_a\,(\ell^I, n^I, m^I) = 0$.

Using the expression of the spacetime metric in NP basis and the internal metric, we can write the tetrad e_a^I on WIH Δ as:

$$e_{\underleftarrow{a}}^{I} \triangleq -n_a\ell^I + m_a m^I \tag{3.8}$$

To calculate the expression of connection on Δ, we shall use the NP coefficients which can be seen in the covariant derivatives of the NP basis. They are as follows:

$$\nabla_{\underleftarrow{a}}\ell^b \triangleq \varpi_a^{(\ell)}\,\ell^b \tag{3.9}$$

$$\nabla_{\underleftarrow{a}}n_b \triangleq -\varpi_a^{(\ell)}\,n_b + U_a^{(\ell,m)}m_b \tag{3.10}$$

$$\nabla_{\underleftarrow{a}}m^b \triangleq U_a^{(\ell,m)}\,\ell^b, \tag{3.11}$$

where, the superscripts on the one-forms $\varpi_a^{(\ell)}$ and $U_a^{(\ell,m)}$ indicate that they depend on the transformations of the corresponding basis vectors. The one-forms used in the eqn. (3.9) are compact expression of the NP coefficients. They are given by:

$$\varpi_a^{(\ell)} \triangleq (-\epsilon\, n_a + \alpha\, m_a) \tag{3.12}$$

$$U_a^{(\ell,m)} \triangleq (-\pi\, n_a + \mu\, m_a) \tag{3.13}$$

We will now demonstrate how the Newmann-Penrose coefficient α is fixed to be real number on Δ using topological arguments. Note from previous discussion that $d\,\varpi^{(\ell)} \triangleq 0$. From the definition (3.50) we have $\underleftarrow{dm} \triangleq -\rho m\wedge n$. But because Δ is expansion-free and ℓ^a is the generator of Δ, $\rho \triangleq 0$. Hence m^a is also closed on Δ (m should not strictly be exact since $\int_{S_\Delta} m \sim$ area of horizon $\neq 0$). Since the first cohomology group of $\Delta \simeq \mathbb{R}\times S^1 \equiv \mathbb{R}$ is non-trivial, we have in general neither $\varpi^{(\ell)}$ nor m_a exact. Hence there exists smooth function ς and a real number s for which

$$\varpi^{(\ell)} \triangleq d\,\varsigma + s\,m \tag{3.14}$$

We now introduce a potential $\psi_{(\ell)}$ for surface gravity (or the acceleration for ℓ^a) $\kappa_{(\ell)} \triangleq \ell^a\varpi_a^{(\ell)} \triangleq \epsilon$ through

$$\pounds_\ell\,\psi_{(\ell)} \triangleq \kappa_{(\ell)}.$$

Since the zeroth law implies constancy of $\kappa_{(\ell)}$ on Δ, $\psi_{(\ell)}$ can only be function of v (could be treated as the affine parameter on Δ) only. Hence $\pounds_m\psi_{(\ell)} \triangleq 0$, which implies on the other hand $d\,\psi_{(\ell)} \triangleq -\epsilon\, n$ and $\varpi \triangleq d\,\psi_{(\ell)} + \alpha\, m$. It is tempting to choose $\varsigma = \psi_{(\ell)}$ by compared with (3.14). That could only be supported if Δ is axisymmetric. (Because even after choosing a triad set for which $\underleftarrow{d}\, n \triangleq 0$, we end up with $\underleftarrow{d\alpha} \wedge m \triangleq 0$, which renders $\underleftarrow{d\alpha} \triangleq 0$ only if α is axisymmetric). For that case, we conclude $\varpi^{(\ell)} \triangleq (\, d\,\psi^{(\ell)} + \alpha\, m\,)$, $\alpha \in \mathbb{R}$.

Now, to calculate the connection, we use two facts. First is that the tetrad is annihilated by the covariant derivative, $\nabla_a\, e_b^I = 0$ and, secondly that partial derivative annihilates the NP internal basis so that

$$\nabla_{\underleftarrow{a}}\, \ell^I \triangleq -\epsilon^{IJK} \omega_{aK}\, \ell_J. \tag{3.15}$$

Using equations (3.12) and (3.15) and $\epsilon_{IJK} = 3!\, \ell_{[I} n_J m_{K]}$, we get the following expression for pulled-back connection on Δ:

$$\omega_{\underleftarrow{a}}^I \triangleq -U_a^{(\ell,m)}\, \ell^I + \varpi_a^{(\ell)}\, m^I. \tag{3.16}$$

The equation (3.16) will be used frequently in what follows.

3.3.3 Differentiablity of the action in presence of Δ

In the chapter 1 we demonstrated that we get the first order Einstein equations of motion even by varying the generalized action (1.10) containing γ. However that result obtained for space-time manifold being boundary-less. The task is now to vary the action to obtain the equations of motion and also to verify that the action principle is obeyed in presence of the boundaries, since we are looking for dynamics in presence of isolated horizon. The variation will be over configurations which satisfy some conditions at infinity and at the inner boundary (see fig. (3.1)). At infinity, they satisfy some asymptotic conditions which are collected in the appendix of [27]. On the inner boundary Δ, they are subjected to the following conditions: (a) the tetrad (e) are such that the vector field $\ell^a = e_I^a \ell^I$ belongs to the equivalence class $[\xi \ell^a]$ and (b) Δ is a WIH. On variation, we shall get equations of motion and some surface terms. The surface terms at infinity vanish because of the asymptotic conditions whereas, as we shall show, those at WIH also vanish because of WIH boundary conditions.

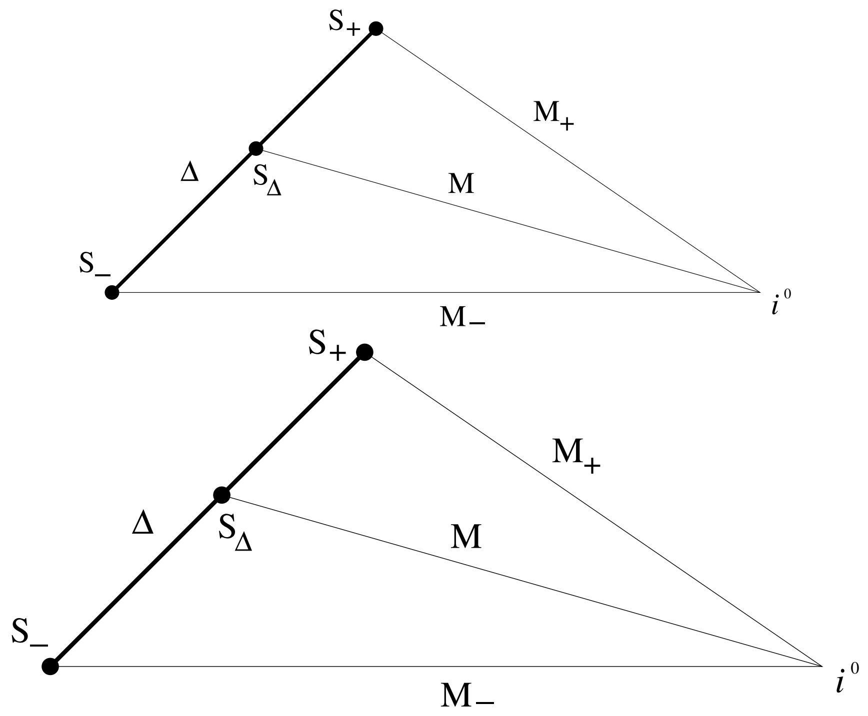

Figure 3.1: Description of space-time. The spacetime is bounded by 2-dimensional surfaces $\Delta, M_{\mp}$ and the infinity. The horizon Δ is a 2-dimensional null surface and $M_{\mp}$ are initial and final hypersurfaces. The infinity is AdS if we work with a spacetime with negative cosmological constant.

Variation of the action with respect to the tetrad (e) and connection ($A\omega$) leads to (for $\gamma^2 \neq 1$):

$$de^I + \epsilon^I{}_{JK}\, e^J \wedge \omega^K = 0 \tag{3.17}$$

$$\text{and} \qquad d\omega_I + \frac{1}{2}\,\epsilon_I{}^{JK}\,\omega_J \wedge \omega_K = -\frac{1}{2l^2}\,\epsilon^{IJK}\, e_I \wedge e_J \wedge e_K. \tag{3.18}$$

The first equation above just points out that the connection ω_I is a spin-connection and the second equation is the Einstein equation. Let us now concentrate on the surface terms. The terms on the initial and final hyper surfaces M_- and M_+ vanish because of action principle. Those at the asymptotic boundary vanish because of the fall-offs at infinity. On Δ, these are given by:

$$\delta\, I = -\int_\Delta (2\, m \wedge \delta\, \varpi^{(\ell)} + \frac{l}{\gamma}\, \varpi^{(\ell)}\, \wedge\, \delta\, \varpi^{(\ell)} + \frac{1}{l\gamma} m\, \wedge\, \delta\, m) \tag{3.19}$$

Our strategy will be to show that the integral is constant on Δ and the integrand is a total derivative so that the integral goes on to the initial and the final boundaries where the variations are zero by assumption. This will then imply that the integral itself vanishes on Δ. Note that in the above equation, $\delta\,\varpi^{(\ell)}$ refers to the variation in $\varpi^{(\ell)}$ among the configurations in the equivalence class $[\xi\ell^a]$. The relation between these are precisely given by eqn. (3.3). Now, we consider the lie derivative of the integrands by $\xi\ell$. Since $dm \triangleq 0$, it follows that $\pounds_{\xi\ell}\, m \triangleq 0$ and $\pounds_{\xi\ell}\,\varpi^{(\ell)} \triangleq d(\pounds_{\xi\ell}\, ln\xi)$. Thus, in the first term, the total contribution is on the initial and final hyper surfaces M_- and M_+ where the variations vanish. Identical arguments for the second and the third integrands also show that the corresponding integral vanishes. Thus, the integral is lie dragged on Δ and since the variations are fixed on the initial and final hyper surfaces, the entire integral vanishes and the action principle remains well-defined.

3.3.4 Covariant Phase Space

Analysis of the dynamics of this theory has been considerably worked out in literature [31, 85] in the canonical framework even in presence of asymptotic boundary. We recall that a covariant phase space analysis for the present theory was described in chapter 1, although in absence of boundaries. As we progress, we will see how apt the covariant analysis is in understanding horizon phenomena and even the conserved charges arising from asymptotic symmetries; using the general ideas of symplectic geometry. We refine the concept of covariant phase space detailed in chapter 1 here as the space of classical solutions (3.17) which satisfy the boundary conditions specified in the previous subsections. In other words, the covariant phase space Γ will consist of of solutions of the field equations which satisfy the boundary conditions of WIH at Δ and have fall-off conditions compatible with asymptotic conditions.

Although outlined earlier, we express the structures on this pre-symplectic manifold again in terms of the explicit variables, which are more suited in the present case. In order to equip this space with a symplectic structure [6], we find the symplectic potential (1.13) from variation of the Lagrangian (1.12), expressed in terms of the variables relevant to

[6]To be more precise, here we will be dealing with the pre-symplectic structure, since the theory has gauge redundancy, which appear as 'degenerate directions' for the symplectic 2-form

gravity:

$$\Theta(\delta) = -2\,(e^I\ \wedge \delta\,\omega_I) - \frac{l}{\gamma}\,(\omega^I\ \wedge \delta\,\omega_I) - \frac{1}{l\gamma}\,(e^I \wedge \delta\,e_I). \tag{3.20}$$

Upon antisymmetrized second variation, it gives the symplectic current J which is a phase-space two-form. For two arbitrary vector fields δ_1 and δ_2 tangent to the space of solutions, the symplectic current for (3.20) is given by following closed two form (cf. (1.14)):

$$\begin{aligned} J(\delta_1,\delta_2) &= \delta_1\,\Theta(\delta_2) - \delta_2\,\Theta(\delta_1) \\ &= -2\left[\left(\delta_1\,e^I \wedge \delta_2\,\omega_I - \delta_2\,e^I \wedge \delta_1\,\omega_I\right) + \frac{l}{\gamma}\,\delta_1\,\omega^I \wedge \delta_2\,\omega_I + \frac{1}{\gamma l}\,\delta_1\,e^I \wedge \delta_2\,e_I\right] \end{aligned} \tag{3.21}$$

Since the symplectic current is closed, $d\,J(\delta_1,\delta_2) = 0$, we define the presymplectic structure on the phase-space by (cf. (1.14)):

$$\Omega\,(\delta_1,\delta_2) = \int_{M_+\cup M_-\cup\Delta\cup i_0} J\,(\delta_1,\delta_2), \tag{3.22}$$

where the terms under the integral show contributions from the various boundaries (refer to figure (3.1)). The surfaces M_+ and M_- are partial Cauchy slices inside the spacetime which meet Δ in S_1 and S_2 respectively. To show that the symplectic structure is independent of the choice of Cauchy surface, we again consider the function $\psi_{(\ell)}$ such that $\pounds_\ell\,\psi_{(\ell)} = \kappa_{(\ell)}$ and $\psi_{(\ell)}$ vanishes on S^1 (where the affine parameter $v = 0$). Choosing an orientation for the boundary, it is not difficult to show that $J\,(\delta_1,\delta_2) \triangleq dj\,(\delta_1,\delta_2)$ so that

$$(\int_{M_1} - \int_{M_2})\,J\,(\delta_1,\delta_2) = (\int_{S_1} - \int_{S_2})\,j\,(\delta_1,\delta_2) \tag{3.23}$$

which establishes the independence of symplectic structure on choice of Cauchy surfaces.The pre-symplectic structure on the space of solutions of the theory in presence of Δ turns out to be

$$\begin{aligned} \Omega\,(\delta_1,\delta_2) &= -2\int_M \left[\left(\delta_1 e^I \wedge \delta_2\omega_I - \delta_2 e^I \wedge \delta_1\omega_I\right) + \frac{l}{\gamma}\,\delta_1\omega^I \wedge \delta_2\omega_I + \frac{1}{\gamma l}\,\delta_1 e^I \wedge \delta_2 e_I\right] \\ &- 2\int_{S^1}\left(\delta_1\psi_{(\ell)}\,\delta_2\left[\left(\frac{l\alpha}{\gamma}+1\right)m\right] - \delta_2\psi_{(\ell)}\,\delta_1\left[\left(\frac{l\alpha}{\gamma}+1\right)m\right]\right) \end{aligned} \tag{3.24}$$

We shall use (3.24) to define conserved quantities like the angular momentum and prove the first law in the next two subsections. We shall also construct the algebra of conserved charges using this symplectic structure and obtain the entropy for black holes in this theory.

3.3.5 Angular Momentum

We shall first introduce the concept of angular momentum starting from the symplectic structure, equation (3.24). Let us consider a fixed vector field φ^a on Δ and all those spacetimes which will have φ^a as the rotational Killing vector field on Δ. The field φ^a is assumed to satisfy certain properties. First, it should lie drag all fields in the equivalence class $[\xi\ell^a]$ ans secondly, it has closed orbits and affine parameter $\in [0, 2\pi)$. To be more precise, we can construct a submanifold Γ_φ of the covariant phase space Γ the points of which are solutions of field equations which admit a WIH $(\Delta, [\xi\ell^a], \varphi^a)$ with a rotational Killing vector field φ^a such that $\pounds_\varphi q_{ab} \triangleq 0$, $\pounds_\varphi \varpi^{(l)} \triangleq 0$. Now, let us choose a vector field ϕ in $\mathcal{M}$ for each point in Γ_φ such that it matches with φ^a on Δ.

We shall now look for phase space realization of diffeomorphisms generated by this vector field ϕ^a on spacetime. Corresponding to the diffeomorphisms on spacetime, we can associate a motion in the phase space Γ_φ which is generated by the vector field $\delta_\varphi = \pounds_\varphi$. It is expected that the vector field δ_ϕ will be Hamiltonian (*i.e.* generate canonical transformations). In that case, the Hamiltonian charge for the corresponding to the rotational Killing vector field can be called the angular momentum [7]. In short, this implies that $\Omega(\delta, \delta_\phi) = \delta J^{(\phi)}$ and the angular momentum is $J^{(\phi)}$ is given by:

$$\begin{aligned} J^{(\phi)} &= -\oint_{S_\Delta} \left[(\varphi\cdot\varpi)m + \frac{l}{2\gamma}(\varphi\cdot\varpi)\varpi + \frac{1}{2\gamma l}(\varphi\cdot m)m \right] \\ &\quad + \oint_{S_\infty} \left[(\phi\cdot A^I)e_I + \frac{l}{2\gamma}(\phi\cdot A^I)\omega_I + \frac{1}{2l\gamma}(\phi\cdot e^I)e_I \right] \\ &= -J_\Delta + J_\infty \end{aligned} \tag{3.25}$$

It is then natural to interpret J_Δ to be the angular momentum on Δ. It is simple to check that for BTZ space-time the expressions for J_Δ and J_∞. It follows that $J_\Delta = (\,J - Ml/\gamma\,) = J_\infty$, leaving $J^{(\phi)} = 0$ (Note that for $\gamma \to \infty$, we get the value of angular momentum of BTZ black hole for GR in $2+1$ dimensions with a negative cosmological constant). That $J_\Delta = J_\infty$ is also supported by the fact that ϕ^a is global Killing vector in BTZ solution. However, if there are electromagnetic fields, the result differs. The value

[7]Since the theory we started with is background independent (has manifest diffeomorphism invariance in bulk) it is natural to expect that Hamiltonians generated by space time diffeomorphisms must consist of boundary terms, if any.

of the angular momentum at infinity J_∞ also gets contribution from the electromagnetic fields and $J^{(\phi)} \neq 0$ [27].

3.3.6 First Law

First law is associated with energy which implies that we should look first for a timelike Killing vector field on spacetime. Let us consider a time-like vector field t^a in $\mathcal{M}$ associated to each point of the phase space (live) which gives the asymptotic time translation symmetry at infinity and becomes $t^a \triangleq \xi\, \ell^a - \mathit{\Omega}_{(t)}\phi^a x$ on Δ, where $\mathit{\Omega}_{(t)}$ is a constant on Δ but may well vary on the space of histories. Just like in the previous subsection, we ask if the associated vector field δ_t on the phase-space Γ_ϕ is a Hamiltonian vector field. The associated function shall be related to the energy. In checking so, we have:

$$\Omega\left(\delta, \delta_t\right) = X^{(t)}(\delta)\,,$$

where,

$$X^{(t)}(\delta) = -2\kappa_{(t)}\,\delta\left((1+\frac{l\,\alpha}{\gamma})\,a_\Delta\right) - 2\mathit{\Omega}_{(t)}\,\delta J_\Delta + X^{(t)}_\infty(\delta) \tag{3.26}$$

and $\kappa_{(t)}$ actually the surface gravity associated with the vector field $\xi\ell^a$. $X^{(t)}_\infty(\delta)$ involves integrals of fields at asymptotic infinity and can be evaluated using asymptotic conditions on the BTZ solution for example. A simple calculation gives:

$$X^{(t)}_\infty(\delta) = \delta\left(M - \frac{J}{\gamma l}\right)$$

Now the evolution along t^a is Hamiltonian only if right hand side of (3.26) is exact on phase space. This implies if the surface gravity is a function of area only and $\mathit{\Omega}_{(t)}$ a function of angular momentum only, there exists a phase space function E^t_Δ such that the *first law* appears:

$$\begin{aligned}\delta E^t_\Delta &= \left[\kappa_{(\xi\ell)}\,\delta\left((1+l\alpha/\gamma)\,a_\Delta\right) + \mathit{\Omega}_{(t)}\,\delta J_\Delta\right] \\ &= \left(\kappa_{(\xi\ell)}\,\delta\tilde{a}_\Delta + \mathit{\Omega}_{(t)}\,\delta J_\Delta\right)\end{aligned} \tag{3.27}$$

where $\tilde{a}_\Delta = \left(1+\dfrac{l\alpha}{\gamma}\right)a_\Delta$. The presence of $\kappa_{(\xi\ell)}$ in the first law indicates that the first law is same for both extremal and non-extremal black holes. A mere choice of the function ξ

can help us interpolate between these class of solutions. We note here that modification in the symplectic structure of the theory leaves its footprint through γ in the first law of (weak) isolated horizon mechanics. The term that plays the role of the 'area' term as it appears in this first law differs from the standard geometrical area of the horizon. If we restrict ourselves to the class of BTZ horizons, we have, [8]

$$\tilde{a}_{\Delta} = 2\pi\left(r_{+} - r_{-}/\gamma\right) = a_{\Delta} - \frac{l\pi J}{\gamma a_{\Delta}} \tag{3.28}$$

3.3.7 Admissible Vector Fields and Horizon Mass

In the previous discussion we used the Hamiltonian evolution of the live time vector field t^a to deduce the first law. It is necessary and sufficient for the existence of the Hamiltonian function E^t_{Δ} as in (3.27) that the functions $\kappa_{(t)}, \Omega_{(t)}$ should be functions of the independent horizon parameters $\tilde{a}_{\Delta}$ and J_{Δ} only and following exactness condition should hold:

$$\frac{\partial \kappa_{(t)}}{\partial J_{\Delta}} = \frac{\partial \Omega_{(t)}}{\partial \tilde{a}_{\Delta}}. \tag{3.29}$$

However, given any vector field, it is not guaranteed that these will be satisfied. In other words, not all vector fields are Hamiltonian. Vector fields t^a for which these conditions are satisfied are admissible and there are infinite of them. We wish to find the class of admissible t^a s by solving (3.29). The essential point is to show the existence of a canonical live vector field. The horizon energy defined by this canonical live vector field is called the horizon mass. In order to proceed, we make the following change of variables for convenience:

$$(\tilde{a}_{\Delta}, J_{\Delta}) \to (R_{+}, R_{-})$$

---

[8] In our conventions, the double roots r_+, r_- of the BTZ lapse polynomial are related with BTZ ($\gamma \mapsto \infty$) mass (M) and angular momentum (J) as

$$M = 2\pi\,\frac{r_+^2 + r_-^2}{l^2} \quad \text{and} \quad J = 4\pi\,\frac{r_+ r_-}{l}$$

with

$$
\begin{aligned}
R_{+} &= \sqrt{\frac{\gamma l}{2\pi\left(\gamma^{2}-1\right)}\left(J_{\Delta}+\frac{\gamma\,\tilde{a}_{\Delta}^{2}}{8\pi l}\right)} \\
R_{-} &= \gamma\sqrt{\frac{\gamma l}{2\pi\left(\gamma^{2}-1\right)}\left(J_{\Delta}+\frac{\gamma\,\tilde{a}_{\Delta}^{2}}{8\pi l}\right)}-\frac{\gamma\tilde{a}_{\Delta}}{4\pi}
\end{aligned}
\tag{3.30}
$$

Now for $\kappa_{(t)}$ we wish to start with a sufficiently smooth function κ_0 of the horizon parameters. In general $\kappa_{(\ell)} \neq \kappa_0$. But we can always find a phase-function ξ in $t^a \triangleq \xi\ell^a - \Omega_{(t)}\varphi$ such that $\kappa_{(\xi\ell)} = \kappa_0$. Again, there is a canonical choice, supplied by the known solution, the BTZ one, in which there is a unique BTZ black-hole for each choice of the horizon parameters. We therefore set $\kappa_0 = \kappa_{(t)}(\text{BTZ})$, where t^a is the global time translation Killing field of the BTZ space time, and express it in terms of the newly introduced coordinates:

$$\kappa_0 = \frac{R_+^2 - R_-^2}{R_+ l^2}$$

The angular velocity $\varpi_{(t)}$ satisfying (3.29) comes out as $\Omega_{(t)} = \dfrac{R_-}{lR_+}$. Using this value of angular velocity and equation (3.30)in (3.27) we have

$$\delta E_{\Delta}^{t} = \delta\left[\frac{2\pi}{l^2}\left(R_+^2 + R_-^2 - 2R_+R_-/\gamma\right)\right] \tag{3.31}$$

Now, from equations (3.30) and (3.31), we have horizon mass in terms of the independent horizon parameters:

$$M_{\Delta}\left(J_{\Delta},\tilde{a}_{\Delta}\right) = \frac{\gamma J_{\Delta}}{l} + \frac{\gamma^2\,\tilde{a}_{\Delta}^2}{8\pi l^2} - \frac{\tilde{a}_{\Delta}}{2l^2}\sqrt{l\gamma\,\left(\gamma^2-1\right)\left(J_{\Delta}+\frac{\gamma\,\tilde{a}_{\Delta}^2}{8\pi l}\right)}.$$

It is not difficult to check that this works for BTZ black hole. Restricting to BTZ values, this reads: $M_{\Delta} = (M - J/\gamma l\,)$. This exactly matches with the asymptotic charge $X_{\infty}^{(t)}(\delta) = \delta\;(M - J/\gamma l)$ associated with asymptotic time translation vector t^a of BTZ space time as would have been expected. We must also note that the deformations of the conserved charges : angular momentum and mass under the influence of the parameter γ are exactly same as those stated in [31,32,46,85] and at the 'chiral point' ($\gamma = 1$) angular momentum and the mass become proportional to each other with opposite sign.

3.4 Covariant phase space realization of asymptotic symmetry algebra

It has been suggested that microscopic details which explain the thermodynamics of black holes is independent of any theory of quantum gravity. If this is taken seriously, it implies that the microstates that describe black hole space-time can be understood from a principle which is expected to govern all quantum gravity theory. It then seems natural to use the arguments of symmetry. Whatever be the theory of quantum gravity, it must at least preserve a part of the symmetries of classical theory. Study of asymptotic symmetries have been advocated to serve this purpose and has achieved striking success in reproducing the Bekenstein-Hawking formula. This issue was first addressed in the context of $2 + 1$ gravity (with negative cosmological constant) by [52].

In this issue we note that diffeomorphisms which are gauges for any theory of gravity become physical symmetry at the boundaries of the space time manifold by physical requirements (boundary conditions). For example, in $3 + 1$ dimensional asymptotically flat space times one naturally identifies a time like vector field at asymptotic infinity as the unique time translation (Killing) as in Minkowski space time and fixes it once and for all. This fixes the diffeomorphisms partially and play the role of a physical symmetry. Only then we can associate a Hamiltonian or Noether charge with time which is the ADM mass. In [19], the authors considered diffeomorphisms generated by asymptotic vector fields which are a bit 'relaxed Killing symmetries' of the asymptotic metric in a 2+1 dimensional space time and showed that they form the pair of affine Witt algebra (2D conformal algebra, or deformation algebra of S^1) as opposed to $SO(2, 2)$, the isometry group of AdS_3. We will show that those vector fields actually generate flows in the phase space which are at least locally Hamiltonian and find the corresponding Hamiltonians (hence qualifying as physical symmetries), *i.e.* charges in the covariant phase space framework. The preference for this frame work is firstly due to its manifest covariant nature and secondly for its immense calculational simplicity, as compared to canonical framework [78].

According to the suggestion mentioned above, this immediately implies that the quantum theory describing the microstates of black holes is a conformal field theory. The simple use of central charges in the Cardy formula determines the asymptotic density of quantum states of black holes which have same mass and angular momentum and approach the asymptotic configuration of a classical BTZ black hole; and eventually the Bekenstein-Hawking result. We shall use the covariant phase-space formulation to compute black hole entropy in this theory.

We would require now the explicit BTZ solution as a case. For that we refer back to (3.6). The asymptotic form of these variables match with the AdS ones as expected upto different orders of $1/r$ [31, 46, 85]. The asymptotic vector fields which generate diffeomorphisms preserving the asymptotic AdS structure (much milder than the BTZ solution) are given by:

$$\xi_n := \exp\left(inx_+\right)\left[l\left(1-\frac{l^2n^2}{2r^2}\right)\partial_t - inr\,\partial_r + \left(1+\frac{l^2n^2}{2r^2}\right)\partial_\phi\right]$$

with n an integer and $x_+ = (t/l + \phi)$. It is easy to check that the vector fields satisfy the affine Witt algebra:

$$[\xi_n, \xi_m] = -i(n-m)\,\xi_{n+m} \tag{3.32}$$

We now want to investigate if the algebra of the vector fields on the space-time manifold is also realised on the phase space *i.e* the Hamiltonian functions (or the generators of diffeomorphisms) corresponding to the vector fields ξ_n^a also satisfy the affine algebra. To see this, we first associate a phase space vector field δ_{ξ_n} to each element ξ_n of the algebra such that δ_{ξ_n} acts as $\pounds_{\xi_n}$ on dynamical variables[9]. Secondly, we need the symplectic structure which will enable us to construct the Hamiltonian functions as has been described in the previous sections (see (3.3.5) and (3.3.6)). Since we are interested in the asymptotic analysis, we will be interested in the contribution to the symplectic structure from the asymptopia or S_∞. If an internal boundary like NEH is present we can assume that the vector fields whose asymptotic forms are as ξ_n^a above vanish on that boundary. From this point of view, for any arbitrary vector field ξ_n^a which vanish on any internal boundary (in this section, we shall reinstate $16\pi G$ but shall choose $c = h = 1$):

$$8\pi G\,\Omega(\delta,\delta_\xi) = \oint_{S_\infty}\left[\left(\xi\cdot e^I\right)\delta\underrightarrow{\omega_I} + \left(\xi\cdot A^I\right)\delta\underrightarrow{e_I} + \frac{l}{\gamma}\left(\xi\cdot A^I\right)\delta\underrightarrow{\omega_I} + \frac{1}{l\gamma}\left(\xi\cdot e^I\right)\delta\underrightarrow{e_I}\right] \tag{3.33}$$

[9]This is because vector fields on the space time manifold work as generators of infinitesimal diffeomorphisms

The under right arrows indicate pull-back of the forms on S_∞. Therefore the second and the fourth term in the integral do not contribute. Only the internal component e_2 (as given above) survives under the pull back which is given by $-rd\phi$. This being a phase space constant, the action of δ on it vanishes. Hence, we get

$$8\pi G\,\Omega(\delta,\delta_\xi) = \oint_{S_\infty}\left[\xi\cdot(e^I + \frac{l}{\gamma}A^I)\right]\delta\underrightarrow{\omega_I} \tag{3.34}$$

for any arbitrary vector field ξ. Using the above expressions of the fields asymptotically, we have

$$8\pi G\,\Omega(\delta,\delta_{\xi_n}) =: \delta H_n = \left(1-\frac{1}{\gamma}\right)\delta\,(\,l\,M + J\,)\ \delta_{n,0}. \tag{3.35}$$

Hence we observe that δ_{ξ_n} are at least locally hamiltonian for all n and has non-zero charge only for $n=0$. We also note using (3.32) that $\delta_{[\xi_n,\xi_m]}$ is also a Hamiltonian vector field with $\delta\,H[\{\xi_n,\xi_m\}]$ given by the right hand side of the following equation

$$8\pi G\,\Omega(\delta,\delta_{[\xi_n,\xi_m]}) = -i(n-m)\left(1-\frac{1}{\gamma}\right)\delta\,(\,l\,M + J\,)\ \delta_{m+n,0} \tag{3.36}$$

We shall now determine the current algebra of the Hamiltonian functions (*i.e.* $\{H_{\xi_n}, H_{\xi_m}\}$) generated by the Hamiltonian vector fields δ_{ξ_n} and δ_{ξ_m} for arbitrary n, m. This will be given by:

$$8\pi G\,\Omega(\delta_{\xi_m},\delta_{\xi_n}) = \oint_{S_\infty}\left[\xi_n\cdot(e^I + \frac{l}{\gamma}A^I)\right]\ \delta_{\xi_m}\underrightarrow{\omega_I} \tag{3.37}$$

It is now important that we first pull back A_I and then calculate the action of δ_{ξ_m} on it as Lie derivative. After some lines of calculation, we find:

$$\begin{aligned} 8\pi G\,\Omega(\delta_{\xi_m},\delta_{\xi_n}) &= -2in\left(1-\frac{1}{\gamma}\right)(\,J + lM\,)\,\delta_{m+n,0} + il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0} + \mathcal{O}\,(\frac{1}{r^2}) \\ &= -i(n-m)\left(1-\frac{1}{\gamma}\right)(\,J + lM\,)\,\delta_{m+n,0} \\ &+\ il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0} \end{aligned} \tag{3.38}$$

Comparing (3.36) and (3.38) we infer that the asymptotic diffeomorphism algebra (3.32) is exactly realized at the canonical level (as a current algebra) except a ‘central term’

$-il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0}$. This is not surprising, although all the vector fields δ_{ξ_n} were Hamiltonian. The second cohomology group of the Witt algebra [10] is not trivial. A theorem of symplectic geometry states that in this case the action of the algebra is not Hamiltonian and moment maps donot exist, which on the other hand implies that the action of the lie algebra on phase space is not hamiltonian [54] [11], *i.e.*

$$\delta\,\Omega\,(\,\delta_{\xi_m}\,,\,\delta_{\xi_n}\,)\neq\Omega\,(\,\delta_{[\xi_m,\xi_n]}\,,\,\delta\,). \tag{3.39}$$

Written in terms of the charges H_n, (3.38) reads as:

$$\{H_m,H_n\}=-i(n-m)\left(1-\frac{1}{\gamma}\right)H_{m+n}+il\pi n^3\left(1-\frac{1}{\gamma}\right)\delta_{m+n,0}. \tag{3.40}$$

Which is isomorphic to the Virasoro algebra.

Now, all of this calculation was done choosing the right moving vector fields. There also are a set of left moving vector fields which preserve the asymptotic structure:

$$\tilde{\xi}_n:=\exp(inx_-)\,\left[l\left(1-\frac{l^2n^2}{2r^2}\right)\partial_t-inr\,\partial_r-\left(1+\frac{l^2n^2}{2r^2}\right)\partial_\phi\right]$$

where $x_-=(t/l-\phi\,)$. Proceeding along the very same route as before, we again end up with the result that canonical realization of this asymptotic symmetries are also realized exactly upto a central term, which now becomes $=-il\pi n^3\left(1+\frac{1}{\gamma}\right)\delta_{m+n,0}$

From the definition of the central charge of Virasoro algebra, which is the centrally extended version of the Witt algebra, we arrive at the exact formulas for the central charges for the right and left moving algebras respectively :

$$c=\frac{3l}{2G}\left(1-\frac{1}{\gamma}\right)\quad\text{and}\quad\tilde{c}=\frac{3l}{2G}\left(1+\frac{1}{\gamma}\right)$$

Once we have the central charges, we can apply the Cardy formula to the BTZ solution to obtain the black hole entropy, by noting from (3.35) that $H_0=\left(1-\frac{1}{\gamma}\right)(lM+J)/8\pi G$ for

[10] For any real lie algebra $\mathcal{G}$ and its dual $\mathcal{G}^*$ a skew symmetric bilinear map $\alpha\in\mathcal{G}^*\wedge\mathcal{G}^*$ is said to be a *cocycle* if $\alpha([A,B],C)+\alpha([B,C],A)+\alpha([C,A],B)=0$ for all $A,B,C\in\mathcal{G}$ and $[,]$ is the usual product on $\mathcal{G}$. The elements $\eth f$ $(f\in\mathcal{G}^*)$ defined via $\eth f(A,B)=\frac{1}{2}f([A,B])$, automatically cocycles by Jacobi identity, are called *coboundary.* Let us define an equivalence $\sim$ as: two cocycles $\alpha\sim\beta$ if $\alpha=\beta+\eth g$ for any $g\in\mathcal{G}^*$. Now one defines $H^2\mathcal{G}$ as the additive group of equivalence classes found through the modulo action of the equivalence relation. All semi simple lie algebras have trivial second cohomology.

[11] If $J[\xi_m]$ and $J[\xi_n]$ are Hamiltonians (calculated in the canonical phase-space) corresponding to the vector fields ξ_m and ξ_n,then $J[[\{\xi_m,\xi_n\}]\neq\{J[\xi_m],J[\xi_n]\}$ where $\{J[\xi_m],J[\xi_n]\}=:\delta_{\xi_n}\,J[\xi_m]=-\delta_{\xi_n}\,J[\xi_m]$.

the right mover and from an almost identical calculation that $\tilde{H}_0 = \left(1+\frac{1}{\gamma}\right)(lM - J)/8\pi G$ for the left mover:

$$\begin{aligned} S &= 2\pi\sqrt{\frac{H_0 c}{6}} + 2\pi\sqrt{\frac{\tilde{H}_0\tilde{c}}{6}} \\ &= \frac{2\pi\, r_+}{4G} - \frac{2\pi\, r_-}{4G\gamma} \qquad \left(\text{Since} \quad M = 2\pi\frac{r_+^2 + r_-^2}{l^2}, \quad J = 4\pi\frac{r_+ r_-}{l}\right) \\ &= \left(a_\Delta - \frac{l\pi J}{\gamma a_\Delta}\right)/4G = \frac{\tilde{a}}{4G} \end{aligned} \tag{3.41}$$

where r_+ and r_- are the radii of the outer and inner horizon, respectively. If we consider the thermodynamic analogy of the first law of black hole mechanics (3.27) (derived for general spacetimes only requiring presence of a weakly isolated horizon only from classical symplectic geometric considerations), we observe that $S \sim \tilde{a}$. Curiously, even in the quantum result (3.41), the entropy-modified area relation continues to hold.

3.5 Conclusion

Let us recollect the main findings presented in this chapter. Firstly, we introduced the concept of WIH in $2+1$ dimensions. The boundary conditions which have been imposed on a 2-dimensional null surface are much weaker than the ones suggested in [27]. Our boundary conditions are satisfied by a equivalence class of null normals which are related by functions, $[\xi\ell^a]$ rather than constants, $[c\ell^a]$ as was first proposed in [27]. The advantage of such generalisation lies in the fact that it becomes possible to include extremal as well as non-extremal solutions in the same space of solutions. Just by choosing the function ξ, one can move from a non-zero $\kappa_{(\ell)}$ to a vanishing $\kappa_{(\xi\ell)}$ (see equation (3.1)) which essentially is like taking extremal limits in phase-space. We also established that the zeroth law (for all solutions in this extended space of solutions) follows quite trivially from the boundary conditions.

Secondly, we have explicitly shown that in presence of an internal boundary satisfying the boundary conditions of a WIH, the variational principle for the generalised $2+1$ dimensional theory remains well-defined. This enable us to take the third step where we have constructed the covariant phase-space of this theory. The covariant phase-space now contains all solutions of the γ-dependent theory which satisfy the WIH boundary conditions at infinity. As expected, extremal as well as non-extremal solutions form a part of this phase-space. We then went on to define the angular momentum as a Hamiltonian function corresponding to the rotational Killing vector field on the horizon. It was also explicitly shown that for the BTZ solution, the angular momentum defined in this manner matches with the expected result.

Thirdly, we established the first law of black hole mechanics directly from the covariant phase-space, for isolated horizons. Instead of the usual horizon area term one encounters in this law, we find a modification due to the γ factor. This is a completely new result in this family of theories. It arose that the first law is the necessarry and sufficient condition for existence of a timelike Hamiltonian vector field on the covariant phase-space. However, not all timelike vector fields are Hamiltonian on phase-space, there exists some which are admissible (there are in fact infinite of them). The canonical choice for these admissible vector fields are constructed too. Quite interestingly, the first law for the WIH formulation, equation (3.27), contains $\kappa_{(\xi\ell)}$. This implies that the first law holds for all solutions, extremal as well as non-extremal. However, the thermodynamic implications of the first law can only be extracted for non-extremal solutions since for the extremal ones, the first law is trivial. But then, since all solutions are equivalent from the point of view of WIH bounhdary conditions, the entropy of both class of black hole solutions will be same. This interpretation of our result is in contradiction to the results of [102–104] who obtain vanishing entropy for extremal black holes. On the other hand, they are in agreement to those of [105].

Using asymptotic analysis, we have calculated the entropy of black holes for the theory under consideration. Contary to the usual approach, we construct the algebra of diffeomorphism generating Hamiltonian functions directly from the covariant phase-space. As usual, we see that the algebra does not match with the Hamiltonian function for the commutator of the asymptotic vector fields. The difference is the central extension. In other words, the algebra of spacetime vector fields is not realised on the covariant phase-space. The Cardy formula then gives the entropy directly which matches with the one expected from the first law. The entropy however not only depends on the geometrical area but also on of other quantities like the parameter of the solution J and the γ-parameter of the theory (equation (3.41)). Keeping the thermodynamic analogy of laws of black hole mechanics in mind and concentrating on the BTZ black hole, one observes that there is a perfect harmony between this result and the modified first law. Also recall that our methods do not rely on existence of bifurcation spheres and applies equally to extremal and non-extremal black holes. To our knowledge, this has not been reproduced earlier since the phase space of Killing horizons which satisfy laws of mechanics do not contain extremal solutions.

Our analysis for the computation of entropy is based on asymptotic symmetry analysis. The principle of using symmetry arguments to determine the density of states for black hole is attractive, it does not depend on the details of quantum gravity. The asymptotic analysis has a major drawback- it seems to be equally applicable for any massive object placed in place of a black hole. Since such objects are not known to behave like black holes, it is not clear where to attribute such large number of density of states. One must directly look at the near-horizon symmetry vector fields for further understanding [106,107]. However, a more interesting step would be to determine the horizon microstates as is done in $3+1$ dimensions. In this case, it arises from classical considerations that the degrees of freedom that reside on a WIH in $3+1$ dimensions is a Chern-Simons theory. Quantization of this theory gives an estimate of the states that contribute to a fixed area horizon and the entropy turns out to be proportional to area. This has not been reproduced in $2+1$ dimensions still and will be investigated in future in order to compliment these new findings already present here.

Appendix:The Newman-Penrose formalism for $2+1$ dimensions

In order to make the calculations done in this chapter self-contained we summarise here the analogue of Newman-Penrose formalism in 2+1 dimensions, which was in detail described in [27]. We will use a triad consisting of two null vectors l^a and n^a and a *real*[12] space-like

[12] All the N-P coefficients appearing in 2+1 dimensions are therefore real unlike in 3+1

vector m^a, subject to:

$$\ell \cdot \ell = n \cdot n = 0, \quad m \cdot m = 1 \tag{3.42}$$

$$\ell \cdot m = n \cdot m = 0 \tag{3.43}$$

$$\ell \cdot n = -1. \tag{3.44}$$

The space-time metric g_{ab} can be expressed as

$$g_{ab} = -2\,\ell_{(a}n_{b)} + m_a m_b\,, \tag{3.45}$$

and its inverse g^{ab} is defined to satisfy

$$g^{ab} = -2\,\ell^{(a}n^{b)} + m^a m^b. \tag{3.46}$$

It is then easy to verify that the expression for the triad is just

$$e_a^I = -\ell_a n^I - n_a \ell^I + m_a m^I. \tag{3.47}$$

Just as in the 3+1 case, we express the connection in the chosen triad basis, the connection coefficients being the new N-P coefficients (the γ defined below is not to be confused with the Barbero-Immirzi parameter):

$$\begin{aligned}\nabla_a \ell_b &= -\epsilon\, n_a \ell_b + \kappa_{\rm NP}\, n_a m_b - \gamma\, \ell_a \ell_b \\ &\quad +\tau\, \ell_a m_b + \alpha\, m_a \ell_b - \rho m_a m_b \end{aligned}\tag{3.48}$$

$$\begin{aligned}\nabla_a n_b &= \epsilon\, n_a n_b - \pi\, n_a m_b + \gamma\, \ell_a n_b \\ &\quad -\nu\, \ell_a m_b - \alpha\, m_a n_b + \mu\, m_a m_b \end{aligned}\tag{3.49}$$

$$\begin{aligned}\nabla_a m_b &= \kappa_{\rm NP}\, n_a n_b - \pi\, n_a \ell_b + \tau\, \ell_a n_b \\ &\quad -\nu\, \ell_a \ell_b - \rho\, m_a n_b + \mu\, m_a \ell_b \end{aligned}\tag{3.50}$$

It then simply follows from the expressions above that $\nabla_a \ell^a = (\epsilon - \rho), \nabla_a n^a = (\mu - \gamma)and\nabla_a m^a = (\pi - \tau)$. Now we wish to expand the connection 1-form ω_a^I in the triad basis with N-P coefficients slated above as coefficients. In order to do so we note that for an arbitrary 1-form v_a which may be mapped uniquely to an $SO(2,1)$ frame element $v_I = v_a e_I^a$. Then, for $\nabla_a v_b = \omega_{a\,J}^{I}\, v^J\, e_{Ib}$, and using $\omega_{a\ I}{}^{J} = -\epsilon_{KI}{}^{J}\omega_a^K$, we arrive at the expression:

$$\begin{aligned}\omega_a^K &= (\pi n_a + \nu \ell_a - \mu m_a)\ell^K + (\kappa_{\rm NP}\, n_a + \tau \ell_a - \rho m_a)\, n^K \\ &\quad +(-\epsilon n_a - \gamma \ell_a + \alpha m_a)\, m^K \end{aligned}\tag{3.51}$$

Chapter 4

Finite 3D de-Sitter Gravity

4.1 Introduction

As we have been discussing throughout this part of the thesis, most of the non-trivial results in 3d gravity including the famous BTZ black hole solution is known for the negative cosmological constant sector. In addition to that there is a definite trace of AdS/CFT correspondence when the space-time is asymptotically AdS. On the other hand study of 3d gravity with positive cosmological constant has generated considerable interest only recently [108], although there is no straight-forward duality with any conformal field theory in this case. This involves evaluation of 1 loop partition function in the metric formulation in order to find the de Sitter vacuum, namely the Hartle Hawking state. It was shown in this work for the first time the equivalence of Chern Simons framework of gravity with Einstein theory up to 1-loop level in the quantum regime. In addition to that, topologically massive gravity (TMG) with positive cosmological constant, has been thoroughly studied in [109]. The main question these studies aim to address is how one can make sense of 3d de Sitter quantum gravity, through finding the vacuum state. Surprisingly, the pure topological gravity theory fails to give any satisfactory answer to it. This can be envisaged in the sense that the partition function (both in 1-loop and nonperturbative computations) tend to diverge unregularizably when one considers the sum over the infinitely large class of allowed classical saddle points. In this case these saddles are lens spaces (typical solutions of Euclidean 3D de Sitter gravity), which are distinguished by their homology properties. On the other hand, the answer for TMG containing local degrees of freedom is in the affirmative. The latter is tame under sum over all saddles.

The pure gravity and TMG calculations have been considered in the Euclidean signature with the motivation that Euclideanized de Sitter gravity is 'thermal'. This has been made precise in terms of the Euclidean de Sitter geometry in [108]. Moreover, in the Einstein-Hilbert theory path integral is sensible in the Euclidean picture. On the other hand if one prefers to study the theory in first order formulation, in the Chern-Simons (CS) framework, Euclideanization is not an obvious idea that one should come across. This is because CS theory is manifestly topological and doesn't rely on background metric as long as perturbative analysis remains not as the primary goal. But once one tries to make contact with metric formulation through $\langle e_\mu, e_\nu \rangle = g_{\mu\nu}$, Euclideanization can be viewed from the choice in the internal metric on the frame bundle (of vielbeins), and hence the structure group. This change reflects upon the choice of gauge group of the CS theory. Gauge group changes from non-compact $SO(3,1)$ to compact $SU(2) \times SU(2)$, thus making the problem tractable from gauge theory perspective. The action then becomes difference of two $SU(2)$ CS theories. We recall that similar situation have been encountered by us in chapter 1, where we used Lorentzian signature and negative cosmological constant. That case however differs from the present one in the sense that we had the non-compact gauge group $SL(2,\mathbb{R})$ over there.

This is the motivation for our purpose to look at Euclideanized version. In this case we don't need a Wick rotation in space time and our partition functions keeps the formal expression

$$Z = \int DA \exp\left(i\frac{k}{4\pi} \int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A^3 \right)\right).$$

where 'tr' stands for the metric over $\mathfrak{su}(2)$. We would see that this form of the path integral will help us in the end so that the trouble of working with imaginary coupling of CS will not also get in our way [1]. Since we would be confined in the first order regime, our concern about the background appears only through its possible topologies. The choice of topology is however motivated strongly from the fact that Euclideanized de Sitter space

[1]In another important work Witten [110] recently pointed out how quantization of CS theory with complex coupling can be carried out by suitably deforming the functional integral contour. However for this case one still has to study the possibility of associating a finite dimensional Hilbert space of CS theory on a compact Riemann surface, which we need for quantization here.

can be identified with S^3, through its metric and topology. We choose it to be of the form S^3/Γ, or lens spaces to be precise. Γ is a suitable discrete group with known action on S^3 as in [108]. Of course feasible solutions are always locally dS.

Now at this point it may seem that we are free to choose any of the standard quantization techniques for this theory. This may involve directly evaluating the partition function or taking recourse to geometric quantization [111]. It is well known that the former is nicely suited for perturbative calculations. In that case one first linearizes the theory around some particular solution and computes the resulting the 1-loop determinant of the elliptic operator. It is expressed of in terms of the analytic torsion, which is a topological invariant of 3 manifold in question. This procedure is clear even for non-compact gauge groups [13]. But once we are interested in nonperturbative results we must investigate whole of the gauge moduli space of solutions, upon which a suitable canonical quantization may be carried out. However, on the given topology of lens space the solution space modulo the gauge transformations give only a collection of finite points, which certainly isn't a symplectic manifold. We therefore use standard surgery and gluing prescription for the construction of the space and using axioms of TQFT find the partition function as [112] [2]

$$Z = \langle\psi|U|\psi\rangle. \tag{4.1}$$

Here $|\psi\rangle \in \mathcal{H}_{T^2}$ is a state of quantized CS theory on the boundary of a solid torus, gluing two of which we construct a lens space. U is an element of the T^2 mapping class group, specifying which gives us a class of lens spaces. This is where 'conventional wisdom' of viewing first order gravity as difference of two $SU(2)$ CS theories fails. This failure becomes manifest when one looks at the CS levels $\pm\frac{l}{8G}$. The problem with equal and opposite couplings is that the CS part corresponding to the negative level is ill-defined and cannot be quantized on T^2 as already noted already for $SO(2,1)$ in chapter 2. We need to extend the theory in a way described in [16, 20, 26, 34] so that the couplings of the CS theories can be tuned to be positive. This is a necessary condition since $\dim(\mathcal{H}_{T^2})$ equals the product of shifted CS couplings. When both the couplings are positive integers we get a situation which we regard as *consistent quantization*. This means, time and again,

[2] Choosing framing of surgery suitably.

the γ comes out as a saviour.

Furthermore, we will show in this chapter, that due to this extension (through introduction of a new dimensionless parameter) we get a finite answer for the partition function, as opposed to [108]. We exhibit the finiteness explicitly at a certain limit of this new parameter. This is certainly an improvement towards finding an answer about how meaningful 3d de Sitter quantum gravity is. Works presented in this chapter follow those in [113].

4.2 The Extended Theory and the Naive Phase-Space

At the classical level, difference between the present scenario and the theory we have been dealing with in this thesis is not much, as long as the structure of the action is concerned. As opposed to the earlier example, here we have cosmological constant *positive* and signature Euclidean. As a result the CS gauge group is different. Referring back to table 1.0.1, we see that we have here with us a compact $SU(2) \times SU(2)$ CS theory. The vielbein and the connection are also on the frame bundle of structure group $SU(2)$. The frame metric is of course $\eta = \mathrm{diag}(1,1,1)$ and will be referred to as δ_{IJ} at places. Keeping this implicit fact back in mind, we read the extended action to be same as (1.10). The apparent signature of the cosmological constant term however can be misleading. It retains same sign because the double sign change made during Euclideanization.

Since we are interested in the nonperturbative evaluation of the partition function, the information about Lens space that suffices is its algebraic topology. This is given by [3] $L(p,q) = S^3/\mathbb{Z}_p$. The physical phase space of this theory containing only flat connections, is given by $(\mathrm{hom} : \pi_1(L(p,q)) \to SU(2)) / \sim$, (moduli space of flat $SU(2)$ connections modulo gauge transformations) where $\sim$ denotes gauge equivalence classes. For lens space $L(p,q)$, the fundamental group is isomorphic to $\mathbb{Z}_p$, which is freely generated by a single generator, say α; ie the group consists of the elements $\{\alpha^n | n = 0, \ldots, p-1\}$. The homomorphisms to $SU(2)$, which we denote by h must satisfy $h[\alpha^p] = (h[\alpha])^p = \mathbb{1}$. In

[3] Role of $q(\mathrm{mod} p)$ coprime to p comes through the action $\mathbb{Z}_p : S^3 \to S^3$. This is most easily viewed by considering S^3 as unit sphere in $\mathbb{C}^2$ and specifying the $\mathbb{Z}_p$ action as $(z_1, z_2) \mapsto \left(\mathrm{e}^{2\pi i/p} z_1, \mathrm{e}^{2\pi i q/p} z_2\right)$.

the defining representation (using the freedom of group conjugation) of $SU(2)$, this gives

$$h[\alpha] = \mathrm{e}^{2\pi i \sigma_3 / p}.$$

Hence the moduli space consists of only p distinct points and therefore can in no way be a symplectic manifold. In physical terms these points represent holonomies of the p disjoint non contractible loops around the p marked points on $L(p, q)$.

In this connection we wish to emphasize that the configuration corresponding to $n = 0$ above, is unique to first order gravity only. It represents the holonomy of the connection $A^{(\pm)} = 0$ or its gauge equivalent class. This means that we are taking the $e = 0 = \omega$ solution in our phase space. These configurations do not give rise to any physically meaningful metric, as elucidated in [20]. But while doing non-perturbative quantization of first order theory we must include them in the phase space.

4.3 Appropriate quantization

4.3.1 $\mathcal{H}_{T^2}$

That we have seen direct attempts to quantize the theory on $L(p, q)$ fails, we should resort to indirect means as exemplified in (4.1). In this respect we construct $L(p, q)$ by gluing two solid tori through their boundaries using an element of the mapping class group

$$U = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{4.2}$$

The quantization strategy [112] as outlined in the introduction requires associating two quantum Hilbert spaces of the CS theory with the boundary of the solid tori. We therefore have to find $\mathcal{H}_{T^2}$. This analysis should be identical for the compact conjugacy class of[4] $SL(2, \mathbb{R})$, which we worked out in 2.3.1. However there are some notational distinctions, for which we present the quantization scheme briefly as follows.

Since we are quantizing CS theory on T^2 (the third dimension may be taken as $\mathbb{R}$, the whole 3 manifold being viewed as a trivial line bundle over T^2), we have as the starting point, the moduli space : $(\mathrm{hom} : \pi_1(T^2) \to SU(2)) \,/\, \sim$.

[4]We also note that there is a single conjugacy class for $SU(2)$.

Now $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ and is a freely generated abelian group with two generators α, β having the relation $\alpha\beta\alpha^{-1}\beta^{-1} = 1$. Taking privilege of the group conjugacy as before we take the 2 dimensional representation of the homomorphism maps as:

$$h[\alpha] = \mathrm{e}^{i\sigma_3\theta} \qquad h[\beta] = \mathrm{e}^{i\sigma_3\phi} \qquad \theta, \phi \in [-\pi, \pi]. \tag{4.3}$$

This endows the two dimensional moduli space $\mathcal{M}$ with the topology of T^2 (parameterized by θ, ϕ). Note that this simple construction of $\mathcal{M}$ is motivated from the rigorous point of viewing it as $\mathcal{M} = T \times T/W$, where T is the torus of maximal dimension (for $SU(2)$ which is 1 and $T = S^1$) and W is the Weyl group with Ad action on the group. Our strategy will be to first quantize $T \times T$ and then take Weyl invariant 'parallel' sections of the line bundle on it.

The 'pushed down' symplectic structure on $\mathcal{M}$ is

$$\omega = \frac{k}{2\pi} d\theta \wedge d\phi.$$

An appeal to Weil's integrality criterion

$$\int_{\mathcal{M}} \frac{\omega}{2\pi} \in \mathbb{Z} \tag{4.4}$$

now assures that k must be an integer. At the stage of prequantization a prequantum line bundle is chosen over $\mathcal{M}$ and before choosing the polarization for this line bundle we pick a complex structure τ for $\mathcal{M}$ (induced by that on the surface of the solid torus). This gives us the holomorphic coordinate: $z = \frac{1}{\pi}(\theta + \tau\phi)$ on $\mathcal{M}$. We re-express

$$\omega = \frac{ik\pi}{4\tau_2} dz \wedge d\bar{z}.$$

We thus work with a Kähler structure on $\mathcal{M}$ and a line bundle on it with a connection whose curvature is $-i\omega$. The rest of the prequantization technique can be analogously constructed as given in [26]. This equips us with prequantized Hamiltonian functions $\hat{\theta}' = -\frac{2i}{k+2}\tau\partial_z + \pi z$ and $\hat{\phi}' = \frac{2i}{k+2}\partial_z$. It is important to note the shift of k by the dual Coxeter number of $SU(2)$ to $k+2$ which originates from the non-trivial Polyakov-Wiegman factor [114] for non-abelian compact gauge groups. In a more rigorous fashion its appearance is explained due to non-anomalous connection construction on the Hilbert bundle in [111], which guarantees finally the quantum Hilbert space to be independent of the complex structure initially chosen for quantization.

We finally impose the quantization conditions on the polarized wavefunctions $\psi(z)$ [5]:

$$\mathrm{e}^{i(k+2)m\hat{\theta}'}\mathrm{e}^{-i(k+2)n\hat{\phi}'}\psi(z) = \psi(z).$$

This is solved by level $r = k+2$ theta functions:

$$\vartheta_{j,r}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left[2\pi i r\tau\left(n+\frac{j}{2r}\right)^2 + 2\pi i r z\left(n+\frac{j}{2r}\right)\right]$$

with $j = -r+1,\ldots,r$ (since $\vartheta_{j+2r,r}(z,\tau) = \vartheta_{j,r}(z,\tau)$). We will now construct the Weyl invariant subspace of this $2r$ dimensional vector space. Weyl invariance on $\mathcal{M}$ means identifiction of z with $-z$ [6]. Observing that $\vartheta_{j,r}(-z,\tau) = \vartheta_{-j,r}(z,\tau)$ we project to the Weyl-odd subspace consisting of the $r-1 = k+1$ vectors:

$$\vartheta^{-}_{j,r}(z,\tau) = \vartheta_{j,r}(z,\tau) - \vartheta_{-j,r}(z,\tau) \quad j = 1,\ldots r-1.$$

As per [111] one should now consider a 'quantum bundle' over the space of complex structures τ with fibres as the Hilbert space we have just found. The physical states should be parallel sections of this new bundle with respect to a projectively flat connection of the 'quantum bundle'. Those vectors turn out to be:

$$\psi_{j,k}(z,\tau) = \frac{\vartheta^{-}_{j+1,r}(z,\tau)}{\vartheta^{-}_{1,2}(z,\tau)} \qquad j = 0,\ldots k \tag{4.5}$$

By taking the ratio of two Weyl-odd function we thus found the Weyl invariant vector space as desired. This space is orthonormal and serves as the required Hilbert space.

4.3.2 Gluing and $L(p,q)$

We know that the mapping class group $SL(2,\mathbb{Z})$ or rather $SL(2,\mathbb{Z})/\mathbb{Z}_2$ of T^2 is 'generated' by two modular transformation elements T, S. Any general element U of $SL(2,\mathbb{Z})$ can be

[5] the apparent operator ordering ambiguity is unphysical, costing only up to a phase in the wavefunction

[6] this is so because the traces of the holonomies (4.3) are gauge invariant rather than $h[\alpha], h[\beta]$ themselves and the traces do not distinguish between (θ,ϕ) and $(-\theta,-\phi)$. This is another statement of Weyl invariance.

expressed as

$$U = S \prod_{s=1}^{t-1} (T^{m_s} S) .$$

In its 2 dimensional representation U produces $L(p,q)$ by gluing two solid tori for [115]

$$U = \begin{pmatrix} q & b \\ p & d \end{pmatrix}$$

The above representation of U in terms of T, S implies the following identity [112]:

$$p/q = -m_{t-1} + \cfrac{1}{m_{t-2} - \cfrac{1}{\cdots - \cfrac{1}{m_1}}} \tag{4.6}$$

The Chern-Simons-Witten invariant or the partition function is given by [3],

$$Z(r)_{L(p,q)} = \langle \psi_{0,k} | U | \psi_{0,k} \rangle$$

and it is independent of the parameters b, d [112]. From the knowledge of action of S and T on theta functions we can evaluate these matrix elements. In the canonical 2-framing this was evaluated to be

$$Z(r)_{L(p,q)} = -\frac{i}{\sqrt{2rp}} \exp\left(6\pi i s(q,p)/r\right) \sum_{\pm} \sum_{n=1}^{p} \exp\left(\frac{2\pi i q r n^2}{p} + \frac{2\pi i n(q \pm 1)}{p} \pm \frac{\pi i}{rp} \right) \tag{4.7}$$

$$\text{where } s(q,p) = \sum_{l=1}^{p-1} \frac{l}{p} \left(\frac{lq}{p} - \left[\frac{lq}{p} \right] - \frac{1}{2} \right)$$

is the Dedekind sum defined in terms of the floor function [ ].

4.3.3 Sum over topologies and finiteness of the partition function

We note from the construction of $\mathcal{H}_{T^2}$ (4.5) that the dimension of the Hilbert space is $r_{(\pm)} - 1$ corresponding respectively to the '+' type and '-' type CS sectors. This is meaningful only when $r_{(\pm)} - 1 \in \mathbb{N}$ (excluding zero). These conditions come out to be stringent and restrict the parameters of the theory. Since $r_{(\pm)} - 2 = k_{(\pm)} = \dfrac{l(1/\gamma \pm 1)}{8G}$, we have (when $\hbar$ and c are restored suitably) [7] the following restrictions

$$a := \frac{l}{8 l_p} = s/2 \quad s \in \mathbb{N} \quad \text{and} \quad \gamma = \frac{a}{(a-1)+t} \quad t \in \mathbb{N}. \tag{4.8}$$

[7] l_p is the three dimensional Planck length $l_p = G\hbar/c^3$

These restrictions are the prototypes of any topological field theory [116]. One may be tempted to compare these with those appearing in [20] for $k_{(\pm)}$, where the unequal CS parameters are prescribed with discrete values in context of gravity. The apparent difference is due the choice of a different background topology used in [20].

These nontrivial restrictions which validate the quantization (through positivity of the dimension of the Hilbert space) does not allow $\gamma \to \infty$ which was again the starting point of the ordinary theory (1.1). It is also interesting to see that the set of allowed value of γ also includes 1, the 'chiral' point for $t = 1$. This motivates us strongly to study the corresponding Chiral limit of the underlying dual-CFT, if any.

Leaving those issues for later discussion we now return to our original problem and express the gravity partition function (henceforth by gravity partition function we mean the partition function for the first order gravity) as the product of the partition functions of '+' type and the '-' type theories (1.7):

$$Z^{\text{Grav}}_{L(p,q)} = Z(r_{(+)})_{L(p,q)} Z(r_{(+)})_{L(p,q)} \tag{4.9}$$

Full gravity partition function would on the other hand be stated after summing over all topologies ie

$$Z^{\text{tot}} = \sum_{p=1}^{\infty} \sum_{\substack{q(\text{mod}p) \\ (q,p)=1}} Z^{\text{Grav}}_{L(p,q)}$$

This final sum is where one encounters the divergence as explained in [108] through sums of kind $\sum_{\substack{q(\text{mod}p) \\ (q,p)=1}} 1 = \phi(p)$, the Euler totient function. For the purpose of comparison with [108] and study the convergence property of our partition function we choose a particular classical saddle for which the sum over n in (4.7) is replaced by a particular value of $n = \frac{q\pm 1}{2}$ respectively for the '+' and the '-' type theory instead of taking the corresponding sum in (4.9). In order to bring in clarity further simplification is made through assuming a to take only integral values and $a/\gamma \in 2\mathbb{N}$. However these simplifications do not alter the final convergence properties of the sum. Using (4.7) in a more illuminating form [8] we

[8] Let A be the set of all such integers $q(\text{mod}p)$ with $(q,p) = 1$. It is easy to see that $\{q^*(\text{mod}p) | qq^* = 1(\text{mod}p)\} = A$. This property has been used.

have explicitly:

$$
\begin{aligned}
Z^{\text{tot}} &= -\frac{1}{2\sqrt{r_{(+)}r_{(-)}}}\sum_{p=1}^{\infty}\frac{1}{p}\sum_{\substack{q(\text{mod}p)\\(q,p)=1}}\exp\left(6\pi i s(q,p)/R_+\right)\exp\left(\frac{\pi i}{p}\left(2a+\left(q+q^*\right)\left(a/\gamma+2\right)\right)\right)\times \\
&\left[\mathrm{e}^{\frac{\pi i}{pR_+}+\frac{2\pi i}{p}(q+1)}+\mathrm{e}^{-\frac{\pi i}{pR_+}+\frac{2\pi i}{p}(q-1)}-\mathrm{e}^{\frac{\pi i}{pR_-}+\frac{4\pi i}{p}}-\mathrm{e}^{\frac{-\pi i}{pR_-}}\right] \qquad (4.10)
\end{aligned}
$$

where $\dfrac{1}{R_\pm}=\dfrac{1}{r_{(+)}}\pm\dfrac{1}{r_{(-)}}$

It is now easy to see that all the terms in the q summand are q dependent and the divergence producing totient function does not occur. However since no closed form of the q sum is available, for the purpose of explicit checking we go to the limit where $\gamma>0$ is small ($\ll 1$). Since the coupling constants become effectively large in this limit the partition function contains the expressions up to one loop. From (4.8) one observes that this limit is consistent with our quantization program by fixing a and pushing the integer t large. In this limit $\dfrac{1}{R_+}\sim\dfrac{2\gamma}{a}$ and $\dfrac{1}{R_-}\sim\dfrac{2\gamma^2}{a}$ are both small. Out of the γ terms appearing as polynomials in the exponentials of (4.10) ie, $\dfrac{1}{\gamma},1,\gamma,\gamma^2$ we keep $\dfrac{1}{\gamma},1$ and neglect the last two. This implies

$$
Z^{\text{tot}}=-\frac{\gamma}{a}\left(1-\frac{2\gamma}{a}\right)\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{e}^{\frac{2\pi i a}{p}}\cos(2\pi/p)\left[S(\frac{a}{2\gamma}+2,\frac{a}{2\gamma}+1;p)-\mathrm{e}^{\frac{2\pi i}{p}}S(\frac{a}{2\gamma}+1,\frac{a}{2\gamma}+1;p)\right] \qquad (4.11)
$$

$$
S(\alpha,\beta;p)=\sum_{\substack{q(\text{mod}p)\\(q,p)=1}}\exp\left(2\pi i(\alpha q+\beta q^*)/p\right)
$$

Expanding the exponential and the cosine functions in the inverse power of p we obtain an infinite series of Kloosterman zeta functions defined by

$$
L(m,n;s)=\sum_{p=1}^{\infty}p^{-2s}S(m,n;p).
$$

Kloosterman zeta function is again analytic in the region $\Re s>1/2$.

Now, as we are in the *small* γ regime, the summand in (4.11) can well be approximated as

$$\begin{aligned}
&\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{e}^{\frac{2\pi i a}{p}}\cos(2\pi/p)\left(1-\mathrm{e}^{\frac{2\pi i}{p}}\right)S(\frac{a}{2\gamma},\frac{a}{2\gamma};p)\\
=&\sum_{m,n,r=0}^{\infty}\frac{(2\pi i)^{r+n+2m+1}}{r+1}\frac{a^n}{(2m)!n!r!}\sum_{p=1}^{\infty}p^{-(r+n+2m+2)}S(\frac{a}{2\gamma},\frac{a}{2\gamma};p)\\
=&\sum_{m,n,r=0}^{\infty}\frac{(2\pi i)^{r+n+2m+1}}{r+1}\frac{a^n}{(2m)!n!r!}L(\frac{a}{2\gamma},\frac{a}{2\gamma};\frac{r+n+2m}{2}+1) \qquad (4.12)
\end{aligned}$$

The good news is that we get a series of $L(\frac{a}{2\gamma},\frac{a}{2\gamma};s)$ with $s\geq 1$. Hence the partition function is free from divergences. Had we set $a/\gamma+2=0$, the second Kloosterman sum would have reduced to the totient function. That is a potential source of singularity, which is obvious since its zeta function is expressed in terms Riemann zeta function and $\zeta(1)$ is singular. We again see that the finiteness of the parameter γ saves us from having a meaningless quantization.

Here we wish to point out that we are evaluating the partition function in the case of small γ. This again corresponds to large CS couplings $k_{((\pm))}$. However quantum CS theory dictates that large coupling means first quantum correction [112]. In that sense (4.11) or (4.12) corresponds to one loop result.

4.4 The metric counterpart and the TMG story

The key relation connecting the first order formalism and metric regime is : $\langle e_\mu, e_\nu\rangle = g_{\mu\nu}$. It should be supplemented with the torsionless condition ensuring the geometry to be Riemannian. If one starts with the action (1.7), one gets this condition (1.3) as an equation of motion. Solving this equation makes (1.9) the well known gravitational Chern Simons and (1.1) the Einstein Hilbert action provided we use only the invertible subset of vielbeins from (1.3). The action (1.7) becomes TMG with γ playing the role of topological mass. It is not surprising that dynamics of TMG and that of (1.7) are quite different; including equations of motion and canonical structures. The most important feature perhaps is that TMG has local degree of freedom which is absent in the theory described by (1.7) and one should not expect similarity in their quantum theories. However TMG being the closest kin to our theory in metric version, for a completion we present a comparative study with quantum TMG focussing its convergence properties as worked out in detail in [109].

To be more precise, we first focus on what is meant by quantum dS TMG. This issue, as we have already mentioned, has been exhaustively studied in [109]. The one loop partition function is showed there to converge. Denoting by E, the contributions coming from pure Einstein Hilbert theory with cosmological constant and by MG, the ones coming from massive graviton modes, they show that:

$$\sum_{p=1}^{\infty}\sum_{\substack{q(\text{mod}p)\\(q,p)=1}} Z_{\text{E}}^{(0)}Z_{\text{MG}}^{(0)}Z_{\text{E}}^{(1)} \sim \sum_{r=0}^{\infty}\frac{(2\pi a)^r}{r!}L(\frac{a}{2\gamma},\frac{a}{2\gamma};\frac{r}{2}+\frac{1}{2}) + \text{trivially analytic terms.} \quad (4.13)$$

One can now compare this with (4.12). The interesting fact is that here the term corresponding to $r = 0$ in the sum of the RHS is the source of divergence since it corresponds to Kloosterman zeta function with $s = 1/2$. But it is also showed in [109] that when one includes $Z_{\text{MG}}^{(1)}$ as the product and then performs the sum over p, the divergence is eaten up. This means that up to one loop calculation they have

$$Z = \sum_{p=1}^{\infty}\sum_{\substack{q(\text{mod}p)\\(q,p)=1}} Z_{\text{E}}^{(0)}Z_{\text{MG}}^{(0)}Z_{\text{E}}^{(1)}Z_{\text{MG}}^{(1)}.$$

The expression of $Z_{\text{MG}}^{(1)}$ as given in [109] is far too complicated for the above expression to be analytically simplified and compared with (4.12). But the mechanism through which the divergence in the above expression is controlled by $Z_{\text{MG}}^{(1)}$ is very similar to the way we showed (4.12) to be finite. In essence both our topological theory of gravity and TMG (dynamical) have finite and *similarly* convergent partition functions. Since these theories are classically different this fact seems to be quite surprising. That TMG is derived as a metric version of our theory may however qualitatively explain this similarity in partition functions up to one loop. We conclude that although the finiteness of TMG could be ascribed to its propagating graviton modes , our theory (1.7), being devoid of massive gravitons still yield a reasonably similar convergent partition function.

4.5 Conclusion

The take home message of our analysis can be summarized as follows :

1. Construction of the associated Hilbert spaces on the torus surfaces is correct only for finite γ. These constructions spell out the set of allowed values of γ and this does not include $\gamma \to \infty$.

2. That finite values of γ can make the partition function divergence free is shown explicitly for $\gamma \ll 1$. This is most important from point of view of the quantization of lens space gravity.

The fact that pure Einstein gravity has divergent partition function even at one loop and TMG is finite may seem to be a lucrative point of discussion in context of the work we present here. One can pass over to TMG (essentially dynamical) from action (1.7), which is topological, by imposing the torsionless condition. Hence they share the same parameter content. In the AdS sector however, this similarity is more pronounced as they have same dual CFTs. Whereas in present case, such an analogy is premature, since dual CFT in 3D de Sitter gravity is yet to be understood. Any progress in this front would surely shade light on the proposed dS/CFT [117] correspondence (which works in 4 dimensions) in 3 dimensions and on its gravitational interpretation.

On the other hand, the finiteness brought in by the gravitational Chern Simons term of TMG also may be interpreted in light of (1.7). This being parity odd, there are phases in the partition function. Control of the divergence can be ascribed to this fact. This explanation works in the perturbative regime for TMG at least, as shown in [109]. Our result being finite is in conformity with TMG.

Another point of interest which we leave for future study is the interpretation of the theory when $\gamma \to 1$. In the AdS paradigm an analogous point in parameter space has been shown to have critical CFT dual [50, 118]. In light of the proposed dS/CFT [117] framework this may serve as an exciting evidence for dual critical CFT.

4D Gravity

Chapter 5

Black Hole Entropy in 4 dimensions

Till now, we have been talking about 3d quantum gravity. As promised in the introduction, we would now be working with 4d gravity. The theme of this thesis is emergence of Chern Simons theory in the quantum gravity perspective, which was observed closely in the first part of this work. In this light, we should expect CS to again act as a useful tool in 4d context. It is not hard to assume that we need again the first order formulation of gravity here in order to see how Chern Simons theory uncovers. As have been outlined earlier in 0.3.3, CS appears as an effective theory of gravitational interactions on a 3d submanifold of the 4 dimensional space-time. That submanifold actually is a boundary, more precisely an isolated horizon, which has again been discussed in chapter 3 in 3d gravity context. It can be thus expected that we would be talking mainly about quantum degrees of freedom describing a black hole horizon. Two immediate questions now crop up in ones mind. Firstly, which quantities do we want to calculate, given that we have a well known theory (CS) that describes horizon dynamics and why will that be important? Secondly, here CS is only restricted on the boundary itself. Bulk dynamics should not be ignored, since horizons are not decoupled from space-time manifolds. In this sense, how will the quantum theory of bulk 4d gravity be accounted for? Answer to the first question involves black hole entropy calculation in the micro-state counting approach, and will be explained in the following section. In connection to the second question, we keep in mind that no completely robust theory of 4d quantum gravity is available till date. However the framework of Loop Quantum Gravity (LQG) comes to be helpful in the scenario of black hole entropy evaluation. The main result we use from LQG, involving spectrum of area operator is kinematical (does not involve quantization of dynamical issues) and is unambiguous. Hence in this chapter we would first display the classical dynamics of the bulk theory and after that, will brief the important result of LQG which would be relevant for black-hole entropy calculation presented in the next chapters of the thesis.

5.1 Black hole entropy

Formation of black holes from collapse of matter and radiation under gravitational pull is an involved topic in astrophysics and we don't have rather the scope of discussing that phenomenology in this thesis. However we start with a relativist's point of view, which perceives black holes as some special solutions of general relativity. With the risk of being non-rigourous, we define black holes as regions of space-time bounded by a two-surface for each spatial foliation. Adding to that, the two-surfaces should not allow any information come out of it. This particular feature seems peculiar in terms of known physics and calls for ascribing the notion of entropy to black hole horizons [119] (the surface that 'hides' information of the black hole). This entropy should be function of macroscopic parameters of the black hole horizon only (for example mass, electromagnetic charge among others). It is Hawking's work on radiation from black hole horizon [120], which associated a temperature to the horizon. This, along with the striking analogy between laws of horizon dynamics and those of thermodynamics gave the formula for entropy as:

$$S_{BH} = \frac{k_B A}{4l_p^2}, \tag{5.1}$$

where $A, k_B, \& \, l_p$ are respectively the horizon area, Boltzmann constant and the Planck length. Since the establishment of this formula, the question about its microscopic origin has been alluring theoreticians, particularly those working on a theory of quantum gravity. In this view, it is a very stringent demand from any candidate theory of quantum gravity to describe the quantum degrees of freedom which account for this entropy from a statistical mechanics point of view.

There are a large number of approaches addressing the above issue. Many do rely upon semi-classical techniques and can reproduce (5.1). Semi-classical studies can produce this result for black-holes with large (as compared to the Planck area) horizon area. Corrections to this formula, if any, is therefore expected from any truly quantum theory.

Euclidean quantum gravity [121] calculations in this direction have been proven to be good candidates, which added corrections logarithmic in the horizon area. There are also a large volume of work done in string theory literature (see [122] and references therein) which also provided similar results from microscopic calculations. These however mostly apply for extremal black holes. In the framework of LQG, it is interesting to see if such corrections do arise or not. (5.1) was checked in the regime of LQG [123] successfully. It was immediately followed by the computations [25, 124] and the issue of black-entropy correction upto logarithmic order was well settled more than a decade ago. However it is due to a more recent resurgence concerning the topic in LQG regime that we have started working in this direction during the period of this thesis. This upsurge is regarding a very subtle issue about the coefficient of the logarithmic correction that one gets in LQG calculation. We would be discussing that later in the next two chapters. This chapter and the next one will be used for making up the stage for that computation.

5.2 A very brief summary of LQG

In this section, we would describe the classical phase space of first order gravity with the Barbero-Immirzi (BI) parameter, which is aimed to be quantized in the programme of LQG. In doing so, we note that this formulation of gravity essentially casts it into a theory consisting of dynamics of connections, ie, a gauge theory (except that we don't have a fixed background manifold now). The immediately next step is to construct a kinematical set-up for the quantum theory and to analyse spectra of geometrical importance. Materials upto this consist of what we require for the entropy calculation, we would briefly remark on LQG dynamics.

5.2.1 Classical phase space

There is a large number of excellent pedagogical reviews and books on the subject including: [60, 63, 125] among others. The starting point of the LQG programme is the Palatini action augmented by a topological term (introduced by Hölst [42]):

$$S = \frac{1}{4k} \int \left(\epsilon^{IJKL} e_I \wedge e_J \wedge F_{KL} - \frac{1}{2\gamma} e_I \wedge e_J \wedge F^{IJ} \right) \tag{5.2}$$

where e is the $SL(2,\mathbb{C})$ tetrad (frame) field and F is the curvature of the frame-connection one-form ω_a^{IJ} (in suitable natural units. Newton's constant is captured with suitable constants in k). The γ constant (non-zero real) however appeared earlier through the analysis of Barbero [126] and Immirzi [127] while they were looking for real connections for canonical gravity (hence known as Barbero-Immirzi parameter commonly) which would replace Ashtekar's complex (half-flat) ones [128]. [1]

The γ augmented term being of topological nature does not alter equations of motion. Moreover the covariant phase space and the (pre)symplectic structure on it are independent of γ. It only induces canonical transformation on the phase space.

In order to study the canonical structure, we first choose a foliation of the space-time as $M\times\mathbb{R}$. As is usual in any canonical formalism, it is required to have a time-like vector field which is related to the normal $\tilde{n}^a$ to M at each foliation (through the lapse scaling and shift vector). Moreover the internal time-like vector field $\tilde{n}_I := e_{Ia}\tilde{n}^a$ is *fixed*, such that

$$\tilde{n}_I\tilde{n}_J\eta^{IJ} = -1 \tag{5.3}$$

(η is the Lie algebra metric). This choice partially fixes the local Lorentz gauge invariance to $SU(2)$ (which is the little group of $SL(2,\mathbb{C})$ with respect to a 'time-like' direction). $\tilde{n}^a$ and $\tilde{n}^I$ will respectively be used as projection operators to tangent spaces associated with points on M and into the obvious (space-like) subspace of the frame vector-space. From now on we will use Greek letters for spatial indices and lower-case Roman for frame space indices.

With this understanding of notation, we redefine our basic dynamical variables:

$$P_i^\alpha = \frac{1}{2k\gamma}e_\beta^j e_\delta^k \epsilon^{\alpha\beta\delta}\epsilon_{ijk} \quad \text{and} \quad A_\alpha^i = \frac{1}{2}\epsilon^i{}_{jk}\omega_\alpha^{jk} + \gamma\omega_\alpha^{i0} \tag{5.4}$$

where $\epsilon_{\alpha\beta\delta}$ represents the volume 3-form on M. Next we identify A as the configuration variable and P its conjugate momentum:

$$\{A_\alpha^i(x), P_j^\beta(y)\} = \delta_j^i\delta_\alpha^\beta\delta(x,y) \tag{5.5}$$

[1]Interestingly, one should be made aware that the $\gamma\to\infty$ limit, which gives the standard first order (Palatini) theory fails to retain the manifest connection dynamics, once canonical analysis is performed.

What remains in the completion of the classical dynamics is the set of constraints. As is expected for a diffeomorphism invariant theory, the Hamiltonian that generates time translation, is a combination of first class constraints. Following are the constraints of the theory:

$$\text{Gauss: } G_i = \partial_\alpha P_i^\alpha + \epsilon_{ij}{}^k A_\alpha^j P_k^\alpha \tag{5.6a}$$

$$\text{Diffeomorphism: } C_\alpha = P^{i\beta} F_{i\alpha\beta} \tag{5.6b}$$

$$\text{Hamiltonian: } C = \left[\gamma^2 F_{\alpha\beta}^i - (1+\gamma^2)\epsilon^i{}_{jk} K_\alpha^j K_\beta^k\right] \frac{\epsilon_{ilm} E^{l\alpha} E^{m\beta}}{\sqrt{|det(\gamma E)|}} \tag{5.6c}$$

Note that the third constraint contains the variable $K_\alpha^i = \frac{1}{\gamma}(A^\alpha - \frac{1}{2}\epsilon^i{}_{jk}\omega_\alpha^{jk}$, where the second term in K_α^i can be expressed as a degree-zero rational of polynomials of P_α^i and its derivative (thanks to the projected part of the torsion-less equation of motion).

Key feature of the constraints displayed above is that all of them are first class and there are no more constraints in the system. The first one, aptly named Gauss constraint, generates $SU(2)$ gauge transformation in the frame space. On the other hand the second one is responsible for the infinite number of diffeomorphisms on M. The last or the Hamiltonian constraint generates diffeomorphism off M. A combination of these three sets do generate required time translation, which is the Hamiltonian of the system. Also note that the algebra of these constraints contains structure 'functions' instead of constants.

5.2.2 Quantum theory: Kinematics

The quantization strategy for the above theory of connections is drastically different from the standard gauge theory quantization where a fixed background is known. The LQG program aims towards a non-perturbative canonical quantization without a fixed background, where a Hilbert space of states is constructed. Action of physical observables and constraints on this Hilber space is studied. First non-triviality that one comes across, while proceeding in this direction, is due to the fact that it is an infinite dimensional system (as opposed to finite dimensional quantum mechanical systems, where canonical quantization is very well understood). The key to handle this uncountably infinite dimensional classical configuration space (of connections) is by using finite sets of probes (analogous to discrete integral transforms), probing the field variables. Finally using arbitrary sets of probes, the requirements of infinite degrees of freedom are met.

To be more concrete, we demonstrate the scheme to study the dynamics. The probes that come to one's mind for a theory which has gauge connections are obviously holonomies. In this case one starts with a graph Γ in M with finite number of edges $(e_1, e_2, \dots e_n)$ and vertices $(v_1, v_2, \dots v_k)$. The holonomies are constructed on the edges. Aiming for a quantum Hilbert space of states, one then constructs the following functions on the infinite dimensional configuration space of connections using the graph as a probe:

$$\Psi = \psi(h_{e_1}(A), h_{e_2}(A) \dots h_{e_n}(A)).$$

These functions are called 'cylindrical' with respect to the particular graph Γ. Here $h_{e_i}(A)$ is the holonomy of A on the edge e_i and is $SU(2)$ valued. This implies that Ψ functions are functions on $SU(2)^n$ group manifold. If one wants a Hilbert space structure on the space of these cylindrical functions one imposes a measure induced by the $SU(2)$ Haar measure. Uniqueness of this Hilbert space metric is again guaranteed by demanding background-independence of the theory.

While constructing the Hilbert space of such functions, the use of results from harmonic analysis, particularly the Peter-Weyl theorem is made. The Hilbert space is then decomposed as a direct sum of irreducible representations of $SU(2)$. Consequently, association of representations of $SU(2)$ with each edge become evident. These representations at edges are again constrained due to the 'Gauss law' (5.6) holding at each vertex through *angular momentum addition*. These states, named aptly as spin-network states form the kinematical Hilbert space, on which the Gauss constraint is already imposed. Finally, through refining graphs by adding more edges, the infinite dimensionality of the configuration space is captured.

5.2.3 Spectrum of geometric operators

The most important operator in relation to our study in black hole entropy calculation is the area operator. Consider a two dimensional surface embedded in M. For simplicity it can be taken as an open disk D. Classical area of it is:

$$A_D = 8\pi G\gamma \int_D \sqrt{P^i_\alpha P^\alpha_i}.$$

Its quantization requires the action of the triad operators P on the kinematical spin-network states described above. Without going into the detail of the regularizing method, we present the result. Any spin-network state associated with a graph is eigenstate of the quantum operator A_D. The spectrum is given by:

$$A_{D,\Gamma} = \gamma l_p^2 \sum_{e \cap D \neq \emptyset} \sqrt{j_e(j_e+1)}. \tag{5.7}$$

Here Γ is the graph, on which the spin-network state is *supported*, e runs over its edges and l_p is the 4d Planck length. Evidently, j_e is the half-integer defining $SU(2)$ irreducible representation identified with the edge e. Interestingly, the spectrum is discrete and there exists a finite 'area-gap' of γl_p^2, comparable to the 'mass-gap', expected in any non-perturbative theory of QCD.

Chapter 6

Local Symmetries of Weak Isolated Horizons

6.1 Introduction

In this brief chapter we explore local symmetries of a non-expanding horizon (NEH) in the first order formulation of gravity. The objective is to establish the stage for the ambitious project of black-hole entropy calculation in quantum gravity framework, which will be the subject matter of the next chapter.

For a detailed definition of NEH see [23, 96]. For our present purpose it is sufficient to characterize NEH to be a light-like hyper-surface Δ imbedded in space-time such that the unique (up to scaling by a function) light-like, real vector field l tangential to Δ is expansion, shear and twist-free. Since l is also normal to Δ, it is geodesic as well. These properties of Δ are independent of the scaling of l [23, 98]. Let us further assume that Δ is topologically equivalent to $S^2 \times \mathbb{R}$ where S^2 is a 2-sphere. It is thus clear, that NEH by definition consists of more general structures than weak isolated horizon discussed in the context of 3d gravity in chapter 3 of this thesis.

In the first order formulation Einstein's theory of gravity is invariant under the local Lorentz group $SL(2, \mathbb{C})$ apart from diffeomorphisms. Here our specific interest is primarily to find out the residual local symmetry of NEH, that is allowed by the boundary conditions on it. Then based on the residual gauge group we wish to propose an effective theory on the horizon whose subsequent quantization would yield the quantum states of a black hole.

As we have been discussing in the introduction of the last chapter, there is a recent upsurge of interest in such effective theories. Lately an $SU(2)$ Chern-Simons [1] theory has been proposed [129], (which was however used successfully long time back in [124], [25]) as the effective quantum theory on the horizon in contrast to a $U(1)$ theory proposed in [24, 64, 123]. Here comes the point of conflict between the two approaches. The quantum theories starting with $SU(2)$ and $U(1)$ invariances respectively give the logarithmic correction to (5.1) with different coefficients, namely -3/2 and -1/2.

In the canonical formulation of loop quantum gravity one gauge fixes the full Lorentz group to its rotation subgroup $SU(2)$ and the canonical theory reduces to a $SU(2)$ gauge theory. This is the main reason for expecting that a $SU(2)$ gauge theory (expectedly a topological one) should play a role as the effective theory on the horizon in this case [25, 130–134]. In the present work we do not gauge fix the theory a priori to $SU(2)$. Rather we consider full $SL(2,\mathbb{C})$ gauge invariance in the bulk. Kinematical and dynamical implications of the boundary conditions of NEH then forces the left-over symmetry on the NEH to be $U(1)$. These results are all in the classical regime. However, in the next chapter we will show how $SU(2)$ invariance comes about inevitably at the quantum level.

it Note: that one of the Newmann-Penrose coefficients is named γ customarily. In order to avoid confusion with the Barbero-Immirzi parameter, which is also usually denoted by the same letter, we add a suffix B to the later in this chapter. This chapter closely follows the work done in [135].

6.2 Kinematics

First, let us see how a NEH Δ reduces the local Lorentz symmetry. We will be using the Newman-Penrose formalism for this purpose. This was exhaustively used in chapter 3 for 3d gravity. The Newman-Penrose connection coefficients for 4-dimensions are more involved and are discussed in great detail in textbooks like [136]. The vector-field congruence l being expansion, shear and twist-free, certain Newman-Penrose coefficients $\kappa_{\rm NP}, \rho, \sigma$ vanish on Δ; $\kappa_{\rm NP}$ vanishes because the null-normal l is a geodesic vector field, ρ vanishes because the expansion of l vanishes and σ vanishes because l is shear-free also.

[1] The use of $SU(2)$ Chern-Simons theory as a boundary theory to derive black hole entropy has a precedence: independently, Krasnov, Rovelli and Smolin (cited in [124]) have considered this possibility, although not using the Isolated Horizon paradigm.

These conditions are satisfied only on Δ. However, the Newman-Penrose coefficients are sensitive to the local Lorentz transformations [136]

$$l \mapsto \xi l, n \mapsto \xi^{-1} n, m \mapsto m, \tag{6.1}$$

$$l \mapsto l, n \mapsto n, m \mapsto e^{i\theta} m, \tag{6.2}$$

$$l \mapsto l, n \mapsto n - cm - \bar{c}\bar{m} + c\bar{c}l, m \mapsto m - \bar{c}l, \tag{6.3}$$

$$l \mapsto l - bm - \bar{b}\bar{m} + b\bar{b}l, n \mapsto n, m \mapsto m - \bar{b}n, \tag{6.4}$$

where ξ, θ, c, b are smooth functions on Δ. Under (6.1), (6.2) and (6.3), $\kappa_{\rm NP}, \rho, \sigma$ transform respectively as

$$\kappa_{\rm NP} \mapsto \xi^2 \kappa_{\rm NP},\ \rho \mapsto \xi\rho,\ \sigma \mapsto \xi\sigma \tag{6.5}$$

$$\kappa_{\rm NP} \mapsto e^{i\theta} \kappa_{\rm NP},\ \rho \mapsto \rho,\ \sigma \mapsto e^{2i\theta}\sigma \tag{6.6}$$

$$\kappa_{\rm NP} \mapsto \kappa_{\rm NP},\ \rho \mapsto \rho - c\,\kappa_{\rm NP},\ \sigma \mapsto \sigma - \bar{c}\,\kappa_{\rm NP}. \tag{6.7}$$

Since they transform homogeneously, their vanishing remain invariant under (6.1)-(6.3). However, under (6.4) they transform inhomogeneously

$$\begin{aligned}
\kappa_{\rm NP} \mapsto \kappa_{\rm NP} &- \bar{b}\rho - b\sigma + |b|^2\tau + 2\bar{b}^2\alpha + 2|b|^2\beta \\
&- 2\bar{b}|b|^2\gamma - 2\bar{b}\epsilon - \bar{b}|b|^2(\mu - \bar{\mu}) + \bar{b}^2|b|^2\nu \\
&+ \bar{b}^2\pi - \bar{b}^3\lambda + D\bar{b} - b\delta\bar{b} - \bar{b}\bar{\delta}\bar{b} + |b|^2\Delta\bar{b} \\
\rho \mapsto \rho &- b\tau - 2\bar{b}\alpha + 2|b|^2\gamma - \bar{b}|b|^2\nu + \bar{b}^2\lambda + \bar{\delta}\bar{b} - b\Delta\bar{b} \\
\sigma \mapsto \sigma &- \bar{b}\tau - 2\bar{b}\beta + 2\bar{b}^2\gamma - \bar{b}^3\nu + \bar{b}^2\mu + \delta\bar{b} - \bar{b}\Delta\bar{b}
\end{aligned} \tag{6.8}$$

where $D = \nabla_l$, $\Delta = \nabla_n$, $\delta = \nabla_m$ and $\bar{\delta} = \nabla_{\bar{m}}$. Clearly, the NEH boundary conditions are satisfied if and only if $b = 0$.

Let us now study the generators of these Lorentz transformations in detail. The Lorentz matrices associated with the transformations (6.1)-(6.3) are respectively

$$\Lambda_{IJ} = -\,\xi l_I n_J - \xi^{-1} n_I l_J + 2m_{(I}\bar{m}_{J)}, \tag{6.9}$$

$$\Lambda_{IJ} = -\,2l_{(I}n_{J)} + (e^{i\theta} m_I \bar{m}_J + c.c.), \tag{6.10}$$

$$\begin{aligned}
\Lambda_{IJ} = &-\, l_I n_J - (n_I - cm_I - \bar{c}m_I + |c|^2 l_I) l_J \\
&+ (m_I - \bar{c}l_I)\bar{m}_J + (\bar{m}_I - cl_I) m_J
\end{aligned} \tag{6.11}$$

and the corresponding generators are respectively

$$B_{IJ} = (\partial\Lambda_{IJ}/\partial\xi)_{\xi=1} = -2l_{[I}n_{J]}, \tag{6.12}$$
$$R_{IJ} = (\partial\Lambda_{IJ}/\partial\theta)_{\theta=0} = 2im_{[I}\bar{m}_{J]}, \tag{6.13}$$
$$P_{IJ} = (\partial\Lambda_{IJ}/\partial\mathrm{Re}\,c)_{c=0} = 2m_{[I}l_{J]} + 2\bar{m}_{[I}l_{J]}, \tag{6.14}$$
$$Q_{IJ} = (\partial\Lambda_{IJ}/\partial\mathrm{Im}\,c)_{c=0} = 2im_{[I}l_{J]} - 2i\bar{m}_{[I}l_{J]}, \tag{6.15}$$

where B, R generate (6.1) and (6.2) respectively and P, Q generate (6.3). A straightforward calculation gives their Lie brackets

$$\begin{aligned} &[R, B] = 0, \quad [R, P] = Q, \quad [R, Q] = -P, \\ &[B, P] = P, \quad [B, Q] = Q, \quad [P, Q] = 0, \end{aligned} \tag{6.16}$$

where $[R, B]_{IJ} = R_{IK}B^K{}_J - B_{IK}R^K{}_J$ and so on. This is the Lie algebra of $ISO(2) \ltimes \mathbb{R}$ where the symbol $\ltimes$ stands for the semidirect product; R, P, Q generate $ISO(2)$ and B generates $\mathbb{R}$.

The complexified Lorentz algebra is isomorphic with $\mathfrak{sl}(2, \mathbb{C})$, which is generated by three elements $\{\sigma_3, \sigma_\pm\}$ such that $[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm$ and $[\sigma_+, \sigma_-] = \sigma_3$. Its *Borel subalgebra* is generated by $\{\sigma_3, \sigma_+\}$, which is isomorphic with (6.16). Explicitly, [137]

$$\begin{aligned} &P = i\sigma_+,\ Q = \sigma_+, \\ &R = i\sigma_+ - \frac{i}{2}\sigma_3,\ B = -\sigma_+ + \frac{1}{2}\sigma_3. \end{aligned} \tag{6.17}$$

It is an elementary exercise to show that P, Q, R, B, as defined by (6.17), are linearly independent in the field of real numbers. Clearly, the NEH boundary conditions are invariant only under this subgroup of the local Lorentz group. We should keep note of the fact that the group $ISO(2) \ltimes \mathbb{R}$ is non-semisimple; its Cartan-Killing metric K is doubly degenerate

$$K = \begin{array}{c} \begin{array}{cccc} R & B & P & Q \end{array} \\ \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{6.18}$$

Let us consider the Palatini connection ω_{IJ} and in the interior of the spacetime let us expand ω_{IJ} in the internal Lorentz basis

$$\begin{aligned}\omega_{IJ} =& -2\mathbb{W}l_{[I}n_{J]} + 2\mathbb{V}m_{[I}\bar{m}_{J]} + 2(\bar{\mathbb{N}}n_{[I}m_{J]} + c.c.) \\ &+ 2(\bar{\mathbb{U}}l_{[I}m_{J]} + c.c.)\end{aligned} \tag{6.19}$$

where $\mathbb{W}, \mathbb{V}, \mathbb{N}, \mathbb{U}$ are connection 1-forms; as defined, $\mathbb{W}$ is real, $\mathbb{V}$ is imaginary and $\mathbb{N}, \mathbb{U}$ are complex (in all, there are six of them associated with the six generators). For the rest of our analysis we will fix an internal Lorentz frame for which $l_I, n_I, m_I, \bar{m}_I$ are constants. However, our results will be unaffected by such a choice.

The pull-back of the Palatini connection to the NEH Δ is of the form

$$\underleftarrow{\omega}_{IJ} \triangleq -2Wl_{[I}n_{J]} + 2Vm_{[I}\bar{m}_{J]} + 2(\bar{U}l_{[I}m_{J]} + c.c.) \tag{6.20}$$

where W, V, U are respectively the pull-backs of $\mathbb{W}, \mathbb{V}, \mathbb{U}$. Clearly, the 1-form N, which is the pull-back of $\mathbb{N}$, vanishes on Δ by the NEH boundary conditions. Proof: The simplest way to show this is to relate the connection 1-forms to the Newman-Penrose coefficients (the constant $l_I, n_I, m_I, \bar{m}_I$ basis simplifies these relations):

$$\mathbb{W} = -(\gamma + \bar{\gamma})l - (\epsilon + \bar{\epsilon})n + (\alpha + \bar{\beta})m + (\bar{\alpha} + \beta)\bar{m} \tag{6.21}$$

$$\mathbb{V} = -(\gamma - \bar{\gamma})l - (\epsilon - \bar{\epsilon})n + (\alpha - \bar{\beta})m + (\beta - \bar{\alpha})\bar{m} \tag{6.22}$$

$$\mathbb{U} = -\bar{\nu}l - \bar{\pi}n + \bar{\mu}m + \bar{\lambda}\bar{m} \tag{6.23}$$

$$\mathbb{N} = \tau l + \kappa_{\rm NP}n - \rho m - \sigma\bar{m}. \tag{6.24}$$

So only four independent connection 1-forms W, V, U survive on Δ. This is consistent with our earlier result that the residual gauge group on Δ is $ISO(2) \ltimes \mathbb{R}$ that has only four generators. However, below we present an independent analysis for the connection to prove this.

Under the local Lorentz transformations (6.1)-(6.4) the Palatini connection (6.19) transform as

$$\omega_{IJ} \mapsto \Lambda_I{}^K\omega_{KL}\Lambda_J{}^L + \Lambda_{IK}d\Lambda_J{}^K \tag{6.25}$$

where Λ_{IJ} are the associated Lorentz matrices (6.9)-(6.11) for (6.1)-(6.3) and for (6.4)

$$\begin{aligned}\Lambda_{IJ} =& -(l_I - bm_I - \bar{b}\bar{m}_I + b\bar{b}n_I)n_J - n_Il_J \\ &+ (m_I - \bar{b}n_I)\bar{m}_J + (\bar{m}_I - bn_I)m_J.\end{aligned} \tag{6.26}$$

A lengthy but straightforward calculation shows that under the Lorentz transformations (6.9)-(6.11) the connection 1-forms transform as

$$\mathbb{W} \mapsto \mathbb{W} - d\ln\xi, \mathbb{V} \mapsto \mathbb{V}, \mathbb{U} \mapsto \xi\mathbb{U}, \mathbb{N} \mapsto \xi^{-1}\mathbb{N}. \tag{6.27}$$

$$\mathbb{W} \mapsto \mathbb{W}, \mathbb{V} \mapsto \mathbb{V} - id\theta, \mathbb{U} \mapsto e^{-i\theta}\mathbb{U}, \mathbb{N} \mapsto e^{-i\theta}\mathbb{N}. \tag{6.28}$$

$$\begin{aligned}\mathbb{W} \mapsto \mathbb{W} - c\mathbb{N} - \bar{c}\bar{\mathbb{N}}, \mathbb{V} \mapsto \mathbb{V} - c\mathbb{N} + \bar{c}\bar{\mathbb{N}}, \\ \mathbb{U} \mapsto \mathbb{U} - d\bar{c} + \bar{c}(\mathbb{W} - \mathbb{V}) - \bar{c}^2\bar{\mathbb{N}}, \mathbb{N} \mapsto \mathbb{N}.\end{aligned} \tag{6.29}$$

Since $\mathbb{N}$ transforms homogeneously, its pull-back $N \triangleq 0$ in one frame implies that it vanishes in all Lorentz frames related by (6.9)-(6.11). However, under (6.26), the connection 1-forms transform as

$$\begin{aligned}\mathbb{W} \mapsto \mathbb{W} + b\mathbb{U} + \bar{b}\bar{\mathbb{U}}, \mathbb{V} \mapsto \mathbb{V} - b\mathbb{U} + \bar{b}\bar{\mathbb{U}} \\ \mathbb{U} \mapsto \mathbb{U}, \mathbb{N} \mapsto \mathbb{N} + d\bar{b} - \bar{b}(\mathbb{W} + \mathbb{V}) - \bar{b}^2\bar{\mathbb{U}}.\end{aligned} \tag{6.30}$$

Clearly, in this case $N \triangleq 0$ if and only if b satisfies the equation $db \triangleq b(W - V + b\bar{U}) =: bY$ where Y is a 1-form. This equation has a nontrivial solution if and only if Y is a closed 1-form. However, we show that the equation admits only the trivial solution, $b = 0$. Proof: Since b is a constant in the phase space of a NEH, it is sufficient to show that Y is not closed for one specific NEH. Consider for example the event horizon of the Schwarzschild solution. In units $G = 1$ and in advanced Eddington-Finkelstein coordinates

$$\begin{aligned}W = \frac{1}{4M}\,dv,\ U = -\frac{1}{\sqrt{2}}(d\theta + i\sin\theta\, d\phi), \\ V = -i\cos\theta\, d\phi.\end{aligned} \tag{6.31}$$

As a result, dV and dU are proportional to the 2-sphere area 2-form and $dW = 0$. However, since Y depends on b, one can ask is there any b for which $dY \triangleq 0$? The answer is explicitly verifiable and one easily finds that $dY \triangleq 0$ if and only if $b = 0$. Since Y is not closed, acting d once more on the equation $db = bY$ one gets

$$0 = db \wedge Y + b\,dY = bY \wedge Y + b\,dY = b\,dY, \tag{6.32}$$

which yields the unique solution $b = 0$. This shows that the connection (6.20) is indeed an $ISO(2) \ltimes \mathbb{R}$ connection. Here we wish to remark that one could also arrive at (6.27)-(6.30) directly using the relations (6.21)-(6.24) and the appropriate Lorentz transformations of the Newman-Penrose coefficients [136].

It is to be noted that unlike the Palatini connection, the Hölst connection $A_{IJ} := \omega_{IJ} - \frac{1}{2}\gamma_B \epsilon_{IJ}{}^{KL}\omega_{Kl}$, where γ_B is the Barbero-Immirzi parameter, does not transform as a connection under any of the local Lorentz transformations (6.1)-(6.4). We also note that under pull-back to any spatial slice M and imposing the time gauge (5.3) makes it the Barbero-Immirzi connection depicted in (5.4).

For later convenience we expand (6.20) in the basis (6.12)-(6.15) of the Lie algebra $\mathfrak{iso}(2) \ltimes \mathbb{R}$:

$$\underleftarrow{\omega}_{IJ} = 2\omega_B B_{IJ} + 2\omega_R R_{IJ} + 2\omega_P P_{IJ} + 2\omega_Q Q_{IJ} \tag{6.33}$$

where $2\omega_B = W$, $2\omega_R = -iV$, $2\omega_P = -\mathrm{Re}\, U$ and $2\omega_Q = \mathrm{Im}\, U$. The connection 1-forms $\omega_B, \omega_R, \omega_P, \omega_Q$ will turn out to be more useful in the context of the effective Chern Simons theory on the horizon.

6.3 Dynamical implications

Let us now turn our attention to the pre-symplectic structure of the theory. We can derive it following the very same approach employed in 3.3.4 starting from the Hölst action (5.2) (in suitable units and e^I is the spacetime tetrad 1-form)

$$\mathbb{J}(\delta_1, \delta_2) = -\frac{1}{4}\mathrm{Tr}\left(\delta_1(e \wedge e) \wedge \delta_2\omega - (1 \leftrightarrow 2)\right). \tag{6.34}$$

We also mention, that the canonical phase space version of it is (5.5), appended with the constraints 5.6. The trace involves the $\mathfrak{sl}(2, \mathbb{C})$ Cartan-Killing metric. The expansion of the tetrad in the null tetrad basis is $e^I = -nl^I - ln^I + m\bar{m}^I + \bar{m}m^I$. So the two-form $e^I \wedge e^J$ pulled back onto Δ and expanded in the $\mathfrak{iso}(2) \ltimes \mathbb{R}$ basis is given by

$$\underleftarrow{e}^I \wedge \underleftarrow{e}^J \triangleq {}^2\epsilon R^{IJ} + \mathrm{Re}(n \wedge m)P^{IJ} - \mathrm{Im}(n \wedge m)Q^{IJ} \tag{6.35}$$

where ${}^2\epsilon = im \wedge \bar{m}$ and left arrows under e denotes its pull-back to Δ. Now the symplectic current (6.34) is a closed space time 3-form $d\,\mathbb{J} = 0$. Integrating $d\,\mathbb{J}$ over $\mathcal{M}$ we find that

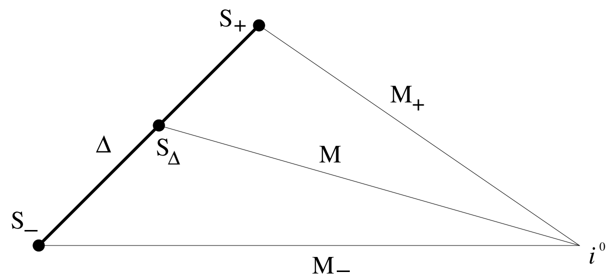

Figure 6.1: $M_{\pm}$ are two partial Cauchy surfaces enclosing a spacetime region $\mathcal{M}$ and intersecting Δ at the two 2-spheres $S_{\pm}$ respectively such that they extend to spatial infinity i^o. M is any intermediate partial Cauchy slice that intersects Δ at S_Δ

the sum-total contribution of the symplectic current from the boundaries of $\mathcal{M}$ must vanish (FIG.1)

$$\int_{M_+ \cup M_- \cup \Delta \cup i^0} \mathbb{J}(\delta_1, \delta_2) = 0 \tag{6.36}$$

We assume that the boundary conditions at infinity are such that the contribution of i^0 to the integral (6.36) vanishes. We must also ensure that the symplectic structure is independent of the choice of our foliation by the partial Cauchy slices. Using (6.33) and (6.35), and that fact that the trace in (6.34) is taken over a degenerate Killing metric (6.18), the pull-back of the symplectic current (6.34) is

$$J(\delta_1, \delta_2) \triangleq \frac{1}{2}\delta_1 {}^2\epsilon \wedge \delta_2(iV + \gamma_B W) - (1 \leftrightarrow 2). \tag{6.37}$$

It is easy to see why only the combination $iV + \gamma_B W$ survives the pull-back. The pull-back of the connection ω, hence also of A, has all the $\mathfrak{iso}(2) \ltimes R$ components. However, the pull-back $e^I \wedge e^J$ is only $\mathfrak{iso}(2)$-valued, as is obvious from (6.35). Furthermore, only the RR and BB components survive the tracing because of the degeneracy of the metric (6.18). Since $e^I \wedge e^J$ has no B-component, only the RR components survive in the symplectic current, which gives rise to the combination in (6.37).

where ${}^2\epsilon$ is the area 2-form of some spherical cross-section of Δ. In the derivation of the symplectic current it is sufficient to assume that the spherical cross-section foliates Δ and is not necessarily a geometric 2-sphere. However, for the rest of our analysis we will restrict ourselves to the unique foliation of Δ in which each leaf is a geometric 2-sphere; this is possible if and only if the isolated horizon Δ is spherically symmetric. For such a horizon with a fixed area $A_{IH} = \int {}^2\epsilon$ [2] the 1-form W is closed and dV is proportional to ${}^2\epsilon$ [90, 99]

$$dW \triangleq 0, \quad dV \triangleq \frac{4\pi i}{A_{IH}}\,{}^2\epsilon \tag{6.38}$$

where d is the exterior derivative intrinsic to Δ. Using (6.38) we find that the symplectic current 3-form is exact on Δ

$$\begin{aligned} &J(\delta_1,\delta_2) \triangleq dj(\delta_1,\delta_2) \text{ where} \\ &j(\delta_1,\delta_2) = -\frac{A_{IH}}{8\pi}\delta_1(iV+\gamma_B W)\wedge\delta_2(iV+\gamma_B W). \end{aligned} \tag{6.39}$$

It is to be noted that in the $\mathfrak{iso}(2) \ltimes \mathbb{R}$ basis the 1-form $iV + \gamma_B W = -2(\omega_R - \gamma_B\omega_B) =: -2\omega_{\text{CS}}$.

We now choose a particular orientation of the relevant space-time boundaries M_+, M_- and Δ, such that the current conservation equation (6.36) reduces to:

$$\left(\int_{M_+} - \int_{M_-}\right)\mathbb{J}(\delta_1,\delta_2) = \frac{A_{IH}}{2\pi}\left(\int_{S_-} - \int_{S_+}\right)(\delta_1\omega_{\text{CS}}\wedge\delta_2\omega_{\text{CS}})$$

This gives a foliation independent symplectic structure, whose boundary part is given by (putting back $4\pi G\gamma_B = 1$)

$$\Omega(\delta_1,\delta_2) = -\frac{A_{IH}}{8\pi^2 G\gamma_B}\int_{S^2}\delta_1\omega_{\text{CS}}\wedge\delta_2\omega_{\text{CS}} \tag{6.40}$$

where $\mathbb{S}$ is the unique spherical cross-section of Δ and $\omega_{\text{CS}} = \omega_R - \gamma_B\omega_B$.

The form (6.40) suggests that on a spherically symmetric isolated horizon one can take the effective boundary theory as a $U(1)$ Chern-Simons theory. Two distinct cases of $U(1)$ arise: `i)` If either the pull-back of ω_B vanishes on $\mathbb{S}$ [138] or one restricts the gauge freedom (6.1) to a constant class ($\xi =$ constant, as has been the original choice [24]) then one gets a compact $U(1)$, `ii)` In general, if no restrictions are imposed, then one gets a noncompact $U(1)$.

We summarize our conclusions as follows:

`1.` Starting from a first order, locally $SL(2,\mathbb{C})$ invariant theory of gravity we find that

[2] We intentionally kep the suffix IH for the area A, for future purpose, when it will be used to denote area of 2-sphere sections of isolated horizons

the gauge invariance *on* a non-expanding horizon reduces to the subgroup $ISO(2) \ltimes \mathbb{R}$. The gauge invariance in the bulk remains $SL(2,\mathbb{C})$; however, the non-$ISO(2) \ltimes \mathbb{R}$ transformations are realized trivially on the horizon.

2. Because the Cartan-Killing form of residual subalgebra $\mathfrak{iso}(2) \ltimes \mathbb{R}$ is degenerate, even a $ISO(2) \ltimes \mathbb{R}$ Chern Simons theory on the horizon can manifest only a $U(1) \times U(1)$ invariance. Here however, only a single $U(1)$ survives for a more subtle reason that has been discussed in the paragraph following (6.37) and the effective $U(1)$ is either compact or noncompact, depending on some choices.

3. Although the $ISO(2) \ltimes \mathbb{R}$ gauge invariance is not manifest at the level of symplectic structure (6.40), it may still play a role in the quantum theory of the horizon.

Chapter 7

Logarithmic Correction to Black-Hole Entropy: Resolving a Contradiction

7.1 Introduction

In the introduction of the last chapter, we discussed the logarithmic correction to the celebrated Bekenstein-Hawking area law (5.1). It was also mentioned that there is a contradiction in computing the logarithmic correction to this formula, in the framework of LQG itself. The purpose of our paper [134] and the present chapter is to show that this contradiction is apparent and how a resolution can be reached.

For the sake of continuity let us recollect that the main idea of this approach involves identifying a 'boundary' (CS) theory characterizing the degrees of freedom on an isolated horizon (of fixed cross-sectional area), consistent with the boundary conditions used to *define* such horizons [23, 90], and then counting the dimension of the Hilbert space of the quantum version of this boundary theory [24, 25, 64, 123, 124]. This dimension is then considered to be the exponential of the *microcanonical* entropy of the isolated horizon (IH). Clearly this is an effective field theory technique. Here the existence of an IH (which is a special case of a weak IH discussed in chapter 3 and of an non-expanding horizon (NEH) discussed in the last chapter), is taken as the starting point. The dynamics of the bulk space-time, by LQG prescription is probed by graphs, as detailed in 5.2. The IH now acts as a null inner boundary of quantum space (on a spatial slice) punctured by edges of the graph (serving as spin-network state in the quantum version) embedded in the spatial slice M. One should keep in mind that in the framework of LQG, the existence of such a boundary is *assumed* from the start, and not derived as a solution of the quantum Einstein equation (the Hamiltonian constraint, in a canonical description). Thus, one has to further make the assumption that the quantum Einstein equation does indeed yield space-times with this assumed property.

Referring back to 5.2, we see that within a canonical formulation, vacuum general relativity is formulated on a partial Cauchy surface M in terms of densitised triads and the Barbero-Immirzi (BI) class of $SU(2)$ Lie-algebra valued connection one-forms (5.4).[1]

We introduce an IH as a null *inner* boundary of space-time with fixed cross sectional area A_{IH} [90]. We have defined a weak IH earlier in 3.2.1. One now adds a stronger condition to that definition to make it an IH: $[\pounds_{\xi\ell}, \mathcal{D}] \triangleq 0$ on tensors on Δ. Here $\mathcal{D}$ is the induced connection on Δ.

In chapter 6 we proved that boundary conditions on a non-expanding horizon (NEH) are strong enough to spell that residual gauge invariance on it is effectively $U(1)$[2]. An IH being a special case of it can at most have that amount of symmetry since the additional conditions for its definition does not deal with symmetry issues. In that sense, (6.40) still holds as the symplectic structure of the theory describing dynamics on the IH. This is the (pre)symplectic structure of an Abelian CS theory. But that is not sufficient for describing horizon degrees of freedom. The gravitational interactions in the bulk are connected to those on the horizon through the equations of motion (6.38). The second one clearly is that of CS theory coupled to a source.

In this chapter we first aim to calculate the above mentioned correction to horizon entropy considering the $U(1)$ theory alone. Then we would move on to investigate what happens if one assumes the theory being $SU(2)$ instead. As it should happen, both give the same result again establishing that physical results do not rely upon gauge choice.

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[1]The reduction in gauge invariance from the full local Lorentz group ($SL(2,\mathbb{C})$) to the group of local rotations ($SU(2)$) is made by gauge-fixing, whereby local Lorentz boosts are fixed on the spatial slice.

[2]However this study was done maintaining the full Lorentz-covariance of the theory, without referring to canonical formulation. Since that computation was limited solely in the classical regime, we technically did not face any obstacle. No reference to spatial foliation and projection of tensors pertaining to it, was needed to be made. This is in contrast to the approach to be taken in this chapter, since here we will be tackling the quantum theory face-on. And as is well-known, quantization of a theory with non-compact gauge group (which $SL(2,\mathbb{C})$ is) is way more difficult than the one with compact group $SU(2)$.

As for notational simplicity, we note that the $\mathfrak{su}(2)$ Lie-algebra valued objects are vectors in 3 dimension (dimension of $SU(2)$). Hence it is apt to use standard 3-vector above-arrows. Dot and cross products between any two such objects should imply the usual scalar product and the antisymmetric product for 3-vectors respectively.

7.2 The $U(1)$ counting

The implementation of the isolated horizon boundary conditions and derivation of the boundary symplectic structure has been accomplished at the classical level in [90] and in a later follow-up work [24]. Now, LQG has inherent $SU(2)$ gauge invariance, after a gauge fixing from $SL(2,\mathbb{C})$. From this point of view it is normal to view boundary theory to have that amount of symmetry. Emergence of only the $U(1)$ part (6.40) should therefore be interpreted as a further fixing of gauge.

Let us illustrate this point in the light of the analysis presented in chapter 6. Consider the Newmann-Penrose null tetrad used there. We can pick them up such that l and n are respectively future and past directed internal tetrad vectors. Consequently $\tilde{n}^I = \frac{l^I - n^I}{\sqrt{2}}$ can be interpreted as the unit time-like vector used in the canonical analysis (5.3) for time-gauge fixing. On the other hand $r^I = \frac{l^I + n^I}{\sqrt{2}}$ is space-like. Once the gauge is fixed to $SU(2)$ imposing (5.3), we use $\tilde{n}$ in projecting internal vectors to the 3d vector space. This vector space furnishes also the adjoint representation of $\mathfrak{su}(2)$. In the notation of our choice, the projected version of r^I defined above will be denoted by $\vec{r}$.

We understand that the spatial foliations M do cut the IH $\Delta(\equiv S^2 \times \mathbb{R})$ at 2-spheres (topologically). For the purpose of illustration of the gauge fixing to $U(1)$, let us impose that condition that $\vec{r}$ should be covariantly constant on the S^2 foliations, with respect to the Barbero-Immirzi connection (5.4). It is obvious such an internal vector field always exists on the S^2, that is,

$$D\vec{r} \equiv d\vec{r} + \vec{A} \times \vec{r} = 0. \tag{7.1}$$

Here $\vec{A}$ is pulled-back to the S^2 of the $SU(2)$ BI connection.The connection admits the decomposition

$$\vec{A} = \vec{r}B + \vec{C} \tag{7.2}$$

with,

$$\begin{aligned} \vec{r} \cdot \vec{C} &= 0 \ , \ \vec{r}^2 = 1 \\ D\vec{r} &= d\vec{r} + \vec{C} \times \vec{r} = 0 \ . \end{aligned} \tag{7.3}$$

Observe that one can solve the second equation above explicitly for $\vec{C}$

$$\vec{C} = -\vec{r} \times d\vec{r} \ . \tag{7.4}$$

The pullback of the curvature two-form to the S^2 is

$$\begin{aligned} \vec{F} &= d\vec{A} + \frac{1}{2}\vec{A} \wedge \vec{A} \\ &= \vec{r}\left(dB - \frac{1}{2}\vec{r} \cdot d\vec{r} \wedge d\vec{r}\right) \ . \end{aligned} \tag{7.5}$$

The projection of this curvature along $\vec{r}$ is given by

$$F \equiv \vec{r} \cdot \vec{F} = dB - \frac{1}{2}\vec{r} \cdot d\vec{r} \wedge d\vec{r} \ . \tag{7.6}$$

The second term in eq. (7.6) is actually a winding number density associated with maps from S^2 to S^2; if we write it as $-d\Xi$, then

$$\frac{1}{8\pi} \int_S d\Xi = N \in \mathbb{Z} \tag{7.7}$$

Thus, we may write the $U(1)$ curvature as

$$F = dB' \tag{7.8}$$

where, $B' \equiv B - \Xi$. Note that this derivation is purely from the topological property of the $SU(2)$ bundle on S^2. This is equivalent to the equation (6.38) with the identification $F = -i\pi\gamma dV$. We derived this as projected part of the equation of motion, which encapsulates dynamical information.

Let us now concentrate on the quantum theory. We note that in LQG, for manifolds with boundary, it is judicious to choose the Hilbert space to be the tensor product $\mathcal{H}_V \otimes \mathcal{H}_{IH}$, with $\mathcal{H}_V$ and $\mathcal{H}_{IH}$ corresponding to the bulk spin-network space and the isolated horizon one respectively. When quantizing (6.38), we note that right hand side accounts for the area 2-form of the horizon S^2. The operator corresponding to area again gets non-zero contribution on its spectrum (5.7) from the points on S^2, which are punctured by edges of the corresponding spin-network state of $\mathcal{H}_V$. It is therefore clear that for the quantum isolated horizon, the $U(1)$ connection B' vanishes locally on the S^2, except on the punctures. Because of the nontrivial winding at each puncture, it is a nontrivial $U(1)$ bundle on S^2. This is the contribution that accumulates to giving the counting of CS of leading to the microcanonical entropy in this approach.

The counting now proceeds by solving (6.38) (in the following form) on $\mathcal{H}_V \otimes \mathcal{H}_{IH}$:

$$\frac{k}{2\pi} F = -{}^2\epsilon, \tag{7.9}$$

using suitable units, $k \equiv A_{IH}/2\pi\gamma$ with γ being the Barbero-Immirzi parameter. This immediately translates, in the quantum version of the theory to

$$\mathcal{I} \otimes \frac{k}{2\pi} \hat{F}(x)|\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} = -\sum_n \delta^{(2)}(x, x_n)^2\epsilon_n \; \vec{r} \cdot \vec{T}_n |\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} \;, \tag{7.10}$$

where, the sum is over a set of punctures carrying $SU(2)$ spin representation T_n for the nth puncture, and ${}^2\epsilon_n$ is the area 2-form for that puncture. It should be kept in mind that upto $O(l_P^2)$, the sum of these areas over the entire set of punctures must equal the fixed classical area A_{IH}. This immediately implies that the states to be counted are $U(1)$ Chern-Simons theory states on the punctured sphere with the net spin projection along $\vec{r}$ vanishing: $\vec{r} \cdot \sum_n \vec{T}_n = 0$. Observe that one can always rotate $\vec{r}$ locally so that this is possible, even though globally this vector corresponds to a nontrivial $U(1)$ bundle on S^2.

This $U(1)$ counting has been done in a variety of ways [133,139]. For example, one can geometrically quantize $U(1)$ CS theory with spatial Riemann surface being a punctured 2-sphere carrying spin projected representations. Most efficient technique however employs Witten's connection [3] between the dimensionality of the Hilbert space of CS theory living on a punctured S^2 ($\times\mathcal{R}$) and the number of *conformal blocks* of the boundary two dimensional conformal field theory level k WZW model on the punctured S^2. One also makes use of the fusion algebra and the Verlinde formula for the representation matrices of that algebra. In terms of the spins $SU(2)$ spins $j_1, j_2, \ldots, j_p$ on punctures, the dimension of the space of $U(1)$ (equivalent $SU(2)$ spin-projected singlet) boundary states is (for

$k \to \infty$)

$$\mathcal{N}(j_1, \ldots, j_p) = \prod_{n=1}^{p} \sum_{m_n=-j_n}^{j_n} \left(\delta_{m_1+\cdots+m_p,0}\right) \ . \tag{7.11}$$

Now the final result for the dimensionality of the $U(1)$ Chern-Simons Hilbert is easily derived from (7.11) (see [25]). For macroscopic ($A_{IH} >> l_P^2$) isolated horizons the corresponding microcanonical entropy is given by

$$S_{IH} = S_{BH} - \frac{1}{2} \log S_{BH} + \cdots \ . \tag{7.12}$$

The leading term offers a fit to the BI parameter γ, which is prototype of the LQG program. The obvious point of the whole conflict between the $U(1)$ and the $SU(2)$ approach is the appearance of a logarithmic LQG correction to the Bekenstein-Hawking area term, *with a coefficient* $-1/2$ instead of $-3/2$ as found above by doing the $SU(2)$ singlet counting [25]. This is obviously because of the apparent gauge reduction enforced by IH boundary condition.

If $SU(2)$ had intact dictated the symmetry of the problem the situation would differ in an interesting fashion. Here we have taken the diagonal $SU(2)$ generator parallel to the covariantly constant internal vector field $\vec{r}$ chosen above. Thus, the generators orthogonal to $\vec{r}$ get vanished, and hence the apparent discrepancy mentioned above. However, one understands that from the bulk $SU(2)$ covariance, the gauge gets fixed to the subgroup $U(1)$ due to boundary conditions. Whenever such a situation arises in canonical quantization, one must not forget to implement the gauge-fixing condition as an operator relation on the Hilbert space of quantum states. In the present case, the boundary condition forces that the curvature of the full $\mathfrak{su}(2)$ valued BI connection *vanishing* projection orthogonal to $\vec{r}$, (cf., the first part of (6.38)) i.e.,

$$\vec{F} \times \vec{r} = 0 \ . \tag{7.13}$$

This can also be derived independently because of the gauge choice in terms of the special internal vector $\vec{r}$ obeying $D(\vec{A})\vec{r} = 0$; one obtains the same constraint

$$[D_a, D_b]\vec{r} = 0 = \vec{F}_{ab} \times \vec{r} \tag{7.14}$$

where a, b are spacetime indices on S^2. This constraint arises as an essential and *inevitable* part of the additional gauge fixing enforced by the boundary condition on S^2, reducing the residual invariance on S^2 (in time gauge) from $SU(2)$ to $U(1)$. The constraint imposes a direct and very significant additional restriction on the class of 'physical' states contributing to the microcanonical entropy, over and above that of $U(1)$-neutrality. If we use the full $\mathfrak{su}(2)$ version of (7.9) and consider the quantum version of the above additional constraint on the spin network bulk and boundary states, we obtain,

$$\sum_n \delta^{(2)}(x, x_n)\ {}^2\epsilon_n\ \epsilon^{ijk} r^j T_n^k |\psi_V\rangle \otimes |\chi_{IH}\rangle_{U(1)} = 0 \tag{7.15}$$

where i, j, k are internal vector indices and $\{T^i\}$ are the $su(2)$ generators. One must now count the dimension of IH states that satisfy the additional constraint (7.15) apart from $U(1)$ neutrality. *This, unfortunately, has not been done in the literature on $U(1)$ counting approaches [139].*

The importance of ((7.15)) is to imply that the representations on the punctures should be such that the net $SU(2)$ spin orthogonal to the direction $\vec{r}$ is zero. This then, along with the net $U(1)$ neutrality is the requirement that all admissible isolated horizon states contributing to the microcanonical entropy are $SU(2)$ singlets. (7.11) therefore should get modified and the CS Hilbert space on the punctured S^2 should exclude the the states satisfying (7.15). Those states, in the conformal field theory perspective should not occur in the space of the corresponding Verlinde blocks. Their contribution in the dimension of the Hilbert space can be read off again as:

$$\tilde{\mathcal{N}}(j_1, \ldots, j_p) = \prod_{n=1}^{p} \sum_{m_n=-j_n}^{j_n} \left(\frac{1}{2}\delta_{m_1+\cdots+m_p,-1} + \frac{1}{2}\delta_{m_1+\cdots+m_p,1} \right) . \tag{7.16}$$

In terms of the bulk spin-network, these overcounted states (with the net azimuthal quantum number equal to zero) comes from net non-zero spin (1, 2, 3, ...) states. Now, the key is to subtract $\tilde{\mathcal{N}}$ from $\mathcal{N}$. This leads exactly to the formula derived in [124]. Following the steps there and in [25], one then sums the above series (taking contributions from spin 1/2, since this suffices in determining the log correction). This leads to the LQG entropy as

$$S_{IH} \ = \ S_{BH} \ - \ \frac{3}{2}\ \log S_{BH} \ + \ const. \ + \ O(S_{BH}^{-1}), \tag{7.17}$$

where, the semi-classical Bekenstein Hawking entropy is given by (5.1).

7.3 Recalling $SU(2)$ singlet state counting

We have now proved what was promised in the outset of this chapter. It is the best time to review the original state counting procedure of [25], where the horizon symmetry was taken to be $SU(2)$. There the complication of boundary conditions and the appearance of the resulting constraints were not present. In that sense the previous section compliments that work.

We would be brief and start with the $\mathfrak{su}(2)$ generalized form of the *equation of motion* (7.9):

$$\frac{k}{2\pi}\vec{F} \;=\; -\vec{\Sigma}\,. \tag{7.18}$$

Here, Σ denotes the area 2-form of the S^2 in question, but is $\mathfrak{su}(2)$ valued and must satisfy $\vec{r}\cdot\vec{\Sigma} = {}^2\epsilon$. This equation is implemented on the bulk-boundary Hilbert space very similarly to (7.10):

$$\mathcal{I}\otimes\frac{k}{2\pi}\hat{\vec{F}}(x)|\psi_V\rangle\otimes|\chi_{IH}\rangle = -\sum_n \delta^{(2)}(x,x_n)\ {}^2\epsilon_n T_n|\psi_V\rangle\otimes|\chi_{IH}\rangle \tag{7.19}$$

where again, the sum is over a set of punctures carrying $SU(2)$ spin representation T_n for the nth puncture, and ${}^2\epsilon_n$ is the area 2-form for that puncture. Given that upto $O(l_P^2)$, the sum of these areas over the entire set of punctures must equal the fixed classical area A_{IH}, one immediately realizes that the set of states to be counted must obey the constraint that they are $SU(2)$ *singlets*.

Exactly as shown in the $U(1)$ projected case, in terms of the spins $j_1, j_2, \ldots, j_p$ on punctures, the dimension of the space of $SU(2)$-singlet boundary states is

$$\mathcal{N}(j_1,\ldots,j_p) = \prod_{n=1}^{p}\sum_{m_n=-j_n}^{j_n}\left(\delta_{m_1+\cdots+m_p,0} - \frac{1}{2}\delta_{m_1+\cdots+m_p,-1} - \frac{1}{2}\delta_{m_1+\cdots+m_p,1}\right)\,. \tag{7.20}$$

The last two terms precisely ensure that the counting is restricted to $SU(2)$ singlet boundary states, since these alone obey the 'Gauss law constraint' which ensures local gauge invariance or 'physicality' of the counted states.

To extract the microcanonical entropy of the isolated horizon, one then follows the footsteps of [25]; the entropy turns out to be

$$S_{IH} \;=\; S_{BH} \;-\; \frac{3}{2}\,\log S_{BH} \;+\; const. \;+\; O(S_{BH}^{-1}), \tag{7.21}$$

There is absolutely no ambiguity in this infinite series, each of whose terms are finite and calculable. This is the old known formula we desired to establish in the $U(1)$ case, bringing the contributions of the constraints.

In this view, we have been successful in resolving the tension between the two results emerging from different amount of gauge invariance.

Chapter 8

Summary and Future Directions

Now we are in a stage to summarize the thesis. In a way to justify the title of the thesis, we have shown how Chern Simons theory comes as a useful handle in different situations arising in quantizing gravity. We however do not claim that all such situations have been studied in the scope of this work. For example there are recent excitements regarding super-conformal CS theory in the context of ABJM and Bagger-Lambert theory, as discussed in the introduction 0.3.1 and in the front of 3d higher-spin theories studied in the context of AdS holographic duality.

On the other hand the examples we worked out, particularly in 3 dimensions (where gravity itself is a CS theory), were interesting in their own right. We introduced a term, which augments the standard action of 3d gravity. This term keeps the topological feature of 3d gravity intact, in contrast to TMG. At least in 2 particular cases (chapters 2 and 4), this helped us perform a complete non-perturbative quantization. With positive cosmological constant (chapter 4), we even went on to produce a finite consistent theory, courtesy to this newly introduced term. Beside these successes, there was this old problem for asymptotically AdS space-times, where we studied our theory. That resulted in extremely interesting developments. The key feature includes a modification of Bekenstein-Hawking area law, BTZ class of black-holes. As a technical advancement, we also made clear the appearance of long known central charge in asymptotic symmetry algebra of AdS_3 space-time, in terms of symplectic geometry.

4d quantum gravity forms another important body of this work. But standard CS theory is defined in 3 dimensions (however one can define CS term in higher odd dimensions). Naturally the question arises about its relevance in 3+1 quantum gravity. However, horizons (for example black hole event horizons or cosmological horizons or there generalizations) of some particular type (of which event horizons are special case) were shown to be described by CS theory. Using this classical result, black hole entropy has been calculated. It is clear, that this calculations involve counting of quantum microstates of the horizon in an ensemble of fixed area horizons. Here quantum CS appears as an effective horizon theory. But these states are *entangled* in a way dictated by bulk quantum gravity states. Here, some results from loop quantum gravity comes as useful. Depending upon the CS gauge group of the horizon however, the microstate counting varies, when one considers true quantum gravity corrections to the celebrated Bekenstein-Hawking area law. This was a point of confusion and tension in the literature, which we resolved in the chapters 6 and 7.

Some of the works, which can be done in this direction, have been planned. Higher spin theories in 3 dimensions being described by CS theory is an important playground. These can be started already based on some results presented in this work. However the necessity and the consistency conditions put by the new parameter are no longer effective in all higher spin interactions. Particular interesting case is the dS−3 case. Here one can look on to find a finite and regularized higher spin quantum gravity starting from pure CS terms.

In a dimension higher things become trickier. One wishes to look for CS theory for horizons of of more general kind or of cosmological importance. Some advancement in this direction too are being done.

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