STUDY IN NONCOMMUTATIVE GEOMETRY INSPIRED PHYSICS

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by
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Dedicated to all my Teachers in Manipur

PUBLICATIONS

The thesis is based on the following publications

1. **Thermal effective potential in two- and three-dimensional non-commutative spaces** Y.C.Devi, K.J.B.Ghosh, B.Chakraborty and F.G.Scholtz, *J. Phys. A: Math. Theor.* **47** (2014) 025302.

2. **Connes distance function on fuzzy sphere and the connection between geometry and statistics** Y.C.Devi, S.Prajapat, A.K.Mukhopadhyay, B.Chakraborty, F.G.Scholtz, *J.Math.Phys.* **56**, 041707 (2015).

3. **Revisiting Connes' Finite Spectral Distance on Non-commutative spaces : Moyal plane and Fuzzy sphere** Y.C.Devi, A.Patil, A.N.Bose, K.Kumar, B.Chakraborty and F.G.Scholtz *arXiv:1608.0527*.

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INTRODUCTION

One of the most challenging problems persisting since 20th century is the quantum theory of gravity. It is expected that it will also unify the four fundamental forces: gravitation, electromagnetic, strong and weak forces. As, quantum mechanics is formulated in terms of operators acting on some Hilbert space realizing operator algebras and gravity, on the other hand, is the theory which describes the geometry and dynamics of the space-time manifold, it is quite conceivable that a future theory of quantum gravity would deal with some generalized geometry containing aspects from both. Non-commutative geometry [1] is such a generalized geometry and is indeed a very promising candidate for a future formulation of quantum gravity.

1.1 NON-COMMUTATIVE/QUANTIZED SPACE-TIME

The idea of quantization of space-time was proposed by Heisenberg in 1930's, not much later than the quantization of phase space, to cure the ultraviolet divergences of quantum field theory. However, the first paper [2] on Lorentz invariant quantized space-time was given by Snyder in 1947. In his second paper [3], Snyder obtained the equation of motion for the electromagnetic field on the Lorentz invaraint quantized space-time.

But it was Bronstein [4] before Snyder who observed that consideration of both quantum mechanics and Einstein's general relativity imposes an upper bound on the density of test body while making a precise measurement of gravitational field and hence a fundamental length scale below which the notion of space-time point i.e. event doesn't make sense. But the success of renormalization techniques rendered the ideas of quantized space-time dormant for many years. However, in 1994 [5, 6], Doplicher *et.al* independently refreshed the idea of quantizing space-time by proposing a Poincare covariant spacetime uncertainty relations at Planck length scales ($l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-33}\ cm$), where $\hbar = 6.582 \times 10^{-16}\ eV.s$ is the reduced Planck constant, $G = 6.674 \times 10^{-8}\ cm^3 g^{-1} s^{-2}$ is the gravitational constant and $c = 2.997 \times 10^{10}\ cm.s^{-1}$ is the speed of light in vacuum. This was suggested by quantum mechanics and Einstein's theory of gravity as quantum measurement of a spacetime event with accuracy $\sim l_p$ need a probing particle with Compton wavelength $\leq l_p$. This means particle with energy $E_{l_p} \geq \frac{2\pi\hbar c}{l_p} \approx 2.557 \times 10^{28}\ eV$ which will generate so strong a gravitational field with the Schwarzschild radius $R_s = \frac{2GE_{l_p}}{c^4} \approx 4.215 \times 10^{-33}\ cm > l_p$ so that a black hole will form and no information will come out. Thus, the idea of quantized space-time is perhaps the main feature of the quantum theory of gravity. Moreover, the non-commutative space-time was found to emerge in string theory [7] at certain low energy limits.

Of the many choices of non-commutativity of the space-time, the simplest one is the canonical one which is the one postulated in [5] and obtained in [7]:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}\ ; \qquad \mu, \nu = 0, 1, 2, 3. \tag{1.1}$$

Such spacetime (1.1) is referred to as the non-commutative Moyal space-time $\mathbb{R}^4_*$ [8] where $\Theta_{\mu\nu}$ is a constant anti-symmetric matrix; and does not transform like a rank 2 anti-symmetric tensor under Lorentz transformation. Rather, the entries are regarded as new constants of Nature like $\hbar, c, G$, etc [9].

Such type of non-commutativity (1.1) appears in condensed-matter theory also. The classic example being the Landau problem [10] where the projected coordinates of the electrons (moving in a plane subjected to an external transverse magnetic field) on the lowest Landau level do not commute. Thus, the theory of electrons in a strong magnetic field can be thought as a non-commutative field theory and hence it is believed that the theory of quantum Hall effect [11] can also be understood in the framework of non-commutative field theory.

1.1.1 *Non-commutative field theory employing star products*

The construction of non-commutative field theory is inspired by the phase space formulation of quantum mechanics (deformation quantization) [12] where the quantization of the phase space can be understood as a deformation of the algebra of observables with a new non-commutative product rule, called star-product [13]. This is also called the Wigner-Weyl quantization[1] [14, 15] where there is one-to-one correspondence, called the Wigner-Weyl correspondence, between quantum operators and classical functions of phase space variables (for a review, one can see [12]).

One then consider the space-time on which we want to formulate quantum field theory to be the same flat Euclidean space-time $\mathbb{R}^4$ but take the algebra of fields $\mathcal{A}_\star(\mathbb{R}^4)$ equipped with the deformed product rule implemented by the star product which can be defined as

$$f \star g = m_\star(f \otimes g) := m_0 \circ F^{-1}(f \otimes g), \tag{1.2}$$

where $m_\star$ is the non-commutative multiplication map called the star product, $m_\star : \mathcal{A}_\star \otimes \mathcal{A}_\star \to \mathcal{A}_\star$ and m_0 is the commutative pointwise multiplication map, $m_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \to \mathcal{A}_0$, i.e.,

1 If $f(x,y)$ is a function defined on $\mathbb{R}^2$, then the Weyl operator $\hat{W}(f)$ of f is given by

$$\hat{W}(f) = \frac{1}{(2\pi)^2} \int dxdy \int dk_x dk_y f(x,y) e^{-i(k_x x + k_y y)}\, e^{i(k_x \hat{x} + k_y \hat{y})}\ ; \quad [\hat{x}, \hat{y}] = i\theta.$$

The inverse of Weyl map gives the Wigner function and Moyal star product appears as $\hat{W}(f \star g) = \hat{W}(f)\hat{W}(g)$.

$m_0(f \otimes g) \mapsto fg$, while F is the invertible, Drinfel'd twist element [16]. The non-commutative field theories can then be constructed in complete analogy with the corresponding commutative theories except using star-product for multiplication between fields [8, 17].

Note that the choice of star product is not unique. Both Moyal [18] and Voros [19] $\star$-products give the same canonical non-commutative space-time:

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\Theta^{\mu\nu}. \tag{1.3}$$

The Moyal star product is given by

$$(f \star g)(x) = f(x) e^{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g(x);\ \overleftarrow{\partial}_\mu \text{ acts on left }, f(x) \text{ and } \overrightarrow{\partial}_\mu \text{ acts on right}, g(x)\ , \tag{1.4}$$

for which the corresponding twist element is

$$F = e^{-\frac{i}{2}\Theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}. \tag{1.5}$$

On the other hand, Voros star product, which can be defined mainly in an even dimensional spaces, has a slightly different structure and will be displayed soon. The Moyal star product (1.4) first appeared as the deformed product of two classical functions of phase space variables which corresponds to the product of the symmetrically ordered quantum operators.

An astonishing feature of non-commutative quantum field theory is that it doesn't completely cures the problems associated with the UV divergences rather it shows a mixing of high energy (UV) and low energy (IR) scales, the so-called *UV/IR mixing* [20, 21] which does not have a commutative counterpart. Also, the Lagrangian of the non-commutative field theory is not Lorentz invariant. This stems from the fact that the basic non-commutative algebra (1.1) is not covariant under the usual relativistic (Lorentz) symmetries of space-time because of the constant matrix $\Theta^{\mu\nu}$. Equivalently, we can consider the transformation property of scalar field $\phi(x)$ under homogeneous Lorentz transformation [22]:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu\, x^\nu \quad \Longrightarrow \quad \phi \to \phi^\Lambda(x) = U(\Lambda)\phi(x) = \phi(\Lambda^{-1}x), \tag{1.6}$$

where $U(\Lambda)$ is the unitary representation of this Lorentz transformation on the space of scalar fields. Then, the automorphism symmetry for a pair of arbitrary scalar fields $\phi_1(x)$ and $\phi_2(x)$ under the above transformation (1.6) is found to be broken:

$$(\phi_1^\Lambda \star_M \phi_2^\Lambda)(x) \neq (\phi_1 \star_M \phi_2)^\Lambda(x). \tag{1.7}$$

However, the automorphism symmetry can be recovered by deforming the co-product $\Delta_0(\Lambda) \to \Delta_\star(\Lambda) = F\Delta_0(\Lambda)F^{-1}$ (see appendix A), with

$$\Delta_0(\Lambda) = U(\Lambda) \otimes U(\Lambda), \tag{1.8}$$

being the undeformed co-product and here the twist element can be written as $F = e^{\frac{i}{2}\Theta^{\mu\nu}P_\mu \otimes P_\nu}$ where P_μ are the generators of translations, so that

$$m_\star\Big(\Delta_\star(\Lambda)(\phi_1 \otimes \phi_2)\Big) = U(\Lambda) \circ \big(m_\star(\phi_1 \otimes \phi_2)\big). \tag{1.9}$$

The resulting algebra of scalar fields is the twisted Poincare algebra [23, 24] with the twisted co-products:

$$\begin{aligned} \Delta_\star(P_\mu) &= \Delta_0(P_\mu) \equiv P_\mu \otimes \mathbb{I} + \mathbb{I} \otimes P_\mu \qquad (1.10)\\ \Delta_\star(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbb{I} + \mathbb{I} \otimes M_{\mu\nu} - \frac{1}{2}\Theta^{\gamma\delta}\big[(\eta_{\gamma\mu}P_\nu - \eta_{\gamma\nu}P_\mu) \otimes P_\delta + P_\gamma \otimes (\eta_{\delta\mu}P_\nu - \eta_{\delta\nu}P_\mu)\big], \end{aligned}$$

where $M_{\mu\nu}$ are the Lorentz generators and $\eta_{\mu\nu}$ is the flat space-time metric. Thus, the symmetries of non-commutative space-time cannot be captured by group structure but by quantum group [25] which is not a group by itself but a twisted Hopf algebra (see appendix A). The twisting of the co-product leads to the notion of twisted statistics in non-commutative physics [26]. Further, it was shown in [27] that the twisted fermions violate the Pauli exclusion principle by computing the two-particle correlation function, employing the Moyal star product, for a canonical ensemble of free gas.

As mentioned above, one can also define another star product, called the Voros star product [19] which comes from the weighted Weyl map[2] [30–32]. This associates product of normal ordered quantum operators to the deformed product(Voros star product) of classical functions on the phase space variables. The inverse of weighted Weyl map is Wigner's distribution function, smoothened with a gaussian function. It is non-negative in all quantum states [33] and is as wide or wider than the minimum uncertainty wave packet [34].

In order to compare the Moyal and Voros $\star$-products, let us consider $2+1$ non-commutative space-time for which $\Theta^{0\alpha} = 0,\ \alpha = 1,2$ and since $\Theta^{00} = 0$ identically we can write $\Theta^{\alpha\beta} = \theta\ \epsilon^{\alpha\beta}$, where $\epsilon^{12} = 1 = -\epsilon^{21}$ and θ is a constant. Here, θ is the non-commutative or deformation parameter of dimension length squared. On such space-time, one can define two deformed algebra of functions viz $\mathcal{A}^M_\star$ and $\mathcal{A}^V_\star$ with the following twist elements respectively for the non-commutative multiplication map (1.2),

$$F^M = e^{\frac{i}{2}\theta(\partial_1\otimes\partial_2 - \partial_2\otimes\partial_1)} \quad \text{and} \quad F^V = e^{\frac{\theta}{2}(\partial_1\otimes\partial_1 + \partial_2\otimes\partial_2)}F^M = F^M e^{\frac{\theta}{2}(\partial_1\otimes\partial_1 + \partial_2\otimes\partial_2)}. \tag{1.11}$$

Thus, the Voros star product between two arbitrary functions can be written as

$$f \star_V g = f\ e^{\frac{i}{2}\left(\Theta^{\alpha\beta}\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta - i\theta\delta^{\alpha\beta}\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta\right)}\ g = f\ e^{\frac{i}{2}\Theta_V^{\alpha\beta}\overleftarrow{\partial}_\alpha\overrightarrow{\partial}_\beta}\ g = f\ e^{\overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_z}\ g\,, \tag{1.12}$$

2 For function $\phi(z,\bar{z})$ on complex plane $\mathbb{C}$, its weighted Weyl map is given by [28, 29]

$$\hat{W}_V(\phi) = \frac{1}{(2\pi)^2}\int dz d\bar{z} \int d\eta d\bar{\eta}\,\phi(z,\bar{z}) e^{-\eta\bar{z}+\bar{\eta}z} e^{\theta\eta\hat{a}^\dagger}\ e^{-\theta\bar{\eta}\hat{a}}\ ; \quad [\hat{a},\hat{a}^\dagger] = \theta\,.$$

The inverse of this map is represented by $\phi(z,\bar{z}) = \langle z|\hat{W}_V(\phi)|z\rangle$ where $\hat{a}|z\rangle = z|z\rangle$ and the Voros star product appears as $\hat{W}_V(\phi \star_V \psi) = \hat{W}_V(\phi)\hat{W}_V(\psi)$ such that $\phi \star_V \psi = \langle z|\hat{W}_V(\phi)\hat{W}_V(\psi)|z\rangle$.

$$\text{where,} \quad \Theta_V^{\alpha\beta} = \theta\epsilon^{\alpha\beta} - i\theta\delta^{\alpha\beta} \quad \text{is the non-commutative Voros matrix ;} \tag{1.13}$$

$$\text{and } z = \frac{1}{\sqrt{2\theta}}(x^1 + ix^2) \,, \quad \bar{z} = \frac{1}{\sqrt{2\theta}}(x^1 - ix^2) \tag{1.14}$$

are the dimensionless complex variables.

These two deformed algebras $\mathcal{A}_\star^M$ and $\mathcal{A}_\star^V$ are connected [35–37] by an non-invertible linear map $T : \mathcal{A}_\star^M \to \mathcal{A}_\star^V$ [38] such that

$$T(f \star_M g)(x) = [(Tf) \star_V (Tg)](x) \;\; \text{and} \;\; T(f^*) = (Tf)^* \quad \text{where} \quad T = e^{\frac{\theta}{4}\nabla^2}. \tag{1.15}$$

It was shown in [38] that this T-map is not invertible on the space of square-integrable or even Schwartz class functions on $\mathbb{R}^2$. This was shown by considering a Gaussian function $f(x) = e^{-\frac{x^2}{\alpha^2}}$ which yields, on the action of T^{-1},

$$T^{-1}f(x) = \frac{\alpha^2}{4\pi}\int d^2p \; e^{-\frac{1}{4}(\alpha^2-\theta)\vec{p}^2 + i\vec{p}.\vec{x}}. \tag{1.16}$$

Clearly, for $\alpha < \sqrt{\theta}$, the above integral does not exist and hence T^{-1} exists only for a class of Schwartz functions which are smooth on the small length scales ($\lesssim \sqrt{\theta}$). Thus, different deformed algebras $\mathcal{A}_\star^M$ and $\mathcal{A}_\star^V$ define inequivalent representations of a quantum system which explain the inequivalence results between non-commutative field theories employing Moyal and Voros $\star$-products obtained in [36, 37]. Moreover, there are infinitely many star products [39] for different non-commutative space-times and so one has to be very careful about the choice of the star product.

Let us now promote the position coordinates of a non-commutative space-time to the level of operators but retaining time coordinate as c-number and consider only the non-commutative space at some time t. In other words, we put $\Theta^{0i} = 0$ for simplicity and not because of any claimed violation of unitarity [40] in the case $\Theta^{0i} \neq 0$. In fact, it has been proved quite conclusively in [41] that there is no violation even when $\Theta^{0i} \neq 0$.

1.1.2 *Non-commutative quantum mechanics: Operator method*

One can define non-commutative space in many ways [42] and some spaces which are relatively simple and have been widely studied are as follows:

1. Moyal Plane $(\mathbb{R}_\star^2)$: $-$ $\quad [\hat{x}^\alpha, \hat{x}^\beta] = i\theta\, \epsilon^{\alpha\beta}$; (note that $\alpha, \beta = 1, 2$) ; (1.17)
2. 3D Moyal space $(\mathbb{R}_\star^3)$: $-$ $\quad [\hat{x}^i, \hat{x}^j] = i\theta^{ij} = i\epsilon^{ijk}\theta^k$; $i, j, k = 1, 2, 3$; (1.18)
3. Fuzzy Sphere $(\mathbb{R}_{\theta_f}^3)$: $-$ $\quad [\hat{x}^i, \hat{x}^j] = i\theta_f\epsilon^{ijk}\hat{x}^k$, θ_f is another constant[3] (1.19)

and a fixed value of the Casimir.

3 Note that θ has the dimension of length squared but θ_f has the dimension of length.

Although non-commutative field theories have been extensively studied (see [8, 17] and references there in) using the star products, quantum mechanics on Moyal plane (1.17) and fuzzy sphere (1.19) were first studied in [43] without using any star product but a complete operator method. However, the complete operatorial formalism of non-commutative quantum mechanics on Moyal plane with the proper concepts of classical configuration space or more precisely, an auxilliary Hilbert space $\mathcal{H}_c$ furnishing a representation of just the non-commutative algebra (1.17). Note that this coordinate algebra is a sub-algebra of the entire noncommutative Heisenberg algebra, obtained by augmenting the above mentioned coordinate algebra with the ones involving linear momentum operators:

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0\ ; \quad [\hat{X}_\alpha, \hat{P}_\beta] = i\delta_{\alpha\beta}\ ; \quad (\text{putting } \hbar = 1). \tag{1.20}$$

On the contrary, the Hilbert space furnishing a representation of the entire Heisenberg algebra is the quantum Hilbert space and was first developed in [44, 45]. In this formalism, the quantum states are represented by Hilbert-Schmidt operators[4] acting on classical configuration space $\mathcal{H}_c$ and has the structure of Boson Fock space. Thus, this operatorial formalism of non-commutative quantum mechanics is referred to as the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics. This formalism for Moyal plane (1.17), 3D Moyal space (1.18) and fuzzy sphere (1.19) will be reviewed in the next chapter 2. The eigenvalue problem for the infinite and finite non-commutative spherical wells in Moyal plane was solved in [44] where the time reversal symmetry was found to be broken. Further, the eigenvalue problem of free particle and harmonic oscillator on Moyal plane were solved in [45, 46]. A very important observation made in [45] in the context of harmonic oscillator is the fact that the spread of the ground state wave packet cannot be squeezed below the length-scale $\sqrt{\theta}$ even in the limit of infinitely steep potential well ($\omega \to \infty$). This displays the fact that the noncommutativity can indeed play a role in preventing a potential gravitational collapse in the localization process as mentioned earlier. Of course, all these solutions reduce to the standard results in the commutaive limit. The generalization of non-commutative quantum mechanics to 3D Moyal space in the Hilbert-Schmidt operator formalism was given in [47] where the breaking of $SO(3)$ symmetry was observed in presence of interaction, despite its restoration at the level of coordinate algebra through the deformed co-product (1.10). The Hilbert-Schmidt operator formalism for the case of fuzzy sphere (1.19), which has an $SO(3)$ symmetry, was first developed in [48] where the Coulomb problem of non-commutative quantum mechanics was solved. In [49], the Schrödinger equation for the free particle, finite and infinite fuzzy wells on fuzzy sphere was solved using the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics.

On a non-commutative space, since the position operators do not commute, the notion of position, at best, can be preserved using the minimal uncertainty state through the use of coherent states [45]. In the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics on Moyal plane, two types of position bases are defined, viz Moyal and

4 Weyl operators corresponding to the square-integrable functions.

Voros position bases [38], corresponding to the Voros and Moyal $\star$-products. The inequivalence between Voros and Moyal position bases in this operator formalism was shown in [38] by demonstrating the non-invertibility of T-map (1.15), as mentioned earlier. And when invertible they are related by a similarity transformation, as Voros basis is not orthonormal, in contrast to Moyal basis. This inequivalence was also demonstrated explicitly by computing the free particle transition amplitudes through path integral formalism given in [50]. It is shown in [45] that the Voros position basis is the only physical basis in Moyal plane as it conforms to the positive operator valued measure (POVM) [51] and it is the minimal uncertainty state in Moyal plane. However, the Moyal position basis is found to be the simultaneous eigenstates of the unphysical, commuting position operator $\hat{X}_c^\alpha$ which is related to the physical position operator $\hat{X}^\alpha$ as

$$\hat{X}_c^\alpha = \frac{1}{2}(\hat{X}_L^\alpha + \hat{X}_R^\alpha) = \hat{X}^\alpha + \frac{\theta}{2}\epsilon^{\alpha\beta}\hat{P}_\beta \,. \tag{1.21}$$

Note that we denote by $\hat{X}_L^\alpha$ and $\hat{X}_R^\alpha$ the left and right action of the position operator $\hat{X}^\alpha$ on the quantum state. The Voros and Moyal position bases on 3D Moyal space (1.18) were introduced in [47] where the Moyal basis is still the eigenstate of the commuting position operator in 3D

$$\hat{X}_c^i = \frac{1}{2}(\hat{X}_L^i + \hat{X}_R^i) = \hat{X}^i + \frac{1}{2}\theta^{ij}\hat{P}_j. \tag{1.22}$$

Thus, we study the properties of Voros position basis and the issue of maximal localization of a single particle in 3D Moyal space in this thesis. This has been pursued by using the symplectic invariant formulation of uncertainty relation in the $2n$-dimensional phase space:

$$\det V \geq \frac{1}{4^n} \;, \tag{1.23}$$

through the computation of the variance matrix V. Since the symplectic invariant uncertainty relation in phase space for non-commutative case is not known, we apply the commutative relation (1.23) after getting commutative variance matrix (V^0) from the non-commutative variance matrix V^θ by using (1.22).

The twisted statistics on Moyal plane have been well studied by using star products [26, 27]. However, this twisted statistics for 3D Moyal space in the Hilbert-Schmidt operator formalism was not studied earlier. We have studied this in this thesis by extending the Hilbert-Schmidt operator formalism on 3D Moyal space to many-particle system and then carrying out the second quantization. Then, there arises two types of bases in the momentum space of many-particle system: the twisted basis and the so-called "quasi-commutative" basis. The twisted basis yields the twisted statistcs but the quasi-commutative basis yields the usual statistics of commutative case and hence the name quasi-commutative. Thus, on 3D Moyal space, we have two momentum bases and also two types of position bases (Voros and Moyal) which can be expressed in terms of these momentum bases. Then considering a canonical ensemble of free gas consisting of a pair of identical particles in 3D Moyal space, we compute the two-particle correlation functions corresponding to twisted Voros and Moyal bases and quasi-commutative Voros and Moyal bases. The inequivalence between Voros and Moyal position bases become more transparent from the expressions of correlation functions for each case apart from a deformation in the thermal wavelength. Also, from these computations the thermal effective potential [52] on 3D Moyal space are plotted for different bases [53] and the comparison between commutative and non-commutative statistics have been made.

1.2 NON-COMMUTATIVE GEOMETRY

After a reliable formulation of quantum mechanics on non-commutative spaces (1.17) and (1.19), we would like to study the geometry of such spaces. Clearly, the usual techniques of Euclidean and Riemannian geometry cannot be applied on such spaces as they lack the notion of points and paths, in particular geodesics. Thus, to define geometry on such spaces we will follow the standard approach of non-commutative geometry formulated by Alian Connes [1, 54]. This is the most generalized mathematical approach to study the geometry of non-commutative spaces by extracting the informations from the algebraic structures defined on the space.

The idea of non-commutative geometry [1] is motivated by the well known Gelfand Naimark theorem [55] and John von Neumann's works on the mathematical theory of quantum mechanics [56]. The Gelfand Naimark theorem gives a duality between topological structure and algebraic structure defined on some set X. According to this theorem [55], *the category of compact Hausdorff topological spaces with the continuous maps as morphisms is dual or anti-equivalent to the category of unital commutative C^*-algebras (algebras of continuous complex-valued functions) with the unital $*$-homomorphisms* (for the definitions, if needed, see appendix C). This is still valid for locally compact spaces for which the morphisms are continuous proper maps corresponding to the proper $*$-homomorphisms between non-unital commutative C^*-algebras [57]. This means given a commutative C^*-algebra $\mathcal{A}$ we can associates a c-number to each element of $\mathcal{A}$ by defining a character $\mu(a)$[5] or spectrum $\sigma(a)$ or state $\omega(a)$, $\forall\ a \in \mathcal{A}$ (see appendix C). Then, the character space $\mathcal{M}(\mathcal{A})$ or spectrum $\Sigma(\mathcal{A})$ or space of pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ (these spaces are all isomorphic to each other for commutative C^*-algebras) will have a relative topology (Gelfand topology) induced by the norm topology defined on the dual of the algebra $\mathcal{A}^*$. Thus, if the commutative C^*-algebra $\mathcal{A}$ is the algebra $C(X)$ of continuous complex-valued functions defined on a Hausdorff topological space X, then we have the Gelfand Naimark duality:

$$\mathcal{M}(\mathcal{A}) \cong X \quad ; \quad C(\mathcal{M}(\mathcal{A})) \cong C(X). \tag{1.24}$$

Further, it is possible to get the information of a vector bundle $E(X)$ on a compact Hausdorff topological space X from the space of sections $\Gamma(E(X))$ which form module over the

5 We have put all the definitions in appendix C

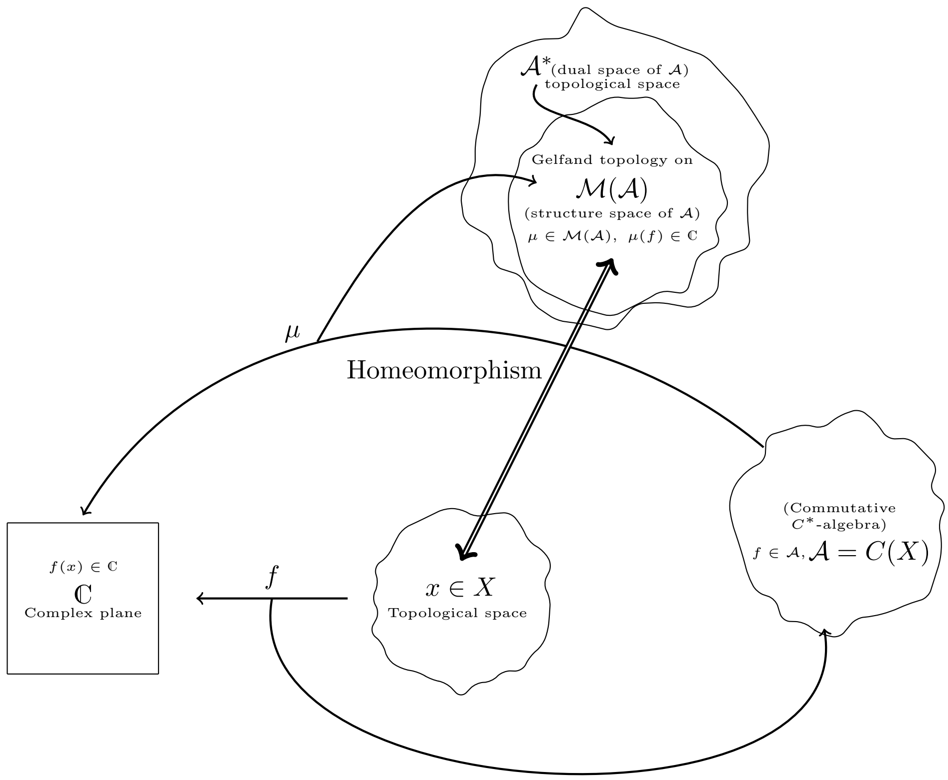

Figure 1.1: Pictorial representation of Gelfand Naimark Theorem.

commutative C^*-algebra $C(X)$ on X. This is given by the Swan theorem [58] which states that *the category of complex vector bundles on a compact Hausdorff space is equivalent to the category of finitely generated projective modules over the unital commutative C^*-algebra* (this statement is taken from [57]).

This duality between topological spaces and commutative C^*-algebras can be extended to include the non-commutative C^*-algebras and hence one can talk about the "non-commutative space" dual to an arbitrary C^*-algebra. In [59, 60], Woronowiz extended the Gelfand Naimark theorem for non-commutative C^*-algebra where he referred the corresponding space as "psuedospace or quantum space" and hence "Non-commutative Gelfand Naimark theorem". Note that a non-commutative C^*-algebra has few characters. For example, the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices have no character at all [61] so that the space corresponding to such algebra has few points. But one can still define the spectrum of $M_n(\mathbb{C})$ matrices and the states on $M_n(\mathbb{C})$ so that the pure states of the C^*-algebra play the role of points on a generalized space. In this way, by studying the algebraic structures defined on some space X, we can extract the topological informations of X or vice versa. Thus, one can have the following dictionary 1.1 between the spaces and the algebras defined on such spaces [62].

By the second Gelfand Naimark theorem, given a separable Hilbert space $\mathcal{H}$ with the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ with an operator norm, any norm closed subalgebra

Table 1.1: Dictionary of duality between space and algebra

Space	Algebra
Compact Hausdorff topological space	Unital commutative C^*-algebra
Locally compact Hausdorff topological space	commutative C^*-algebra
Continuous map	$*$-homomorphism
Homomorphism	Automorphism
Open set	Ideal
Closed set	Quotient algebra
Point	Pure state
Vector bundle	Finitely generated projective module
Non-commutative space	Non-commutative C^*-algebra

of $\mathcal{B}(\mathcal{H})$ is a separable C^*-algebra. Thus, we can describe all C^*-algebras as operators acting on some Hilbert space.

In this way, given a C^*-algebra $\mathcal{A}$ we get the non-commutative topology on the space $\mathcal{S}(\mathcal{A})$ of states but to study the geometry of $\mathcal{S}(\mathcal{A})$, we have to define the differential structures on $\mathcal{S}(\mathcal{A})$. Getting hint from Milnor's works [63] where he found that all the information about a compact Riemannian manifold M cannot be extracted just from the eigenvalues of the Laplace operator on M. Rather, the Dirac operator $\mathcal{D}$ [6] plays a more fundamental role. Thus, Connes introduced the so-called "Spectral triple" $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [64] consisting of an involutive algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, through a representation π, with an unbounded, self adjoint operator $\mathcal{D}$ which is the generalized Dirac operator. The spectral triple with certain axioms [65] will generalize the notions of manifolds (topological spaces with additional structures) and differential geometry. Thus, a spectral triple defines a 'non-commutative manifold', where the algebra $\mathcal{A}$ will give the information about the topological properties and the Dirac operator $\mathcal{D}$ will give the geometric informations. In this way, one can talk about the "quantized differential calculus" called "Non-commutative differential geometry" [64] of a quantized space using algebraic objects defined on the space. Moreover, Connes gave a distance formula called the "Spectral distance" which generalizes the notion of geodesic distance without invoking any path but given in terms of the normalized positive functionals i.e. states ω's on the algebra, under certain conditions (this has been reviewed in chapter 5), as

$$d(\omega', \omega) = \sup_{a \in \mathcal{A}} \{ |\omega'(a) - \omega(a)| \ : \ \| [\mathcal{D}, \pi(a)] \| \leq 1 \}. \tag{1.25}$$

This exactly gives the geodesic distance of the compact Riemannian spin manifold M when computed between the pure states $\delta(x) = \omega_x(f)$ and $\delta(y) = \omega_y(f)$ of the algebra $\mathcal{A} =$

6 Dirac operator is obtained by contracting the covariant derivative on the spinor module via the Clifford multiplication and is more fundamental than the Laplace operator since by Lichnerowicz formula, $\mathcal{D}^2 = \Delta + \frac{s}{4}$ where Δ is the Laplacian and s is the scalar curvature.

$C^\infty(M)$, $(f \in \mathcal{A})$ of smooth functions on M where the commutative spectral triple consists of $(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M,S), \mathcal{D} = \not{D})$ with $L^2(M,S)$ being the Hilbert space of square integrable spinors, S is the spinor bundle over M and $\not{D}$ is the usual Dirac operator of compact Riemannian spin manifold M. Thus, without introducing any concept of space, points, curves, surfaces (which may not simply exist at the length scale of quantum gravity where we believe the natural structure of Minkowski space-time will break down) but by just studying the algebra of observables which is a C^*-algebra, we would be able to extract all the possible geometric information about the quantized space-time.

One should note that the language of non-commutative geometry has been successfully used to prove that the Hall conductivity is quantized in integer quantum Hall effect [66]. More significantly a non-commutative geometrical formulation unifying the Einstein's general theory of relativity with the Standard model of particle physics as a pure gravity theory of a non-commutative manifold called an 'almost commutative manifold' $M \times F$ where M is the compact Riemanniam spin manifold and F is an internal "discrete" space and described by finite rank matrices having a non-trivial inner-automorphism group, which in turn describes the gauge contents of the particle theory as was shown in [65, 67]. This was eventually used to post-dict the Higgs mass of 125 GeV [68]. The idea of introducing an almost commutative manifold is somewhat like the Kaluza-Klein theory [69] which succcesfully unified the Einstein's gravity theory with that of electromagnetism with group $U(1)$ or for that matter any gauge theory with group G at the classical level by introducing an internal compact manifold, in the form of a circle $\mathbb{S}^1$ or a suitable coset space G/H [70, 71]. In contrast, Connes's idea is to replace the internal compact manifold in each point of the spacetime by the discrete space F described by finite rank matrices and of zero dimension and carrying the non-commutative structure. This stems from the fact that the total action

$$S_t = S_E + S_{SM}\,;\;\; S_E \rightarrow \text{ Einstein-Hilbert action and } \; S_{SM} \rightarrow \text{ Standard model action,} \quad (1.26)$$

of the particle theory living on a general space-time has the symmetry group, given by the semi-direct product $\mathcal{G} \rtimes Diff(M)$, of the gauge group $\mathcal{G}$ of all maps, i.e., $\mathcal{G} : M \rightarrow SU(3) \times SU(2) \times U(1)$ of S_{SM} and $Diff(M)$ being the symmetry of S_E. By a mathematical result obtained in [72, 73], we know that $Diff(M)$ of a differential manifold M cannot have a non-trival normal subgroup. If there exists a manifold X for which the diffeomorphism group is $\mathcal{G} \rtimes Diff(M)$, then clearly X cannot be a usual differentiable manifold as it contains the normal subgroup $\mathcal{G}$. However, if we consider a "non-commutative manifold" corresponding to a non-commutative algebra, clearly the automorphism group of the algebra contains a non-trival normal subgroup called the inner automorphism group and hence corresponds to non-trival normal subgroup of the symmetry of the non-commutative manifold. This captures the gauge content of the theory. Hence, unification of the Einstein's gravity and the Standard model of the particle physics can be made possible only if the total space where both theories live is non-commutative. One should note that this unification is purely classical in the sense that we are taking Einstein's gravity and the quantization of this pure gravity theory is yet to be obtained. It is generally believed that the quantization of such theory can be achieved if we consider a truely non-commutative manifold $M_\star \times F$, where $M_\star$ is a non-commutatve space-time. This motivates us to study the non-commutative space-time and its geometry from the standard approach of non-commutative geometry. However, in this thesis we would like to study the metric properties of the most widely studied non-commutative spaces viz Moyal plane and Fuzzy sphere.

As we have mentioned earlier, the way (mostly taken) for studying non-commutative spaces is by considering it as a deformation of commutative ones. For example, Moyal plane $\mathbb{R}^2_\star$ is the deformation of Euclidean plane $\mathbb{R}^2$ and fuzzy sphere $\mathbb{S}^2_\star$ that of the 2-sphere S^2. It was shown in [74] that $\mathbb{A}^2_\star = (S(\mathbb{R}^2), \star_M)$ is the corresponding deformed algebra of the Moyal plane $\mathbb{R}^2_\star$, where $S(\mathbb{R}^2)$ is the space of Schwartz functions (smooth and rapidly decreasing functions on $\mathbb{R}^2$) and $\star_M$ is the Moyal star product. The corresponding spectral triple using Moyal star product for the Moyal plane has been obtained in [75]:

$$(\mathcal{A}_{\star_M} = (S(\mathbb{R}^2), \star_M)\ ,\ \ \mathcal{H}_{\star_M} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2\ ,\ \ \mathcal{D}_{\star_M} = -i\sigma_\alpha \partial_\alpha), \tag{1.27}$$

where $\mathcal{D}_{\star_M}$ is the Dirac operator of ordinary Euclidean plane $\mathbb{R}^2$ as this spectral triple (1.27) is just the isospectral deformation of the commutative spectral triple of $\mathbb{R}^2$ [76]. Then the Connes' spectral distance on the Moyal plane is computed in [77, 78] using the spectral triple (1.27). As we have already mentioned that the different choices of star products may lead to different results, the algorithm (adaptable to the framework of Hilbert-Schimdt operator formalism of non-commutative quantum mechanics) of computing infinitesimal Connes' spectral distance between any pair of normal states represented by density matrices given in [79] seem to be a better approach to determine the metric properties of Moyal plane.

The algorithm of finding Connes' spectral distance given in [79] defines the distance between pure density matrices which represent the pure normal states so that the generalization to mixed states can be easily made. The infinitesimal distance between two neighbouring pure density matrices using the simplified formula given in [79] has the same structure as that of the induced metric from the Hilbert space inner product obtained in [80]. Further, in this approach [79], vectors $|\psi\rangle$'s on the classical configuration space $\mathcal{H}_c$ have one-to-one correspondence with the corresponding pure density matrices $\rho_c = |\psi\rangle\langle\psi|$ acting on $\mathcal{H}_c$. Since the quantum Hilbert space $\mathcal{H}_q$ has the tensor product structure $\mathcal{H}_c \otimes \mathcal{H}_c^*$, the elements of $\mathcal{H}_q$ can have either diagonal form $|\psi,\psi) \equiv ||\psi\rangle\langle\psi|)$[7] or off-diagonal form $|\psi,\phi) \equiv ||\psi\rangle\langle\phi|)$. This implies that there is one-to-one correspondence of diagonal states $|\psi,\psi)$ on $\mathcal{H}_q$ with the pure density matrices $\rho_q = |\psi,\psi)(\psi,\psi|$ acting on $\mathcal{H}_q$ and also with the states $|\psi\rangle \in \mathcal{H}_c$ but the off-diagonal states $|\psi,\phi)$ on $\mathcal{H}_q$ have one-to-one correspondence with the pure density matrices $\rho'_q = |\psi,\phi)(\psi,\phi|$ but many-to-one correspondence with states $|\psi\rangle, |\phi\rangle \in \mathcal{H}_c$. This feature of non-commutative quantum mechanics doesn't have commutative counterparts and hence makes the quantum Hilbert space interesting. This amounts to the presence of additional degrees of freedom on the quantum Hilbert space $\mathcal{H}_q$ of a non-commutative system [81].

7 We denote element of $\mathcal{H}_c$ by angle ket $|\psi\rangle$ and that of $\mathcal{H}_q$ by round ket $|\psi)$

We can briefly describe this additional degrees of freedom by considering states which have off-diagonal form, particularly, $|z,\phi) = |z\rangle\langle\phi|$, where $\langle\phi|$ is arbitrary. For such states, we have $\langle\hat{X}_1\rangle_{|z,\phi)} = \sqrt{\frac{\theta}{2}}(z+\bar{z})$ $\left(z = \frac{1}{\sqrt{2\theta}}(x_1+ix_2)\right.$ which indicates the insensitivity of position measurements on the right sector of the position eigenstates $|z,\phi)$. Moreover, as shown in [81] this variable ϕ cahracterizing the right sector of the position eigenstates $|z,\phi)$ can be used to decompose the Voros star product as follows:

$$|z) \star_V (z| = \sum_{\phi} |z,\phi)(z,\phi|. \tag{1.28}$$

Thus, the additional structure of non-commutative quantum system is manifested in this operator formalism.

In [79], computation of infinitesimal Connes' spectral distance between mixed states $\rho_q(z) = \sum_{n=0}^{\infty}|z,n)(z,n|$ representing the position of a particle localized at a generalized point (x_1,x_2)) on the quantum Hilbert space of Moyal plane is done using the simplified formula. The effect of additional degrees of freedom shows up in this spectral distance on $\mathcal{H}_q$ which manifests a connection between the statistics and geometry on Moyal plane. This feature has been checked on fuzzy sphere [82] showing that this is a generic feature of non-commutative spaces.

However, this algorithm works for only infinitesimal distance between discrete normal states and give the exact infinitesimal distance between coherent states up to a numerical factor. Besides, in the absence of any geodesic in a generic non-commutative space, the infinitesimal distance cannot simply be integrated to compute finite distance. This motivates us to improve the algorithm [79] of computing finite Connes' spectral distance for any pair of normal states without using any star products and check the interesting consequences on quantum Hilbert space of fuzzy sphere. Thus, we extend this algorithm to compute finite distance Connes' spectral distance between any pair of normal states on non-commutative spaces, both for Moyal plane and fuzzy sphere.

Fuzzy sphere was first introduced in [83]. It can be understood by studying the algebraic properties of the commutative 2-sphere $\mathbb{S}^2$. Let $C(\mathbb{S}^2)$ be the algebra of complex-valued functions on $\mathbb{S}^2$ where each element $f(x^i)$ has a polynomial expansion in x^i:

$$f(x^i) = f_0 + f_i x^i + \frac{1}{2} f_{ij} x^i x^j + \ldots.. \;; \quad \text{where} \;\; f_{ijk\ldots} \;\; \text{are some constants.} \tag{1.29}$$

Here, one can construct a sequence of approximations to function $f(x^i) \in C(\mathbb{S}^2)$ and such approximations will yield the fuzzy sphere. For example, if we truncate (1.29) to constant term f_0 the algebra $C(\mathbb{S}^2)$ reduces to the algebra $\mathcal{A}_1 = \mathbb{C}$ yielding the geometry of a single point. The truncation of (1.29) after f_i term can lead us to two ways of getting the algebra $\mathcal{A}_2$, which is a $4D$-vector space, from $C(\mathbb{S}^2)$. If we choose $\mathcal{A}_2 = \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$, this algebra is commutative and it will describes the geometry of sphere with four points where there is no rotational invariance. But to keep the rotational invariance, we can choose the algebra $\mathcal{A}_2 = M_2(\mathbb{C})$, the algebra of complex 2×2 matrices. Clearly, this algebra $M_2(\mathbb{C})$ is non-commutative and describes the geometry of sphere with two points: North pole and South pole. This is the fuzzy sphere with maximal fuzziness but with rotational symmetry. The next truncation of (1.29) after f_{ij} term gives a set of functions $\mathcal{A}_3$ which is nine-dimensional vector space because of the constraint $x^i x_i = r^2$. One can define a new product in x^i such that we get the algebra $\mathcal{A}_3 = M_3(\mathbb{C})$ [83]. This algebra describes the fuzzy sphere where we can distinguish three points: Equator, North pole and South pole. In this way, we can get a series of algebras $\mathcal{A}_{2j+1}$ with $j \in \mathbb{Z}^+/2$ describing a series of fuzzy spheres $\{\mathbb{S}^2_j\}$ such that the commutative sphere $\mathbb{S}^2$ is recovered when $j \to \infty$.

Note that the quantization of commutative 2-sphere $\mathbb{S}^2$ i.e. fuzzy sphere $\mathbb{S}^2_\star$ (1.19) can be represented from the phase-space formulation of quantum mechanics for spinning particles [84]. In the ordinary formulation of quantum mechanics, the j-spin is represented by operators acting on Hilbert space $\mathbb{C}^{2j+1}$, $j \in \mathbb{Z}^+/2$. We know that $\mathbb{S}^2$ is an orbit of $SU(2)$ and $\mathbb{C}^{2j+1}$ carries an irreducible representation of $SU(2)$ such that an extended Wigner-Weyl correspondence [85] can be established between operators acting on $\mathbb{C}^{2j+1}$ and functions on $\mathbb{S}^2$ just like in case of Moyal plane. Similarly, here we can consider two bases on $\mathbb{C}^{2j+1}$: the eigenvectors $|j, m\rangle$ of the Casimir operator $\hat{\vec{x}}^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2$ and $\hat{x}_3$ which we will refer to as the discrete basis of fuzzy sphere; and the coherent states of $SU(2)$ [86], referred to as the continuous basis of fuzzy sphere. We will study the properties of the fuzzy sphere using these two type of bases.

In order to define a legimate spectral triple on fuzzy sphere, we need to define the Dirac operator for fuzzy sphere. Fuzzy sphere is the isospectral deformation of commutative 2-sphere [76] which can be understood from the extended Wigner-Weyl correspondence. Using the coherent states of $SU(2)$, i.e. the continuous basis of fuzzy sphere, the construction of non-commutative manifold as a non-commutative analog of homogeneous space[8] was obtained in [87]. Moreover, the Dirac operator of fuzzy sphere was obtained in [88] by defining the fuzzy analog of spinor bundle over 2-sphere. Then a legimate spectral triple on fuzzy sphere is given in [89] where the metric properties of fuzzy sphere are studied using Connes' spectral distance formula. However, the computation of Connes' spectral distance is generally very involved. So we would like to first modify the simpler algorithm [79] to enable us to compute finite Connes' spectral distance between normal states on fuzzy sphere which is adaptable to the Hilbert-Shmidt operator formalism of non-commutative quantum mechanics.

Once again we should note that on non-commutative spaces, the notion of points is served by the pure states of the algebra in the spectral triple and the role of geodesic distance is played the Connes' spectral distance. And the integration of infinitesimal Connes' spectral distance between continuous pure states may/may not yield the finite Connes' spectral distance as interpolating states may turn out to be mixed, rather than pure. Consequently, the notion of classical geodesic may not survive in non-commutative spaces.

8 A space X with transitive group action by a Lie group G, i.e. for any $x, y \in X$ there exists $g \in G$ such that $gx = y$.

1.3 PLAN OF THE THESIS

The thesis is divided into two parts after the introduction. The part I deals with the non-commutative quantum mechanics studied in a pure operator method called the Hilbert-Schmidt operator formalism where the use of any star product is avoided. The part II deals with the studies of metric properties of non-commutative spaces using the power of spectral triple method of non-commutative geometry which is adapted to the above mentioned operator formalism of non-commutative quantum mechanics.

In chapter 2 of part I, we have reviewed the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics for Moyal plane, 3D Moyal space and fuzzy sphere where necessary distinctions between the classical configuration space and quantum Hilbert space are made. The difference of two position bases corresponding to two choices of star products viz Moyal and Voros star products on Moyal plane and 3D Moyal space is highlighted.

In chapter 3 of part I, we have first reviewed the symplectic invariant uncertainty relation in $2n$-dimensional phase space. Then, we have computed the non-commutative variance matrix in the Voros position basis on 6D phase space for 3D Moyal space. By transforming it into the corresponding commutative variance matrix using the relation (1.22), we have shown that the Voros position basis on 3D Moyal space is maximally localized on 6D phase space but not in 3D coordinate space as there exists non-Voros states for which the uncertainty in position coordinates is lower than that one computed using Voros states.

In chapter 4 of part I, the extension of operator method of non-commutative quantum mechanics to many-particle system has been discussed. Due to the deformation of co-product (1.10) acting on the multi-particle states under the symmetry transformation of 3D Moyal space, the notion of twisted bases and the corresponding twisted statistics arise. This leads to the violation of Pauli exclusion principle for twisted fermions which can be understood by computing the two-particle correlation functions of canonical free gas. However, one can also construct a quasi-commutative basis which recovers the Pauli exclusion principle, along with rotational $SO(3)$ symmetry.

Chapter 5 of part II gives a brief review of spectral triple formalism of non-commutative geometry. We have provided a brief summary of spin geometry of a compact Riemannian manifold discussed extensively in [90, 91]. We have also reviewed the computation of Connes' spectral distance on a compact Riemannian spin manifold which exactly gives the geodesic distance. Also, we have given some examples of computation for spaces like Real line $\mathbb{R}$, two-point space which consists of only two points and also $\mathbb{CP}^1$, the space of states of complex 2×2 matrices. Moreover, here we have reviewed the construction of coherent states for Moyal plane and fuzzy sphere as we will compute the spectral distances on their respective homogeneous spaces.

In chapter 6 of part II, we have re-visted the spectral triple of Moyal plane defined in [79], adaptable to the Hilbert-Schmidt operator formalism. Here, we have obtained the most generalized Connes' distance formula for any pair of normal states. However, this formula is not user-freindly except for some cases. So, we follow up the procedures given in [77, 78] to compute the finite spectral distance in Moyal plane. In [78], the spectral distance between a pair of coherent states on Moyal plane is found to be bounded above by the geodesic distance on the complex plane. An optimal element a_s which saturates this bound is found to be one which does not belong to the algebra but which is a limit point of a sequence of algebra elements and can be regarded as an element belonging to the multiplier algebra. In our approach, we have obtained such a sequence of elements as projected elements on the representation space of the aglebra which is spanned by the eigenspinors of Dirac operator. Such projected elements are trace-class operators and satisfy the ball condition. Hence, we find that the spectral distance between a pair of coherent states on Moyal plane is the geodesic distance on the complex plane. Moreover, we can get the finite distance by integrating the infinitesimal ones and thus the notion of conventional geodesic still exists [92]. Further, we have reviewed the infinitesimal spectral distance between discrete mixed states on the quantum Hilbert space of Moyal plane, obtained in [79], which reveals a connection between statistics and geometry.

Chapter 7 of part *II* deals with the computation of Connes' spectral distance on fuzzy sphere $\mathbb{S}^2_j$ with a fixed radius $r_j = \theta_f \sqrt{j(j+1)}$. For these, we have followed up the same prescription done for Moyal plane. However, it is worth noting that the computation of operator norm, which gives the ball condition, using the eigenspinors of Dirac operator gets very much simplified here. We have obtained both the infinitesimal and finite Connes' spectral distances between any pair of discrete states on the configuration space of fuzzy sphere $\mathbb{S}^2_j$. We have found that the distance between north and south poles $\mathbb{S}^2_j$ reduces to that of $\mathbb{S}^2$ only in the limit $j \to \infty$ [82, 89, 92]. This gives us a hint that the spectral distance between a pair of coherent states on the homogeneous space of fuzzy sphere $\mathbb{S}^2_j$ will be always bounded above by the geodesic distance on $\mathbb{S}^2$. For any finite j, this bound is never going to saturate. Even though we cannot obtain an exact distance for a pair of coherent states on fuzzy sphere $\mathbb{S}^2_j$, we can give a lower bound to the spectral distance between a pair of infinitesimally separated coherent states on $\mathbb{S}^2_j$. Moreover, we get the exact distance on fuzzy sphere $\mathbb{S}^2_{\frac{1}{2}}$ with maximal fuzziness. Here, we try to provide the spectral distance between a given mixed state with its nearest pure state as an alternative measure of "mixedness" of a state. Further, we have obtained an analytic estimate of spectral distance a pair of coherent states on fuzzy sphere $\mathbb{S}^2_1$ under some constraints and we have found that this result exaclty agrees with the numerical results obtained by using the generalized Connes' distance formula up to 6th decimal places.

Lastly, we provide the conclusion of the thesis in chapter 8. Also, we put the necessary definitions in the appendices A, B, C and some relevant reviews like construction of Dirac operators on Moyal plane and fuzzy sphere in the appendix D.

Part I

Non-relativistic quantum physics on Non-commutative spaces

2

HILBERT-SCHMIDT OPERATOR FORMALISM

In this chapter, we review the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics on non-commutative spaces. This formalism is a pure operatorial formalism and is independent of any star product which usually plays an important role in a non-commutative space. However, different star products can be related to different position representations defined in this operatorial formalism and give rise to some in-equivalent physics [35, 36]. This formalism without star product is therefore necessary in order to avoid those ambiguities arise from using different star products.

2.1 2D MOYAL PLANE $\mathbb{R}^2_\star$ [44, 45]

A Moyal plane is a 2D noncomutative space with the commutation relation: $[\hat{x}_1, \hat{x}_2] = i\theta$, where θ is the non-commutative parameter. Taking analogy with the 1D quantum harmonic oscillator where $[\hat{x}, \hat{p}] = i\hbar$, we can define the following creation and annihilation operators:

$$\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2) \quad \text{and} \quad \hat{b}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2) \quad \text{with} \quad [\hat{b}, \hat{b}^\dagger] = 1. \tag{2.1}$$

The classical configuration space for the Moyal plane $\mathbb{R}^2_\star$, which is usually $\mathbb{R}^2$ for usual commutative quantum mechanics on $\mathbb{R}^2$, is now a Hilbert space itself and can be obtained as the boson Fock space:

$$\mathcal{H}_c = \text{span}\left\{ |n\rangle = \frac{(\hat{b}^\dagger)^n}{\sqrt{n!}}|0\rangle \right\}_{n=0}^{\infty}; \quad \text{where the span is over } \mathbb{C}. \tag{2.2}$$

This just furnishes a representation of the coordinate algebra (1.17). In the ordinary formalism of quantum mechanics, the physical states of a quantum system is described by square-integrable functions $\psi(x, y) \in \mathcal{L}^2(\mathbb{R}^2)$. We can generalize this to non-commutative quantum mechanics so that Hilbert-Schmidt operators acting on non-commutative configuration space $\mathcal{H}_c$ will represent the physical states of the quantum system. The quantum Hilbert space can thus be constructed as

$$\mathcal{H}_q = \left\{ |\psi(\hat{x}_\alpha)) = \sum_{m,n} \psi_{mn} |m\rangle\langle n| \in \mathcal{H}_c \otimes \mathcal{H}_c^* \middle| \, \mathrm{tr}_c(\psi^\dagger \psi) < \infty \right\}, \tag{2.3}$$

with the inner product

$$(\psi|\phi) = \mathrm{tr}_c(\psi^\dagger \phi) \ , \quad \text{where} \ \ \mathrm{tr}_c \ \ \text{implies trace over} \ \ \mathcal{H}_c \ . \tag{2.4}$$

Note that by definition, Hilbert-Schmidt operators are bounded and trace-class (see appendix C). Moreover, we denote $\dagger$ for Hermitian conjugation on $\mathcal{H}_c$ and $\ddagger$ for Hermitian conjugation on $\mathcal{H}_q$.

We can then construct a unitary representation of non-commutative Heisenberg algebra:

$$[\hat{X}^1, \hat{X}^2] = i\theta \ , \quad [\hat{X}^\alpha, \hat{P}_\beta] = i\delta^\alpha_\beta \ , \quad [\hat{P}_\alpha, \hat{P}_\beta] = 0 \ ; \ \ (\hbar = 1) \ , \tag{2.5}$$

on $\mathcal{H}_q$, analogous to the usual Scrödinger representation as follows:

$$\hat{X}_\alpha |\psi) = |\hat{x}_\alpha \psi); \quad \hat{P}_\alpha |\psi) = \frac{1}{\theta} \epsilon_{\alpha\beta} \, |[\hat{x}_\beta, |\psi)]). \tag{2.6}$$

Here, $\hat{X}_\alpha$'s are the observables representing the position coordinates of a quantum particle moving in a Moyal plane while $\hat{P}_\alpha$'s are the observables representing the momenta of the particle. Note that $\mathcal{H}_c$ just furnishes a representation of the coordinate algebra $[\hat{x}_1, \hat{x}_2] = i\theta$, which cannot furnish a representation of the entire non-commutative Heisenberg algebra (2.5), as an adjoint action required for $\hat{P}_\alpha$ cannot be defined there.

Since the momentum operators $\hat{P}_\alpha$ commute, we can find its simultaneous eigenstates which are orthonormal and complete:

$$|\vec{p}) = \sqrt{\frac{\theta}{2\pi}} \, e^{i\vec{p}.\hat{\vec{x}}}; \ \ \hat{P}_i |\vec{p}) = p_i |\vec{p}), \quad (\vec{p}|\vec{p}') = \delta^2(\vec{p} - \vec{p}') \ \ \text{and} \ \ \int d^2p |\vec{p})(\vec{p}| = 1_q. \tag{2.7}$$

Note that we denote operators acting on $\mathcal{H}_c$ by small letters $\hat{b}, \hat{b}^\dagger$ with the corresponding operators on $\mathcal{H}_q$ denoted by capital letters $\hat{B}, \hat{B}^\ddagger$ as

$$\hat{B}/\hat{B}^\ddagger |\psi) = |\hat{b}/\hat{b}^\ddagger \psi) \ ; \quad \text{where} \ \ \hat{B}^\ddagger = \frac{1}{\sqrt{2\theta}} (\hat{X}_1 - i\hat{X}_2), \ \ \hat{B} = \frac{1}{\sqrt{2\theta}} (\hat{X}_1 + i\hat{X}_2). \tag{2.8}$$

With this, we can define the complex momentum operators as

$$\hat{P} = \hat{P}_1 + i\hat{P}_2 \quad \text{and} \quad \hat{P}^\ddagger = \hat{P}_1 - i\hat{P}_2 \ \ \text{such that} \ \ \hat{P}^2 = \hat{P}_1^2 + \hat{P}_2^2 = \hat{P}^\ddagger \hat{P} = \hat{P}\hat{P}^\ddagger, \tag{2.9}$$

whose actions on $\mathcal{H}_q$ are given by

$$\hat{P}|\psi) = -i\sqrt{\frac{2}{\theta}} \, |[\hat{b}, |\psi)]) \quad \text{and} \quad \hat{P}^\ddagger |\psi) = i\sqrt{\frac{2}{\theta}} \, |[\hat{b}^\dagger, |\psi)]). \tag{2.10}$$

2.1.1 *Position bases*

Due to the non-commutativity of the position operators $\hat{X}_\alpha$, we have the following uncertainty relation:

$$\Delta\hat{X}_1\Delta\hat{X}_2 \geq \frac{\theta}{2}\,, \quad \text{where} \quad \Delta\hat{X}_\alpha = \sqrt{(\psi|\hat{X}_\alpha^2|\psi) - (\psi|\hat{X}_\alpha|\psi)^2}. \tag{2.11}$$

This implies a precise measurement of $\hat{X}_1$ leads to total uncertainty in $\hat{X}_2$ or vice versa. However, we can preserve the standard notion of position by considering states which saturate the above uncertainty relation (2.11). Such states can be constructed out of the normalized coherent states of $\mathcal{H}_c$:

$$|z\rangle = e^{-z\bar{z}/2}\, e^{z\hat{b}^\dagger}|0\rangle \in \mathcal{H}_c\,, \quad \hat{b}|z\rangle = z|z\rangle\,; \quad \text{note that (1.14) define } z, \bar{z}. \tag{2.12}$$

Corresponding to each $|z\rangle \in \mathcal{H}_c$, we have a quantum state $|z) = |z\rangle\langle z| \in \mathcal{H}_q$ which provide an overcomplete and non-orthogonal basis on $\mathcal{H}_q$, iff they are composed with a Voros star product:

$$\frac{1}{\pi}\int d^2z|z)\, e^{\overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_z}(z| \equiv \frac{1}{\pi}\int d^2z|z) \star_V (z| = 1_q \quad \text{and} \quad (z_1|z_2) = e^{-|z_1-z_2|^2}. \tag{2.13}$$

Here $\star_V$ is the Voros star product (1.12) and we refer to this state $|z) = |\vec{x})_V$ as the Voros position basis of $\mathcal{H}_q$. We can easily check that on such states $|z) \in \mathcal{H}_q$, we have

$$\Delta\hat{X}_1\Delta\hat{X}_2 = \frac{\theta}{2}\,, \tag{2.14}$$

saturating the uncertainty relation. Note that we can define un-normalized projection operators $\pi_z = \frac{1}{2\pi\theta}|\vec{x})_V *_V {}_V(\vec{x}|$,

$$\pi_z = \frac{1}{2\pi\theta}|z) *_V (z| = \frac{1}{2\pi\theta}\sum_\phi |z,\phi)(z,\phi| \quad ; \quad \pi_z^2 \propto \pi_z \tag{2.15}$$

which are positive i.e. $(\psi|\pi_z|\psi) \geq 0\;\forall\;|\psi) \in \mathcal{H}_q$ and form a complete set, i.e. $\int dx_1 dx_2 \pi_z = 1_q$. As π_z is regarded as a mixed state with equal apriori probability, which is independent of $|\phi\rangle$ in (1.28),(2.15). This forces us to relax the strong measurement *a la* von Neumann to a weak one, as these operators provide a Positive Operator Valued Measure (POVM) so that we can provide a consistent probability interpretation by assigning the probability of finding the outcome of a position measurement to be (x_1, x_2). For example, if the system is in a pure state $\Omega = |\psi)(\psi|$, then

$$P(x_1,x_2) = \mathrm{tr}_q(\pi_z\Omega) = \frac{1}{2\pi\theta}\sum_\phi(\psi|z,\phi)(z,\phi|\psi) = \frac{1}{2\pi\theta}\sum_\phi|(z,\phi|\psi)|^2. \tag{2.16}$$

This clearly goes into the corresponding commutative result in the limit $\theta \to 0$. The overlap of Voros position basis $|\vec{x})_V$ with the momentum eigenstate takes the form

$$_V(\vec{x}|\vec{p}) \equiv (z|\vec{p}) = \sqrt{\frac{\theta}{2\pi}}\, e^{-\frac{\theta p^2}{4}}\; e^{i\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)} = \sqrt{\frac{\theta}{2\pi}}\, e^{-\frac{\theta p^2}{4}}\; e^{i\vec{p}.\vec{x}}\;; \tag{2.17}$$

with $p = p_1 + ip_2$ and $p^2 = |\vec{p}|^2 = (p_1)^2 + (p_2)^2$. Then the expansion of Voros position basis in terms of momentum eigenstates can be written as

$$|\vec{x})_V = \sqrt{\frac{\theta}{2\pi}} \int d^2p\; e^{-\frac{\theta p^2}{4}}\; e^{-i\vec{p}.\vec{x}}\; |\vec{p}) = \int \frac{d^2p}{2\pi}\, \theta\, e^{-\frac{\theta p^2}{4}}\; e^{i\vec{p}.(\hat{\vec{x}}-\vec{x})}. \tag{2.18}$$

One can also introduce a Moyal position basis $|\vec{x})_M$ [38] which can be expanded as

$$|\vec{x})_M = \int \frac{d^2p}{2\pi}\, e^{-i\vec{p}.\vec{x}}|\vec{p}) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{2\pi}\, e^{i\vec{p}.(\hat{\vec{x}}-\vec{x})}. \tag{2.19}$$

This basis form a complete orthonormal basis as

$$\int d^2x\; |\vec{x})_M *_M {}_M(\vec{x}| = \int d^2x\; |\vec{x})_{MM}(\vec{x}| = 1_q\;, \quad {}_M(\vec{x}|\vec{x}')_M = \delta^2(\vec{x}-\vec{x}') \tag{2.20}$$

and its overlap with the momentum basis is found to be

$$(\vec{p}|\vec{x})_M = \frac{1}{2\pi} e^{-i\vec{p}.\vec{x}}. \tag{2.21}$$

These Moyal position basis states are found to be the simultaneous eigenstates for the commuting unphysical position operators: $\hat{X}^c_\alpha\; |\vec{x})_M = x_\alpha |\vec{x})_M$, where $\hat{X}^c_\alpha$ is defined in (1.21). However, these states do not conform to the POVM [38]. All these facts suggest that Moyal basis is just a mathematical construction and is devoid of any physical significance. Nevertheless, we can check that in the commutative limit $\theta \to 0$, the difference between these two position bases disappear as

$$_V(\vec{x}'|\vec{x})_M = \sqrt{\frac{2}{\pi\theta}}\, e^{-\frac{(\vec{x}'-\vec{x})^2}{\theta}} \xrightarrow{\theta\to 0} \delta^2(\vec{x}'-\vec{x}). \tag{2.22}$$

Note that the position representation of a state $|\psi) \in \mathcal{H}_q$ is simply either $_V(\vec{x}|\psi) \in \mathbb{C}$ for Voros basis or $_M(\vec{x}|\psi) \in \mathbb{C}$ for Moyal basis. At this stage only, we can make the comparision of the non-commutative features with the corresponding commutative features. Moreover, the appearance of star products can be understood by imposing an algebra structure on the quantum Hilbert space $\mathcal{H}_q$. We can define a multiplication map $m : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q$:

$$m(|\psi \otimes |\phi)) = |\psi\phi)\;, \tag{2.23}$$

in the sense of usual operator product. By expanding any generic state $|\psi) \in \mathcal{H}_q$ in terms of the momentum eigenstates as

$$|\psi) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{2\pi}\, \psi(\vec{p})\, e^{i\vec{p}.\hat{\vec{x}}}, \text{ where } \psi(\vec{p}) \in \mathcal{L}^2(\mathbb{R}) \text{ as } (\psi|\psi) = \mathrm{tr}_c(\psi^\dagger\psi) < \infty, \tag{2.24}$$

we can expand a product state $|\psi\phi) \in \mathcal{H}_q$ as

$$|\psi\phi) = \frac{\theta}{2\pi} \int\int \frac{d^2p\, d^2p'}{(2\pi)^2}\, \psi(\vec{p})\phi(\vec{p}')\, e^{i(\vec{p}+\vec{p}').\hat{\vec{x}}}\, e^{-\frac{i}{2}p_i p'_j \theta\epsilon_{ij}}. \tag{2.25}$$

Then the following composition rules implemented by the following star products can be easily verified

$$_M(\vec{x}|\psi\phi) = \sqrt{2\pi\theta}\; _M(\vec{x}|\psi) *_M {}_M(\vec{x}|\phi) \text{ and } _V(\vec{x}|\psi\phi) = 4\pi^2\; _V(\vec{x}|\psi) *_V {}_V(\vec{x}|\phi), \tag{2.26}$$

where

$$_M(\vec{x}|\psi) = \int \frac{d^2p}{(2\pi)^2}\, \psi(\vec{p})\, e^{i\vec{p}.\vec{x}} \text{ and } _V(\vec{x}|\psi) = \sqrt{\frac{\theta}{2\pi}} \int \frac{d^2p}{(2\pi)^2}\, \psi(\vec{p})\, e^{-\frac{\theta p^2}{4}} e^{i\vec{p}.\vec{x}} = \sqrt{\frac{\theta}{2\pi}}\, e^{\frac{\theta\nabla^2}{4}}\; _M(\vec{x}|\psi). \tag{2.27}$$

This reflects the fact that wave function representation of the composite state $|\psi\phi)$ in Moyal/Voros basis is obtained by composing the representations of individual states $|\psi)$ and $|\phi)$ using Moyal/Voros star product. Furthermore, the connection between Voros representation and Moyal representation of a state $|\psi) \in \mathcal{H}_q$ is through the T-map (1.15) which is invertible only for a class of Schwartz functions which are smooth on small length scales $\lesssim \sqrt{\theta}$. The Hilbert space of functions in Voros representation consists of all that satisfy smoothness condition at this scale $\sim \sqrt{\theta}$ and any oscillations with wavelength smaller than this is automatically suppressed exponentially. The wave functions in this representation therefore capture the noncommutative features correctly in contrast to the representation in the Moyal basis. In this sense, we can say that it is the Voros basis which serve as the physical framework to describe a localized particle in Moyal plane.

2.2 3 D MOYAL SPACE [47]

The 3D non-commutative Moyal space is such that the three coordinate operators $\hat{x}^i$ with $i = 1, 2, 3$ satisfy the folowing algebra:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} = i\epsilon^{ijk}\theta_k; \quad \text{where} \quad \theta_k = \frac{1}{2}\epsilon_{ijk}\theta^{ij} \text{ is a vector dual to } \theta^{ij}. \tag{2.28}$$

We can always formally choose a suitable $\bar{R} \in SO(3)$ rotation [47] such that $\hat{x}^i \to \hat{\bar{x}}^i = \bar{R}^i_j \hat{x}^j$ where

$$\bar{R} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta\cos\alpha & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{pmatrix} \; ; \quad \text{with} \quad \vec{\theta} = \theta \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix} . \tag{2.29}$$

After this rotation, the coordinate algebras reduce to the following with the non-commutative matrix in barred frame as

$$[\hat{\bar{x}}^1, \hat{\bar{x}}^2] = i\theta \, , \quad [\hat{\bar{x}}^\alpha, \hat{\bar{x}}^3] = 0 \, ; \quad \text{with} \;\; \bar{\Theta} = \bar{R}\Theta\bar{R}^T = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.30}$$

It should be emphasised at this stage that the original $\Theta = \{\theta^{ij}\}$ matrix is a constant under rotation [47]; indeed the deformed co-product (see (2.57) below) is precisely employed to restore automorphism symmetry which in turn, ensures the constancy of Θ. Therefore there does not exist any rotated frame, for which the noncommutative matrix is given by $\bar{\Theta}$ matrix (2.30). It is only for simplified construction of quantum Hilbert space that such a formal rotation $\bar{R}$ (2.29) has been introduced. Clearly, the $\hat{\bar{x}}^3$ coordinate essentially becomes commutative here so that we can construct the auxilliary Hilbert space as the configuration space for 3D Moyal space as a tensor product space of Moyal plane $\mathcal{H}_c$ and a Hilbert space spanned by the eigenstates of $\hat{\bar{x}}^3$ i.e. $|\bar{x}^3\rangle$ $\left(\hat{\bar{x}}^3|\bar{x}^3\rangle = \bar{x}^3|\bar{x}^3\rangle\right)$:

$$\mathcal{H}_c^3 = \text{span}\{|n, \bar{x}^3\rangle = |n\rangle \otimes |\bar{x}^3\rangle\} \; ; \tag{2.31}$$

where n labels eigenstates of $b^\dagger b$ as in (2.2) and $\bar{x}^3$ labels eigenstates of $\hat{\bar{x}}^3$. The action of the original (unbarred) coordinates $\hat{x}^i$ on these basis states can be obtained as

$$\hat{x}^i|n, \bar{x}^3\rangle \;\; = \;\; \{(\bar{R}^{-1})^i_j \hat{\bar{x}}^j\} \; |n, \bar{x}^3\rangle = (\bar{R}^{-1})^i_\alpha \hat{\bar{x}}^\alpha |n, \bar{x}^3\rangle + (\bar{R}^{-1})^i_3 \bar{x}^3 |n, \bar{x}^3\rangle. \tag{2.32}$$

Likewise, the physical states will be represented by the Hilbert-Schmidt operators i.e., essentially the set of bounded trace-class operator acting on $\mathcal{H}_c^3$:

$$\mathcal{H}_q^3 = \text{span}\left\{|\psi(\hat{\bar{x}}^i)) : \int \frac{d\bar{x}^3}{\sqrt{\theta}} \; \text{tr}'_c \psi^\dagger \psi < \infty\right\}, \tag{2.33}$$

where tr'_c denotes the restricted trace over $\mathcal{H}_c$ of Moyal plane. Since $|\psi(\hat{\bar{x}}^i)) \in \mathcal{H}_q^3$ leave the subspace $\mathcal{H'}_c^3 = \text{span}\{|n, \bar{x}^3\rangle \;:\; \text{fixed } \bar{x}^3\}$ of $\mathcal{H}_c^3$ invariant, the quantum Hilbert space of 3D

Moyal space will be given by the set of Hilbert-Schmidt operators acting on $\mathcal{H}_c^3$ with the constraint $[\hat{\bar{x}}^3, |\psi(\bar{x}^i))] = 0$:

$$\mathcal{H}_q^3 = \text{span}\Big\{|\psi(\bar{x}^i)) \ : \ [\hat{\bar{x}}^3, |\psi(\bar{x}^i))] = 0 \ ; \ \int \frac{d\bar{x}^3}{\sqrt{\theta}} \ \text{tr}_c'\psi^\dagger\psi < \infty\Big\}. \tag{2.34}$$

Clearly, the inner product on $\mathcal{H}_q^3$ is defined as

$$(\phi|\psi) = \text{tr}_c(\phi^\dagger\psi) = \int \frac{d\bar{x}_3}{\sqrt{\theta}} \ \text{tr}_c'(\phi^\dagger\psi). \tag{2.35}$$

In order to define momentum operators $\hat{P}_i$ and its adjoint action on $\mathcal{H}_q^3$, we need to introduce another barred coordinate '$\hat{\bar{x}}^4$' such that $[\hat{\bar{x}}^j, \hat{\bar{x}}^4] = i\theta\delta^{j3}$. That is, $\hat{\bar{x}}^4$ commutes with $\hat{\bar{x}}^1, \hat{\bar{x}}^2$ and is conjugate to $\hat{\bar{x}}^3$ so that $\hat{\bar{x}}^4 = i\theta\epsilon^{43}\frac{\partial}{\partial\hat{\bar{x}}^3}$ where ϵ^{34} is the anti-symmetric tensor in $\hat{\bar{x}}^3 - \hat{\bar{x}}^4$ plane, with $\epsilon^{34} = 1$. With this, we can define the adjoint action of $\hat{\bar{P}}_\mu$ on $\mathcal{H}_q^3$ as

$$\hat{\bar{P}}_\mu|\psi) = \frac{1}{\theta}\Gamma_{\mu\nu}[\hat{\bar{x}}^\nu, |\psi)] \ ; \quad \text{where} \quad \Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.36}$$

Note that by the constraint $[\hat{\bar{x}}^3, |\psi(\bar{x}^i))] = 0$ on $\mathcal{H}_q^3$, clearly $\hat{\bar{P}}_4|\psi) = 0$ so that there are only three non-trivial momenta $\hat{\bar{P}}_i$. The action of momentum operators $\hat{P}_i$ in the original un-barred frame can then be obtained through linearity as

$$\hat{P}_i|\psi) \quad = \quad (\bar{R}^{-1})_i^j\hat{\bar{P}}_j|\psi) = \frac{\hbar}{\theta}(\bar{R}^{-1})_i^j\Gamma_{j\mu}[\hat{\bar{x}}^\mu, |\psi)]. \tag{2.37}$$

With this, we can define the left action of position operators $\hat{X}^i$ on $\mathcal{H}_q^{(3)}$ as

$$\hat{X}^i|\psi) = |\hat{x}^i\psi). \tag{2.38}$$

Thus, the non-commutative Heisenberg algebra on 3D Moyal space has the following form:

$$[\hat{X}^i, \hat{X}^j] = i\theta^{ij} \ , \quad [\hat{X}^i, \hat{P}_j] = i\delta_j^i \ , \quad [\hat{P}_i, \hat{P}_j] = 0. \tag{2.39}$$

The simultaneous eigenstates of the above commuting momentum operators $\hat{P}_i$ are given by

$$|\vec{p}) = \frac{\theta^{\frac{3}{4}}}{2\pi} \ e^{i\vec{p}.\hat{\vec{x}}} = \frac{\theta^{\frac{3}{4}}}{2\pi} \ e^{i\bar{p}_\alpha\hat{\bar{x}}_\alpha}e^{i\bar{p}_3\hat{\bar{x}}_3} \ , \quad p_i\hat{x}^i \ \ \text{is scalar under} \ \ \bar{R} \in SO(3). \tag{2.40}$$

These momentum eigenstates serve as a complete orthonormal basis on $\mathcal{H}_q^{(3)}$

$$(\vec{p}'|\vec{p}) = \frac{\theta^{\frac{3}{2}}}{(2\pi)^2}\,\mathrm{tr}_c(e^{-i\vec{p}.\hat{\vec{x}}}e^{i\vec{p}.\hat{\vec{x}}}) = \int \frac{d\bar{x}_3}{\sqrt{\theta}}\,[e^{-i(\bar{p}_3'-\bar{p}_3)\bar{x}^3}]\,\mathrm{tr}_c'\,[e^{-i(\bar{p}_\alpha'-\bar{p}_\alpha)\hat{\bar{x}}^\alpha}] = \delta^3(\vec{p}'-\vec{p}) \quad (2.41)$$

and

$$\int d^3p\,|\vec{p})(\vec{p}| = 1_q. \quad (2.42)$$

In the complete analogy with Moyal plane the Voros position basis states $|\vec{x})_V$ on $\mathcal{H}_q^{(3)}$ are defined as [47]

$$|\vec{x})_V = \frac{\theta^{\frac{3}{4}}}{\sqrt{2\pi}}\int d^3p\,e^{-\frac{\theta p^2}{4}}e^{-i\vec{p}.\vec{x}}\,|\vec{p}) = \left(\frac{\theta}{2\pi}\right)^{\frac{3}{2}}\int d^3p\,e^{-\frac{\theta p^2}{4}}\,e^{i\vec{p}.(\hat{\vec{x}}-\vec{x})}. \quad (2.43)$$

Note that $p^2 \equiv \vec{p}^2 = (p_1)^2+(p_2)^2+(p_3)^2$. These states satisfy the following similar relations:

$$\int \frac{d^3x}{(2\pi)^2\theta^{\frac{3}{2}}}\,|\vec{x})_V *_V {}_V(\vec{x}| \;=\; 1_q\,, \quad \text{and} \quad {}_V(\vec{x}'|\vec{x})_V = \sqrt{2\pi}\,e^{-\frac{1}{2\theta}(\vec{x}'-\vec{x})^2}, \quad (2.44)$$

with the overlap with the momentum eigenstates being

$${}_V(\vec{x}|\vec{p}) = \frac{\theta^{\frac{3}{4}}}{\sqrt{2\pi}}\,e^{-\frac{\theta p^2}{4}}\,e^{i\vec{p}.\vec{x}}. \quad (2.45)$$

However, there is no obvious connection between the Voros basis $|\vec{x})_V$ and coherent states, as the latter cannot simply be defined in the odd-dimensional case. Thus we investigate whether Voros basis states give minimum uncertainty product between position operators and also between phase space operators in the following chapter 3.

Similarly, the Moyal position basis on 3D Moyal space is introduced as

$$|\vec{x})_M = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}e^{-i\vec{p}.\vec{x}}|\vec{p}). \quad (2.46)$$

They satisfy the completeness and orthonormality relations:

$$\int d^3x\,|\vec{x})_M *_M {}_M(\vec{x}| = \int d^3x\,|\vec{x})_{MM}(\vec{x}| = 1_q \quad \text{and} \quad {}_M(\vec{x}'|\vec{x})_M = \delta^3(\vec{x}'-\vec{x}), \quad (2.47)$$

with its overlap with the monetum eigenstates as

$$(\vec{p}|\vec{x})_M = \frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{-i\vec{p}.\vec{x}}. \quad (2.48)$$

They are the simultaneous eigenstates of the commuting position operators $\hat{X}_i^{(c)}$ (1.22):

$$\hat{X}_c^i|\psi) \equiv \frac{1}{2}[\hat{X}_L^i+\hat{X}_R^i]\,|\psi)\,, \quad \text{with} \quad \hat{X}_L^i|\psi) = |\hat{x}^i\psi) \quad \text{and} \quad \hat{X}_R^i|\psi) = |\psi\hat{x}^i). \quad (2.49)$$

As defined in [38] on $\mathcal{H}_q$, we can impose an additional structure of algebra on the quantum Hilbert space $\mathcal{H}_q^3$ [47] by defining a multiplication map:

$$m(|\psi) \otimes |\phi)) = |\psi\phi). \tag{2.50}$$

We can expand a generic state $|\psi) \in \mathcal{H}_q^3$ in terms of momentum eigenstates and then express in barred frame as

$$|\psi) = \frac{\theta^{\frac{3}{4}}}{2\pi} \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\ \psi(\vec{p})\ e^{ip_i\hat{x}_i} = \frac{\theta^{\frac{3}{4}}}{2\pi} \int \frac{d^3\bar{p}}{(2\pi)^{\frac{3}{2}}}\ \psi(\vec{\bar{p}})\ e^{i\bar{p}_i\hat{\bar{x}}_i} \tag{2.51}$$

Making similar expansion for $|\phi)$ and the product state $|\psi\phi)$, we can compute the composition rules when the product state is represented in the Moyal and the Voros basis:

$$_M(\vec{x}|\psi\phi) = 2\pi\theta^{\frac{3}{4}}\ _M(\vec{x}|\psi) *_M {}_M(\vec{x}|\phi) \quad \text{and} \quad _V(\vec{x}|\psi\phi) =_V (\vec{x}|\psi) *_V {}_V(\vec{x}|\phi) \tag{2.52}$$

with

$$_M(\vec{x}|\psi) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\ \psi(\vec{p})\ e^{ip_ix_i} \quad \text{and} \quad _V(\vec{x}|\psi)\frac{\theta^{\frac{3}{4}}}{\sqrt{2\pi}} \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\ \psi(\vec{p})\ e^{-\frac{\theta p^2}{4}}\ e^{ip_ix_i}. \tag{2.53}$$

Let us now see the effect of $R \in SO(3)$ rotation on a generic state $|\psi) \in \mathcal{H}_q^{(3)}$. The state $|\psi)$ transforms under R as

$$|\psi) \to |\psi^R) = \int d^3p\ \psi(\vec{p})e^{i\vec{p}.(R^{-1}\hat{\vec{x}})} = \int d^3p\ \psi(\vec{p})e^{i(R\vec{p}).\hat{\vec{x}}}, \tag{2.54}$$

and the composite state $|(\psi\phi))$ transforms under R as

$$|(\psi\phi)^R) = U(R)[m(|\psi) \otimes |\phi))] = \int d^3p\ d^3p'\ \psi(\vec{p})\phi(\vec{p}')e^{i\{R(\vec{p}+\vec{p}')\}.\hat{\vec{x}}}e^{-\frac{i}{2}p_ip'_j\theta^{ij}}. \tag{2.55}$$

However, this rotated composite state is found to be inequivalent with the composite of rotated states $|\psi)^R$ and $|\phi)^R$ if we use the undeformed co-product $\Delta_0(R) = U(R) \otimes U(R)$ to implement the rotation on the tensor product state $|\psi) \otimes |\phi) \in \mathcal{H}_q^3 \otimes \mathcal{H}_q^3$ as

$$|\psi^R\phi^R) = m[\Delta_0(R)(|\psi) \otimes |\phi))] = \int d^3p\ d^3p'\ \psi(\vec{p})\phi(\vec{p}')e^{iR(\vec{p}').\hat{\vec{x}}}e^{-\frac{i}{2}(Rp)_i(Rp')_j\theta^{ij}}. \tag{2.56}$$

If we redefine the composite of rotated states $|\psi)^R$ and $|\phi)^R$ using the deformed co-product:

$$\Delta_\theta(R) = F\Delta_0(R)F^{-1} \quad \text{where} \quad F = e^{\frac{i}{2}\theta^{ij}\hat{P}_i \otimes \hat{P}_j}\ , \tag{2.57}$$

we get

$$|\psi^R\phi^R) = m[\Delta_\theta(R)(|\psi) \otimes |\phi))] = \int d^3pd^3p'\psi(\vec{p})\phi(\vec{p}')e^{-\frac{i}{2}\theta^{ij}p_ip'_j}e^{iR(\vec{p}+\vec{p}').\hat{\vec{x}}} = |(\psi\phi)^R). \tag{2.58}$$

Thus, the automorphism symmetry on the quantum Hilbert space $\mathcal{H}_q^3$ is restored by using deformed co-product and hence the symmetry of non-commutative quantum mechanics on 3D Moyal space is captured by the quantum group structure or it is not an exact symmetry but a twisted symmetry. This is due to the fact that the noncommutative Θ matrix is constant (as we can see from (3.26) below) under the action of deformed co-product (2.57). Such twisted symmetry have serious effects on the multi-particle states which have been shown in chapter 4 by computing the two-particle correlation functions for a non-relativistic free gas, at finite temperature.

2.3 FUZZY SPHERE [48]

Here, the position operators $\hat{x}^i$ satisfy the $su(2)$ Lie algebra: $[\hat{x}^i, \hat{x}^j] = i\theta_f \epsilon^{ijk}\hat{x}^k$. By the Jordan-Schwinger map,

$$\hat{x}_i = \hat{\chi}^\dagger \sigma_i \hat{\chi} = \hat{\chi}_\alpha^\dagger \sigma_i^{\alpha\beta} \hat{\chi}_\beta \; ; \quad \text{where } \sigma_i \text{ are the Pauli matrices} \; , \tag{2.59}$$

we can recover the $su(2)$ Lie algebra through a pair of independent harmonic oscillators $\hat{\chi}_\alpha / \hat{\chi}_\alpha^\dagger$ satisfying the following commutation relations:

$$[\hat{\chi}_\alpha, \hat{\chi}_\beta^\dagger] = \frac{1}{2}\theta_f \delta_{\alpha\beta} \; , \quad [\hat{\chi}_\alpha, \hat{\chi}_\beta] = 0 = [\hat{\chi}_\alpha^\dagger, \hat{\chi}_\beta^\dagger] \; ; \qquad \alpha, \beta = 1, 2. \tag{2.60}$$

We can now label a generic harmonic oscillator state by the pair of integers n_1 and n_2 as

$$|n_1, n_2\rangle = \sqrt{\frac{(2/\theta_f)^{n_1+n_2}}{n_1! n_2!}} \chi_1^{\dagger n_1} \chi_2^{\dagger n_2} |0\rangle \; , \quad \text{with} \;\; \hat{N}|n_1, n_2\rangle = \frac{\theta_f}{2}(n_1 + n_2)|n_1, n_2\rangle, \tag{2.61}$$

where

$$\hat{N} = \hat{\chi}_\alpha^\dagger \hat{\chi}_\alpha \; , \quad \text{is the number operator.} \tag{2.62}$$

These states (2.61) are the simultaneous eigenstates of the radius squared operator $\hat{\vec{x}}^2$ and $\hat{x}^3$,

$$\hat{\vec{x}}^2 |n_1, n_2\rangle = \theta_f^2 \, j(j+1)|n_1, n_2\rangle \quad \text{and} \quad \hat{x}_3 |n_1, n_2\rangle = \theta_f \; m|n_1, n_2\rangle \; , \tag{2.63}$$

where $j = \frac{n_1+n_2}{2} \in \mathbb{Z}^+/2$ and $m = \frac{n_1-n_2}{2}$, $-j \leq m \leq j$ so that we can relabel the states as $|n_1, n_2\rangle \equiv |j, m\rangle$. We have the ladder operators $\hat{x}_\pm = \hat{x}^1 \pm i\hat{x}^2$ which satisfy the following relations:

$$[\hat{x}^3, \hat{x}_\pm] \;=\; \pm\theta_f \hat{x}_\pm \; , \quad [\hat{x}_+, \hat{x}_-] = 2\theta_f \hat{x}^3 \; ; \tag{2.64}$$

$$\hat{x}_\pm |j, m\rangle \;=\; \theta_f \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle. \tag{2.65}$$

With this, the classical configuration space $\mathcal{F}_c$ of fuzzy space of type (1.19) can be constructed as

$$\mathcal{F}_c = \text{span}\{|j,m\rangle \mid \forall\, j \in \mathbb{Z}/2,\ -j \leq m \leq j\} \ . \tag{2.66}$$

Each j corresponds to the fixed fuzzy sphere $\mathbb{S}^2_j$ of radius $r_j = \theta_f\sqrt{j(j+1)}$ such that for a fixed j the Hilbert space is restricted to a finite $(2j+1)$-dimensional sub-space

$$\mathcal{F}_j = \text{span}\{|j,m\rangle \ \mid \ j \text{ is fixed}, -j \leq m \leq j\}. \tag{2.67}$$

Then the quantum Hilbert space, in which the physical states are represented, consists, however, of those operators generated by coordinate operators only and, since these commute with the Casimir, the elements of the quantum Hilbert space must in addition commute with the Casimir, i.e., must be diagonal in j. Therefore, the quantum Hilbert space $\mathcal{H}^f_q$ of the fuzzy space of type (1.19) splits into the following direct sum:

$$\mathcal{H}^f_q = \{\Psi \in Span\{|j,m\rangle\langle j,m'|\} : \ \text{tr}_c(\Psi^\dagger\Psi) < \infty\} = \bigoplus_j \mathcal{H}_j \ , \tag{2.68}$$

$$\text{where } \ \mathcal{H}_j = \{\Psi \in Span\{|j,m\rangle\langle j,m|\} \equiv |m,m') : \ \text{tr}_c(\Psi^\dagger\Psi) < \infty \text{ with fixed } j\}, \tag{2.69}$$

represents the quantum Hilbert space of a fuzzy sphere $\mathbb{S}^2_j$ with fixed radius r_j.

The quantum Hilbert space $\mathcal{H}^f_q$ furnishes a unitary representation of analogue of non-commutative Heisenberg algebra (2.5) and given by

$$[\hat{X}^i, \hat{X}^j] = i\theta_f \epsilon^{ijk}\hat{X}^k \ , \quad [\hat{J}^i, \hat{X}^j] = i\epsilon^{ijk}\hat{X}^k \ , \quad [\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{J}^k \ , \tag{2.70}$$

where the linear momentum operators is replaced by the angular momentum operators $\hat{J}_i$ satisfying the $su(2)$ Lie algebra. The position operators $\hat{X}^i$ acts on $\mathcal{H}^f_q$ by left multiplication and the angular momentum operators $\hat{J}^i$ acts adjointly on $\mathcal{H}^f_q$:

$$\hat{X}^i|\psi) = |\hat{x}^i\psi) \quad \text{and} \quad \hat{J}^i|\psi) = \frac{1}{\theta_f}[\hat{x}^i, |\psi)] \ , \quad \forall\ |\psi) \in \mathcal{H}^f_q. \tag{2.71}$$

Note that $\hat{J}^i$ are the non-commutative analogs of the differential operators or more precisely the vector fields on $\mathbb{S}^3$ which are discussed in appendix D.

3

VOROS POSITION BASIS STATES ON 3D MOYAL SPACE

In 2D Moyal plane, the Voros position basis states are the maximally localized states, i.e. given by the normalized coherent states. But in 3D Moyal space, the Voros position basis states have no obvious connection with the coherent states. They are not maximally localized states in 3D configuration space even though they are maximally localized states in 6D phase space. We show this by using the symplectic invariant uncertainty relation in phase space.

3.1 REVIEW OF SYMPLECTIC INVARIANT UNCERTAINTY RELATIONS

The most widely understood uncertainty relation in quantum mechanics is the Robertson uncertainty relation which takes the form:

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\;;\quad \Delta\hat{A} = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2} \tag{3.1}$$

where $\hat{A}$ and $\hat{B}$ denote the quantum Hermitian operators and $\Delta\hat{A}$ is the standard deviation of $\hat{A}$ in a state $|\psi\rangle$. However, we also have the Schrödinger uncertainty relation which has the form:

$$\Delta\hat{A}\,\Delta\hat{B} \geq \sqrt{\left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2}\;, \tag{3.2}$$

where $\{.,.\}$ denotes anti-commutator and $[.,.]$ denotes commutator. The latter (3.2) is the most generalized form of uncertainty relation in quantum mechanics [93] (see appendix B).

3.1.1 *Variance matrix and Williamson theorem*

Let us recast the Schrödinger's uncertainty relation in terms of Variance matrix by renaming the position and momentum operators by a single phase-space operator $\hat{Z}$ such that $\hat{Z}_i = \hat{X}^i_c$ with $i = 1, 2, 3.$ and $\hat{Z}_{i+3} = \hat{P}_i$. Here, we are considering the commutative quantum mechanics with the Heisenberg algebra:

$$[\hat{X}^i_c, \hat{X}^j_c] = 0\;;\quad [\hat{X}^i_c, \hat{P}_j] = i\delta^i_j\;;\quad [\hat{P}_i, \hat{P}_j] = 0\;. \tag{3.3}$$

Then the Schrödinger uncertainty relation for the phase-space operators will be given by

$$\Delta \hat{Z}_a \Delta \hat{Z}_b = \sqrt{(V^c_{ab})^2 + (\Omega^c_{ab})^2} \;\; ; \;\; a, b = 1, 2, \ldots, 6, \tag{3.4}$$

where V^c_{ab} is the ab-th element of the commutative variance matrix V^c:

$$V^c_{ab} = \frac{1}{2}\langle\{\hat{Z}_a, \hat{Z}_b\}\rangle - \langle\hat{Z}_a\rangle\langle\hat{Z}_b\rangle \;\Rightarrow\; V^c = \{V^c_{ab}\} = \begin{pmatrix} V^c_{XX} & V^c_{XP} \\ V^c_{PX} & V^c_{PP} \end{pmatrix} \tag{3.5}$$

and Ω^c_{ab} is the ab-th element of the symplectic matrix Ω^c:

$$\Omega^c_{ab} = \frac{1}{2i}\langle[\hat{Z}_a, \hat{Z}_b]\rangle \Rightarrow \Omega^c = \{\Omega^c_{ab}\} = \frac{1}{2}\begin{pmatrix} \mathbf{0_3} & \mathbb{I}_3 \\ -\mathbb{I}_3 & \mathbf{0_3} \end{pmatrix} \;;\; \mathbb{I}\text{-unit matrix}\;,\; \mathbf{0}\text{-null matrix.} \tag{3.6}$$

Here, the expectation values are taken in a certain state $|\psi\rangle$.

Let us now consider a generalized variance matrix V which is $2n \times 2n$ square matrix corresponding to a $2n$-dimensional phase space (above, we consider $n = 3$). By Williamson's theorem [94], we know that any arbitrary Variance matrix V^c can be brought to a diagonal form by a symplectic transformation i.e., $V^d = SV^cS^T$, where $S \in Sp(2n, R)$ is a symplectic matrix satisfying $\Omega^c = S\Omega^cS^T$. This 'symplectically' diagonalized Variance matrix V^d comprises of the the symplectic eigenvalues $\nu_j/2$ (up to the orderings of ν_j) of V^c, which are at least doubly degenerate:

$$V^d = \text{diag}(\nu_1/2, ..., \nu_n/2, \nu_1/2, ..., \nu_n/2). \tag{3.7}$$

This symplectic spectrum is not equal to the ordinary spectrum but can be obtained through the ordinary spectrum of $|2i\Omega^cV^c|$, as the composite object (Ω^cV^c) undergoes a similarity transformation, if V^c undergoes a symplectic transformation [95]. Correspondingly, the density matrix $\rho = |\psi\rangle\langle\psi|$ transforms a $\rho \rightarrow U(S)\rho U^\dagger(S)$, where $\hat{U}(S)$ is a unitary operator implementing the symplectic transformation. This implies that this diagonal V^d in $2n$-dimensional phase space splits into n-copies of independent 2-dimensional phase space. It is therefore convenient to consider the Schrödinger's uncertainty relation (3.4) for 2D phase space re-written as,

$$\Delta\hat{Z}_\alpha\Delta\hat{Z}_\beta \geq \sqrt{(V^c_{\alpha\beta})^2 + (\Omega^c_{\alpha\beta})^2}\,. \tag{3.8}$$

Identifying $\hat{Z}_1 = \hat{X}^1_c$ and $\hat{Z}_2 = \hat{P}_1$, this inequality is equivalent to $\Delta\hat{X}^{1\,2}_c \geq \frac{\nu_1}{2}$; $\Delta\hat{P}_1^{\,2} \geq \frac{\nu_1}{2}$ whereas $\Delta\hat{X}_1\Delta\hat{P}_1 \geq \frac{1}{2}$. Here we have taken $V^d_{11} = V^d_{22} = \frac{\nu_1}{2}$ and $V^d_{12} = V^d_{21} = 0$ without loss of generality, so that the spread $\Delta\hat{X}_1$ and $\Delta\hat{P}_1$ are equal. Further, we have used the symplectic invariant form of Ω^0 : $\Omega^c_{12} = -\Omega^c_{21} = \frac{1}{2}$ and $\Omega^c_{11} = \Omega^c_{22} = 0$. Compatibility among these three inequalities implies

$$\nu_1 \geq 1 \quad \text{or,} \quad \text{equivalently} \quad \det V^c \geq \frac{1}{4}. \tag{3.9}$$

Thus for a bonafide Variance marix V^c we must have the symplectic spectrum to be such that $\nu_j \geq 1\ \forall j$ or more generally

$$\det V^c \geq \frac{1}{4^n} \tag{3.10}$$

for the general $2n$-dimensional phase space. This provides a symplectic $Sp(2n, R)$ invariant formulation of the uncertainty relation. Finally, note that both Robertson and Schrödinger form of uncertainty relations become equivalent in this diagonal form.

3.2 COMPUTATION OF NON-COMMUTATIVE VARIANCE MATRIX

For a quantum system on non-commutative 3D Moyal space, the above formalism of obtaining a symplectic invariant form of uncertainty relation is not directly applicable as it is not known whether the Williamson's theorem remains valid or not for non-commutative quantum mechanics ($\theta \neq 0$). Here, we denote the single space operator by $\hat{Z}^\theta$:

$$\hat{Z}^\theta_i = \hat{X}^i\ ,\ \ \hat{Z}^\theta_{i+3} = \hat{P}_i\ ; \quad \text{with the commutation relations (2.39) among them.} \tag{3.11}$$

Thus, we denote non-commutative variance matrix by V^θ and the non-commutative symplectic matrix by Ω^θ. Since the commutative position operators $\hat{X}^i_c$ is related to the non-commutative position operators $\hat{X}^i$ by (1.22), we can define the following transformation matrix M:

$$\hat{X}^i_c = M^i_j \hat{X}^j\ ;\ \ \text{where}\ \ M = \begin{pmatrix} \mathbb{I}_3 & M^\theta_3 \\ \mathbf{0}_3 & \mathbb{I}_3 \end{pmatrix},\ \ \text{with}\ \ M^\theta_3 = \begin{pmatrix} 0 & \frac{\theta_3}{2} & -\frac{\theta_2}{2} \\ -\frac{\theta_3}{2} & 0 & \frac{\theta_1}{2} \\ \frac{\theta_2}{2} & -\frac{\theta_1}{2} & 0 \end{pmatrix}. \tag{3.12}$$

With this, the non-commutative variance matrix V^θ and symplectic matrix Ω^θ can be transformed into commutative ones V^c (3.5) and Ω^c (3.6) as follows:

$$V^c = MV^\theta M^T \quad \text{and} \quad \Omega^c = M\Omega^\theta M^T. \tag{3.13}$$

The ab-th element of non-commutative variance matrix V^θ and symplectic matrix Ω^θ have the similar forms as those of commutative ones (3.5) and (3.6):

$$V^\theta_{ab} = \frac{1}{2}\langle\{\hat{Z}^\theta_a, \hat{Z}^\theta_b\}\rangle - \langle\hat{Z}^\theta_a\rangle\langle\hat{Z}^\theta_b\rangle \quad \text{and} \quad \Omega^\theta_{ab} = \frac{1}{2i}\langle[\hat{Z}^\theta_a, \hat{Z}^\theta_b]\rangle. \tag{3.14}$$

We can also rewrite the non-commutative variance matrix V^θ and symplectic matrix Ω^θ

$$V^\theta = \begin{pmatrix} V^\theta_{XX} & V^\theta_{XP} \\ V^\theta_{PX} & V^\theta_{PP} \end{pmatrix} \quad \text{and} \quad \Omega^\theta = -\frac{i}{2}\begin{pmatrix} \langle[\hat{X}_i,\hat{X}_j]\rangle & \langle[\hat{X}_i,\hat{P}_j]\rangle \\ \langle[\hat{P}_i,\hat{X}_j]\rangle & \langle[\hat{P}_i,\hat{P}_j]\rangle \end{pmatrix}, \tag{3.15}$$

where the block matrices of V^θ are given by

$$V^\theta_{XX} = \frac{1}{2}\langle\{\hat{X}_i,\hat{X}_j\}\rangle - \langle\hat{X}_i\rangle\langle\hat{X}_j\rangle \; ; \quad V^\theta_{XP} = \frac{1}{2}\langle\{\hat{X}_i,\hat{P}_j\}\rangle - \langle\hat{X}_i\rangle\langle\hat{P}_j\rangle \tag{3.16}$$

$$V^\theta_{PX} = \frac{1}{2}\langle\{\hat{P}_i,\hat{X}_j\}\rangle - \langle\hat{P}_i\rangle\langle\hat{X}_j\rangle \; ; \quad V^\theta_{PP} = \frac{1}{2}\langle\{\hat{P}_i,\hat{P}_j\}\rangle - \langle\hat{P}_i\rangle\langle\hat{P}_j\rangle. \tag{3.17}$$

In order to study the properties of Voros basis we compute the above expectation values in the physical Voros position basis $|\vec{x})_V$ (2.43):

$$|\vec{x})_V = \frac{\theta^{\frac{3}{4}}}{\sqrt{2\pi}} \int d^3p \; e^{-\frac{\theta p^2}{4}} e^{-i\vec{p}.\vec{x}} \; |\vec{p}) \; ; \quad {}_V(\vec{x}|\vec{x})_V = 1. \tag{3.18}$$

First of all, we compute the block V^θ_{XX}. For that, we compute the expectation values of $\hat{X}_i$ and the composite $\hat{X}_i\hat{X}_j$ in the normalized Voros states (3.18). Since going to the barred frame through $\bar{R} \in SO(3)$ rotation (2.29) minimizes the non-commutativity of 3D Moyal space, the computation of expectation values also get simplified. For example,

$${}_V(\vec{x}|\hat{X}_i|\vec{x})_V = \bar{R}^{-1}_{ij} \; {}_V(\vec{x}|\hat{\bar{X}}_j|\vec{x})_V = \bar{R}^{-1}_{ij} \; {}_V(\vec{x}|\hat{\bar{x}}_j|\vec{x})_V \; ; \quad \hat{\bar{X}}_i|\psi) = |\hat{\bar{x}}_i\psi) \; , \;\; \forall \; |\psi) \in \mathcal{H}^3_q. \tag{3.19}$$

We find that

$${}_V(\vec{x}|\hat{\bar{x}}_j|\vec{x})_V = \frac{\theta^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \int\int d^3p \; d^3p' e^{-\frac{\theta}{4}(\vec{p}^2+\vec{p}'^2)} \; e^{-i(\vec{p}-\vec{p}').\vec{x}} (\vec{p}'|\hat{\bar{x}}_j|\vec{p}) \; , \; \text{with} \tag{3.20}$$

$$(\vec{p}'|\hat{\bar{x}}_\alpha|\vec{p}) = -i\delta(p_3 - p'_3)\delta(p_\beta - p'_\beta) e^{-\frac{\theta}{4}(p_\alpha - p'_\alpha)^2 + \frac{i}{2}\theta\epsilon_{\alpha\beta}(p_\alpha - p'_\alpha)p_\beta} \frac{\partial}{\partial p_\alpha}\delta(p_\alpha - p'_\alpha); \tag{3.21}$$

$$(\vec{p}'|\hat{\bar{x}}_3|\vec{p}) = -i\delta(p_1 - p'_1)\delta(p_2 - p'_2)\frac{\partial}{\partial p_3}\delta(p_3 - p'_3) \tag{3.22}$$

Thus, we obtain

$${}_V(\vec{x}|\hat{\bar{X}}_j|\vec{x})_V = \bar{x}_j \;\Rightarrow\; {}_V(\vec{x}|\hat{X}_i|\vec{x})_V = \bar{R}^{-1}_{ij} \; \bar{x}_j = x_i \; . \tag{3.23}$$

Further, we need to calculate the expectation values of the composite $\hat{X}_i\hat{X}_j$ and then its symmetrized $\langle\{\hat{X}_i,\hat{X}_j\}\rangle$ and anti-symmetrized expectation values $\langle[\hat{X}_i,\hat{X}_j]\rangle$. One should note that the composite $\hat{X}_i\hat{X}_j$ doesn't transform as a second rank tensor under $\bar{R} \in SO(3)$, as was shown in [47]. As already reviewed in chapter 2, if the operator $\hat{x}_i$ transforms under $R \in SO(3)$ rotation as $\hat{x}_i^{\bar{R}} \equiv \hat{\bar{x}}_i = \bar{R}_{ij}\hat{x}_j$, then the composite operators $\hat{x}_i\hat{x}_j$ transform as

$$(\hat{x}_i\hat{x}_j)^{\bar{R}} = m[\Delta_\theta(\bar{R})(\hat{x}_i \otimes \hat{x}_j)] \; , \quad \text{to preserve the automorphism symmetry (2.58).} \tag{3.24}$$

After computation, we get

$$(\hat{x}_i\hat{x}_j)^{\bar{R}} = m[\Delta_\theta(\bar{R})(\hat{x}_i \otimes \hat{x}_j)] = \hat{x}_i^{\bar{R}}\hat{x}_j^{\bar{R}} + \frac{i}{2}\theta_{ij} - \frac{i}{2}\bar{R}_{ik}\theta_{kl}(\bar{R}^T)_{lj}. \quad (3.25)$$

With this, we have the rotated anti-commutator and commutator of $\hat{x}_i,\ \hat{x}_j$:

$$(\hat{x}_i\hat{x}_j)^{\bar{R}} + (\hat{x}_j\hat{x}_i)^{\bar{R}} = \hat{x}_i^{\bar{R}}\hat{x}_j^{\bar{R}} + \hat{x}_j^{\bar{R}}\hat{x}_i^{\bar{R}}, \quad \text{while} \quad (\hat{x}_i\hat{x}_j)^{\bar{R}} - (\hat{x}_j\hat{x}_i)^{\bar{R}} = i\theta_{ij}\ . \quad (3.26)$$

This means that the anti-commutator transforms as a tensor, whereas the commutator transforms as an invariant $SO(3)$ scalar, as was observed earlier [47]. Thus, under the rotation $\bar{R}$, we have

$$_V(\vec{x}|\{\hat{X}_i, \hat{X}_j\}|\vec{x})_V = \bar{R}^{-1}_{im}\bar{R}^{-1}_{jn}\ {}_V(\vec{x}|\{\hat{\bar{X}}_m, \hat{\bar{X}}_n\}|\vec{x})_V \quad (3.27)$$

This gives after a straightforward computation

$$_V(\vec{x}|\hat{X}_i\hat{X}_j|\vec{x})_V = x_i x_j + \frac{\theta}{2}\delta_{ij} - \frac{\theta_i\theta_j}{4\theta} + \frac{i}{2}\theta_{ij}. \quad (3.28)$$

so that upon symmetrization and anti-symmetrization, we respectively obtain

$$\frac{1}{2}\langle\{\hat{X}_i, \hat{X}_j\}\rangle = x_i x_j + \frac{\theta}{2}\delta_{ij} - \frac{\theta_i\theta_j}{4\theta} \quad \text{and} \quad \frac{1}{2}\langle[\hat{X}_i, \hat{X}_j]\rangle = \frac{i}{2}\theta_{ij}. \quad (3.29)$$

We then get

$$V^\theta_{XX} = \begin{pmatrix} \frac{\theta}{2} - \frac{\theta_1^2}{4\theta} & -\frac{\theta_1\theta_2}{4\theta} & -\frac{\theta_1\theta_3}{4\theta} \\ -\frac{\theta_1\theta_2}{4\theta} & \frac{\theta}{2} - \frac{\theta_2^2}{4\theta} & -\frac{\theta_2\theta_3}{4\theta} \\ -\frac{\theta_1\theta_3}{4\theta} & -\frac{\theta_2\theta_3}{4\theta} & \frac{\theta}{2} - \frac{\theta_3^2}{4\theta} \end{pmatrix} \quad \text{and} \quad \Omega^\theta_{XX} = \frac{1}{2}\begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}. \quad (3.30)$$

Now to compute the second block matrices V^c_{XP} and Ω^c_{XP}, we have to calculate the expectation values of $\hat{P}_i$ and the composite $\hat{X}_i\hat{P}_j$. We can easily obtain

$$_V(\vec{x}|\hat{P}_i|\vec{x})_V = \frac{\theta^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int d^3p\ p_i\ e^{-\frac{\theta}{2}\vec{p}^2} = 0\ . \quad (3.31)$$

In the same way as above, we go to the barred frame to calculate the expectation value of the composite,

$$_V(\vec{x}|\{\hat{X}_i, \hat{P}_j\}|\vec{x})_V = \bar{R}^{-1}_{im}\bar{R}^{-1}_{jn}\ {}_V(\vec{x}|\{\hat{\bar{X}}_m, \hat{\bar{P}}_n\}|\vec{x})_V\ , \quad (3.32)$$

$$\text{where} \quad {}_V(\vec{x}|\hat{\bar{X}}_m\hat{\bar{P}}_n|\vec{x})_V = \frac{\theta^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}}\int\int d^3\bar{p}\ d^3\bar{p}' e^{-\frac{\theta}{4}(\vec{\bar{p}}^2+\vec{\bar{p}}'^2)}\ e^{-i(\vec{\bar{p}}-\vec{\bar{p}}').\vec{x}}\bar{p}_n(\vec{\bar{p}}'|\hat{x}_m|\vec{\bar{p}}). \quad (3.33)$$

Putting (3.21), (3.22) in the above equation (3.33) and then integrating, we obtain

$$_V(\vec{x}|\{\hat{\hat{X}}_m, \hat{\hat{P}}_n\}|\vec{x})_V = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.34}$$

Using the expression of $\bar{R}_{ij}$ in (2.29), and the fact that $\langle \hat{P}_i \rangle = 0$, we get the second block matrices as

$$V^{\theta}_{XP} = \frac{1}{2}\langle\{\hat{X}_i, \hat{P}_j\}\rangle = \begin{pmatrix} 0 & -\frac{\theta_3}{2\theta} & \frac{\theta_2}{2\theta} \\ \frac{\theta_3}{2\theta} & 0 & -\frac{\theta_1}{2\theta} \\ -\frac{\theta_2}{2\theta} & \frac{\theta_1}{2\theta} & 0 \end{pmatrix} \text{ and } \Omega^{\theta}_{XP} = \frac{1}{2}\,\mathbb{I}_3. \tag{3.35}$$

Similarly, the third block matrices can be easily obtained as

$$V^{\theta}_{PX} = \frac{1}{2}\langle\{\hat{P}_i, \hat{X}_j\}\rangle = \begin{pmatrix} 0 & \frac{\theta_3}{2\theta} & -\frac{\theta_2}{2\theta} \\ -\frac{\theta_3}{2\theta} & 0 & \frac{\theta_1}{2\theta} \\ \frac{\theta_2}{2\theta} & -\frac{\theta_1}{2\theta} & 0 \end{pmatrix} \text{ and } \Omega^{\theta}_{PX} = -\frac{1}{2}\,\mathbb{I}_3. \tag{3.36}$$

Finally, the last block matrices are easily obtained as

$$V^{\theta}_{PP} = \frac{1}{2}\langle\{\hat{P}_i, \hat{P}_j\}\rangle - \langle\hat{P}_i\rangle\langle\hat{P}_j\rangle = \frac{1}{\theta}\,\mathbb{I}_3 \quad \text{and} \quad \Omega^{\theta}_{PP} = \mathbf{0_3}. \tag{3.37}$$

With these (3.30),(3.35), (3.36) and (3.37), the complete non-commutative variance matrix V^θ and symplectic matrix Ω^θ are obtained as

$$V^{\theta} = \begin{pmatrix} \frac{\theta}{2} - \frac{\theta_1^2}{4\theta} & -\frac{\theta_1\theta_2}{4\theta} & -\frac{\theta_1\theta_3}{4\theta} & 0 & -\frac{\theta_3}{2\theta} & \frac{\theta_2}{2\theta} \\ -\frac{\theta_1\theta_2}{4\theta} & \frac{\theta}{2} - \frac{\theta_2^2}{4\theta} & -\frac{\theta_2\theta_3}{4\theta} & \frac{\theta_3}{2\theta} & 0 & -\frac{\theta_1}{2\theta} \\ -\frac{\theta_1\theta_3}{4\theta} & -\frac{\theta_2\theta_3}{4\theta} & \frac{\theta}{2} - \frac{\theta_3^2}{4\theta} & -\frac{\theta_2}{2\theta} & \frac{\theta_1}{2\theta} & 0 \\ 0 & \frac{\theta_3}{2\theta} & -\frac{\theta_2}{2\theta} & \frac{1}{\theta} & 0 & 0 \\ -\frac{\theta_3}{2\theta} & 0 & \frac{\theta_1}{2\theta} & 0 & \frac{1}{\theta} & 0 \\ \frac{\theta_2}{2\theta} & -\frac{\theta_1}{2\theta} & 0 & 0 & 0 & \frac{1}{\theta} \end{pmatrix} ; \Omega^{\theta} = \frac{1}{2}\begin{pmatrix} 0 & \theta_3 & -\theta_2 & 1 & 0 & 0 \\ -\theta_3 & 0 & \theta_1 & 0 & 1 & 0 \\ \theta_2 & -\theta_1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.38}$$

The corresponding commutative variance matrix V^c and the symplectic matrix Ω^c can be obtained from the above respective non-commutative matrices by linear transformations (3.13) [96] as

$$V^c = \begin{pmatrix} \frac{\theta}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\theta}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\theta}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\theta} \end{pmatrix} \quad \text{and} \quad \Omega^c = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.39}$$

We then calculate the symplectic eigenvalues of V^c i.e. the ordinary eigenvalues of $|2i\Omega^0 V^0|$ [95] and obtain it as $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ i.e. 6-fold degenerate. These are the three pairs of symplectic eigenvalues, each of the form of $(\frac{1}{2}, \frac{1}{2})$. This can be obtained simply from the corresponding single mode of V^c: $(\frac{\theta}{4}, \frac{1}{\theta})$ (3.39), which occurs symmetrically in all the directions x_1, x_2, x_3, by a simple canonical transformation $(x_1, p_1) \to (\lambda x_1, \frac{1}{\lambda} p_1)$ for suitable $\lambda \neq 0$. In any case it is simple to see that both V^θ (3.38) and V^c (3.39) satisfy the saturation condition (3.10) (since $\det M = 1$)

$$\det V^\theta = \det V^0 = \frac{1}{4^3}. \tag{3.40}$$

This indicates that the Voros basis represents a maximally localized state in phase space. But note that the Voros basis (2.43), can be factorized by going to the barred frame as

$$|\vec{x})_V = \left(\frac{\theta}{2\pi}\int d^2\bar{p}\; e^{-\frac{\theta}{4}(\bar{p}_1^2+\bar{p}_2^2)} e^{i\bar{p}_\alpha(\hat{\bar{x}}_\alpha - \bar{x}_\alpha)}\right)\left(\sqrt{\frac{\theta}{2\pi}}\int d\bar{p}_3 e^{-\frac{\theta}{4}\bar{p}_3^2} e^{i\bar{p}_3(\hat{\bar{x}}_3-\bar{x}_3)}\right). \tag{3.41}$$

Here, the first factor is the 2D Voros basis (2.18) and the second : $\int d\bar{p}_3 e^{-\frac{\theta}{4}\bar{p}_3^2} e^{i\bar{p}_3(\hat{\bar{x}}_3-\bar{x}_3)} \sim e^{-\frac{1}{\theta}(\hat{\bar{x}}_3-\bar{x}_3)^2}$ represents a one-dimensional Gaussian state centered at $\bar{x}_3$ with a spread $\Delta\bar{X}_3 \sim \sqrt{\theta}$. Clearly, this $\Delta\bar{X}_3$ can be made as small as we like by a suitable scaling factor and scaling up $\Delta\bar{P}_3$ appropriately to preserve the saturation condition (obviously the resulting state is a non-Voros one). Even if $\Delta\bar{X}_3$ is squeezed to the extreme such that $\Delta\bar{x}_3 = 0$ we can still have $\Delta\bar{X}_1\Delta\bar{X}_2 \geq \frac{\theta}{2}$ for such a non-Voros state. More generally, we can write in this case

$$\Delta\bar{X}_1\Delta\bar{X}_2 + \Delta\bar{X}_2\Delta\bar{X}_3 + \Delta\bar{X}_3\Delta\bar{X}_1 \geq \frac{\theta}{2} \tag{3.42}$$

It is therefore quite interesting to see the form of analogous inequality in the original fiducial frame for the Voros basis. Using $\Delta\hat{X}_i = \sqrt{\frac{\theta}{2} - \frac{\theta_i^2}{4\theta}}$ from (3.38), we have

$$\begin{aligned}\Delta\hat{X}_1\Delta\hat{X}_2 + \Delta\hat{X}_2\Delta\hat{X}_3 + \Delta\hat{X}_1\Delta\hat{X}_3 \;&=\; \frac{1}{4\theta}\Big[\sqrt{(2\theta^2-\theta_1^2)(2\theta^2-\theta_2^2)} + \sqrt{(2\theta^2-\theta_2^2)(2\theta^2-\theta_3^2)} \\ &\quad + \sqrt{(2\theta^2-\theta_1^2)(2\theta^2-\theta_3^2)}\Big]\end{aligned} \tag{3.43}$$

where the vector $\vec{\theta}$ points in arbitrary direction. We can see that the expression (3.43) attains its minimum value $\frac{\theta}{2}(1+\sqrt{2})$ when the vector $\vec{\theta}$ points in one of the three axes (e.g. $\theta = \theta_3; \ \ \theta_1 = \theta_2 = 0$). So we have the following condition

$$\Delta\hat{X}_1\Delta\hat{X}_2 + \Delta\hat{X}_2\Delta\hat{X}_3 + \Delta\hat{X}_1\Delta\hat{X}_3 \geq \frac{\theta}{2}(1+\sqrt{2}). \tag{3.44}$$

Thus, there exist non-Voros states on which the space-space uncertainty is lower than that of Voros states even though the phase-space uncertainty is saturated by the Voros states.

Finally, we would like to point out some subtle features of the commutative limit ($\theta \to 0$). Firstly, note that one cannot take naively the commutative ($\theta \to 0$) limit in (3.38). Although there is no problem in taking this limit in the block diagonal parts where $V^{\theta}_{XX} \to \mathbf{0_3}$, $\Omega^{\theta}_{XX} \to \mathbf{0_3}$ and $V^{\theta}_{PP} \to \infty$, $\Omega^{\theta}_{PP} \to \mathbf{0_3}$ as defined in (3.30) and (3.37), the off diagonal blocks V^{θ}_{XP} and V^{θ}_{PX} do not vanish in this limit, where one is left with terms involving trigonometric functions if one parametrizes $\vec{\theta}$ as in (2.29). The source of this difficulty actually stems from the fact that the Voros basis written in the barred frame (3.41) is constructed by taking Fourier transform of a Gaussian state in momentum space which has the spread $\sim \frac{1}{\sqrt{\theta}}$ i.e. in terms of the non-commutative parameter θ itself. Consequently, (3.34) which is the expression of V^{θ}_{XP} in the barred frame turns out to be completely θ independent, whereas the corresponding expression (3.32) in the unbarred fiducial frame is obtained by carrying out $SO(3)$ rotation (2.29) and thus leaving the trigonometric functions as remnants. Had we considered momentum space wave function:

$$\psi(\vec{p};\alpha) = \frac{1}{\sqrt{\pi\alpha}} e^{-\frac{1}{2\alpha}\vec{p}^2} , \tag{3.45}$$

having a spread given by an independent parameter α - distinct from θ as was done in [96], the corresponding variance matrix in two-dimensional space would have the form :

$$V_{XP} = \begin{bmatrix} 0 & -\frac{\theta\alpha}{4} \\ \frac{\theta\alpha}{4} & 0 \end{bmatrix} , \tag{3.46}$$

which reduces to the corresponding 2×2 diagonal block in (3.34) if $\alpha = \frac{2}{\theta}$. One can now see easily that the entire V^{θ}_{XP} matrix vanishes in the commutative ($\theta \to 0$) limit. It is therefore convenient to take the commutative limit from (3.39) itself which is the variance matrix for $X^c - P$ space, where the off-diagonal block vanishes. Thus, in this commutative limit $\Delta\hat{X}_i \to 0$ and $\Delta\hat{P}_i \to \infty$. Finally, regarding the symplectic matrix (3.38), it has a well-defined commutative limit as its value is independent of the state.

4

MULTI-PARTICLE SYSTEM ON 3D MOYAL SPACE

In this chapter, we discuss the extension of Hilbert-Schimdt operator formalism of non-commutative quantum mechanics to multi-particle system. After carrying out the second quantization by introducing basis independent field operators, we compute the thermal correlation functions for a pair of free particles and then analyse the thermal effective potential between the pair of particles.

4.1 TWO-PARTICLE SYATEM

In commutative quantum mechanics, we consider a two particle system in terms of wave-functions defined over R^6 and therefore to think of classical configuration space as a tensor product $R^3 \otimes R^3$. One may therefore be tempted to take the same approach in non-commutative quantum mechanics and to introduce the non-commutative 3D configuration space for a two particle system as a tensor product of two single particle configuration spaces, i.e., $\mathcal{H}_c^{(2)} = \mathcal{H}_c \otimes \mathcal{H}_c$. The quantum Hilbert space being the space of operators generated by the $\hat{\vec{x}}_1$ and $\hat{\vec{x}}_2$, with the subscripts referring to the non-commutative coordinates of the two particles. That is, the elements of the quantum Hilbert space are operators $\psi(\hat{\vec{x}}_1, \hat{\vec{x}}_2)$. This is essentially the approach adopted in [97] where a Moyal-like star product between the functions $\Psi(\vec{x}_1, \vec{x}_2)$ and $\Phi(\vec{x}_1, \vec{x}_2)$ was introduced as

$$(\Psi \star \Phi)(\vec{x}_1, \vec{x}_2) = \Psi(\vec{x}_1, \vec{x}_2) e^{\frac{i}{2}\theta_{ij}(\frac{\overleftarrow{\partial}}{\partial x_{1i}} + \frac{\overleftarrow{\partial}}{\partial x_{2i}})(\frac{\overrightarrow{\partial}}{\partial x_{1j}} + \frac{\overrightarrow{\partial}}{\partial x_{2j}})} \Phi(\vec{x}_1, \vec{x}_2) \tag{4.1}$$

which yields the following commutation relations which was also obtained in the approach of braided twisted symmetry [98]:

$$[\hat{x}_{\alpha i}, \hat{x}_{\beta j}] = i\theta_{ij} ; \quad \text{where} \quad \alpha, \beta \ \text{ are the particle labels}. \tag{4.2}$$

This is, however, a bit un-natural, as the x_1 component of one particle does not commute with the x_2 component of second particle, and that too it is independent of their separation! We therefor want to follow a different point of view by thinking of a two particle commutative system as two sets of 3D labels in the same 3D configuration space. We know that in non-commutative quantum mechanics, the notion of coordinates does not exists in classical configuration space but arise through the state $|\psi(\hat{\vec{x}}))$ or, operator $\psi(\hat{\vec{x}})$ acting on $\mathcal{H}_c$, and describes the quantum state of the system. For example, in 2D Moyal plane, the state in quantum Hilbert space describing a single particle localized at $z = (x_1 + ix_2)/\sqrt{2\theta}$ is given by $|z,\phi) = |z\rangle\langle\phi| \in \mathcal{H}_q \equiv \mathcal{H}_c \otimes \mathcal{H}_c^*$, $\forall\ |\phi\rangle \in \mathcal{H}_c$. Such state $|z,\phi)$ is an eigenstate of $\hat{B}$ (2.8), i.e. $\hat{B}|z,\phi) = z|z,\phi)$ and it is a minimum uncertainty state. Thus, the position measurement of the particle localized at z is insensitive to the right hand sector $|\phi\rangle$ of the state $|z,\phi)$ as discussed earlier (2.16) [99]. More explicitly, let us calculate the Voros position representation of this state $|z,\phi)$,

$$_V(\vec{x}|z,\phi) \equiv (w|z,\phi) = e^{-\frac{1}{2}|z-w|^2} e^{\frac{1}{2}(\bar{w}z - w\bar{z})}\langle\phi|w\rangle\ ; \quad \text{where } |\vec{x})_V \equiv |w) = |w\rangle\langle w|. \tag{4.3}$$

Setting $\xi = w - z$, we get

$$_V(\vec{x}|z,\phi) \equiv (w|z,\phi) = e^{-\frac{1}{2}|\xi|^2} e^{\frac{1}{2}(z\bar{\xi} - \bar{z}\xi)}\langle\phi|z+\xi\rangle. \tag{4.4}$$

Clearly, the state $|z,\phi)$ is localized at z with non-local corrections deriving from an expansion in ξ. Restoring dimensions, ξ is of the order $\sqrt{\theta}$, demonstrating that the non-local corrections are of the order of the length scale set by the non-commutative parameter. Keeping in mind that $|z,\phi)$ is an operator on $\mathcal{H}_c$, and therefore an element of the algebra generated by the $\hat{x}_i$, it can be written in the form $|z,\phi) = |\psi_{z,\phi}(\hat{\vec{x}})) \equiv |\psi_{\vec{x},\phi}(\hat{\vec{x}}))$. Note that $\vec{x}$ is a label and $\hat{\vec{x}}$ are operators. The two particle state is now described as the tensor product state:

$$|\vec{x}_1,\phi_1;\vec{x}_2,\phi_2) = |\psi_{\vec{x}_1,\phi_1}(\hat{\vec{x}})) \otimes |\psi_{\vec{x}_2,\phi_2}(\hat{\vec{x}})) = \psi_{\vec{x}_1,\phi_1}(\hat{\vec{x}}) \otimes \psi_{\vec{x}_2,\phi_2}(\hat{\vec{x}}). \tag{4.5}$$

Thus, we retain the classical configuration space for multi-particle 3D non-commutative system as $\mathcal{H}_c^3$ but take the quantum Hilbert space of multi-particle non-commutative system as the tensor product of single particle quantum Hilbert space:

$$\mathcal{H}_q^{3(2)} = \mathcal{H}_q^3 \otimes \mathcal{H}_q^3\ \ni\ \psi_{\vec{x}_1,\phi_1}(\hat{\vec{x}}) \otimes \psi_{\vec{x}_2,\phi_2}(\hat{\vec{x}}) = \psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}})\ . \tag{4.6}$$

The single particle coordinates, which are operators on quantum Hilbert space $\mathcal{H}_q^{3(2)}$, are given by $\hat{\hat{X}}_1 = \hat{\hat{X}} \otimes 1_q$ and $\hat{\hat{X}}_2 = 1_q \otimes \hat{\hat{X}}$. It can be easily checked that the action of $\hat{\hat{X}}_1$ (respectively $\hat{\hat{X}}_2$) (or more precisely $\hat{B}_1 = \hat{B} \otimes 1_q$ (respectively $\hat{B}_2 = 1_q \otimes \hat{B}$) on the two-particle state (4.5) returns the coordinates of the first (respectively second) particle as z_1 (respectively z_2).

To define the action of these two-particle states $\psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}}) \in \mathcal{H}_q^{3(2)}$ on the classical configuration space $\mathcal{H}_c^3$, let us recall the multiplication map $m : \mathcal{H}_q^3 \otimes \mathcal{H}_q^3 \to \mathcal{H}_q^3$ introduced in (2.50) as

$$m[|\psi_1(\hat{\vec{x}})) \otimes |\psi_2(\hat{\vec{x}}))] = |(\psi_1\psi_2)(\hat{\vec{x}})). \tag{4.7}$$

With this, the quantum Hilbert space is endowed with the structure of an algebra [38]. Any diffeomorphism symmetry and in particular the $SO(3)$ symmetry in our case, should therefore correspond to the automorphism symmetry of this algebra. As mentioned already in chapter 2, this automorphism symmetry of the algebra of states $\{\psi(\hat{\vec{x}})\}$ is attained at the cost of twisting the co-product of $SO(3)$ rotation generators $\hat{J}_i$ as $\Delta_\theta(\hat{J}_i) = F\Delta_0(\hat{J}_i)F^{-1}$, where $\Delta_0(\hat{J}_i) = \hat{J}_i \otimes I + I \otimes \hat{J}_i$ and $F = e^{\frac{i}{2}\theta^{ij}\hat{P}_i \otimes \hat{P}_j}$. That is,

$$\hat{J}_i \left[m \left(|\psi_1(\hat{\vec{x}})) \otimes |\psi_2(\hat{\vec{x}})) \right) \right] = m \left[\Delta_\theta(\hat{J}_i) \left(|\psi_1(\hat{\vec{x}})) \otimes |\psi_2(\hat{\vec{x}})) \right) \right], \tag{4.8}$$

where

$$\Delta_\theta(\hat{J}_i) = \Delta_0(\hat{J}_i) + \frac{1}{2} \left[\hat{P}_i \otimes (\vec{\theta} \cdot \vec{P}) - (\vec{\theta} \cdot \vec{P}) \otimes \hat{P}_i \right]. \tag{4.9}$$

This forces us to twist the exchange operation $\Sigma : \mathcal{H}_q^3 \otimes \mathcal{H}_q^3 \to \mathcal{H}_q^3 \otimes \mathcal{H}_q^3$, where $\Sigma : A \otimes B \to B \otimes A$, to $\Sigma_\theta = F\ \Sigma\ F^{-1}$ as we found that

$$\Sigma[\Delta_\theta(\hat{J}_i)(\psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}}))] \neq \Delta_\theta(\hat{J}_i)[\Sigma(\psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}})]. \tag{4.10}$$

This deformation of the exchange operation is essential because we must have $[\Sigma_\theta, \Delta_\theta] = 0$. Otherwise, under a transformation like rotation the statistics of the physical state can get altered; a pure bosonic/fermionic state, obtained by projecting into symmetric/antisymmetric subspace by the projector $P^\pm = \frac{1}{2}(I \pm \Sigma)$ will yield an admixture of bosonic/fermionic states under rotation. But this cannot be allowed as implied by the super-selection rules which says that a system of fermions or bosons should remain as the one under any transformation. Thus, corresponding to this deformed exchange operator, the deformed projection operator can be constructed as

$$P_\theta^\pm = \frac{1}{2}(I \pm \Sigma_\theta) = \frac{1}{2}\left(I \pm e^{i\theta_{ij}\hat{P}_i \otimes \hat{P}_j}\Sigma \right) \quad , \quad \text{since } \ \Sigma\ F^{-1} = F\ \Sigma \Rightarrow \Sigma_\theta = F^2\Sigma\ . \tag{4.11}$$

We then obtain the twisted two-particle state as

$$|\psi_1(\hat{x}), \psi_2(\hat{x}))_{\pm\theta} = P_\theta^\pm \left(\psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}}) \right) = \frac{1}{2}\left[\psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}}) \pm e^{i\theta_{ij}\hat{P}_i \otimes \hat{P}_j} (\psi_2(\hat{\vec{x}}) \otimes \psi_1(\hat{\vec{x}})) \right]. \tag{4.12}$$

We refer to $P_\theta^\pm$ as the twisted symmetric(+)/ antisymmetric(-) projection operator which give the twisted symmetric/ antisymmetric states corresponding to twisted bosons/ fermions system.

4.2 MULTI-PARTICLE STATES

The construction of N-particle states proceed in complete analogy with the two particle states. The quantum Hilbert space of N-particle non-commutative system will be given by the following N-fold tensor product:

$$\mathcal{H}_q^{3(N)} = \mathcal{H}_q^3 \otimes \mathcal{H}_q^3 \otimes .... \otimes \mathcal{H}_q^3 \otimes \mathcal{H}_q^3. \tag{4.13}$$

The elements of $\mathcal{H}_q^{3(N)}$ are of the form:

$$|\psi_{\vec{x}_1,\phi_1}(\hat{\vec{x}})) \otimes |\psi_{\vec{x}_2,\phi_2}(\hat{\vec{x}})) \ldots \otimes |\psi_{\vec{x}_N,\phi_N}(\hat{\vec{x}})) = \psi_1(\hat{\vec{x}}) \otimes \psi_2(\hat{\vec{x}}) \ldots \otimes \psi_N(\hat{\vec{x}}). \tag{4.14}$$

As before the multiplication map can be extended to arbitrary N-particle state by making successive pair-wise composition and making use of the associative property of operator products to get:

$$m\left[\prod_{a=1}^{N} \otimes |\psi_a(\hat{\vec{x}}))\right] = |(\psi_1\psi_2 \ldots \psi_N)(\hat{\vec{x}})). \tag{4.15}$$

The corresponding transformation property of multi-particle states ((4.14)) can be easily obtained by making use of co-associativity of the co-product (see for example [100, 101]). The twisted N-particle symmetric and antisymmetric states [26] can be constructed as

$$|\psi_1, \psi_2, ..., \psi_N)_{\pm\theta} = P^N_{\pm\theta}(\psi_1 \otimes \psi_2 \otimes ... \otimes \psi_N) \tag{4.16}$$

We need to find the form of the deformed projection operator $P^N_{\pm\theta}$ for N-particle system. First of all, let us obtain the form of $P^3_{\pm\theta}$ for 3-particle system. Here, we encounter with two deformed nearest neighbor exchange operators: $\Sigma_\theta^{12} = \Sigma_\theta \otimes 1$ and $\Sigma_\theta^{23} = 1 \otimes \Sigma_\theta$ acting on $\mathcal{H}_q^3 \otimes \mathcal{H}_q^3 \otimes \mathcal{H}_q^3$, where Σ_θ^{12} exchanges the first and the second slots keeping the third slot fixed and Σ_θ^{23} exchanges the second and the third slots keeping the first slot fixed. With this, we can write the deformed projection operator for 3-particle physical states as

$$P^3_{\pm\theta} = \frac{1}{3!}[1 \pm \{\Sigma_\theta^{12} + \Sigma_\theta^{23}\} (\pm 1)^2 \{\Sigma_\theta^{12}\,\Sigma_\theta^{23} + \Sigma_\theta^{23}\,\Sigma_\theta^{12}\} (\pm 1)^3\, \Sigma_\theta^{12}\,\Sigma_\theta^{23}\,\Sigma_\theta^{12}] \tag{4.17}$$

where we should note that Σ_θ's satisfy $(\Sigma_\theta^{12})^2 = 1 = (\Sigma_\theta^{23})^2$ and the braid equation:

$$\Sigma_\theta^{12}\,\Sigma_\theta^{23}\,\Sigma_\theta^{12} = \Sigma_\theta^{23}\,\Sigma_\theta^{12}\,\Sigma_\theta^{23}\,. \tag{4.18}$$

In this way, for N-particle system we have $(N-1)$ deformed nearest neighbor exchange operators $\Sigma_\theta^{n,n+1}$, $n = 1, 2....(N-1)$, acting on $\mathcal{H}_q^{3(N)}$, which exchanges only the entries at the nth and $(n+1)$th slots, keeping all the entries at other slots fixed. That is,

$$\Sigma_\theta^{n,n+1} = 1 \otimes 1 \otimes .... \otimes \Sigma_\theta \otimes .... \otimes 1 \otimes 1, \quad \Sigma_\theta \text{ is at the } n\text{th position.} \tag{4.19}$$

For an N-particle system, we have $(\Sigma_\theta^{n,n+1})^2 = 1$, and the braid equation:

$$\Sigma_\theta^{n,n+1}\,\Sigma_\theta^{n+1,n+2}\,\Sigma_\theta^{n,n+1} = \Sigma_\theta^{n+1,n+2}\,\Sigma_\theta^{n,n+1}\,\Sigma_\theta^{n+1,n+2}. \tag{4.20}$$

Thus, the deformed projection operator $P_{\pm\theta}^N$ for N-particle system is given by

$$P_{\pm\theta}^N = \frac{1}{N!}\sum_{n=1}^{N-1}[1 \;\pm\; \Sigma_\theta^{n,n+1}\;(\pm 1)^2\,\Sigma_\theta^{n,n+1}\,\Sigma_\theta^{n+1,n+2}\;(\pm 1)^3....(\pm 1)^N\,\Sigma_\theta^{n,n+1}\Sigma_\theta^{n+1,n+2}\Sigma_\theta^{n+2,n+3}.....] \tag{4.21}$$

Since $\Sigma_\theta = F^2\Sigma = e^{i\theta^{ij}\hat{P}_i\otimes\hat{P}_j}\Sigma$, so in the above expression we have a phase factor for each deformed exchange operator. Thus in the last term of the above equation there are N phase factors for N deformed exchange operators. Needless to say that $(\pm 1)^N = \pm$, depending on whether N is even or odd. In this way, we can define a twisted symmetric/ antisymmetric state corresponding to the twisted bosons/fermions for N-particle system on the quantum Hilbert space $\mathcal{H}_q^{(N)}$.

4.2.1 *Momentum Basis on $\mathcal{H}_q^{3(N)}$*

Given a momentum basis $|\vec{p}) \in \mathcal{H}_q^3$, we can construct the twisted symmetric/antisymmetric momentum basis on $\mathcal{H}_q^{3(N)}$ as

$$|\vec{p}_1,\vec{p}_2,.....,\vec{p}_N)_{\pm\theta} = P_{\pm\theta}^N\Big(|\vec{p}_1)\otimes|\vec{p}_2)\otimes.....\otimes|\vec{p}_N)\Big)\;. \tag{4.22}$$

For simplicity, let us first consider the momentum basis on $\mathcal{H}_q^{3(2)}$:

$$|\vec{p}_1,\vec{p}_2)_{\pm\theta} = P_\theta^{\pm}\Big(|\vec{p}_1)\otimes|\vec{p}_2)\Big) = \frac{1}{2}\Big[|\vec{p}_1)\otimes|\vec{p}_2) \pm e^{ip_2\wedge p_1}|\vec{p}_2)\otimes|\vec{p}_1)\Big]\;. \tag{4.23}$$

where the wedge '$\wedge$' between the momenta $\vec{p}_1$ and $\vec{p}_2$ simply denotes the following factor:

$$p_2\wedge p_1 = \theta^{ij}p_{2i}p_{1j}. \tag{4.24}$$

Because of the phase factor $e^{ip_2\wedge p_1}$, these momentum states do not satisfy the orthogonality condition:

$${}_{\pm\theta}(\vec{p}_1',\vec{p}_2'|\vec{p}_1,\vec{p}_2)_{\pm\theta} = \frac{1}{2}[\delta^3(\vec{p}_1'-\vec{p}_1)\delta^3(\vec{p}_2'-\vec{p}_2) \pm e^{ip_2\wedge p_1}\delta^3(\vec{p}_1'-\vec{p}_2)\delta^3(\vec{p}_2'-\vec{p}_1)]. \tag{4.25}$$

But satisfy the following completeness relation:

$$\int d^3p_1 d^3p_2 |\vec{p}_1,\vec{p}_2)_\theta\;{}_\theta(\vec{p}_1,\vec{p}_2| = \mathbf{1}_q^2. \tag{4.26}$$

In this way, we can write the twisted symmetric/antisymmetric momentum states as

$$|\vec{p}_1, .., \vec{p}_n, \vec{p}_{n+1}.., \vec{p}_N)_\theta = \frac{1}{N!}\sum_{n=1}^{N-1}[|\vec{p}_1) \otimes .. \otimes |\vec{p}_n) \otimes |\vec{p}_{n+1}) \otimes .. \otimes |\vec{p}_N)$$
$$\pm e^{ip_{n+1}\wedge p_n}|\vec{p}_1) \otimes ... \otimes |\vec{p}_{n+1}) \otimes |\vec{p}_n) \otimes ... \otimes |\vec{p}_N)$$
$$(\pm 1)^2 e^{ip_{n+1}\wedge p_n} e^{ip_{n+2}\wedge p_{n+1}}\{|\vec{p}_1) \otimes .. \otimes |\vec{p}_{n+2}) \otimes |\vec{p}_n) \otimes |\vec{p}_{n+1}) \otimes .. \otimes |\vec{p}_N)$$
$$+ |\vec{p}_1) \otimes .. \otimes |\vec{p}_{n+1}) \otimes |\vec{p}_{n+2}) \otimes |\vec{p}_n) \otimes .. \otimes |\vec{p}_N)\}..........................$$
$$(\pm 1)^N e^{ip_{n+1}\wedge p_n} e^{ip_{n+2}\wedge p_{n+1}}.........|\vec{p}_N) \otimes .... \otimes |\vec{p}_{n+1}) \otimes |\vec{p}_n) \otimes .... \otimes |\vec{p}_1)] \,. \tag{4.27}$$

These states satisfy the following relations:

$$_\theta(\vec{p}\,'_1, \vec{p}\,'_2...., \vec{p}\,'_N|\vec{p}_1, \vec{p}_2, .., \vec{p}_N)_\theta = \frac{1}{N!}\sum_{n=0}^{(N-1)}[\delta^3(\vec{p}\,'_1 - \vec{p}_1)\delta^3(\vec{p}\,'_2 - \vec{p}_2)...\delta^3(\vec{p}\,'_N - \vec{p}_N) \tag{4.28}$$
$$\pm e^{ip_{n+1}\wedge p_n}\{\delta^3(\vec{p}\,'_1 - \vec{p}_1).........\delta^3(\vec{p}\,'_n - \vec{p}_{n+1}).........\delta^3(\vec{p}\,'_{n+1} - \vec{p}_n).........\delta^3(\vec{p}\,'_N - \vec{p}_N)\}$$
$$(\pm 1)^2 e^{ip_{n+1}\wedge p_n} e^{ip_{n+2}\wedge p_{n+1}}\{\delta(\vec{p}\,'_1 - \vec{p}_1)........\delta^3(\vec{p}\,'_n - \vec{p}_{n+2})\delta^3(\vec{p}\,'_{n+1} - \vec{p}_n)\delta^3(\vec{p}\,'_{n+2} - \vec{p}_{n+1})$$
$$.......\delta^3(\vec{p}\,'_N - \vec{p}_N) + \delta^3(\vec{p}\,'_1 - \vec{p}_1)............\delta^3(\vec{p}\,'_n - \vec{p}_{n+1})\delta^3(\vec{p}\,'_{n+1} - \vec{p}_{n+2})\delta^3(\vec{p}\,'_{n+2} - \vec{p}_n)$$
$$...............\delta^3(\vec{p}\,'_N - \vec{p}_N)\} + ...(\pm 1)^N e^{ip_{n+1}\wedge p_n} e^{ip_{n+2}\wedge p_{n+1}}.................$$
$$\times\delta^3(\vec{p}\,'_1 - \vec{p}_N)\delta^3(\vec{p}\,'_2 - \vec{p}_{N-1})...\delta^3(\vec{p}\,'_{N-1} - \vec{p}_2)\delta^3(\vec{p}\,'_N - \vec{p}_1)].$$

and

$$\int d^3p_1 d^3p_2....d^3p_N|\vec{p}_1, \vec{p}_2..., \vec{p}_N)_\theta\ {}_\theta(\vec{p}_1, \vec{p}_2...., \vec{p}_N| = \mathbf{1}_q^N \,. \tag{4.29}$$

In addition to such non-orthogonal twisted symmetric/anti-symmetric momentum basis, we can introduce a new momentum basis which is different from the twisted one by a phase factor and it can be seen that such basis satisfies the orthonormality as well as completeness relations. Explicitly, let us introduce this new momentum basis on $\mathcal{H}_q^{2(2)}$ as $|\vec{p}_1, \vec{p}_2))$ (denoted henceforth by a "double ket"), given by

$$|\vec{p}_1, \vec{p}_2)) = e^{\frac{i}{2}p_1\wedge p_2}|\vec{p}_1, \vec{p}_2)_{\pm\theta} = \frac{1}{2}[e^{\frac{i}{2}p_1\wedge p_2}|\vec{p}_1) \otimes |\vec{p}_2) \pm e^{\frac{i}{2}p_2\wedge p_1}|\vec{p}_2) \otimes |\vec{p}_1)] \,, \tag{4.30}$$

with the following relations:

$$((\vec{p}\,'_1, \vec{p}\,'_2|\vec{p}_1, \vec{p}_2)) = \frac{1}{2}[\delta^3(\vec{p}\,'_1 - \vec{p}_1)\delta^3(\vec{p}\,'_2 - \vec{p}_2) \pm \delta^3(\vec{p}\,'_1 - \vec{p}_2)\delta^3(\vec{p}\,'_2 - \vec{p}_1)] \tag{4.31}$$

and

$$\int d^3p_1 d^3p_2\ |\vec{p}_1, \vec{p}_2))((\vec{p}_1, \vec{p}_2| = \mathbf{1}_q^2. \tag{4.32}$$

This basis will be referred to as "quasi-commutative" basis as it is found to be symmetric under the undeformed exchange operation Σ and so no notion of twisted statistics arises.

In the similar way we can define the new basis for 3-particle and so on as

$$|\vec{p}_1, \vec{p}_1, \vec{p}_3))_{\pm} = \frac{1}{3!}[e^{\frac{i}{2}(p_1\wedge p_2+p_2\wedge p_3+p_1\wedge p_3)}|\vec{p}_1) \otimes |\vec{p}_2) \otimes |\vec{p}_3) \pm e^{\frac{i}{2}(p_2\wedge p_1+p_2\wedge p_3+p_1\wedge p_3)}|\vec{p}_2) \otimes |\vec{p}_1) \otimes |\vec{p}_3)$$
$$\pm e^{\frac{i}{2}(p_1\wedge p_2+p_3\wedge p_2+p_1\wedge p_3)}|\vec{p}_1) \otimes |\vec{p}_3) \otimes |\vec{p}_2)(\pm 1)^2 e^{\frac{i}{2}(p_1\wedge p_2+p_3\wedge p_2+p_3\wedge p_1)}|\vec{p}_3) \otimes |\vec{p}_1) \otimes |\vec{p}_2)$$
$$(\pm 1)^2 e^{\frac{i}{2}(p_2\wedge p_1+p_2\wedge p_3+p_3\wedge p_1)}|\vec{p}_2) \otimes |\vec{p}_3) \otimes |\vec{p}_1)(\pm 1)^3 e^{\frac{i}{2}(p_2\wedge p_1+p_3\wedge p_2+p_3\wedge p_1)}|\vec{p}_3) \otimes |\vec{p}_2) \otimes |\vec{p}_1)] ; \quad (4.33)$$

where ,

$$|\vec{p}_1, \vec{p}_2, \vec{p}_3)_\theta = e^{-\frac{i}{2}(p_1\wedge p_2+p_2\wedge p_3+p_1\wedge p_3)}|\vec{p}_1, \vec{p}_1, \vec{p}_3))_{\pm} . \quad (4.34)$$

These states satisfying the following orthogonality and completeness relations:

$$_{\pm}((\vec{p}_1', \vec{p}_2', \vec{p}_3'|\vec{p}_1, \vec{p}_2, \vec{p}_3))_{\pm} = \frac{1}{3!}[\delta^3(\vec{p}_1' - \vec{p}_1)\delta^3(\vec{p}_2' - \vec{p}_2)\delta^3(\vec{p}_3' - \vec{p}_3) \pm \delta^3(\vec{p}_1' - \vec{p}_2)\delta^3(\vec{p}_2' - \vec{p}_1)\delta^3(\vec{p}_3' - \vec{p}_3)$$
$$\pm \delta^3(\vec{p}_1' - \vec{p}_1)\delta^3(\vec{p}_2' - \vec{p}_3)\delta^3(\vec{p}_3' - \vec{p}_2)(\pm 1)^2\delta^3(\vec{p}_1' - \vec{p}_3)\delta^3(\vec{p}_2' - \vec{p}_1)\delta^3(\vec{p}_3' - \vec{p}_2)$$
$$(\pm 1)^2\delta^3(\vec{p}_1' - \vec{p}_2)\delta^3(\vec{p}_2' - \vec{p}_3)\delta^3(\vec{p}_3' - \vec{p}_1)(\pm 1)^3\delta^3(\vec{p}_1' - \vec{p}_3)\delta^3(\vec{p}_2' - \vec{p}_2)\delta^3(\vec{p}_3' - \vec{p}_1)] \quad (4.35)$$

and

$$\int d^3p_1 d^3p_2 d^3p_3 \; |\vec{p}_1, \vec{p}_2, \vec{p}_3))((\vec{p}_1, \vec{p}_2, \vec{p}_3| = \mathbf{1}_q^3. \quad (4.36)$$

In this way, we can get the symmetrized/anti-symmetrized quasi-commutative momentum basis for any arbitrary number of particles which satisfy the usual orthogonality and completeness relations.

4.3 SECOND QUANTIZATION

The second quantization method allows to create and destroy particles in a quantum system consisting of arbitrary number of indistinguishable particles. Let us consider the physical, full quantum Hilbert space $\mathcal{H}_Q^3$ which is just the direct sum of quantum Hilbert spaces with all possible number of particle states:

$$\mathcal{H}_Q^3 \equiv \mathcal{H}_q^{3(0)} \oplus \mathcal{H}_q^{3(1)} \oplus \mathcal{H}_q^{3(2)} \oplus ... \oplus \mathcal{H}_q^{3(n)} \oplus ..... , \quad (4.37)$$

where $\mathcal{H}_q^{3(0)}$ is the quantum Hilbert space with no particle and corresponds to the so called vacuum state. Clearly, $\mathcal{H}_q^{3(n)}$ is the quantum Hilbert space with n-particles. We can then construct the raising/lowering operators which act on $\mathcal{H}_Q^3$ creating a particle : $\mathcal{H}_q^{3(n)} \to \mathcal{H}_q^{3(n+1)}$ or destroying one: $\mathcal{H}_q^{3(n)} \to \mathcal{H}_q^{3(n-1)}$. We have seen that dealing with the momentum basis on a particular quantum Hilbert space $\mathcal{H}_q^{3(N)}$ with N-particles makes it easier. Moreover, we have two types of momentum basis in the multi-particle quantum system: the twisted symmetric/anti-symmetric momentum basis and the quasi-commutative mometum basis.

Following in the similar way as given in [102] for second quantization formalism of commutative quantum mechanics, we construct the two types of creation and annihilation operators in momentum space, corresponding to the above two types of momentum basis on $\mathcal{H}_q^{3(N)}$. Note that these ladder operators on twisted momentum basis have already been introduced on non-commutative quantum system on Moyal plane in [26, 97, 103, 104] where the use of Moyal star product is involved. Here, we proceed without using any position basis at first. We first construct the ladder operators in momentum basis, then the abstract ladder operators and at last we define the ladder operators in position basis. This will enable us to define the multi-particle states at different positions.

4.3.1 *Creation and Annihilation operators in twisted momentum basis*

The twisted symmetrized/anti-symmetrized momentum basis on the full quantum Hilbert space $\mathcal{H}_Q^3$ resolves to the identity operator as

$$\mathbf{I}_Q = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3p_1 d^3p_2 ....d^3p_n |\vec{p}_1, \vec{p}_2, ..., \vec{p}_n)_{\pm\theta} \;{}_{\pm\theta}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n|. \tag{4.38}$$

The overlap of any two such basis states with different number of particles on $\mathcal{H}_Q$ vanishes. That is,

$$_{\pm\theta}(\vec{p}_1', \vec{p}_2', ...\vec{p}_N'|\vec{p}_1, \vec{p}_2, ...\vec{p}_M)_{\pm\theta} = \delta_{NM} \;{}_{\pm\theta}(\vec{p}_1', \vec{p}_2', ...\vec{p}_N'|\vec{p}_1, \vec{p}_2, ...\vec{p}_N)_{\pm\theta}. \tag{4.39}$$

The creation and annihilation operators in this twisted momentum basis can then be defined as

$$\hat{a}^\ddagger(\vec{p}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3p_1 d^3p_2 ....d^3p_n \; |\vec{p}, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n)_{\pm\theta} \;{}_{\pm\theta}(\vec{p}_1, \vec{p}_2, ..., \vec{p}_n| \; ; \tag{4.40}$$

$$\text{and} \quad \hat{a}(\vec{p}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3p_1 d^3p_2 ....d^3p_n \; |\vec{p}_1, \vec{p}_2, ..., \vec{p}_n)_{\pm\theta} \;{}_{\pm\theta}(\vec{p}, \vec{p}_1, \vec{p}_2, ..., \vec{p}_n| \; . \tag{4.41}$$

An arbitrary state on $\mathcal{H}_Q$ containing N-particles can be created by the N-fold action of the creation operators on the vacuum state as

$$|\vec{p}_1, \vec{p}_2, ..., \vec{p}_N)_{\pm\theta} = \hat{a}^\ddagger(\vec{p}_1)\hat{a}^\ddagger(\vec{p}_2)....\hat{a}^\ddagger(\vec{p}_N)|0) \; ; \quad |0) \in \mathcal{H}_q^{3(0)}. \tag{4.42}$$

Note that any further action of the creation operator on an arbitrary state is taken as

$$\hat{a}^\ddagger(\vec{p}) \; |\vec{p}_1, \vec{p}_2, ...\vec{p}_N)_{\pm\theta} = |\vec{p}, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N)_{\pm\theta} \; . \tag{4.43}$$

Unlike the commutative quantum mechanics, the creation operator creating a new particle at the first slot is different from creating a particle at the last slot as these twisted field operators obey the deformed commutation relations:

$$\hat{a}^{\ddagger}(\vec{p})\,\hat{a}^{\ddagger}(\vec{p}\,') = \pm\, e^{ip'\wedge p}\,\hat{a}^{\ddagger}(\vec{p}\,')\,\hat{a}^{\ddagger}(\vec{p}) \tag{4.44}$$

$$\hat{a}(\vec{p})\,\hat{a}(\vec{p}\,') = \pm\, e^{ip'\wedge p}\,\hat{a}(\vec{p}\,')\,\hat{a}(\vec{p}) \tag{4.45}$$

$$\text{and} \qquad \hat{a}(\vec{p})\,\hat{a}^{\ddagger}(\vec{p}\,') = \delta^3(\vec{p}-\vec{p}\,') \pm\, e^{ip\wedge p'}\,\hat{a}^{\ddagger}(\vec{p}\,')\,\hat{a}(\vec{p}). \tag{4.46}$$

These equations essentially reproduce the results obtained in [26, 97, 103, 104]. On the other hand, the action of annihilation operator on an arbitrary state can be defined as

$$\hat{a}(\vec{p})\,|\vec{p}_1,\vec{p}_2,...,\vec{p}_N)_\theta = \sum_{a=0}^{N}(\pm 1)^{a-1}\, e^{\frac{i}{2}(p\wedge p_1+...+p\wedge p_{a-1}+p\wedge p_{a+1}+...+p\wedge p_n)}\delta^3(\vec{p}-\vec{p}_a)|\vec{p}_1,...\vec{p}_{a-1},\vec{p}_{a+1},...,\vec{p}_n)_\theta. \tag{4.47}$$

4.3.2 *Creation and Annihilation operators in quasi-commutative mometum basis*

Similarly, the symmetrized/anti-symmetrized quasi-commutative orthonormal basis $|\vec{p}_1,\vec{p}_2,....,\vec{p}_n))$ resolve to the identity operator on the full quantum Hilbert space $\mathcal{H}_Q^3$ as

$$\mathbf{I}_Q = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^3p_1 d^3p_2....d^3p_n |\vec{p}_1,\vec{p}_2,....,\vec{p}_n))((\vec{p}_1,\vec{p}_2,....,\vec{p}_n|\;, \tag{4.48}$$

with the overlap between any two such states on $\mathcal{H}_Q$ with distinct number of particles as

$$_\pm((\vec{p}\,'_1,\vec{p}\,'_2,...\vec{p}\,'_N|\vec{p}_1,\vec{p}_2,...\vec{p}_M))_\pm = \delta_{NM}\;_\pm((\vec{p}\,'_1,\vec{p}\,'_2,...\vec{p}\,'_N|\vec{p}_1,\vec{p}_2,...\vec{p}_M))_\pm\;. \tag{4.49}$$

In an analogous way, we can construct the creation and annihilation operators in the quasi-commutative momentum basis as

$$\hat{c}^{\ddagger}(\vec{p}) = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^3p_1 d^3p_2....d^3p_n\;|\vec{p},\vec{p}_1,\vec{p}_2,....,\vec{p}_n))((\vec{p}_1,\vec{p}_2,....,\vec{p}_n|\;; \tag{4.50}$$

$$\text{and} \qquad \hat{c}(\vec{p}) = \sum_{n=0}^{\infty}\frac{1}{n!}\int d^3p_1 d^3p_2....d^3p_n\;|\vec{p}_1,\vec{p}_2,....,\vec{p}_n))((\vec{p},\vec{p}_1,\vec{p}_2,....,\vec{p}_n|\;, \tag{4.51}$$

where these new operators are related to the ones of twisted basis (4.40, 4.41) as

$$\hat{a}(\vec{p}) = \hat{c}(\vec{p})\,e^{\frac{i}{2}p_i\theta^{ij}\hat{P}_j} \qquad \text{and} \qquad \hat{a}^{\ddagger}(\vec{p}) = e^{-\frac{i}{2}p_i\theta^{ij}\hat{P}_j}\,\hat{c}^{\ddagger}(\vec{p})\;. \tag{4.52}$$

Here, $\hat{P}_j$ is the total momentum. Note that similar expression also occurs in [26], but there $\hat{c}(\vec{p})$ and $\hat{c}^{\ddagger}(\vec{p})$ stand for entirely commutative case ($\theta = 0$), in contrast to ours, where θ-dependence persists in their defining expressions (4.50, 4.51) through the quasi-commutative basis. The fact that their (anti) commutation relations (see (4.53),(4.55) below) are just like their commutative ($\theta = 0$) counterparts, which however develops θ-deformation in the correlation function through deformed thermal wavelength in the more physical Voros basis, as will be shown subsequently, is the reason behind adopting the terminology "quasi-commutative basis". The similar actions of these new creation/annihilation operators are given by

$$|\vec{p}_1, \vec{p}_2, ..., \vec{p}_N)) = \hat{c}^\ddagger(\vec{p}_1)\hat{c}^\ddagger(\vec{p}_2)....\hat{c}^\ddagger(\vec{p}_N)|0) \quad \text{with} \quad \hat{c}^\ddagger(\vec{p})\,|\vec{p}_1, \vec{p}_2, ...\vec{p}_N)) = |\vec{p}, \vec{p}_1, \vec{p}_2, ..., \vec{p}_N))\;;$$
$$\text{and} \quad \hat{c}(\vec{p})\,|\vec{p}_1, \vec{p}_2, ..., \vec{p}_N)) = \sum_{a=0}^{N} \eta^{a-1}\delta^3(\vec{p}-\vec{p}_a)|\vec{p}_1, ...\vec{p}_{a-1}, \vec{p}_{a+1}, ..., \vec{p}_n)).$$

We can easily verify that these field operators in quasi-commutative momentum basis obey the usual (i.e. like $\theta = 0$) (anti) commutation relations:

$$\hat{c}^\ddagger(\vec{p})\,\hat{c}^\ddagger(\vec{p}\,') = \pm\,\hat{c}^\ddagger(\vec{p}\,')\,\hat{c}^\ddagger(\vec{p}) \tag{4.53}$$
$$\hat{c}(\vec{p})\,\hat{c}(\vec{p}\,') = \pm\,\hat{c}(\vec{p}\,')\,\hat{c}(\vec{p}) \tag{4.54}$$
$$\text{and} \quad \hat{c}(\vec{p})\,\hat{c}^\ddagger(\vec{p}\,') = \delta^3(\vec{p}-\vec{p}\,') \pm \hat{c}^\ddagger(\vec{p}\,')\,\hat{c}(\vec{p}). \tag{4.55}$$

Before we conclude this section, we would like to point out that a basis analogous to the quasi-commutative basis (4.30) was also occurred earlier in the literature (see e.g. the first formula of section 5 in [105]), by obtaining the images of the commutative (anti) symmetric wave functions under the generalized Weyl map-which is nothing but a unitary transformation from commutative to non-commutative wave functions. The emergence of the commutative ($\theta = 0$) (anti) commutation relations (4.53-4.55) is therefore somewhat expected.

4.4 FIELD OPERATORS

The abstract field operators, i.e. the basis independent ladder oneartors can be constructed as

$$\hat{\Psi} \equiv \hat{\Psi}(\hat{\vec{x}}) = \int d^3p\,\left(\hat{a}(\vec{p}) \otimes |\vec{p})\right) \quad \text{and} \quad \hat{\Psi}^\ddagger \equiv \hat{\Psi}^\ddagger(\hat{\vec{x}}) = \int d^3p\,\left(\hat{a}^\ddagger(\vec{p}) \otimes (\vec{p}|\right). \tag{4.56}$$

We should note that the first slot of tensor product is an operator acting on a particular quantum Hilbert space $\mathcal{H}_q^{3(n)}$ to give an element of $\mathcal{H}_q^{3(n-1)}/\mathcal{H}_q^{3(n+1)}$ corresponding to one less or one more number of particles, while the second slot of the tensor product is the momentum eigenstate belonging to quantum Hilbert space $\mathcal{H}_q^3$. Thus, we can take the position representations in Voros $|\vec{x})^V$ (2.43) or Moyal $|\vec{x})^M$ (2.46) bases of the field operators (4.56) as

$$\hat{\Psi}^\ddagger(\vec{x}^{M/V}) = \hat{\Psi}^\ddagger(\hat{\vec{x}})\left(1 \otimes |\vec{x})^{M/V}\right) = \int d^3p\,\hat{a}^\ddagger(\vec{p})\,(\vec{p}|\vec{x})^{M/V}\;; \tag{4.57}$$
$$\text{and} \quad \hat{\Psi}(\vec{x}^{M/V}) = \left(1 \otimes {}^{M/V}(\vec{x}|\right)\hat{\Psi}(\hat{\vec{x}}) = \int d^3p\,\hat{a}(\vec{p})\;{}^{M/V}(\vec{x}|\vec{p}). \tag{4.58}$$

We can clearly reproduce the momentum representations of the field operators (4.56) as the ones we defined in (4.40)-(4.41):

$$\hat{\Psi}^{\ddagger}(\vec{p}) = \hat{\Psi}^{\ddagger}(\hat{\vec{x}})(1 \otimes |\vec{p})) = \int d^3p' \, \hat{a}^{\ddagger}(\vec{p}') \, \delta^3(\vec{p}-\vec{p}') = \hat{a}^{\ddagger}(\vec{p}) \; ; \tag{4.59}$$

$$\hat{\Psi}(\vec{p}) = (1 \otimes (\vec{p}|)\hat{\Psi}(\hat{\vec{x}}) = \int d^3p' \, \hat{a}(\vec{p}') \, \delta^3(\vec{p}-\vec{p}') = \hat{a}(\vec{p}) \tag{4.60}$$

Similarly for the new oscillators $\hat{c}(\vec{p})$ (4.50) and $\hat{c}^{\ddagger}(\vec{p})$ (4.51), we have the field operators defined as

$$\hat{\Psi}_c \equiv \hat{\Psi}_c(\hat{\vec{x}}) = \int d^3p \Big(\hat{c}(\vec{p}) \otimes |\vec{p})\Big) \quad \text{and} \quad \hat{\Psi}_c^{\ddagger} \equiv \hat{\Psi}_c^{\ddagger}(\hat{\vec{x}}) = \int d^3p \Big(\hat{c}^{\ddagger}(\vec{p}) \otimes (\vec{p}|\Big) \tag{4.61}$$

with their actions defined in the analogous manner and the respective position and momentum representations can be obtained as

$$\hat{\Psi}_c^{\ddagger}(\vec{x}^{M/V}) = \hat{\Psi}_c^{\ddagger}(\hat{\vec{x}})\Big(1 \otimes |\vec{x})^{M/V}\Big) = \int d^3p \; \hat{c}^{\ddagger}(\vec{p}) \; (\vec{p}|\vec{x})^{M/V} \; ; \tag{4.62}$$

$$\hat{\Psi}_c(\vec{x}^{M/V}) = \Big(1 \otimes \,^{M/V}(\vec{x}|\Big) \; \hat{\Psi}_c(\hat{\vec{x}}) = \int d^3p \; \hat{c}(\vec{p}) \;^{M/V}(\vec{x}|\vec{p}) \; ; \tag{4.63}$$

$$\hat{\Psi}_c^{\ddagger}(\vec{p}) = \hat{c}^{\ddagger}(\vec{p}) \; ; \quad \text{and} \quad \hat{\Psi}_c(\vec{p}) = \hat{c}(\vec{p}) \; . \tag{4.64}$$

Thus, we can write the twisted and the quasi-commutative two-particle Voros/Moyal bases with positions $\vec{x}_1$ and $\vec{x}_2$ on $\mathcal{H}_q^{3(2)} \in \mathcal{H}_Q^3$:

$$|\vec{x}_1, \vec{x}_2)_{\pm\theta}^{V/M} = \hat{\Psi}^{\ddagger}(\vec{x}_1^{V/M})\hat{\Psi}^{\ddagger}(\vec{x}_2^{V/M})|0) = \int d^3p_1 \; d^3p_2 \; (\vec{p}_1|\vec{x}_1)^{V/M} \; (\vec{p}_2|\vec{x}_2)^{V/M} \; |\vec{p}_1, \vec{p}_2)_{\pm\theta} \; ; \tag{4.65}$$

$$|\vec{x}_1, \vec{x}_2))^{V/M} = \hat{\Psi}_c^{\ddagger}(\vec{x}_1^{V/M})\hat{\Psi}_c^{\ddagger}(\vec{x}_1^{V/M})|0) = \int d^3p_1 \; d^3p_2 \; (\vec{p}_1|\vec{x}_1)^{V/M} \; (\vec{p}_2|\vec{x}_2)^{V/M} \; |\vec{p}_1, \vec{p}_2)) \; . \tag{4.66}$$

Let us remind the overlaps of the Voros/Moyal position basis with the mometum eigenstate (2.45)/(2.48) as

$$(\vec{p}|\vec{x})^V = \left(\frac{\theta}{2\pi}\right)^{\frac{3}{4}} e^{-\frac{\theta}{4}p^2} \, e^{-i\vec{p}.\vec{x}} \quad \text{and} \quad (\vec{p}|\vec{x})^M = \frac{1}{(2\pi)^{\frac{3}{2}}} \, e^{-i\vec{p}.\vec{x}}. \tag{4.67}$$

Putting these, we have all the possible two-particle states with positions $\vec{x}_1, \vec{x}_2$ and momenta $\vec{p}_1, \vec{p}_2$ (in all position bases and momentum bases that we encounter in non-commutative multi-particle system) as

$$|\vec{x}_1, \vec{x}_2)_{\pm\theta}^V = \left(\frac{\theta}{2\pi}\right)^{\frac{3}{2}} \int d^3p_1 \; d^3p_2 \; e^{-\frac{\theta}{4}(\vec{p}_1^2+\vec{p}_2^2)} \; e^{-i(\vec{p}_1.\vec{x}_1+\vec{p}_2.\vec{x}_2)} \; |\vec{p}_1, \vec{p}_2)_{\pm\theta} \; ; \tag{4.68}$$

$$|\vec{x}_1, \vec{x}_2)_{\pm\theta}^M = \frac{1}{(2\pi)^3} \int d^3p_1 \; d^3p_2 \; e^{-i(\vec{p}_1.\vec{x}_1+\vec{p}_2.\vec{x}_2)} \; |\vec{p}_1, \vec{p}_2)_{\pm\theta} \; ; \tag{4.69}$$

$$|\vec{x}_1, \vec{x}_2))^V = \left(\frac{\theta}{2\pi}\right)^{\frac{3}{2}} \int d^3p_1 \; d^3p_2 \; e^{-\frac{\theta}{4}(\vec{p}_1^2+\vec{p}_2^2)} \; e^{-i(\vec{p}_1.\vec{x}_1+\vec{p}_2.\vec{x}_2)} \; |\vec{p}_1, \vec{p}_2)) \; ; \tag{4.70}$$

$$|\vec{x}_1, \vec{x}_2))^M = \frac{1}{(2\pi)^3} \int d^3p_1 \; d^3p_2 \; e^{-i(\vec{p}_1.\vec{x}_1+\vec{p}_2.\vec{x}_2)} \; |\vec{p}_1, \vec{p}_2)) \; ; \tag{4.71}$$

4.5 two-particle correlation function

In order to see the effects of such extension of non-commutative quantum mechanics to many particle systems, let us consider a canonical ensemble of free gas. The application of this pure operator formalism, where we encounter so many types of multi-particle states, can be made vivid by calculating the two-particle correlation function for a free gas, consisting of a pair of non-interacting particles:

$$C(\vec{x}_1,\vec{x}_2) = \frac{1}{Z}(\vec{x}_1,\vec{x}_2|e^{-\beta H}|\vec{x}_1,\vec{x}_2)\ , \quad \text{where} \quad Z = \int d^3\vec{x}_1\ d^3\vec{x}_2\ (\vec{x}_1,\vec{x}_2|e^{-\beta H}|\vec{x}_1,\vec{x}_2). \quad (4.72)$$

That is, Z is the canonical partition function; β is the inverse of thermodynamical temperature T of the canonical system with the Boltzmann constant $k_B = 8.617 \times 10^{-5}\ eV.K^{-1}$; and H is the non-relativistic Hamiltonian for a pair of free particles, each of mass m. That is,

$$\beta = \frac{1}{k_B T} \quad \text{and} \quad H = \frac{1}{2m}(\vec{p}_1^{\,2} + \vec{p}_2^{\,2})\ . \quad (4.73)$$

The two-particle correlation function $C(\vec{x}_1,\vec{x}_2)$ gives us the probability of finding a particle at a position $\vec{x}_1$, given another particle at position $\vec{x}_2$. Thus, from the computation of this correlation function in different bases, we can check the nature of the twisted fermions/bosons and quasi-commutative fermions/bosons. It was found in [27] that the twisted fermions on 2D Moyal plane violates the Pauli exclusion principle. However, this result was obtained in the framework where only Moyal star product is employed and the one which employed the Voros star product is given in [106]. Thus, we compute this twisted two-particle correlation function on 3D Moyal space using the pure operatorial methods of non-commutative quantum mechanics, the Hilbert-Schimdt operator method. The difference between the two position bases: Voros and Moyal, corresponding to the two star products, should also get manifested from this computation. Moreover, we have defined another momentum basis (quasi-commutative) on the non-commutative maulti-particle system which shows commutative behaviour. So, from the computation of quasi-commutative two-particle correlation function, we expect to recover the Pauli's exclusion principle on non-commutative space.

4.5.1 *Twisted correlation function*

The twisted two-particle correlation functions for a canonical ensemble of a pair of free particles on 3D Moyal space will be of the form:

$$C_\theta(\vec{x}_1,\vec{x}_2) = \frac{1}{Z_\theta}\ {}_{\pm\theta}(\vec{x}_1,\vec{x}_2|e^{-\beta H}|\vec{x}_1,\vec{x}_2)_{\pm\theta}\ , \quad (4.74)$$

where Z_θ is the non-commutative twisted partition function:

$$Z_\theta = \int d^3\vec{x}_1\ d^3\vec{x}_2\ {}^{V/M}_{\pm\theta}(\vec{x}_1,\vec{x}_2|e^{-\beta H}|\vec{x}_1,\vec{x}_2)^{V/M}_{\pm\theta}\ , \quad (4.75)$$

β is the same (4.73) and H is also the same, except for the notation of momenta:

$$H = \frac{1}{2m}(\vec{P}_1^2 + \vec{P}_2^2) \,, \tag{4.76}$$

as we denote observables of non-commutative quantum system by capital letters. Using the completeness relation $\int d^3k_1\, d^3k_2\, |\vec{k}_1, \vec{k}_2)_{\pm\theta}\, {}_{\pm\theta}(\vec{k}_1, \vec{k}_2| = \mathbf{1}_q^2$, we can write

$$C_\theta(\vec{x}_1, \vec{x}_2) = \frac{1}{Z_\theta} \int d^3k_1\, d^3k_2\, e^{-\frac{\beta}{2m}(\vec{k}_1^2+\vec{k}_2^2)} \left|{}_{\pm\theta}(\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)_{\pm\theta}\right|^2. \tag{4.77}$$

Let us take the twisted Voros (4.68) and Moyal (4.69) bases for the above position basis and we then get followings after computation:

$$\left|{}^{V}_{\pm\theta}(\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)_{\pm\theta}\right|^2 = \frac{1}{2}\left(\frac{\theta}{2\pi}\right)^3 e^{-\frac{\theta}{2}(\vec{k}_1^2+\vec{k}_2^2)} \left[1 \pm Re.\left\{e^{i\, k_2\wedge k_1}\, e^{-i(\vec{k}_1-\vec{k}_2).(\vec{x}_1-\vec{x}_2)}\right\}\right] ; \tag{4.78}$$

$$\left|{}^{M}_{\pm\theta}(\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)_{\pm\theta}\right|^2 = \frac{1}{2}\left(\frac{1}{2\pi}\right)^6 \left[1 \pm Re.\left\{e^{i\, k_2\wedge k_1}\, e^{-i(\vec{k}_1-\vec{k}_2).(\vec{x}_1-\vec{x}_2)}\right\}\right]. \tag{4.79}$$

We know that, on 3D Moyal space where θ^{ij} has a dual vector θ_k, we can always perform an $SO(3)$ rotation, $\bar{R}$ (2.29) on the classical configuration space $\mathcal{H}_c^3$. This rotation $\bar{R}$ will implement a unitary transformation on the single particle state $|\psi) \in \mathcal{H}_q^3$ and the two-particle state $|\psi_1) \otimes |\psi_2) \in \mathcal{H}_q^{3(2)}$ through the deformed co-product $\Delta_\theta(\bar{R})$ as

$$\hat{x}^i \;\rightarrow\; \hat{\bar{x}}^i = \bar{R}^i_j\, \hat{x}^j \;\Rightarrow\; |\psi) \;\rightarrow\; |\psi)^{\bar{R}} = U(\bar{R})|\psi) \,; \tag{4.80}$$

$$\text{and} \qquad |\psi_1) \otimes |\psi_2) \;\rightarrow\; |\psi_1^{\bar{R}}) \otimes |\psi_2^{\bar{R}}) = \Delta_\theta(\bar{R})\big(|\psi_1) \otimes |\psi_2)\big) \,. \tag{4.81}$$

Let us remind ourselves that $\Delta_\theta(\bar{R}) = F\, \Delta_0(\bar{R})\, F^{-1}$ (2.57), where $\Delta_0(\bar{R}) = U(\bar{R}) \otimes U(\bar{R})$ is the undeformed co-product and $F = e^{\frac{i}{2}\theta^{ij}\hat{P}_i\otimes\hat{P}_j}$ is the Drinfeld twist. With this, we have

$$\begin{aligned} {}_{\pm\theta}(\vec{x}_1, \vec{x}_2|e^{-\beta H}|\vec{x}_1, \vec{x}_2)_{\pm\theta} \;\rightarrow\; & {}_{\pm\theta}(\vec{\bar{x}}_1, \vec{\bar{x}}_2|e^{-\beta H}|\vec{\bar{x}}_1, \vec{\bar{x}}_2)_{\pm\theta} = {}_{\pm\theta}(\vec{x}_1, \vec{x}_2|\Big(F\, U^\ddagger(\bar{R}) \otimes\; U^\ddagger(\bar{R})\; F^{-1}\Big) \\ & \times\, e^{-\beta H}\Big(F\; U(\bar{R}) \otimes\; U(\bar{R})\; F^{-1}\Big)|\vec{x}_1, \vec{x}_2)_{\pm\theta} \end{aligned} \tag{4.82}$$

Since the Hamiltonian of free particle is $H = \frac{1}{2m}(\hat{\vec{P}} \otimes \mathbb{I} + \mathbb{I} \otimes \hat{\vec{P}})$, it commutes with the twist F, i.e. $[H, F] = 0$. This implies that we can just take the factor $e^{-\beta H}$ on the left or right side of F (4.82) and clearly we have

$$\Big(F\, U^\ddagger(\bar{R}) \otimes\; U^\ddagger(\bar{R})\; F^{-1}\Big)\Big(F\; U(\bar{R}) \otimes\; U(\bar{R})\; F^{-1}\Big) = \mathbb{I}. \tag{4.83}$$

Thus, there is overall no effect of rotation $\bar{R}$ on the twisted two-particle correlation function for a pair of free particle, as we have

$${}_{\pm\theta}(\vec{x}_1, \vec{x}_2|e^{-\beta H}|\vec{x}_1, \vec{x}_2)_{\pm\theta} = {}_{\pm\theta}(\vec{\bar{x}}_1, \vec{\bar{x}}_2|e^{-\beta H}|\vec{\bar{x}}_1, \vec{\bar{x}}_2)_{\pm\theta} \,, \quad \forall\, H \;:\; [H, F] = 0 \,. \tag{4.84}$$

With this, let us go back to the computation of twisted two-particle correlation functions in Voros and Moyal bases. After the rotation $\bar{R}$, we get $\theta_1 = \theta_2 = 0$ and $\theta_3 = \theta_{12} = \theta$. This simplifies the phase factor: $e^{i\ k_2 \wedge k_1}$ in (4.78),(4.79) as

$$i\, k_2 \wedge k_1 = i\, \theta^{ij}\, k_{2i}\, k_{1j} = i\, \theta\, k_{2x_1} k_{1x_2} \ \Rightarrow\ e^{i\ k_2 \wedge k_1} = e^{i\ \theta\ k_{2x_1} k_{1x_2}}. \tag{4.85}$$

Denoting the interparticle distance as $\vec{r} = \vec{x}_1 - \vec{x}_2$, we have

$$\left| {}^{V}_{\pm\theta}(\vec{x}_1, \vec{x}_2 | \vec{k}_1, \vec{k}_2)_{\pm\theta} \right|^2 = \frac{1}{2}\left(\frac{\theta}{2\pi}\right)^3 e^{-\frac{\theta}{2}(\vec{k}_1^2 + \vec{k}_2^2)} \left[1 \pm\ Re.\left\{e^{i\ \theta\ k_{2x_1} k_{1x_2}}\ e^{-i(\vec{k}_1 - \vec{k}_2).\vec{r}}\right\}\right]\ ; \tag{4.86}$$

$$\left| {}^{M}_{\pm\theta}(\vec{x}_1, \vec{x}_2 | \vec{k}_1, \vec{k}_2)_{\pm\theta} \right|^2 = \frac{1}{2}\left(\frac{1}{2\pi}\right)^6 \left[1 \pm\ Re.\left\{e^{i\ \theta\ k_{2x_1} k_{1x_2}}\ e^{-i(\vec{k}_1 - \vec{k}_2).\vec{r}}\right\}\right]. \tag{4.87}$$

Putting these in the expression of correlation functions, we get

$$C^V_\theta(\vec{r}) = \frac{1}{Z^V_\theta}\frac{1}{2}\left(\frac{\theta}{2\pi}\right)^3 \int d^3k_1\, d^3k_2\, e^{-\left(\frac{\beta + m\theta}{2m}\right)(\vec{k}_1^2 + \vec{k}_2^2)} \left[1 \pm\ Re.\left\{e^{i\theta k_{2x_1} k_{1x_2}}\ e^{-i(\vec{k}_1 - \vec{k}_2).\vec{r}}\right\}\right]; \tag{4.88}$$

$$C^M_\theta(\vec{r}) = \frac{1}{Z^M_\theta}\frac{1}{2}\left(\frac{1}{2\pi}\right)^6 \int d^3k_1\, d^3k_2\, e^{-\frac{\beta}{2m}(\vec{k}_1^2 + \vec{k}_2^2)} \left[1 \pm\ Re.\left\{e^{i\theta k_{2x_1} k_{1x_2}}\ e^{-i(\vec{k}_1 - \vec{k}_2).\vec{r}}\right\}\right]\ . \tag{4.89}$$

In the following, we introduce the mean thermal wavelength $\lambda = \sqrt{\frac{2\pi\beta}{m}}$; and $r_\perp = \sqrt{r_1^2 + r_2^2}$ and $r_\parallel = r_3$ representing the relative separations along the transverse and longitudinal directions respectively, as determined by the $\vec{\theta}$ vector. After integrating the above equations, we now obtain

$$C^V_\theta(\vec{r}) = \frac{1}{2Z^V_\theta}\left(\frac{2\pi\theta}{\lambda^2 + 2\pi\theta}\right)^3 \left[1 \pm \frac{1}{1 + \frac{4\pi^2\theta^2}{(\lambda^2 + 2\pi\theta)^2}} \exp\left\{-\frac{2\pi r_\perp^2}{(\lambda^2 + 2\pi\theta) + \frac{4\pi^2\theta^2}{(\lambda^2 + 2\pi\theta)^2}} - \frac{2\pi r_\parallel^2}{\lambda^2 + 2\pi\theta}\right\}\right] \tag{4.90}$$

$$C^M_\theta(\vec{r}) = \frac{1}{Z^M_\theta}\frac{1}{2\lambda^6}\left[1 \pm \frac{1}{1 + \frac{4\pi^2\theta^2}{\lambda^4}}\ \exp\left\{-\frac{2\pi r_\perp^2}{\lambda^2 - \frac{4\pi^2\theta^2}{\lambda^2}} + \frac{2\pi r_\parallel^2}{\lambda^2}\right\}\right]\ . \tag{4.91}$$

The corresponding partition functions can be obtained as follows:

$$Z^V_\theta = \int d^3\vec{x}_1\, d^3\vec{x}_2\, C^V_\theta(\vec{r}) = \frac{1}{2}\left(\frac{2\pi\theta}{\lambda^2 + 2\pi\theta}\right)^3 \left[V^2 \pm V\left(\frac{\lambda^2 + 2\pi\theta}{2}\right)^{\frac{3}{2}}\right]\ ; \tag{4.92}$$

$$Z^M_\theta = \int d^3\vec{x}_1\, d^3\vec{x}_2\, C^M_\theta(\vec{r}) = \frac{1}{2\lambda^6}\left[V^2 \pm V\frac{\lambda^3}{2\sqrt{2}}\right]\ , \tag{4.93}$$

where V is the volume of the system. In the thermodynamic limit $V \to \infty$, we have

$$Z^V_\theta = \frac{1}{2}\left(\frac{2\pi\theta}{\lambda^2 + 2\pi\theta}\right)^3 V^2 \quad \text{and} \quad Z^M_\theta = \frac{1}{2\lambda^6}V^2. \tag{4.94}$$

Putting these in the expressions for twisted correlation functions (4.90) and (4.91), in the thermodynamic limit $V \to \infty$, we finally obtain the twisted two-particle correlation functions in Voros and Moyal bases as

$$C_\theta^V(\vec{r}) = \frac{1}{V^2}\left[1 \pm \frac{1}{1+\frac{4\pi^2\theta^2}{(\lambda^2+2\pi\theta)^2}} \exp\left\{-2\pi\left\{\frac{r_\perp^2}{(\lambda^2+2\pi\theta)+\frac{4\pi^2\theta^2}{(\lambda^2+2\pi\theta)^2}} + \frac{r_\parallel^2}{\lambda^2+2\pi\theta}\right\}\right\}\right] \; ; \quad (4.95)$$

$$C_\theta^M(\vec{r}) = \frac{1}{V^2}\left[1 \pm \frac{1}{1+\frac{4\pi^2\theta^2}{\lambda^4}} \exp\left\{-2\pi\left\{\frac{r_\perp^2}{\lambda^2+\frac{4\pi^2\theta^2}{\lambda^2}} + \frac{r_\parallel^2}{\lambda^2}\right\}\right\}\right] . \quad (4.96)$$

With this, we can now to discuss the above results. First of all, compare the two expressions (4.95) and (4.96) of two-particle correlation functions respectively for twisted Voros and Moyal basis. The two forms look exactly the same except for $(\lambda^2 + 2\pi\theta)$ in Voros case wherever λ^2 is there in Moyal case. This extra term $2\pi\theta$ arises from the Gaussian term $e^{-\frac{\theta}{2}\vec{p}^2}$ in Voros basis (2.43). This deformation of the mean thermal wavelength in Voros case, $\lambda_V^2 = (\lambda^2 + 2\pi\theta)$, implies a natural cut-off in the mean thermal wave-length because of the non-commutativity of space. Let us remind that in usual canonical system, the quantum nature of particles with average interparticle distance r and mean thermal wavelength $\lambda = \sqrt{\frac{2\pi\beta}{m}}$ is given by the condition that $r \leq \lambda$ where temperature T and mass m play the dominant role. However, in the non-commutative system, specially when represented in the Voros basis which is a physical position basis, we found that the non-commutative parameter θ significantly contribute to this condition $r \leq \lambda \mapsto r \leq \lambda_V$ for determining the quantum nature of particles. Clearly, the mean thermal wavelength for non-commutative space system cannot be made smaller than $\sqrt{2\pi\theta}$. This is an essential feature of the non-commutative space where oscillations with wavelngth $\lesssim \sqrt{\theta}$ should be suppressed exponentially [38]. With this, the relative distance r occurring in the expression (4.95) can only be regarded as the physical distance *a la* Connes for the Voros case only. However, in both cases, there is a breaking of $SO(3)$ symmetry which can be attributed non-orthogonality of the twisted momentum basis (4.25) which carries an extra phase factor $e^{i\theta^{ij}p_{2i}p_{1j}}$ because of the twist.

Now let us discuss the crucial parts of the results. Let us put $r = 0$ and take the '$-$' sign for fermions in the expressions (4.95) and (4.96). We then get

$$C_\theta^V(\vec{r}) = \frac{1}{V^2}\left[1 - \frac{1}{1+\frac{4\pi^2\theta^2}{(\lambda^2+2\pi\theta)^2}}\right] \neq 0 \quad ; \quad C_\theta^M(\vec{r}) = \frac{1}{V^2}\left[1 - \frac{1}{1+\frac{4\pi^2\theta^2}{\lambda^4}}\right] \neq 0, \qquad (4.97)$$

which implies that there is finite probability that two fermions can sit on top of the other. This is the violation of Pauli exclusion principle. Note that in the commutative limit $\theta \to 0$ and the low temperature limit $\lambda \gg \sqrt{\theta}$, we recover the commutative results [52].

4.5.2 *Quasi-commutative Basis*

In quasi-commutative basis, we can write the two-particle correlation function as

$$C_c(\vec{x}_1, \vec{x}_2) = \frac{1}{Z_c} \int d^3k_1 \, d^3k_2 \, e^{-\frac{\beta}{2m}(\vec{k}_1^2 + \vec{k}_2^2)} \left|((\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)))\right|^2 , \tag{4.98}$$

where in the Voros and Moyal bases, we have

$$| \, {}^V((\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)) \, |^2 = \frac{1}{2}\left(\frac{\theta}{2\pi}\right)^3 e^{-\frac{\theta}{2}(\vec{k}_1^2+\vec{k}_2^2)} \left[1 \pm \, Re.\left\{e^{-i(\vec{k}_1-\vec{k}_2).(\vec{x}_1-\vec{x}_2)}\right\}\right] \; ; \tag{4.99}$$

$$| \, {}^M((\vec{x}_1, \vec{x}_2|\vec{k}_1, \vec{k}_2)) \, |^2 = \frac{1}{2}\left(\frac{1}{2\pi}\right)^6 \left[1 \pm \, Re.\left\{e^{-i(\vec{k}_1-\vec{k}_2).(\vec{x}_1-\vec{x}_2)}\right\}\right] . \tag{4.100}$$

Putting these (4.99) and (4.100) in (4.98) for different position bases, we get the following with same partition functions in the thermodynamic limit $V \to \infty$ as

$$C_c^V(r) = \frac{1}{Z_c}\frac{1}{2}\left(\frac{2\pi\theta}{\lambda^2 + 2\pi\theta}\right)^3 \left[1 \pm e^{-\frac{2\pi}{\lambda^2+2\pi\theta}r^2}\right] \quad \text{and} \quad C_c^M(r) = \frac{1}{Z_c}\frac{1}{2\lambda^6}\left[1 \pm e^{-\frac{2\pi}{\lambda^2}r^2}\right], \tag{4.101}$$

$$Z_c^V = \frac{1}{2}\left(\frac{2\pi\theta}{\lambda^2 + 2\pi\theta}\right)^3 V^2 \quad \text{and} \quad Z_c^M = \frac{1}{2\lambda^6}V^2 . \tag{4.102}$$

With this, our final expressions for the quasi-commutative two-particle correlation functions look like

$$C_c^V(r) = \frac{1}{V^2}\left[1 \pm e^{-\frac{2\pi}{\lambda^2+2\pi\theta}r^2}\right] \quad \text{and} \quad C_c^M(r) = \frac{1}{V^2}\left[1 \pm e^{-\frac{2\pi}{\lambda^2}r^2}\right]. \tag{4.103}$$

Again there is structural similarity between Moyal and Voros cases, except that in Voros case, the mean thermal wavelength gets deformed as before. Moreover, the $SO(3)$ symmetry and Pauli's exclusion principle are recovered. Note that the quasi-commutative two-particle correlation function in Moyal basis is exactly the commutative two-particle correlation function [52]. Hence, the unphysical nature of Moyal position basis regarding the non-commutativity of configuration space is revealed again by the computation of two-particle correlation function.

4.5.3 *Thermal Effective Potential*

We can then compute the effective potential for each case by putting the above expressions in the relation $V(\vec{r}) = -k_B T \ln C(\vec{r})$. For the convenience of comparison, we recast these expressions in terms of the dimensionless variables $(\frac{r_\perp}{\lambda})$, $(\frac{r_\parallel}{\lambda})$, $(\frac{r}{\lambda})$ and $(\frac{\theta}{\lambda^2})$, involving the un-deformed thermal wavelength λ:

$$\begin{aligned} V_\theta^V(r_\perp, r_\parallel) &= -k_B T \ln C_\theta^V \\ &= -k_B T \ln \left[1 + \frac{1}{\left\{1 \pm \frac{4\pi^2 \frac{\theta^2}{\lambda^4}}{(1+2\pi\frac{\theta}{\lambda^2})^2}\right\}} e^{-\left\{ \frac{2\pi}{(1+2\pi\frac{\theta}{\lambda^2})\left\{1+\frac{4\pi^2\frac{\theta^2}{\lambda^4}}{(1+2\pi\frac{\theta}{\lambda^2})^2}\right\}} \frac{r_\perp^2}{\lambda^2} + \frac{2\pi}{(1+2\pi\frac{\theta}{\lambda^2})} \frac{r_\parallel^2}{\lambda^2} \right\}} \right] \end{aligned} \quad (4.104)$$

$$\text{and} \quad V_\theta^M(r_\perp, r_\parallel) = -k_B T \ln C_\theta^M = -k_B T \ln \left[1 \pm \frac{1}{1 + \frac{4\pi^2\theta^2}{\lambda^4}} e^{-\left\{ \frac{2\pi}{\left(1+\frac{4\pi^2\theta^2}{\lambda^4}\right)} \frac{r_\perp^2}{\lambda^2} + 2\pi \frac{r_\parallel^2}{\lambda^2} \right\}} \right] . \quad (4.105)$$

For the quasi-commutative Moyal and Voros bases, we have

$$V_c^V(r) = -k_B T \ln C_c^V = -k_B T \ln \left[1 \pm e^{-\frac{2\pi}{1+2\pi\frac{\theta}{\lambda^2}} \frac{r^2}{\lambda^2}} \right] ; \quad (4.106)$$

$$\text{and} \quad V_c^M(r) = -k_B T \ln C_c^M = -k_B T \ln \left[1 \pm e^{-2\pi \frac{r^2}{\lambda^2}} \right] . \quad (4.107)$$

The respective plots of the thermal effective potentials versus the interparticle distance $\frac{r}{\lambda}$ are given in the following by taking the value $\frac{\theta}{\lambda^2} = 0.1$. Clearly, the these potential for twisted cases behave differently from the commutative potential [52], which is exactly the plot of $V_c^M(r)$. The difference is not significance for the twisted bosons from the usual bosons, however, the behaviour of twisted fermions is in quite constrast to that of usual fermions.

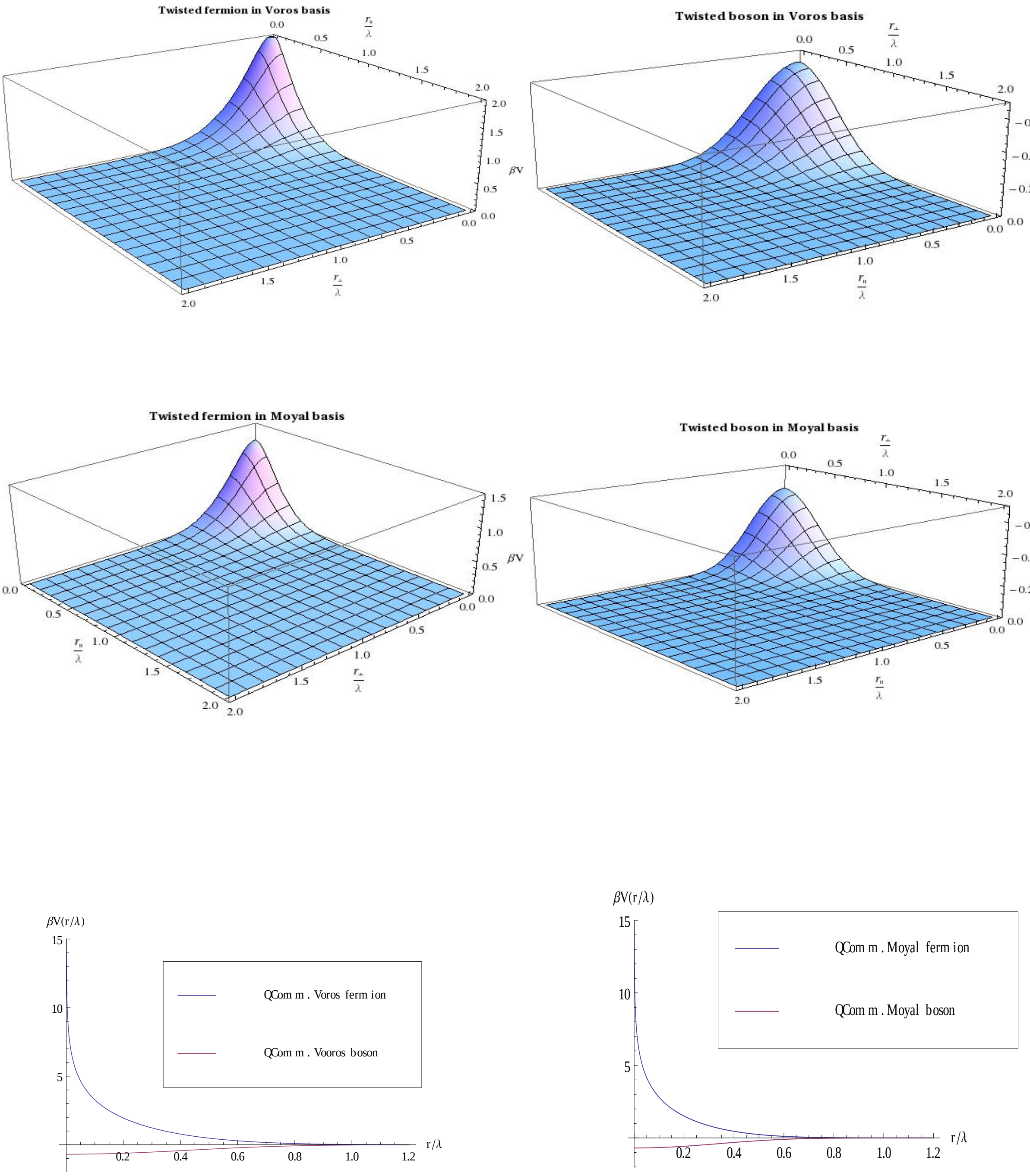

Figure 4.1: **Thermal effective potential vs distance for $\frac{\theta}{\lambda^2} = 0.1$ for each case in three-dimension. Note that, in the twisted case, this depends functionally on $r_\perp = \sqrt{r_x^2 + r_y^2}$ and $r_\parallel$, in contrast to quasi commutative case, where it depends only on r.**

Part II

Spectral distances on Non-commutative spaces

5

SPECTRAL TRIPLE AND CONNES' SPECTRAL DISTANCE

Non-commutative geometry is the most generalized form of geometry which even take care of discrete space and fractals as well [1] where the notion of points and paths are generalized with the means of algebraic structures associated with the corresponding space.

5.1 SPECTRAL TRIPLE METHOD OF NON-COMMUTATIVE GEOMETRY

As mentioned already, Connes' approach to non-commutative geometry [54] replaces the notion of a compact differentiable manifold by the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. This is given by an involutive unital algebra $\mathcal{A}$ of operators acting on a Hilbert space $\mathcal{H}$ through a representation, denoted say by π and a self-adjoint operator $\mathcal{D}$ in $\mathcal{H}$ with the conditions:

1. the resolvent $(\mathcal{D} - \lambda)^{-1}$, $\lambda \notin \text{Sp}.\mathcal{D}$ of $\mathcal{D}$ is compact (if $\mathcal{A}$ is non-unital corresponding to a locally compact Hausdorff space, the condition is $\pi(a)(\mathcal{D} - \lambda)^{-1}$,$\forall\ a \in \mathcal{A}$ is compact [107]),

2. the commutators $[\mathcal{D}, \pi(a)]$ are bounded, $\forall\ a \in \mathcal{A}$.

With this, $\mathcal{D}^{-1}$ (if $0 \notin \text{Sp}.\mathcal{D}$) or, in general, $(\mathcal{D} - \lambda)^{-1}$ plays the role of infinitesimal ds and the pure states of the algebra $\mathcal{A}$ play the role of points such that the generalized distance between a pair of pure states ω and ω' is given by

$$d(\omega, \omega') \doteq \sup_{a \in \mathcal{A}} \{ |\omega(a) - \omega'(a)| \ : \ \|[\mathcal{D}, \pi(a)]\|_{\text{op}} \leq 1 \}, \tag{5.1}$$

called the spectral distance. This reduces to the usual geodesic distance between two points on the commutative differentiable manifold.

A spectral triple is said to be **even** if there exists a self adjoint unitary operator called grading or chirality operator γ on $\mathcal{H}$ which commutes with $\pi(a)$, $\forall\ a \in \mathcal{A}$ and anticommutes with the Dirac operator $\mathcal{D}$. Moreover, the spectral triple is said to be **real** if there exists an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that the representation π^0 of the opposite algebra $\mathcal{A}^0 = \{a^0 = Ja^*J^\dagger \ : \ a \in \mathcal{A}$ with product $a^0 b^0 = (ba)^0\}$ commutes with $\pi(\mathcal{A})$. Moreover, J should commute/anticommute with γ and $\mathcal{D}$ according to the KO-dimension [54, 90].

There are more axioms for the spectral triple to satisfy so that it gives a complete description of topological and geometrical properties of a general space. However, for the computation of spectral distance these additional axioms are not necessary. The most important thing about the spectral distance is the computation of the operator norm $\|[\mathcal{D}, \pi(a)]\|_{\mathrm{op}}$ which is usually difficult. This is the challenge we take up in this thesis. In a spectral triple, it is the Dirac operator which gives the geometric information and hence plays an important role in studying the metric properties of the given C^*-algebra.

To understand the spectral triple formalism in a better way, let us consider the commutative spin geometry. Then we will go through the non-commutative generalization of spin manifold by introducting an spectral triple and compare the geometric properties.

5.2 SPIN GEOMETRY

In this section, we are reviewing the definition of a spin manifold and its corresponding Dirac operator. Here, we provide a brief outline of the spin geometry given in [90, 91, 108].

5.2.1 *Clifford algebras*

A real vector space V is said to be equipped with a nondegenerate symmetric bilinear form g if we have

$$g : V \times V \to \mathbb{R} \ ; \quad g(u, v) = g(v, u) \ \ \forall \, u, v \ \in \ V \, . \tag{5.2}$$

We can then define a real Clifford algebra $Cl(V)$ over V as an associative algebra generated by V where the Clifford product '.' is subjected to the relation:

$$u.v + v.u = 2g(u, v) \quad \forall \, u, v \ \in \ V. \tag{5.3}$$

As a vector space, this Clifford algebra $Cl(V)$ is isomorphic with the exterior algebra $\Lambda(V)$ with the exterior product '$\wedge$' subject to the relation: $u \wedge v + v \wedge u = 0 \ , \ \forall \ u, v \ \in \ V$ and $dim(Cl(V)) = dim(\Lambda(V)) = 2^n$ if $dim(V) = n$.

Given a unital real algebra A and $f : V \to A$ a real linear map satisfying $[f(v)]^2 = g(v, v).\mathbb{I}_A$, $\forall \, v \in V$, we have a unique algebra homomorphism $\tilde{f} : Cl(V) \to A$. If $A = Cl(V)$, then the linear map $v \to -v$ on V extends to an automorphism $\chi \in \mathrm{Aut}(Cl(V))$, given by

$$\chi(v_1.....v_n) := (-1)^n v_1.....v_n \ , \quad \text{satisfying} \quad \chi^2 = \mathrm{id}_A. \tag{5.4}$$

This χ gives the $\mathbb{Z}_2$-grading spliting of $Cl(V)$ into an even subalgebra (which consists of all linear combination of Clifford product of an even number of elements of V) and odd subspace:

$$Cl(V) = Cl(V)^+ \oplus Cl(V)^- \ , \ \text{where} \ \ Cl(V)^{\pm} \ \text{denotes the} \ (\pm)\text{-eigenspace of} \ \chi. \tag{5.5}$$

Complexification of Clifford algebras $Cl(V)$ is obtained by complexifying the vector space V as $V^{\mathbb{C}} = V \oplus iV$ extending the symmetric bilinear form g on V as $g(u_1 + iv_1, u_2 + iv_2) = g(u_1, v_1) + ig(u_1, v_2) + ig(v_1, u_2) - g(v_1, v_2)$ on $V^{\mathbb{C}}$. Thus, the complex Clifford algebra $\mathbb{C}l(V) = Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$ over V is an associative algebra generated by $V^{\mathbb{C}}$ with the Clifford product '.', subjected to the relation: $u.v + v.u = 2g(u,v) \quad \forall\, u, v \;\in\; V^{\mathbb{C}}$. Such complex Clifford algebras $\mathbb{C}l(V)$ are found to be isomorphic to the matrix algebras as

$$\mathbb{C}l(V) \cong \begin{cases} M_{2^m}(\mathbb{C}) & \text{if} \quad dim.V = n = 2m, \;\; \text{even;} \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) & \text{if} \quad dim.V = n = 2m+1, \;\; \text{odd.} \end{cases} \tag{5.6}$$

The relationship between the two copies of $M_{2^m}(\mathbb{C})$ for odd n is clarified by the introduction of the chirality element:

$$\gamma = (-1)^m \; e_1.....e_n \;; \quad \text{with either } n = 2m \text{ or } n = 2m+1 \text{ and } m = \left[\frac{n}{2}\right], \tag{5.7}$$

where $\{e_i\}$ is an orthonormal basis of V. We can regard a matrix algebra $M_{2^m}(\mathbb{C})$ as the space of endomorphisms of an 2^m-dimensional complex vector space $\mathbb{S}$. Thus, we have the following isomorphisms:

$$\mathbb{C}l(V) \cong \begin{cases} \text{End}(\mathbb{S}) & \text{if} \quad dim.V = n = 2m, \;\; \text{even;} \\ \text{End}(\mathbb{S}^+) \oplus \text{End}(\mathbb{S}^-) & \text{if} \quad dim.V = n = 2m+1, \;\; \text{odd,} \end{cases} \tag{5.8}$$

where $\mathbb{S}, \mathbb{S}^+, \mathbb{S}^-$ are 2^m-dimensional complex vector spaces. This yields a representation of the Clifford algebra $\mathbb{C}l(V)$ on the spinor space S for even n and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ for odd n, given by $c : \mathbb{C}l(V) \to \text{End}(\mathbb{S})$. With this, we can define the Clifford multiplication in $\mathbb{C}l(V)$ as a map $\hat{c} : \mathbb{C}l(V) \otimes \mathbb{S} \to \mathbb{S}$.

5.2.2 *Spin Manifold*

Let us consider an n-dimensional differentiable manifold M with a metric $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$. Then we have the tangent space $T_x M$ and co-tangent space $T_x^* M$ which are vector spaces at each point $x \in M$, equipped with the symmetric bilinear forms as $g(x)$ and $g^{-1}(x)$ respectively. We can then define real Clifford algebras $Cl(T_x^* M)$ over each point $x \in M$ of the manifold. We can complexify our co-tangent spaces $T_x^* M$ so that we get the complex Clifford algebras $\mathbb{C}l(T_x^* M)$. The **Clifford bundle** $\mathbb{C}l(M) \to M$ is then defined as the bundle of complex Clifford algebras generated by the co-tangent bundle $T^* M \to M$ with the symmetric bilinear form g^{-1} and is denoted by

$$\mathbb{C}l(M) := \mathbb{C}l(T^* M) = \bigcup_{x \in M} \mathbb{C}l(T_x^* M). \tag{5.9}$$

At each point $x \in M$, the Clifford algebra $\mathbb{C}l(T_x^*M)$ has a representation on a spinor space S_x. Even though all these Clifford algebras $\mathbb{C}l(T_x^*M)$ at every point $x \in M$ join to form the Clifford bundle, one question arises whether these spinor spaces S_x at every point $x \in M$ join to form an appropriate vector bundle. The answer is in affirmative iff they satisfy certain conditions. Let $A := C(M)$ represents an algebra of continuous functions on M and the space of sections of the Clifford bundle be

$$B := \begin{cases} \Gamma(M, \mathbb{C}l(T^*M)) & , n = \text{even}; \\ \Gamma(M, \mathbb{C}l^+(T^*M)) & , n = \text{odd}. \end{cases} \tag{5.10}$$

Let $\mathcal{A}$ and $\mathcal{B}$ denote the restrictions of A and B to smooth functions and smooth sections respectively. The spinor spaces S_x at each point $x \in M$ can then be joined to form a vector bundle called spinor bundle S if the Dixmier-Douady class $\delta(\underline{B})$[1] and the second Stiefel-Whitney class $\kappa(B)$ both vanish identically [90, 91]. The manifold satisfying these conditions is called a **spin manifold**. If M is a spin manifold without boundary M, we can define the spinor module as the B-$A-$bimodule $\mathcal{S} := \Gamma(M, S)$ such that the vector space $S_x \cong \mathbb{S}$ at each point $x \in M$ is an irreducible representation of the Clifford algebra $B_x \cong \mathbb{C}l(T_x^*M)$. Therefore, any spinor module $\mathcal{S}$ has a partner $\mathcal{S}^\sharp = \text{Hom}_A(\mathcal{S}, A)$ such that $\mathcal{S} \otimes_A \mathcal{S}^\sharp \simeq B$ and $\mathcal{S}^\sharp \otimes_B \mathcal{S} \simeq A$. Note that $\mathcal{S}^\sharp \simeq \Gamma(M, S^*)$ where $S^* \to M$ is the dual vector bundle to the spinor bundle $S \to M$. Given a complex vector bundle $E \to M$ and the space of sections $\mathcal{E} := \Gamma(E)$ is equipped with the $C^\infty(M)$-bilinear form $g^E : \mathcal{E} \otimes \mathcal{E} \to C^\infty(M)$, we can define a $C^\infty(M)$-valued Hermitian pairing $(s|t) = g^E(s^*, t)$, $\forall\, s, t \in \mathcal{E}$ with the scalar product on $\mathcal{E}$ as $\langle s|t \rangle = \int_M (s|t) v_g$, where v_g is the volume form on M. With such scalar product $\langle s|t \rangle$, the square-integrable spinors form a Hilbert space $L^2(M, S)$ as the completion of $\{\psi \in \mathcal{S} = \Gamma(M, S)\ ;\ \langle s|t \rangle < \infty\}$ in the norm $\|\psi\| = \sqrt{\langle \psi|\psi \rangle}$, $\forall\, \psi \in \mathcal{S}$. Then, there is a B-$A-$ bimodule isomorphism $\mathcal{S}^\sharp \simeq \mathcal{S}$ if and only if there is an antilinear endomorphism C of $\mathcal{S}$ which becomes antiunitary operator called the **charge conjugation** on the Hilbert space completion $L^2(M, S)$ of $\mathcal{S}$.

5.2.3 *Spin Connection*

A connection ∇^E on a vector bundle $E \to M$ with smooth sections $\mathcal{E} = \Gamma(M, E)$ is a linear map $\nabla^E : \mathcal{E} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{E}$, where $\mathcal{A}^1(M) = \Gamma(M, T^*M)$, satisfying the Leibniz rule,

$$\nabla^E(\sigma f) = \nabla^E(\sigma) f + \sigma \otimes df\ ,\ \ \forall\, \sigma \in \mathcal{E}\ ,\ f \in \mathcal{A} = C^\infty(M)\ ;\ d \text{ is the exterior derivative.} \tag{5.11}$$

We can equivalently define a map $\nabla_X^E : \mathfrak{X}(M) \otimes \mathcal{E} \to \mathcal{E}$ called covariant derivative, which includes a vector field $X \in \mathfrak{X}(M)$ satifying the Leibniz rule:$\nabla_X^E(\sigma f) = \nabla_X^E(\sigma) f + \sigma X(f)$. If

1 $\underline{B}$ denotes the collection of fibres: $\underline{B} := \begin{cases} \{B_x = \mathbb{C}l(T^*M) : x \in M\}, & \text{if } dim.M \text{ is even} \\ \{B_x = \mathbb{C}l^+(T^*M) : x \in M\}, & \text{if } dim.M \text{ is odd} \end{cases}$ which is locally trivial.

$\mathcal{E} = \mathfrak{X}(M) = \Gamma(M, TM)$, then the connection which is torsion-free: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, $\forall\, X, Y \in \mathfrak{X}(M)$ and compatible with the metric g: $g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = Z(g(X, Y))$ is called the **Levi-Civita connection** ∇^g. The dual connection on the co-tangent bundle T^*M with smooth sections $\mathcal{A}^1(M)$ is referred to as the Levi-Civita connection on 1-forms and is also denoted by ∇^g. If $\mathcal{E}$, which is an $\mathcal{A}$-module, is equipped with an $\mathcal{A}$-valued Hermitian pairing $(.|.)$ then the connection ∇ is said to be Hermitian if $(\nabla s|t) + (s|\nabla t) = d(s|t)$.

To define the connection[2] on the spinor bundle S with the finitely generated projective spinor module $\mathcal{S} = \Gamma(M, S)$, let us extend the Clifford multliplication $\hat{c} : \mathcal{B}_x \otimes S_x \to S_x$ to the spinor module $\mathcal{S}$ which carries an action of $\mathcal{B}$ as

$$\hat{c}(b \otimes \psi)(x) = \hat{c}\big(b(x) \otimes \psi(x)\big)\ ,\ \forall\, b \in \mathcal{B} \text{ and } \psi \in \mathcal{S} \ \Leftrightarrow\ \hat{c}(b \otimes \psi) = c(b)\psi. \tag{5.12}$$

Using the inclusion $\mathcal{A}^1(M) \hookrightarrow \mathcal{B}$, we can define a map $\hat{c} : \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \to \mathcal{S}$. In odd dimensional case, $\mathcal{B}$ consists of only even elements and the inclusion is given by $c(\alpha) := c(\alpha\gamma)$, $\forall\, \alpha \in \mathcal{A}^1(M)$. On a local basis dx^μ of $\mathcal{A}^1(M)$, let us introduce $\gamma^\mu := c(dx^\mu)$.

The **spin connection** on the spinor bundle $S \to M$ is a Hermitian connection $\nabla^S : \mathcal{S} \to \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S}$ which is compatible with the action of $\mathcal{B}$ in the following way

$$\nabla^S\big(c(\alpha)\psi\big) = c(\nabla^g\alpha)\psi + c(\alpha)\nabla^S\psi\ , \quad \forall\, \alpha \in \mathcal{A}^1(M),\ \psi \in \mathcal{S}\ ; \tag{5.13}$$

where ∇^g is the Levi-Civita connection on $\mathcal{A}^1(M)$ and $\nabla^S_X C = C\nabla^S_X$, where $\nabla^S_X : \mathfrak{X}(M) \otimes \mathcal{S} \to \mathcal{S}$, for real $X \in \mathfrak{X}(M)$.

The **Dirac operator** $\not{D} : \mathcal{S} \to \mathcal{S}$ is a first-order partial differential operator associated to the spin connection ∇^S as

$$\not{D} := -i\hat{c} \circ \nabla^S\ , \qquad \text{where} \quad \mathcal{S} \xrightarrow{\nabla^S} \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{S} \xrightarrow{\hat{c}} \mathcal{S}. \tag{5.14}$$

In local coordinates, where the spin connection is given locally as $\nabla^S = dx^\mu \otimes \nabla^S_\mu$, by applying the Clifford multiplication $\hat{c}$ on dx^μ we can locally write the Dirac operator as

$$\not{D} = -i\gamma^\mu\nabla^S_\mu\ , \quad \text{where} \quad \nabla^S_\mu = \partial_\mu + \frac{1}{2}\omega^{ab}_\mu\gamma_a\gamma_b. \tag{5.15}$$

Note that $\nabla^g_\mu e_a = \omega^b_{\mu a}e_b$ where $\{e_a\}$ is the local veilbein such that $c(e_a) = \gamma_a$.

5.3 connes' spectral triple of a compact spin manifold

Given a compact orientable Riemannian spin manifold (M, g), we can define the following spectral triple:

$$\big(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(M, S), \not{D} = -i\gamma^\mu\nabla^S_\mu\big). \tag{5.16}$$

2 the lift of the Levi-Civita connection

The action π of $\mathcal{A}$ on $\mathcal{H}$ is defined through the pointwise multiplication : $(\pi(a)\psi)(x) = a(x)\psi(x),\ \forall\, a \in \mathcal{A}, \psi \in \mathcal{H}$.

The pure states of $\mathcal{A}$ are the evaluations at points: $\delta_x(a) = a(x)$ and the commutator $[D\!\!\!/, \pi(a)]$ can be obtained as

$$\begin{aligned}[D\!\!\!/, \pi(a)] &= -i\hat{c}\big(\nabla^S(a\psi)\big) + ia\hat{c}(\nabla^S\psi) \\ &= -i\hat{c}\{a\nabla^S\psi + da\otimes\psi\} + ia\hat{c}(\nabla^S\psi)\ , \quad \text{where } \ \hat{c}(a) = a \\ &= -i\hat{c}(da)\psi\ ,\ \ \forall\ \psi \in \mathcal{H}.\end{aligned}$$

Thus, $[D\!\!\!/, \pi(a)] = -i\hat{c}(da) = -i\gamma^\mu\partial_\mu a$ so that its operator norm is

$$\|[D\!\!\!/, \pi(a)]\|^2 = \sup_{x\in M}\|c_x\big(da(x)\big)\|^2 = \sup_{x\in M}\|\gamma^\mu\partial_\mu a\|^2;\ \ \|\gamma^\mu\partial_\mu a\|^2 = (\gamma^\mu\partial_\mu a|\gamma^\nu\partial_\nu a), \tag{5.17}$$

where $(\gamma^\mu\partial_\mu a|\gamma^\nu\partial_\nu a) = (\partial_\mu a)^*\partial_\nu a\ g^{-1}\big(\gamma^\mu, \gamma^\nu\big) = \partial_\mu a^*\partial_\nu a\ g^{\mu\nu} = \|\text{grad } a|_x\|^2$, where g^{-1} is the symmetric bilinear form on T^*M and grad $a|_x \in \mathfrak{X}(M)$ is the gradient of $a \in C^\infty(M)$ at point $x \in M$. That is,

$$\|[D\!\!\!/, \pi(a)]\|^2 = \sup_{x\in M}\|c_x\big(da(x)\big)\|^2 = \sup_{x\in M}\|\text{grad } a|_x\|^2 = \|\text{grad } a\|^2_\infty. \tag{5.18}$$

Now the Connes' spectral distance formula between two pure states δ_x and δ_y is given by

$$\begin{aligned} d_{D\!\!\!/}(\delta_x, \delta_y) &= \sup_{a\in\mathcal{A}}\Big\{|\delta_x(a) - \delta_y(a)| : \|[D\!\!\!/, \pi(a)]\| \le 1\Big\} \\ &= \sup_{a\in\mathcal{A}}\Big\{|a(x) - a(y)| : \|\text{grad } a\|_\infty \le 1\Big\}. \end{aligned} \tag{5.19}$$

Let us introduce a curve $\lambda : [0,1] \to M$ such that $\lambda(0) = x$ and $\lambda(1) = y$. We define the geodesic distance between two points x and y on the Riemannian manifold (M, g) as

$$d_{geo}(x,y) = \inf_{\lambda\in M}\{\text{length of curve } \lambda \text{ from } x \text{ to } y\} = \inf_{\lambda\in M}\int_0^1\sqrt{g_{\mu\nu}dx^\mu dx^\nu}. \tag{5.20}$$

Let us now see how the Connes' spectral distance between two pure states δ_x and δ_y reduces to the geodesic distance between two points x and y. Clearly, we can write

$$|a(x) - a(y)| = |a(\lambda(0)) - a(\lambda(1))| = \Big|\int_0^1\frac{d}{dt}\{a(\lambda(t))\}dt\Big| = \Big|\int_0^1 \text{grad } a|_{\lambda(t)}\dot{\lambda}(t)dt\Big|. \tag{5.21}$$

Since $\Big|\int_0^1 \text{grad } a|_{\lambda(t)}\dot{\lambda}(t)dt\Big| \le \int_0^1|\text{grad } a|_{\lambda(t)}|\dot{\lambda}(t)|dt \le \|\text{grad } a\|_\infty\int_0^1|\dot{\lambda}(t)|dt$, we get

$$|a(x) - a(y)| \le \|\text{grad } a\|_\infty\ \{\text{Length of } \lambda \text{ from } x = \lambda(0) \text{ to } y = \lambda(1)\}. \tag{5.22}$$

Thus we can see that the Connes' spectral distance is bounded above by the geodesic distance as

$$d_{\not{D}}(\delta_x, \delta_y) = \sup_{a \in \mathcal{A}} \left\{ |a(x) - a(y)| : \|\text{grad } a\|_\infty \leq 1 \right\} \leq d_{geo}(x, y). \tag{5.23}$$

To prove that this inequality gets saturated for some $a \in C^\infty(M)$ such that Connes' spectral distance is indeed a geodesic distance on a Riemannian spin manifold, we first note that our search here can be restricted to real functions only [110]. We therefore introduce some function $a'_p(x) = d_{geo}(x, p)$ for a fixed given point $p \in M$. We know that the geodesic distance function saturates the triangle inequality: $|a'_p(x) - a'_p(y)| \leq d_{geo}(x, y)$ for p lying on the geodesic connecting the points x and y and $\|\text{grad } a'_p\|_\infty = \|[\not{D}, \pi(a'_p)]\| = 1$. This implies that $a'_p(x) = d_{geo}(x, p) \in \mathcal{A}$ saturates the inequality (5.23). Hence, the Connes' spectral distance between two pure states δ_x and δ_y for the commutative spectral triple (5.16) gives the geodesic distance between two points x and y of the manifold M:

$$d_{\not{D}}(\delta_x, \delta_y) = d_{geo}(x, y). \tag{5.24}$$

5.4 SPECTRAL DISTANCE ON REAL LINE $\mathbb{R}^1$

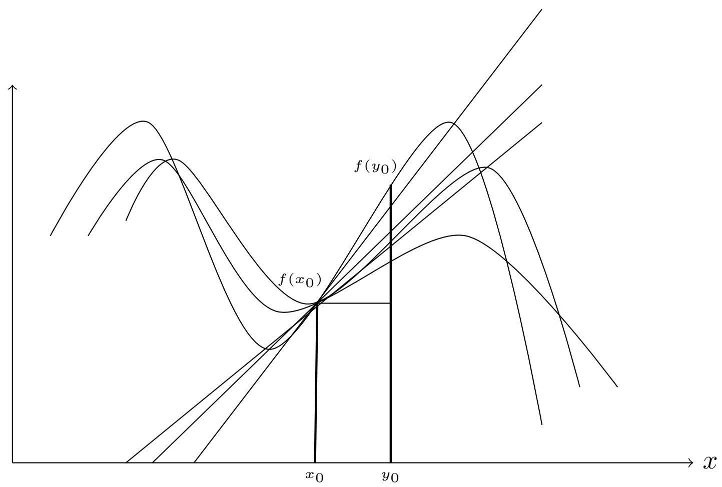

Figure 5.1: Computation of Connes' spectral distance in 1D

To illustrate the ideas perhaps in the simplest possible way, let us consider a "down-to-earth" example of the 1D real line, where the spectral triple is $(\mathcal{A} = C_0^\infty(\mathbb{R}^1), \mathcal{H} = L^2(\mathbb{R}^1), \mathcal{D} = -i\frac{d}{dx})$ with the action of $\mathcal{A}$ on $\mathcal{H}$: $(\pi(f)\psi)(x) = f(x)\psi(x), \ \forall \ \psi \in L^2(\mathbb{R}^1)$, we have the commutator $[\mathcal{D}, \pi(f)]\psi = -i\left(\frac{df}{dx}\right)\psi$ yielding the ball condition $\|[\mathcal{D}, \pi(f)]\| =$

$\|\frac{df}{dx}\| \leq 1$. The pure states of $f \in \mathcal{A}$ are just evaluations at point $x \in \mathbb{R}^1$: $\delta_x(f) = f(x)$. The Connes' spectral distance in 1D real space $\mathbb{R}^1$ is then given by

$$\begin{aligned} d(\delta_x, \delta_y) &= \sup_{f \in \mathcal{A}} \left\{ |\delta_{x_0}(f) - \delta_{y_0}(f)| \ : \ \|[\mathcal{D}, \pi(f)]\| \leq 1 \right\} \\ &= \sup_{f \in \mathcal{A}} \left\{ |f(x_0) - f(y_0)| \ : \ \left\| \frac{df}{dx} \right\| \leq 1 \right\} = |x_0 - y_0| \, . \end{aligned} \tag{5.25}$$

In the spirit of Gelfand and Naimark [55], this can be regarded as the distance between the points x and y.

Note that $\forall \ f \in C_0^\infty(\mathbb{R}^1)$, as shown in fig. 5.1, the ball condition $\|\frac{df}{dx}\| \leq 1$ chooses those functions for which the slope $|\frac{df}{dx}| \leq 1$ such that the supremum of the difference of the functions is bounded by the distance between the points at which the functions are evaluated. The functions which saturate this bound are $f(x_0) = \pm x_0 + k$, where k is some constant.

5.5 SPECTRAL DISTANCE ON TWO-POINT SPACE, A DISCRETE SPACE [107, 109]

Now, let us consider the simplest example of finite space which is the space consisting of only two points $X_2 = (a, b)$. For this space, the algebra is $\mathcal{A}_2 = \mathbb{C} \oplus \mathbb{C}$ with the elements f which are given by a pair of complex numbers $f(a)$ and $f(b)$ which are the evaluations of the function $f \in \mathcal{A}_2$ at the points a, b. A pair f_1 and f_2 of such functions necessarily satisfy [107]

$$(f_1 f_2)(a) = f_1(a) f_2(a) \, , \quad (f_1 f_2)(b) = f_1(b) f_2(b) \, . \tag{5.26}$$

Clearly, $\mathcal{A}_2$ can be written as the diagonal sub-algebra of $\mathcal{A}_2$ and itself can be defined to be the representation π of $f \in \mathcal{A}_2$ which in turn acts on $\mathcal{H}_2$, taken to be also as $\mathbb{C}^2$ as,

$$f \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \pi(f) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \, , \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in \mathbb{C}^2 \equiv \mathcal{H}_2 \, . \tag{5.27}$$

The Dirac operator can be taken as an off-diagonal matrix [109]:

$$\mathcal{D}_2 = \mathcal{D}_2^\dagger = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix} \, , \quad \Lambda \in \mathbb{C} \, . \tag{5.28}$$

The commutator $[\mathcal{D}_2, \pi(a)] = (f(a) - f(b)) \begin{pmatrix} 0 & \Lambda \\ -\bar{\Lambda} & 0 \end{pmatrix}$ such that the operator norm is $\|[\mathcal{D}_2, \pi(a)]\| = |f(a) - f(b)||\Lambda|$. The spectral distance between these two pure (normal) states $\omega_a(f) = \mathrm{Tr}\{\rho_a \pi(f)\} = f(a)$ and $\omega_b(f) = \mathrm{Tr}\{\rho_b \pi(f)\} = f(b)$ representing the two points a and b on X_2 in terms of the associated density matrices $\rho_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is given by

$$d_{\mathcal{D}_2}(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}_2} \left\{ |f(a) - f(b)| \; : \; |f(a) - f(b)||\Lambda| \leq 1 \right\} = \frac{1}{|\Lambda|}. \tag{5.29}$$

If $\Lambda = 0$, then the spectral distance becomes infinite.

5.6 SPECTRAL DISTANCE ON THE STATE SPACE $\mathbb{CP}^1$ OF $M_2(\mathbb{C})$ [82, 110]

Let us now consider a non-commutative spectral triple i.e., for which the algebra is non-commutative. This is perhaps the simplest example of non-commutative spectral triples where the algebra is the algebra of complex 2×2 matrices :

$$\mathcal{A} = M_2(\mathbb{C}) \; , \quad \mathcal{H}_2 = \mathbb{C}^2 \; , \quad \mathcal{D}_2 = \begin{pmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{pmatrix} = |\Lambda| \begin{pmatrix} 0 & e^{i\lambda} \\ e^{-i\lambda} & 0 \end{pmatrix} \; ; \; \Lambda = |\Lambda| e^{i\lambda} \; . \tag{5.30}$$

The Dirac operator $\mathcal{D}_2$, being a Hermitian matrix, can be brought to a diagonal form:

$$\mathcal{D}_2^U = U^\dagger \mathcal{D}_2 U = \begin{pmatrix} -|\Lambda| & 0 \\ 0 & |\Lambda| \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \; ; \; \text{where} \; U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\lambda} \\ -e^{-i\lambda} & 1 \end{pmatrix} . \tag{5.31}$$

Clearly, $D_1 = -|\Lambda|$ and $D_2 = |\Lambda|$ are the real and distinct eigenvalues of $\mathcal{D}$.

Now any normalized vector $\chi = \begin{pmatrix} \chi_1 = x_1 + i x_2 \\ \chi_2 = x_3 + i x_4 \end{pmatrix} \in \mathbb{C}^2$ with the normalization condition $\chi^\dagger \chi = |\chi_1|^2 + |\chi_2|^2 = 1$ can be used to define a map $\omega : M_2(\mathbb{C}) \to \mathbb{C}$ such that $\forall \; a \in \mathcal{M}_2(\mathbb{C}), \; a \mapsto \chi^\dagger a \chi$. Such maps $w(a) = \chi^\dagger a \chi \in \mathbb{C}$ are referred to as the vector states of $a \in M_2(\mathbb{C})$ and are gauge invariant under the $U(1)$ transformation $\chi \to \chi' = e^{i\theta}\chi$. This implies that the space of vector states $\mathcal{S}(M_2(\mathbb{C})) \simeq \mathbb{CP}^1$. Then choosing a local gauge, say $\chi_2 = \chi_2^* = x_3$, the above normalization condition $\chi^\dagger \chi = 1$ reduces from $\mathbb{S}^3$ $(x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1)$ to $\mathbb{S}^2$ $(x_1^2 + x_2^2 + x_3^2 = 1)$ so that we can identify $\mathbb{CP}^1 \simeq \mathbb{S}^2$. We can then use the local parametrization $\chi = \begin{pmatrix} \sin \frac{\theta}{2} e^{i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix}$ with the further association of $\chi \in CP^1$ to $\vec{x} \in S^2$ satisfying $\vec{x}^2 = 1$ where the respective components are $x_1 = \sin\theta \cos\phi, \; x_2 = \sin\theta \sin\phi, \; x_3 = \cos\theta$.

We can now associate a density operator ρ on $\mathcal{S}(M_2(\mathbb{C}))$ as $\rho = \chi\chi^\dagger$ so that we can define the normal state on the algebra $M_2(\mathbb{C})$ as $\omega_\rho(a) = \text{Tr}(\rho a)$. The spectral distance between two normal states can then be written as

$$d_{\mathcal{D}_2}(\omega_{\rho'}, \omega_\rho) = \sup_{a \in M_2(\mathbb{C})} \{|\omega_{\rho'}(a) - \omega_\rho(a)| \ : \ \|[\mathcal{D}_2, \pi(a)]\|_{\text{op}} \leq 1\}. \tag{5.32}$$

It is shown in [110] that the above supremum value will be attained by the positive and hermitian elements of algebra. This implies that we can restrict our search to the subset of positive and hermitian elements only. And since such elements remain so after a unitary transformation like (5.31). Parametrising this as $a \to U^\dagger a U \equiv a^U = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix}$, we get the simplified commutator as

$$[\mathcal{D}_2, \pi(a)] \xrightarrow{U} [\mathcal{D}_2^U, \pi(a^U)] = (D_1 - D_2) \begin{pmatrix} 0 & a_{12} \\ -a_{12}^* & 0 \end{pmatrix}. \tag{5.33}$$

Clearly, the ball condition gives the following bound on the off-diagonal elements of a^U only

$$\|[\mathcal{D}_2, \pi(a)]\|_{\text{op}} = \|[\mathcal{D}_2^U, \pi(a^U)]\|_{\text{op}} = |a_{12}||D_1 - D_2| \leq 1 \ \Rightarrow \ |a_{12}| \leq \frac{1}{|D_1 - D_2|}, \tag{5.34}$$

leaving the diagonal elements a_{11} and a_{22} completely unconstrained. This can result in divergent distances between certain pair of states, as we shall see.

Now, to find the difference between two normal states $\omega_{\rho'}(a)$ and $\omega_{\rho'}(a)$ with the density operators ρ' and ρ respectively, let us introduce $\Delta\rho = \rho' - \rho$ which can be written as a matrix

$$\Delta\rho = \chi'\chi'^\dagger - \chi\chi^\dagger = \begin{pmatrix} \chi'_1\chi'^*_1 - \chi_1\chi_1^* & \chi'_1\chi'^*_2 - \chi_1\chi_2^* \\ \chi'_2\chi'^*_1 - \chi_2\chi_1^* & \chi'_2\chi'^*_2 - \chi_2\chi_2^* \end{pmatrix}. \tag{5.35}$$

We then have

$$|\omega_{\rho'}(a) - \omega_\rho(a)| = |\text{Tr}(\Delta\rho\ a)| = \Big| \sum_{i,j=1}^{2} (\chi'_i\chi'^*_j - \chi_i\chi_j^*) a_{ij} \Big| \ = \ |a_{11}(|\chi'_1|^2 - |\chi_1|^2) + a_{22}(|\chi'_2|^2 - |\chi_2|^2)$$
$$+ \ 2\text{Re.}\{a_{12}(\chi'_1\chi'^*_2 - \chi_2\chi_1^*)\}|. \tag{5.36}$$

To proceed further, we parametrize χ and χ' as $\chi = \begin{pmatrix} \chi_1 = \sin\frac{\theta}{2} e^{i\varphi} \\ \chi_2 = \cos\frac{\theta}{2} \end{pmatrix}$ and $\chi' = \begin{pmatrix} \chi'_1 = \sin\frac{\theta'}{2} e^{i\varphi'} \\ \chi'_2 = \cos\frac{\theta'}{2} \end{pmatrix}$. With this, the computation of the spectral distance can be carried on for two different cases:

1. Distance between a pair of states with different lattitudes:

Here, $\theta \neq \theta'$ so that $|\chi'_1|^2 \neq |\chi_1|^2$ and $|\chi'_2|^2 \neq |\chi_2|^2$. Since the spectral distance is the supremum of (5.36) with the condition (5.34), which imposes no bound on a_{11} and a_{22} and hence the spectral distance between two points lying on different lattitudes on $\mathbb{S}^2$ with the Dirac operator $\mathcal{D}_2$ diverges.

2. Distance between a pair of states lying on the same lattitude:

Here, $|\chi'_1|^2 = |\chi_1|^2$ and $|\chi'_2|^2 = |\chi_2|^2$ so that (5.36) reduces to

$$|\omega_{\rho'}(a) - \omega_\rho(a)| = |\mathrm{Tr}(\Delta\rho)| = |2\mathrm{Re.}\{a_{12}(\chi'_1\chi'^*_2 - \chi_2\chi_1^*)\}| \,, \tag{5.37}$$

which depends on just a_{12}, as the diagonal entries a_{11} and a_{22} get eliminated.

Taking $a_{12} = |a_{12}|e^{i\alpha}$, we get

$$a_{12}(\chi'_1\chi'^*_2 - \chi_2\chi_1^*) = |a_{12}| \cos\frac{\theta}{2} \sin\frac{\theta}{2}(e^{i\varphi'} - e^{-i\varphi})e^{i\alpha}. \tag{5.38}$$

Choosing $\alpha = \frac{1}{2}(\pi + \varphi - \varphi')$, we can write

$$a_{12}(\chi'_1\chi'^*_2 - \chi_2\chi_1^*) = |a_{12}| \cos\frac{\theta}{2} \sin\frac{\theta}{2} i\left(e^{\frac{i}{2}(\varphi'-\varphi)} - e^{-\frac{i}{2}(\varphi'-\varphi)}\right) = -|a_{12}| \sin\theta \sin\left(\frac{\varphi' - \varphi}{2}\right), \tag{5.39}$$

which take real values. Thus, we have using (5.34),

$$|\omega_{\rho'}(a) - \omega_\rho(a)| = |\mathrm{Tr}(\Delta\rho)| = 2|a_{12}||\sin\theta|\left|\sin\left(\frac{\varphi'-\varphi}{2}\right)\right| \leq \frac{2\sin\theta}{|D_1 - D_2|}\left|\sin\left(\frac{\varphi'-\varphi}{2}\right)\right|. \tag{5.40}$$

Hence, the spectral distance between a pair of points (θ, φ) and (θ, φ') lying on the same latitude of $CP^1 \approx S^2$ or more precisely between the associated states $\omega_{\rho'}$ and ω_ρ is obtained as

$$d_{\mathcal{D}_2}(\omega_{\rho'}, \omega_\rho) = \frac{2\sin\theta}{|D_1 - D_2|}\left|\sin\left(\frac{\phi - \phi'}{2}\right)\right| = \frac{\sin\theta}{|\Lambda|}\left|\sin\left(\frac{\phi - \phi'}{2}\right)\right| . \tag{5.41}$$

This can be identified as the chordal distance.

As a corollary, the infinitesimal spectral distance can be obtained by replacing $\varphi' - \varphi \to d\varphi$ as

$$d_{\mathcal{D}_2}(\omega_{\rho + d\rho}, \omega_\rho) = \frac{\sin\theta \, d\varphi}{2|\Lambda|} \,. \tag{5.42}$$

However, the important point to emphasise at this stage is that this process cannot be inverted. Unlike the case of Riemannian differentiable manifold this infinitesimal distance cannot be integrated to yield finite distance (5.41). This is related to the fact that the interpolating states of the chord can be shown to correspond to the mixed states in contrast to the extremal points, associated to the pure states.

With this, we will follow up in the next chapters the computation of Connes' spectral distance on non-commutative spaces: Moyal plane and fuzzy sphere by introducing appropriate spectral triples which is adaptable to the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics, which we reviewed in chapter 2.

We have seen that the configuration spaces of a non-commutative spaces like Moyal plane and Fuzzy sphere, which are themselves Hilbert sapces and furnish the unitary representaions of the respective coordinate algebras of types (1.17),(1.19) are analogous to the Hilbert space of a commutative quantum system. In [80], it was shown that the quantum state space of a commutative quantum system has a natural Riemannian metric and other geometrical properties like curvature tensor was presented for the generalized coherent states manifolds. For this, we will review the construction of coherent states on Moyal plane $\mathbb{R}^2_\star$ and fuzzy sphere $\mathbb{S}^2_\star$ so that the corresponding manifolds of these coherent states can be compared with the respective undeformed spaces: Euclidean plane $\mathbb{R}^2$ and commutative 2-sphere $\mathbb{S}^2$.

5.7 COHERENT STATES

Let a Lie group G be a dynamical symmetry group of a given quantum system described by the quantum Hilbert space $\mathcal{H}$. Let $T(g)$ be a unitary irreducible representation of $g \in G$ acting on the Hilbert space $\mathcal{H}$. We can then define the system of coherent states $\{|\psi_g\rangle\}$ by the action of $T(g)$ on a fixed normalized reference state $|\psi_0\rangle \in \mathcal{H}$ [111]:

$$|\psi_g\rangle = T(g)|\psi_0\rangle. \tag{5.43}$$

Let $H \subset G$ be a subgroup with elements h which leave reference state $|\psi_0\rangle$ invariant up to a phase factor,

$$T(h)|\psi_0\rangle = e^{i\phi(h)}|\psi_0\rangle \ , \quad |e^{i\phi(h)}| = 1 \ \ \forall\, h \in H. \tag{5.44}$$

This implies that every element $g \in G$ can be decomposed as

$$g = \Omega h \ ; \ \ h \in H \quad \text{and} \quad \Omega \in X = G/H, \ \ \text{Coset space}. \tag{5.45}$$

Then, elements g and g' of G with different h and h' but with same Ω produce coherent states which differ only by a phase factor:

$$|\psi_g\rangle = e^{i\delta}|\psi_{g'}\rangle \ , \quad \text{where} \quad \delta = \phi(h) - \phi(h'). \tag{5.46}$$

Thus, the corresponding density matrices are equal : $|\psi_g\rangle\langle\psi_g| = |\psi_{g'}\rangle\langle\psi_{g'}|$ and characterises the space of equivalence classes i.e., points of coset space $X = G/H$. A coherent state $|\Omega\rangle \equiv |\psi_\Omega\rangle$ is determined by a point $\Omega = \Omega(g)$ in the coset space $X = G/H$.

In case, if $G = H_3$, the Heisenberg-Weyl group then Glauber coherent states are defined on the complex plane $\mathbb{C} = H_3/U(1)$ which serve as the homogeneous space of Moyal plane. If $G = SU(2)$, the spin coherent states are defined on the unit sphere $\mathbb{S}^2 = SU(2)/U(1)$ which serve as the homogeneous space of fuzzy sphere.

5.7.1 *Moyal plane*

Let us remind that the coherent states (2.12) on the configuration space $\mathcal{H}_c$ (2.2) of Moyal plane are quantum states which are the eigenstates of the annihilation operator $\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2)$ (2.1) as

$$|z\rangle = U(z)|0\rangle = e^{-z\bar{z}/2}\, e^{z\hat{b}^\dagger}|0\rangle = e^{-z\bar{z}/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}z^n|n\rangle\ ;\ \hat{b}|z\rangle = z|z\rangle\ ,\ \ \langle z|\hat{b}^\dagger = \bar{z}\langle z|. \quad (5.47)$$

where

$$U(z) \;=\; e^{z\hat{b}^\dagger - \bar{z}\hat{b}} \;=\; e^{-z\bar{z}/2}\, e^{z\hat{b}^\dagger} e^{-\bar{z}\hat{b}}\ . \quad (5.48)$$

These coherent states are the quantum states which saturates the Heisenberg's uncertainty relation: $\Delta\hat{x}_1\Delta\hat{x}_2 = \frac{\theta}{2}$. Each state $|z\rangle$ which is obtained by the action of a unitary operator $U(z)$ on some fixed vector, say the vacuum state $|0\rangle$, is specified by a complex number z. It is well-known that $\||z\rangle - |z'\rangle\| \to 0$ as $|z - z'| \to 0$ and hence implies that coherent states have the properties of the classical states.

They admit a resolution of the identity on $\mathcal{H}_c$ but they are not orthogonal

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = \mathbb{1}_c \quad \text{and} \quad \langle z_1|z_2\rangle = e^{-z_1\bar{z}_1/2 - -z_2\bar{z}_2/2 + z_2\bar{z}_1} \neq \delta(z_1 - z_2)\ , \quad (5.49)$$

as they furnish an over-complete basis.

5.7.2 *Fuzzy sphere*

Let us review the construction of generalized coherent states of $SU(2)$ group [86] such that the non-commutative analog of the homogeneous space of the fuzzy sphere can be constructed by using the Perelomov's $SU(2)$ coherent states [87]. Note that the Heisenberg uncertainty relations for fuzzy sphere is given by

$$\Delta\hat{x}_1\Delta\hat{x}_2 \geq \frac{\theta_f}{2}|\langle\hat{x}_3\rangle| \Longrightarrow \Delta\hat{x}_1\Delta\hat{x}_2 = \frac{1}{2}\theta_f^2 j \quad \text{for both} \quad |j, j\rangle,\ |j, -j\rangle \quad (5.50)$$

Thus, we can choose the "vacuum state" as the highest weight state $|j,j\rangle$ associated to the north pole and the Perelomov $SU(2)$ coherent states can be obtained by the action of representation $T(g)$ of $g \in SU(2)$ acting on $|j,j\rangle$. Note that any action of $\hat{X}_3$ on the vacuum state does not change it so that the group element generated by $\hat{X}_3$ is the stability subgroup $U(1)$ of $SU(2)$. This implies that the set of generalized coherent states of $SU(2)$ is topologically isomorphic to the coset space $SU(2)/U(1) \simeq S^2$. However, geometrically it reduces to S^2 only in the limit $j \to \infty$.

We know that the operator $T(g)$, $\forall g \in SU(2)$ can be expressed in terms of Euler angles as $T(g) = e^{-i\frac{\phi}{\theta_f}\hat{X}_3} e^{-i\frac{\theta}{\theta_f}\hat{X}_2} e^{-i\frac{\psi}{\theta_f}\hat{X}_3}$ [86]. For S^2, $\psi = 0$ locally such that a generic spin coherent state specified by a point on S^2 is given by

$$|z\rangle \;=\; e^{-i\frac{\phi}{\theta_f}\hat{X}_3} e^{-i\frac{\theta}{\theta_f}\hat{X}_2} |j,j\rangle \;. \tag{5.51}$$

Further, setting the azimuthal angle $\phi = 0$, we get

$$|z\rangle \;=\; e^{-i\frac{\theta}{\theta_f}\hat{X}_2}|j,j\rangle \;=\; e^{\frac{\theta}{2\theta_f}(\hat{X}_- - \hat{X}_+)}|j,j\rangle \;. \tag{5.52}$$

This parametrises the points along the geodesic (meridian) connecting north and south pole. Note that z is the stereographic variable of S^2 projected from the south pole to the complex plane $z = 1$ and $z = -\tan\frac{\theta}{2}e^{i\phi}$. We can also express it as

$$|z\rangle \equiv |\theta\rangle = \sum_{m=-j}^{j} \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(\sin\frac{\theta}{2}\right)^{j-m} |j,m\rangle. \tag{5.53}$$

These spin coherent states $|z\rangle$ [92] for a given spin j belongs to the configuration space $\mathcal{F}_j$ of a fuzzy sphere with radius $r_j = \theta_f\sqrt{j(j+1)}$. To every $|z\rangle \in \mathcal{F}_j$, we can define a density matrix $\rho_z = |z\rangle\langle z| \in \mathcal{H}_j$ and this pure quantum states ρ_z plays the role of generalized points on the homogeneous space of fuzzy sphere.

6

CONNES' SPECTRAL DISTANCE ON MOYAL PLANE

In this chapter, we study the metric properties of Moyal plane by introducing a spectral triple adaptable to the Hilbert-Schmidt operator formalism of non-commutative quantum mechanics, reviewed in chapter 2. As mentioned earlier, a legitimate spectral triple (1.27) for Moyal plane was first given in [74]. Using this triple (1.27), the spectral distance on the configuration space of Moyal plane was obtained in [77, 78] where the Moyal star product (1.4) is employed. But different star products and hence different position bases associated to them for a given quantum system is not always equivalent, as explained in [38] through the examples of Moyal and Voros bases. Indeed, it is the Voros position basis which is the physical basis on Moyal plane as this conforms to POVM. This motivates us to introduce an algorithm [79] by introducing an appropriate spectral triple, adaptable to the Hilbert-Schmidt operator formalism and hence independent of any star products. One of the advantages of this algorithm is that we can extend it to other non-commutative space like fuzzy sphere also. In fact, for the example like fuzzy sphere, where the concerned Hilbert space is finite dimensional, this algorithm is readily applicable-at least in principle. In contrast, if the concerned Hilbert space is of infinite dimension, like in the Moyal plane this algorithm turns out to be not so user-friendly-as we explain in the sequel. Nevertheless, it can definitely be used to improve the estimate over and above the lower bound - as follows from this algorithm itself. Moreover, employing this algorithm, it is possible to compute infinitesimal spectral distance (up to a numerical factor) on the quantum Hilbert space of Moyal plane and fuzzy sphere by exploiting the presence of additional degrees of freedom, indicating a deep connection between geometry and statistics.

We would like to point out in this context that the above mentioned lower bound yields the exact distance for a pair of infinitesimally separated discrete basis states and an almost exact distance i.e., up to a constant numerical factor for infinitesimally separated coherent states [79, 82]. We thus need to modify this formula to compute the finite distance, since as pointed out earlier, that in generic noncommutative space the notion of conventional geodesic does not exist preventing one from computing finite distance by integrating infinitesimal distance. Here, we have proposed a generalized distance formula [92] which should, in principle, yield the exact distance between any pair of states. We take up the case of Moyal plane in this chapter and derive our general algorithm taking this as a prototype example. We show, however, why this formula turns out to be not so user-friendly-as mentioned above.

Consequently, we are forced to adopt a different approach to tackle this problem. And it is only in the next chapter that this algorithm will find its non-trivial application in the computation of finite spectral distance on fuzzy sphere.

The construction of Dirac operator $\mathcal{D}_M$ on Moyal plane is given in appendix D. It is one of the major ingredients of the corresponding spectral triple and plays the most important role in the computation of spectral distance.

6.1 EIGENSPINORS OF DIRAC OPERATOR

The Dirac operator on Moyal plane takes the following hermitian form (see appendix D):

$$\mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}, \tag{6.1}$$

acting on $\Psi = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \in \mathcal{H}_c \otimes \mathbb{C}^2$ through the left multiplication as

$$\mathcal{D}\Psi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} b^\dagger |\psi_2\rangle \\ b\,|\psi_1\rangle \end{pmatrix}. \tag{6.2}$$

We can then easily obtain the following normalized eigen-spinors of $\mathcal{D}_M$:

$$|0\rangle\rangle := \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \in \mathcal{H}_c \otimes \mathbb{C}^2 \;\; ; \;\; |m\rangle\rangle_\pm := \frac{1}{\sqrt{2}} \begin{pmatrix} |m\rangle \\ \pm |m-1\rangle \end{pmatrix} \in \mathcal{H}_c \otimes \mathbb{C}^2 \;\; ; \;\; m = 1, 2, 3, \cdots \tag{6.3}$$

with the eigenvalues λ_m for any state $|m\rangle\rangle_\pm$ as,

$$\lambda_0 = 0 \quad ; \quad \lambda_m^\pm = \pm\sqrt{\frac{2m}{\theta}}$$

Further, they furnish a complete and orthonormal set of basis for $\mathcal{H}_c \otimes \mathbb{C}^2$, so that the resolution of identity takes the form

$$\mathbb{1}_{\mathcal{H}_q \otimes M_2(\mathbb{C})} = |0\rangle\rangle\langle\langle 0| + \sum_{m=1}^{\infty} \Big(|m\rangle\rangle_{+\,+}\langle\langle m| + |m\rangle\rangle_{-\,-}\langle\langle m| \Big); \tag{6.4}$$

$$\text{with} \quad {}_\pm\langle\langle m|n\rangle\rangle_\pm = \delta_{mn}; \;\; {}_\pm\langle\langle m|n\rangle\rangle_\mp = 0. \tag{6.5}$$

6.2 SPECTRAL TRIPLE FOR MOYAL PLANE $\mathbb{R}^2_\star$ AND THE SPECTRAL DISTANCE

To study the metric properties of the Moyal plane, we work with the same spectral triple that was introduced in [79]:

$$\mathcal{A}_M = \mathcal{H}_q \ , \ \ \mathcal{H}_M = \mathcal{H}_c \otimes \mathbb{C}^2 \ \text{ and } \ \mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{bmatrix} \tag{6.6}$$

The module of the algebra $\mathcal{A}_M$ is given by the space of sections of appropriate "spinor bundle" defined through $\mathcal{H}_M$ on which it acts through the diagonal representation π:

$$\pi(a)\Psi = \pi(a) \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \begin{pmatrix} a\,|\psi_1\rangle \\ a\,|\psi_2\rangle \end{pmatrix} \ ; \ \ \Psi \in \mathcal{H}_M. \tag{6.7}$$

In order to define the Connes' spectral distance on $\mathcal{H}_c$, we have to define the pure states on the algebra $\mathcal{A}_M$ which play the role of generalized points on $\mathbb{R}^2_\star$. Note that the algebra $\mathcal{A}_M \equiv \mathcal{H}_q \subset \mathcal{B}(\mathcal{H}_c)$ has the tensor product structure: $\mathcal{H}_q \equiv \mathcal{H}_c \otimes \mathcal{H}_c^*$, where $\mathcal{H}_c^*$ denotes the dual of $\mathcal{H}_c$. This implies that $\mathcal{H}_q$ is self dual. This motivates us to consider the normal states [112, 113] on $\mathcal{H}_q$ which are given by

$$\omega_\rho(a) := \text{tr}_c(\rho a) \ , \ \ \forall\ a \in \mathcal{H}_q \ ; \ \ \rho^\dagger = \rho \in \mathcal{H}_q \ , \ \ \rho \geq 0 \ , \ \ \text{tr}_c(\rho) = 1. \tag{6.8}$$

Note that ρ is a density matrix as viewed from $\mathcal{H}_c$ but is valued in $\mathcal{H}_q$. And this feature is the reason behind employing the Hilbert-Schmidt formalism, which facilitates the present geometrical analysis. If the density matrix is of the form of rank-one projection operator $\rho = |\psi\rangle\langle\psi| \in \mathcal{H}_q$ for some $|\psi\rangle \in \mathcal{H}_c$, then the normal state ω_ρ is a pure state. In constrast, a mixed state is given by a convex sum of such pure states.

Now, we can define Connes' spectral distance (1.25) between two normal states on $\mathcal{H}_a$ as

$$d(\omega_{\rho'}, \omega_\rho) = \sup_{a \in B} |\omega_{\rho'}(a) - \omega_\rho(a)| = \sup_{a \in B} |\text{tr}_c(\Delta\rho a)| \ ; \ \ \Delta\rho = (\rho' - \rho). \tag{6.9}$$

where B is the set of those elements of $\mathcal{H}_q$ which satisfy the ball condition, i.e.

$$B = \{a \in \mathcal{A} \ : \ \|[\mathcal{D}_M, \pi(a)]\|_{\text{op}} \leq 1\} \ , \quad \text{where} \quad \|A\|_{\text{op}} = \sup_{\phi \in \mathcal{H}} \left\{ \frac{\|A\phi\|}{\|\phi\|} \right\}. \tag{6.10}$$

Let us impose the following condition:

$$|\omega_{\rho'}(a) - \omega_\rho(a)| = 0 \ , \ \forall\ a \in V_0 \ ; \quad V_0 = \{a \in \mathcal{H}_q \ : \ \|[\mathcal{D}_M, \pi(a)]\|_{op} = 0\}. \tag{6.11}$$

Note that $\forall\ a \in \mathcal{H}_q$, we can write $a = \|a\|_{\text{tr}}\ \hat{a}$ where $\|\hat{a}\|_{\text{tr}} = 1$. If the condition (6.11) is not obeyed i.e. $|\omega_{\rho'}(a) - \omega_\rho(a)| \neq 0$ for some $a \in V_0$, then the ball condition (6.10) gives no upper bound on $\|a\|_{\text{tr}}$ such that the supremum of $|\omega_{\rho'}(a) - \omega_\rho(a)| \neq 0$ will just diverge. This is just reminescent of the divergent distance between a pair of states at different latitudes in the $\mathbb{CP}^1$ model introduced in section 5.6. Hence spectral distance between normal states of $a \notin V_0$ can be infinite making it a pseudo-distance[1] [114]. Moreover, we can see that this condition (6.11) is obeyed by all elements of $\mathcal{H}_q$. Since the operator norm of an operator A is given by the square-root of the highest eigenvalue of $A^\dagger A$, we then have

$$[\mathcal{D}_M, \pi(a)] = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [\hat{b}^\dagger, a] \\ [\hat{b}, a] & 0 \end{pmatrix} \; ; \quad [\mathcal{D}_M, \pi(a)]^\dagger = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [\hat{b}, a]^\dagger \\ [\hat{b}^\dagger, a]^\dagger & 0 \end{pmatrix}, \tag{6.12}$$

$$[\mathcal{D}_M, \pi(a)]^\dagger [\mathcal{D}_M, \pi(a)] = \frac{2}{\theta} \begin{pmatrix} [\hat{b}, a]^\dagger [\hat{b}, a] & 0 \\ 0 & [\hat{b}^\dagger, a]^\dagger [\hat{b}^\dagger, a] \end{pmatrix}. \tag{6.13}$$

This shows that $\|[\mathcal{D}_M, \pi(a)]\|_{\text{op}} = 0 \Rightarrow \|[[\hat{b}^\dagger, a]]\|_{\text{op}} = 0 = \|[\hat{b}, a]\|_{\text{op}} \Rightarrow [\hat{b}^\dagger, a] = 0 = [\hat{b}, a]$ which is true only for $a \propto \mathbb{1}_c$. This implies that $\|[\mathcal{D}_M, \pi(a)]\|_{\text{op}} = 0$ for $a \propto \mathbb{1}_c$ but $\mathbb{1}_c \notin \mathcal{A}_M = \mathcal{H}_q$ as this is a non-unital algebra. Thus, the set V_0 contains only the null element. Here, we can introduce an orthogonal complement $V_0^\perp$ of V_0, i.e., $V_0^\perp = \{a \in \mathcal{H}_q \;:\; \|[\mathcal{D}_M, \pi(a)]\|_{\text{op}} \neq 0\}$. It is shown in [110] that the positive elements $a^+ \in \mathcal{A}^+$ of $\mathcal{A}$ reach the supremum in the distance formula (5.1). With this, we will consider the following subspace B_s of $\mathcal{H}_q$:

$$B_s = \{a \in \mathcal{H}_q \;:\; a^\dagger = a = cc^\dagger,\; c \in \mathcal{H}_q \;;\; 0 < \|[\mathcal{D}_M, \pi(a)]\|_{\text{op}} \leq 1\}, \tag{6.14}$$

such that the spectral distance on $\mathbb{R}^2_\star$ is given by

$$d(\omega_{\rho'}, \omega_\rho) = \sup_{a \in B_s} |\omega_{\rho'}(a) - \omega_\rho(a)| = \sup_{a \in B_s} |\text{tr}_c(\Delta\rho a)| = \sup_{a \in B_s} \{\|a\|_{\text{tr}} |\text{tr}_c(\Delta\rho \hat{a})|\}. \tag{6.15}$$

Clearly, the ball condition (6.10) yields upper bound on $\|a\|_{\text{tr}}$:

$$\|a\|_{\text{tr}} \leq \frac{1}{\|[\mathcal{D}_M, \pi(\hat{a})]\|_{\text{op}}} \leq \sup_{a \in B_s} \left\{ \frac{1}{\|[\mathcal{D}_M, \pi(\hat{a})]\|_{\text{op}}} \right\} = \frac{1}{\inf_{a \in B_s} \|[\mathcal{D}_M, \pi(\hat{a})]\|_{\text{op}}}. \tag{6.16}$$

We can also write $\Delta\rho = \|\Delta\rho\|_{\text{tr}} \, \widehat{\Delta\rho}$ with $\|\widehat{\Delta\rho}\|_{\text{tr}} = 1$ such that we can decompose $\hat{a} \in B_s$ into "longitudinal" ($\widehat{\Delta\rho}$) and "transverse" ($\widehat{\Delta\rho}_\perp$) components as

$$\hat{a} = \cos\theta \; \widehat{\Delta\rho} + \sin\theta \; \widehat{\Delta\rho}_\perp \;, \quad \text{where} \quad \|\widehat{\Delta\rho}\|_{\text{tr}} = \|\widehat{\Delta\rho}_\perp\|_{\text{tr}} = 1 \quad \text{and} \quad \text{tr}_c(\widehat{\Delta\rho}_\perp \widehat{\Delta\rho}) = 0. \tag{6.17}$$

Here, we can choose θ to be $0 \leq \theta < \frac{\pi}{2}$ as only positive values of $\sin\theta$ and $\cos\theta$ are required. This is implied by the distance formula (6.9) itself which includes the absolute value. We

1 which satisfy all the properties of a distance but can be infinite.

exclude $\theta = \frac{\pi}{2}$ as for this case $\hat{a} = \widehat{\Delta\rho}_\perp$ and hence the distance would collapse to zero as $|\mathrm{tr}_c(\widehat{\Delta\rho}\ \widehat{\Delta\rho}_\perp)| = 0$. Now, using this decomposition (6.17), we have

$$\|[\mathcal{D}_M, \pi(\hat{a})]\|_{\mathrm{op}} = \|[\mathcal{D}_M\ ,\ \cos\theta\ \pi(\widehat{\Delta\rho}) + \sin\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}. \tag{6.18}$$

Putting (6.16) and (6.18) in the distance formula (6.15), we get

$$d(\omega_{\rho'}, \omega_\rho) = \sup_{a\in B_s} \frac{\|\Delta\rho\|_{\mathrm{tr}}|\cos\theta|}{\|[\mathcal{D}_M, \pi(\hat{a})]\|_{\mathrm{op}}} = \frac{\|\Delta\rho\|_{\mathrm{tr}}}{\inf_{a\in B_s}\|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}}. \tag{6.19}$$

By using the traingle inequality, we have

$$\|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}} \leq \|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho})]\|_{\mathrm{op}} + |\tan\theta|\|[\mathcal{D}_M\ , \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}, \tag{6.20}$$

such that

$$\begin{aligned}\inf_{a\in B_s}\|[\mathcal{D}_M, \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}} &\leq \inf_{\theta\in[0,\frac{\pi}{2})}\left\{\|[\mathcal{D}_M, \pi(\widehat{\Delta\rho})]\|_{\mathrm{op}} + |\tan\theta|\|[\mathcal{D}_M, \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}\right\} \\ &= \|[\mathcal{D}_M, \pi(\widehat{\Delta\rho})]\|_{\mathrm{op}}\ . \end{aligned} \tag{6.21}$$

This implies that

$$\frac{1}{\inf_{a\in B_s}\left\{\|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}\right\}} \geq \frac{1}{\|[\mathcal{D}_M, \pi(\widehat{\Delta\rho})]\|_{\mathrm{op}}}\ . \tag{6.22}$$

That is, we find an exact spectral distance formula:

$$\begin{aligned} d(\omega_{\rho'}, \omega_\rho) &= \frac{\|\Delta\rho\|_{\mathrm{tr}}}{\inf_{\theta\in[0,\frac{\pi}{2}),\widehat{\Delta\rho}_\perp}\|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}} = \tilde{N}\|\Delta\rho\|_{\mathrm{tr}}\ ; \quad (6.23) \\ &= \frac{\|\Delta\rho\|^2_{\mathrm{tr}}}{\inf_{\theta\in[0,\frac{\pi}{2}),\Delta\rho_\perp}\|[\mathcal{D}_M\ ,\ \pi(\Delta\rho) + \kappa\ \pi(\Delta\rho_\perp)]\|_{\mathrm{op}}} = N\|\Delta\rho\|^2_{\mathrm{tr}}\ , \quad (6.24) \\ \text{where}\quad \kappa &= \frac{\|\Delta\rho\|_{\mathrm{tr}}}{\|\Delta\rho_\perp\|_{\mathrm{tr}}}\ \tan\theta\ ; \quad (6.25) \\ \tilde{N} &= \frac{1}{\inf_{\theta\in[0,\frac{\pi}{2}),\widehat{\Delta\rho}_\perp}\|[\mathcal{D}_M\ ,\ \pi(\widehat{\Delta\rho}) + \tan\theta\ \pi(\widehat{\Delta\rho}_\perp)]\|_{\mathrm{op}}}\ ; \quad (6.26) \\ N &= \frac{1}{\inf_{\theta\in[0,\frac{\pi}{2}),\Delta\rho_\perp}\|[\mathcal{D}_M\ ,\ \pi(\Delta\rho) + \kappa\ \pi(\Delta\rho_\perp)]\|_{\mathrm{op}}}\ . \quad (6.27) \end{aligned}$$

This exact distance formula has a lower bound:

$$d(\omega_{\rho'}, \omega_\rho) \geq \frac{\|\Delta\rho\|_{\mathrm{tr}}}{\|[\mathcal{D}_M, \pi(\widehat{\Delta\rho})]\|_{\mathrm{op}}} = \frac{\|\Delta\rho\|^2_{\mathrm{tr}}}{\|[\mathcal{D}_M, \pi(\Delta\rho)]\|_{\mathrm{op}}}\ . \tag{6.28}$$

This lower bound was obtained in [79] to compute the infinitesimally separated pure states on the configuration space of Moyal plane. Note at this stage that this lower bound will yield the exact distance iff the optimal element $a_s \propto \Delta\rho$.

Note that the Connes' distance function depend on $\Delta\rho$ as it can be seen from (6.9) and under a unitary transformation U, we have

$$d(\omega_{\rho+\Delta\rho}, \omega_\rho) = d(\Delta\rho) \xrightarrow{U} d(U\Delta\rho U^\dagger) = d(\Delta\rho)\,. \tag{6.29}$$

Since $\|U\Delta\rho U^\dagger\|_{\mathrm{tr}} = \|\Delta\rho\|_{\mathrm{tr}}$, we have

$$\tilde{N}(U\Delta\rho U^\dagger) = \tilde{N}(\Delta\rho), \quad N(U\Delta\rho U^\dagger) = N(\Delta\rho)\,. \tag{6.30}$$

From (6.26) and (6.27), we can see that $\tilde{N}$and N depend on the 'direction' of $\Delta\rho$, in the sense that even if $\|\Delta\rho'\|_{\mathrm{tr}} = \|\Delta\rho\|_{\mathrm{tr}}$, $N(\Delta\rho') \neq N(\Delta\rho)$. However, if $\|\Delta\rho'\|_{\mathrm{tr}} = \|\Delta\rho\|_{\mathrm{tr}}$ implies $\Delta\rho' = U\Delta\rho U^\dagger$, this dependence disappears using (6.30), $\tilde{N}$ is a constant as $\widehat{\Delta\rho}'$ and $\widehat{\Delta\rho}$ both have norm one and $N = \frac{\tilde{N}}{\|\Delta\rho\|_{\mathrm{tr}}}$. This is the case for the coherent state basis in the Moyal plane, where equality of the trace norms implies that $\Delta\rho'$ and $\Delta\rho$ differ by a rotation of the form $R = e^{i\phi b^\dagger b}$. This explains why the Connes' distance on the Moyal plane is proportional to the trace norm, which is simply the Euclidean distance, infinitesimally and for finite distances. We corroborate this result in the next section through a more explicit calculation.

6.3 SPECTRAL DISTANCE ON THE CONFIGURATION SPACE OF MOYAL PLANE

From the above discussion, we see that it is quite simple to compute and matches with the exact distance if we consider the "infinitesimally separated" discrete pure states, i.e. taking $\rho' = |n+1\rangle\langle n+1|$ and $\rho = |n\rangle\langle n|$ where $|n\rangle \in \mathcal{H}_c$ is referred to as the "harmonic oscillator state". However, for finite distance, computation the exact distance formula (6.24) is not favourable but we can obtain the exact distance from the first principle.

In case, we consider infinitesimally separated coherent pure states i.e. taking $\rho' = |z + dz\rangle\langle z+dz|$ and $\rho = |z\rangle\langle z|$ $\big(|z\rangle \in \mathcal{H}_c$ (2.12)$\big)$, the lower bound (6.28) result differs from the exact distance by a numerical factor. However, the exact distance can be obtained through the introduction of projection operators on the Hilbert space $\mathcal{H}_q \otimes M_2(\mathbb{C})$ which is spanned by the eigen-spinor basis (6.3) of Dirac operator $\mathcal{D}_M$. This will be discussed in next upcoming sections.

6.3.1 *Ball condition with the Eigenspinors of $\mathcal{D}_M$*

The most important part for the computation of Connes' spectral distance is the computation of $\|[\mathcal{D}_M, \pi(a)]\|_{op}$ which we will compute using the eigenspinors (6.3) of $\mathcal{D}_M$. That is, $\forall\; m, m' = 0, 1, 2, \cdots,$ we have

$$
\begin{aligned}
{}_\pm\langle\langle m|[\mathcal{D}, \pi(a)]|m'\rangle\rangle_\pm &= \pm(\lambda^\pm_m - \lambda^\pm_{m'})_\pm\langle\langle m|\pi(a)|m'\rangle\rangle_\pm = \pm\sqrt{\frac{2}{\theta}}(\sqrt{m} - \sqrt{m'})A^\pm_{mm'}\;; && (6.31)\\
{}_\pm\langle\langle m|[\mathcal{D}, \pi(a)]|m'\rangle\rangle_\mp &= \pm(\lambda^\pm_m - \lambda^\mp_{m'})_\pm\langle\langle m|\pi(a)|m'\rangle\rangle_\mp = \pm\sqrt{\frac{2}{\theta}}(\sqrt{m} + \sqrt{m'})B^\pm_{mm'}\;; && (6.32)\\
\text{where,}\quad A^\pm_{ll'} &= \frac{1}{2}\left(a_{l,l'} + a_{l-1,l'-1}\right)\;;\;\; B^\pm_{ll'} = \frac{1}{2}\left(a_{l,l'} - a_{l-1,l'-1}\right)\;;\;\; l, l' = 1, 2, \cdots && (6.33)\\
E^\pm_{0l} &\equiv \langle\langle 0|\pi(a)|l\rangle\rangle_\pm = \frac{1}{\sqrt{2}}\,a_{0l}\;; && (6.34)\\
E^\pm_{l0} &\equiv {}_\pm\langle\langle l|\pi(a)|0\rangle\rangle = \frac{1}{\sqrt{2}}\,a_{l0} = \frac{1}{\sqrt{2}}\,a^*_{0l}\,. && (6.35)
\end{aligned}
$$

With these, we can write the commutator $[\mathcal{D}_M, \pi(a)]$ where the rows and coloumns for each four block matrices are labeled by $|0\rangle\rangle, |1\rangle\rangle_+, |1\rangle\rangle_-, |2\rangle\rangle_+, |2\rangle\rangle_-, \cdots$:

$$
[\mathcal{D}_M, \pi(a)] = \sqrt{\frac{2}{\theta}}\left(\begin{array}{c|c} (\sqrt{m} - \sqrt{m'})\;{}_+\langle\langle m|\pi(a)|m'\rangle\rangle_+ & (\sqrt{m} + \sqrt{m'})\;{}_+\langle\langle m|\pi(a)|m'\rangle\rangle_- \\ \hline -(\sqrt{m} + \sqrt{m'})\;{}_-\langle\langle m|[\pi(a)|m'\rangle\rangle_+ & -(\sqrt{m} - \sqrt{m'})\;{}_-\langle\langle m|\pi(a)|m'\rangle\rangle_- \end{array}\right). \tag{6.36}
$$

6.3.2 *Distance between discrete/harmonic oscillator states*

In this section we compute the Connes' spectral distance between discrete/harmonic oscillator states. Since the formal algorithm (6.24) and (6.27), devised in the preceding section, is not very user-friendly, as $\Delta\rho_\perp$ depends upon infinitely large number of parameters so that looking for the infimum (6.27) is an extremely difficult job. However, we can follow up the prescription (which is used in [77]) performed in section 5.3 of chapter 5 where we first found an upper bound (5.23) to the distance and then searched for an optimal element which saturates the upper bound. If we can identify at least one a_s (note that this may not be unique!) then we can identify the upper bound to be the true distance. Note that we also have a lower bound (6.28) to the spectral distance and it can be easily computed [79]. It may happen in some situations that both upper and lower bounds coincide. In this case, their common value can be identified as the distance. For example, for the spectral distances between infinitesimally separated discrete pure states in Moyal plane [79] and fuzzy sphere [82] the upper and lower bounds coincide, as we shall see below.

6.3.2.1 *Infinitesimal distance between discrete/harmonic oscillator states*

Let us consider a pair of pure states represented by the respective density matrices: $\rho_{n+1} = |n+1\rangle\langle n+1|$ and $\rho_n = |n\rangle\langle n|$. Then, the spectral distance is

$$d(\omega_{n+1}, \omega_n) = \sup_{a \in B_s} |\mathrm{tr}(\rho_{n+1} a) - \mathrm{tr}(\rho_n a)| = \sup_{a \in B_s} |\langle n+1|a|n+1\rangle - \langle n|a|n\rangle|. \tag{6.37}$$

By simply using $\hat{b}|n+1\rangle = \sqrt{n+1}|n\rangle$ and $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$, we can easily get

$$\langle n+1|a|n+1\rangle = \frac{1}{n+1}\langle n|\hat{b}a\hat{b}^\dagger|n\rangle = \frac{1}{\sqrt{n+1}}\langle n|[\hat{b}, a]|n+1\rangle + \langle n|a|n\rangle. \tag{6.38}$$

Thus, the distance (6.37) reduces to

$$d(\omega_{n+1}, \omega_n) = \sup_{a \in B_s} \frac{1}{\sqrt{n+1}} |\langle n|[\hat{b}, a]|n+1\rangle| = \sup_{a \in B_s} \frac{1}{\sqrt{n+1}} |[\hat{b}, a]_{n,n+1}|. \tag{6.39}$$

Using the Bessel's inequality:

$$\|A\|_{\mathrm{op}}^2 \geq \sum_i |A_{ij}|^2 \geq |A_{ij}|^2 \tag{6.40}$$

We get the distance as

$$d(\omega_{n+1}, \omega_n) = \sup_{a \in B_s} \frac{1}{\sqrt{n+1}} |[\hat{b}, a]_{n,n+1}| = \frac{1}{\sqrt{n+1}} \sup_{a \in B_s} \|[\hat{b}, a]\|_{\mathrm{op}}. \tag{6.41}$$

From (6.13), we find that $\|[\mathcal{D}_M, \pi(a)]\|_{\mathrm{op}} = \sqrt{\frac{2}{\theta}}\|[\hat{b}, a]\|_{\mathrm{op}} = \sqrt{\frac{2}{\theta}}\|[\hat{b}^\dagger, a]\|_{\mathrm{op}}$ so that the ball condition yields

$$\|[\mathcal{D}_M, \pi(a)]\|_{\mathrm{op}} \leq 1 \Longrightarrow \|[\hat{b}, a]\|_{\mathrm{op}} \leq \sqrt{\frac{\theta}{2}}. \tag{6.42}$$

Hence, we get an upper bound on the infinitesimal spectral distance as

$$d(\omega_{n+1}, \omega_n) \leq \sqrt{\frac{\theta}{2(n+1)}}. \tag{6.43}$$

Let us now hold the search for optimal element which saturates the above in-equality (6.43) and compute the lower bound (6.28). For the computation of operator norm $\|[\mathcal{D}_M, \pi(d\rho)]\|_{\mathrm{op}}$, we can use the Dirac eigenspinors (6.3). Here $d\rho = |n+1\rangle\langle n+1| - |n\rangle\langle n|$ is a diagonal matrix so that the block matrices of the commutator $[\mathcal{D}_M, \pi(d\rho)]$ (6.36) will be zero as $m = m'$ in (6.31). With this, we get

$$[\mathcal{D}, \pi(d\rho)] = \sqrt{\frac{2}{\theta}} \left(\begin{array}{c|c} 0 & B \\ \hline -B^\dagger & 0 \end{array} \right); \quad \text{where} \quad B = \begin{pmatrix} -\sqrt{n} & 0 & 0 \\ 0 & 2\sqrt{n+1} & 0 \\ 0 & 0 & -\sqrt{n+2} \end{pmatrix} \tag{6.44}$$

with the rows and columns labeled from top to bottom and left to right respectively by $|n\rangle\rangle_+, |n+1\rangle\rangle_+, |n+2\rangle\rangle_+$ and $|n\rangle\rangle_-, |n+1\rangle\rangle_-, |n+2\rangle\rangle_-$. From this, we get the operator norm

$$\|[\mathcal{D}, \pi(d\rho)]\|_{op} = \sqrt{\frac{2}{\theta}}\, \|B\|_{\text{op}} = 2\sqrt{\frac{2(n+1)}{\theta}}\ . \tag{6.45}$$

Since $\text{tr}(d\rho)^2 = 2$, we have

$$d(\omega_{n+1}, \omega_n) = \frac{\text{tr}(d\rho)^2}{\|[\mathcal{D}, \pi(d\rho)]\|_{op}} = \sqrt{\frac{\theta}{2(n+1)}}\ , \tag{6.46}$$

which is exactly the upper bound (6.43). That is, the true distance is

$$d(\omega_{n+1}, \omega_n) = \sqrt{\frac{\theta}{2(n+1)}}. \tag{6.47}$$

Clearly, the optimal element $a_s \in B_s$ is

$$a_s = \frac{d\rho}{\|[\mathcal{D}_M, \pi(d\rho)]\|_{\text{op}}}\ . \tag{6.48}$$

This implies that the spectral distance between infinitesimally separated discrete pure states ρ_{n+1} and ρ_n is

$$d(\omega_{n+1}, \omega_n) = |\omega_{n+1}(a_s) - \omega_n(a_s)| = \frac{\text{tr}(d\rho)^2}{\|[\mathcal{D}, \pi(d\rho)]\|_{op}} = \sqrt{\frac{\theta}{2(n+1)}}\ . \tag{6.49}$$

This has been obtained in [77] where Moyal star product is employed to define the spectral triple and also in [79] within the context of Hilbert-Schmidt operator formalism.

6.3.2.2 *Finite distance between discrete/harmonic oscillato states*

Let us consider a pair of "finitely separated" discrete pure states ω_m and ω_n with the corresponding density matrices $\rho_m = |m\rangle\langle m|$ and $\rho_n = |n\rangle\langle n|$ with $k \equiv m - n \geq 2$. Now the distance between these states can be written as

$$d(\omega_m, \omega_n) = \sup_{a \in B_s} |\text{tr}\{(\rho_m - \rho_n)a\}| = \sup_{a \in B_s} |\text{tr}(\rho_{n+k}a) - \text{tr}(\rho_n a)|. \tag{6.50}$$

Clearly, we can write

$$|\text{tr}(\rho_{n+k}a) - \text{tr}(\rho_n a)| = \left|\sum_{i=1}^{k} \text{tr}\Big[\{\rho_{n+i} - \rho_{n+(i-1)}\}a\Big]\right| = \left|\sum_{p=n}^{m-1} \text{tr}\{(\rho_{p+1} - \rho_p)a\}\right|, \tag{6.51}$$

where we relabel the summing index as $p = n + i - 1$ so that we can easily make the comparision with the infinitesimal case. We had

$$\mathrm{tr}\{(\rho_{p+1} - \rho_p)a\} = \frac{1}{\sqrt{p+1}}|\langle p|[\hat{b}, a]|p+1\rangle| \leq \frac{1}{\sqrt{p+1}}\|[\hat{b}, a]\|_{\mathrm{op}} \,, \tag{6.52}$$

along with the ball condition (6.42). This implies that the finite spectral distance has the upper bound which is just sum over each upper bound of infinitesimal distances:

$$d(\omega_m, \omega_n) \leq \sqrt{\frac{\theta}{2}} \sum_{p=n}^{m-1} \frac{1}{\sqrt{p+1}}. \tag{6.53}$$

Now, we need to find an optimal element $a_s \in B$ which saturates the above inequality.

Let us demand $a_s \in B_s$ is the one such that

$$\Big|\sum_{p=n}^{m-1} \frac{1}{\sqrt{p+1}}\langle p|[\hat{b}, a_s]|p+1\rangle\Big| \equiv \Big|\sum_{p=n}^{m-1}(a_s)_{p+1,p+1} - (a_s)_{p,p}\Big| = |(a_s)_{m,m} - (a_s)_{n,n}| = \sqrt{\frac{\theta}{2}} \sum_{p=n}^{m-1} \frac{1}{\sqrt{p+1}}.$$

Taking $(a_s)_{m,m} = 0$ which implies that $|(a_s)_{n,n}| = \sqrt{\frac{\theta}{2}} \sum_{p=n}^{m-1} \frac{1}{\sqrt{p+1}}$, the optimal element can be constructed as

$$a_s = \sqrt{\frac{\theta}{2}} \sum_{p'=n}^{m-1} \sum_{i=1}^{m-n} \frac{1}{\sqrt{n+i}}|p'\rangle\langle p'|. \tag{6.54}$$

This gives the finite spectral distance between ω_m and ω_n as

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{p=n}^{m-1} \frac{1}{\sqrt{p+1}} \;>\; \frac{\mathrm{tr}\{(\Delta\rho)^2\}}{\|[\mathcal{D}_M, \pi(\Delta\rho)]\|_{\mathrm{op}}} \;; \;\; \Delta\rho = \rho_m - \rho_n \,, \tag{6.55}$$

where the RHS of the in-equality is the lower bound (6.28) and the reason behind the strict greater than sign is that a_s is not proportional to $\Delta\rho$ unlike the infinitesimal case. We can, however, easily check that this spectral distance (6.55) saturates the triangle inequality [77]:

$$d_{\mathcal{D}_M}(\omega_m, \omega_n) = d_{\mathcal{D}_M}(\omega_m, \omega_l) + d_{\mathcal{D}_M}(\omega_l, \omega_n), \quad n \leq l \leq m. \tag{6.56}$$

6.3.3 *Distance between coherent states*

Coherent states are maximally localized states which have the properties similar to the classical states so that we expect the spectral distance between any pair of pure coherent states should give the same metric data of the classical manifold. Note that every coherent state $|z\rangle \in \mathcal{H}_c$ is obtained by the action of unitary transformation $U(z) = e^{z\hat{b}^\dagger - \bar{z}\hat{b}}$ on the "vacuum state" $|0\rangle \in \mathcal{H}_c$. Let us consider a pair of density matrices $\rho_z = |z\rangle\langle z|$ and $\rho_w = |w\rangle\langle w|$ such that the distance between them is

$$d(\omega_z, \omega_w) = \sup_{a\in B_s} |\omega_z(a) - \omega_w(a)| = \sup_{a\in B_s} |\mathrm{tr}(\Delta\rho a)| \ , \quad \Delta\rho = (\rho_z - \rho_w) \ . \tag{6.57}$$

Let us apply a unitary transformation $U(z')$ which represents a symmetry transformation on the homogeneous space of coherent states such that $\Delta\rho \to \Delta\rho^U = U(z')\Delta\rho U(z')^\dagger$. Then, the distance with the transformed $\Delta\rho^U$ is

$$d(\omega_{(\rho+\Delta\rho)^U}, \omega_{\rho^U}) = \sup_{a\in B_s} \left\{|\mathrm{tr}\{(\Delta\rho)^U a\}|\right\} = \sup_{a^U\in B_s^U} \left\{|\mathrm{tr}\{\Delta\rho a^U\}|\right\} ; \ a^U = U(z')^\dagger a U(z'). \tag{6.58}$$

Here, $B_s^U = \{a^U \in \mathcal{H}_q \ : \ a^{\dagger U} = a^U \ , \ 0 < \|[\mathcal{D}_M, \pi(a^U)]\|_{\mathrm{op}} \leq 1\}$. Now, we can check that $\|[\mathcal{D}_M, \pi(a^U)]\|_{\mathrm{op}} = \|[\mathcal{D}_M^U, \pi(a)]\|_{\mathrm{op}}$ where $\mathcal{D}_M^U = U(z')\mathcal{D}_M U^\dagger(z')$ since

$$\mathcal{D}_M^U \equiv e^{z'\hat{b}^\dagger - \bar{z}'\hat{b}} \mathcal{D}_M e^{-z'\hat{b}^\dagger + \bar{z}'\hat{b}} = \sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & \hat{b}^\dagger - \bar{z}' \\ \hat{b} - z' & 0 \end{bmatrix} = \mathcal{D}_M - \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \bar{z}' \\ z' & 0 \end{pmatrix} \ . \tag{6.59}$$

This implies $[\mathcal{D}_M^U, \pi(a)] = [\mathcal{D}_M, \pi(a)]$. This, in turn, implies that the ball remains invariant under unitary transformation : $B_s = B_s^U$. Consequently, the distance between a pair of states remains invariant if both the states are subjected to same unitary transformation, as the supremum in (6.58) is computed by varying the algebra elements in the same ball.

$$d(\omega_{(\rho+\Delta\rho)^U}, \omega_{\rho^U}) = \sup_{a^U\in B_s^U} \left|\mathrm{tr}\{\Delta\rho a^U\}\right| = \sup_{a\in B_s} |\mathrm{tr}(\Delta\rho \ a)| = d(\omega_{\rho+\Delta\rho}, \omega_\rho). \tag{6.60}$$

We can now choose z' such that $U(z')\rho_w U^\dagger(z') = \rho_0 \equiv |0\rangle\langle 0|$, i.e. $z' = -w$ and then $U(-w)\rho_z U^\dagger(-w) = |z-w\rangle\langle z-w|$ so that the spectral distance depends on $\Delta\rho = \rho_z - \rho_w \equiv U(\rho_{z-w} - \rho_0)U^\dagger$:

$$d(\omega_z, \omega_w) \equiv d(\omega_{z-w}, \omega_0) \ , \tag{6.61}$$

indicating the translational invariance in the complex plane, parametrizing coherent states. This implies we can always choose, without loss of generality, a pure state to be $\rho_0 = |0\rangle\langle 0|$ at the origin and another being $\rho_z = |z\rangle\langle z| = U(z)\rho_0 U^\dagger(z)$, translated state of ρ_0 by the action of $U(z)$. The spectral distance can then be written as

$$d(\omega_z, \omega_0) = \sup_{a\in B_s} |\omega_z(a) - \omega_0(a)| = \sup_{a\in B_s} |\langle 0|U^\dagger(z) a U(z) - a|0\rangle|. \tag{6.62}$$

Note that this is the same thing as done in [79] where we can define new translated boson operators as $\hat{\tilde{b}} = \hat{b} - z'$ and $\hat{\tilde{b}}^\dagger = \hat{b}^\dagger - \bar{z}'$ with the new "vacuum": $|z'\rangle = |\tilde{0}\rangle$ for which $\hat{\tilde{b}}|\tilde{0}\rangle = 0$, $\hat{\tilde{b}}^\dagger|\tilde{0}\rangle = |\tilde{1}\rangle$, etc.

This distance (6.61) can be thought as the maximum change in the expectation values of an operator a and the translated operatore $U^\dagger(z)aU(z)$ in the same state $|0\rangle$ [78]. This reminds us of the transition from the Schrödinger to Heisenberg picture, where the operators are subjected to the unitary evolution in time through an adjoint action of the unitary operator, while the states are held frozen in time. Thus, we can introduce a one-parameter family of pure states ω_{zt} [78] given by the pure density matrices

$$\rho_{zt} = |zt\rangle\langle zt| \ , \quad \text{with} \quad t \in [0,1] \tag{6.63}$$

. With this, we can define a function $W : [0,1] \to \mathbb{R}$ [78] as

$$W(t) = \omega_{zt}(a) = \text{tr}(\rho_{zt}a) = \langle 0|U^\dagger(zt)aU(zt)|0\rangle \ ; \ \text{with} \ W(0) = \omega_0(a), \ W(1) = \omega_z(a). \tag{6.64}$$

We can then write

$$|\omega_z(a) - \omega_0(a)| = \left| \int_0^1 \frac{dW(t)}{dt}\, dt \right| \leq \int_0^1 \left| \frac{dW(t)}{dt} \right| dt. \tag{6.65}$$

Using the Hadamard identity with $U(zt) = e^{G(zt)}$ where $G(zt) = zt\hat{b}^\dagger - \bar{z}t\hat{b}$, we have

$$U(zt)aU^\dagger(zt) = e^{G(zt)}\ a\ e^{-G(zt)} = a + [G(zt), a] + \frac{1}{2!}[G(zt), [G(zt), a]] + \cdots , \tag{6.66}$$

$$\begin{aligned}\Rightarrow \ \frac{dW(t)}{dt} &= \langle 0|[G(z), a]|0\rangle + t\langle 0|[G(z), [G(z), a]]|0\rangle + \frac{t^2}{2!}\langle 0|[G(z), [G(z), [G(z), a]]]|0\rangle + \cdots \\ &= \langle 0|e^{G(zt)}\ [G(z), a]\ e^{-G(zt)}|0\rangle = \omega_{zt}([G(z), a]) = \bar{z}\omega_{zt}([b, a]) + z\omega_{zt}([b, a]^\dagger)\end{aligned} \tag{6.67}$$

Using the Cauchy-Schwarz inequality, we have

$$|\bar{z}\omega_{zt}([b,a]) + z\omega_{zt}([b,a]^\dagger)| \leq \sqrt{2}|z|\sqrt{|\omega_{zt}([b,a])^2 + |\omega_{zt}([b,a]^\dagger)^2|}. \tag{6.68}$$

Since for every state ω, we have $\omega(a) \leq \|a\|$ and $\|a\| = \|a^\dagger\|$, we further have

$$\sqrt{2}|z|\sqrt{|\omega_{zt}([b,a])^2 + |\omega_{zt}([b,a]^\dagger)^2|} \leq \sqrt{2}|z|\sqrt{\|[b,a]\|_{\text{op}}^2 + \|[b,a]^\dagger\|_{\text{op}}^2} = 2|z|\|[b,a]\|_{\text{op}}. \tag{6.69}$$

With these results and the ball condition (6.42), we finally get the spectral distance as

$$d_{\mathcal{D}_M}(\omega_z, \omega_0) = \sup_{a \in B_s} |\omega_z(a) - \omega_0(a)| \leq 2|z| \sup_{a \in B_s} \|[b,a]\|_{\text{op}} \leq \sqrt{2\theta}|z| \ . \tag{6.70}$$

As follows from (6.66), the optimal element a_s which gives the exact distance should satisfy

$$[G(z), a_s] = \sqrt{2\theta}|z| \ \text{ and } \ [G(z), [G(z), a_s]] = 0. \tag{6.71}$$

A simple inspection suggests the following form for optimal element a_s:

$$a_s = \sqrt{\frac{\theta}{2}}\left(\hat{b}e^{-i\alpha} + \hat{b}^\dagger e^{i\alpha}\right) \ , \quad \text{where} \quad z = |z|e^{i\alpha} \ , \tag{6.72}$$

reproducing the result of [78]. But note here that although $\|[\mathcal{D}_M, \pi(a_s)]\|_{op} = 1$, we found that $\|a_s\|^2_{tr} = \sqrt{\frac{\theta}{2}} \sum_{n=0}^{\infty}(2n+1) = \infty$ which means $a_s \notin \mathcal{H}_q = \mathcal{A}_M$, but can be thought of as belonging to the multiplier algebra[2]. However, one can obtain this a_s (6.72) as the limit point of a sequence (by Proposition 3.5 of [78]):

$$a_n = \sqrt{\frac{\theta}{2}} \left(b e^{-i\alpha} e^{-\lambda_n b^\dagger b} + e^{-\lambda_n b^\dagger b} b^\dagger e^{i\alpha} \right) \in \mathcal{A}_M \,. \tag{6.73}$$

We briefly review the proof of proposition 3.5 of [78] which says 'Let $z = |z|e^{\,i\alpha}$ be a fixed translation and $\lambda > 0$. Define $a = \sqrt{\frac{\theta}{2}} \left(\hat{b}' + \hat{b}'^\dagger \right)$, where $\hat{b}' = \hat{b} e^{-i\alpha} \left(e^{-\lambda \hat{b}^\dagger \hat{b}} \right)$. Then there exists a $\gamma > 0$ s.t. $a \in B_s$ for any $\lambda \leq \gamma$. Using this proposition, any generic element of the sequence (6.73) can be written in terms of above $\hat{b}'$ as $a = \sqrt{\frac{\theta}{2}} (\hat{b}' + \hat{b}'^\dagger) \in B_s$ with $\lambda \leq \gamma$. Now, it can be easily shown that $\omega_0(a) = 0$ and $\omega_z(a) = \sqrt{2\theta}\, |z| \exp\left(-|z|^2 (1 - e^{-\lambda})\right)$. Therefore, we have

$$d(\omega_z, \omega_0) = \lim_{n \to \infty} |\omega_z(a) - \omega_0(a)| = \lim_{\lambda \to 0} \sqrt{2\theta}\, |z| \exp\left(-|z|^2 (1 - e^{-\lambda}) \right) = \sqrt{2\theta}\, |z|. \tag{6.74}$$

But here we propose an alternative approach [92] to obtain the optimal element. For that we define

$$\pi^N(a) = \mathbb{P}_N \pi(a) \mathbb{P}_N \,, \;\; \mathbb{P}_N = |0\rangle\rangle\langle\langle 0| + \sum_{n=1}^{N} \left\{ |n\rangle\rangle_{+\,+}\langle\langle n| + |n\rangle\rangle_{-\,-}\langle\langle n| \right\}, \tag{6.75}$$

where $|n\rangle\rangle_\pm$ are the eigenspinors of $\mathcal{D}_M$ (6.3) and $\pi^N(a) \in (2N+1)$-dimensional subspace of $\mathcal{H}_q \otimes M_2(\mathbb{C})$ and obtained by projecting using the projector $\mathbb{P}_N$. As it turns out that in the case of Moyal plane that the computation of infinitesimal distance is quite adequate to compute finite distance, as we shall discuss shortly.

Let us now make use of the eigenbasis (6.3) of Dirac opeartor $\mathcal{D}_M$ such that the infinitesimal separation $\Delta\rho = d\rho = \rho_{dz} - \rho_0 = d\bar{z}|0\rangle\langle 1| + dz|1\rangle\langle 0|$ can be represented by a 5 dimensional subspace of $\mathcal{H}_q \otimes M_2(\mathbb{C})$ as

$$\pi(d\rho) \equiv \begin{pmatrix} d\rho & 0 \\ 0 & d\rho \end{pmatrix} = \begin{pmatrix} 0 & \frac{d\bar{z}}{\sqrt{2}} & \frac{d\bar{z}}{\sqrt{2}} & 0 & 0 \\ \frac{dz}{\sqrt{2}} & 0 & 0 & \frac{d\bar{z}}{2} & -\frac{d\bar{z}}{2} \\ \frac{dz}{\sqrt{2}} & 0 & 0 & -\frac{d\bar{z}}{2} & \frac{d\bar{z}}{2} \\ 0 & \frac{dz}{2} & -\frac{dz}{2} & 0 & 0 \\ 0 & -\frac{dz}{2} & \frac{dz}{2} & 0 & 0 \end{pmatrix} \in \mathcal{H}_q \otimes M_2(\mathbb{C}), \tag{6.76}$$

where the columns and rows are labelled by $|0\rangle\rangle, |1\rangle\rangle_+, |1\rangle\rangle_-, |2\rangle\rangle_+, |2\rangle\rangle_-$ of (6.3) respectively.

2 Multiplier algebra $M = M_L \cap M_R$ where $M_L = \{T \mid \psi T \in \mathcal{H}_q \;\forall\, \psi \in \mathcal{H}_q\}$ and $M_R = \{T \mid T\psi \in \mathcal{H}_q \;\forall\, \psi \in \mathcal{H}_q\}$

Since $\pi(d\rho)$ lives on 5-dimensional subspace of $\mathcal{H}_q \otimes M_2(\mathbb{C})$, let us now put $N = 2$ in (6.75) which gives

$$\pi^2(a_s) \equiv \mathbb{P}_2 \pi(a_s) \mathbb{P}_2 = \sqrt{\frac{\theta}{2}} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{\sqrt{2}+1}{2} & \frac{\sqrt{2}-1}{2} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}+1}{2} \\ 0 & \frac{\sqrt{2}+1}{2} & \frac{\sqrt{2}-1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}+1}{2} & 0 & 0 \end{pmatrix} . \tag{6.77}$$

We then get

$$[\mathcal{D}_M, \pi^2(a_s)] \equiv \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} ; \text{ yielding}$$

$$[\mathcal{D}_M, \pi^2(a_s)]^\dagger [\mathcal{D}_M, \pi^2(a_s)] \equiv \left(\begin{array}{c|c} \mathbb{1}_{3\times 3} & 0_{3\times 2} \\ \hline 0_{2\times 3} & B_{2\times 2} \end{array}\right) , \text{ where } B_{2\times 2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} . \tag{6.78}$$

For any N, we find that

$$[\mathcal{D}_M, \pi^N(a_s)]^\dagger [\mathcal{D}_M, \pi^N(a_s)] = \left(\begin{array}{c|c} \mathbb{1}_{(2N-1)\times(2N-1)} & O_{(2N-1)\times 2} \\ \hline O_{2\times(2N-1)} & B \end{array}\right) ; \ B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} . \tag{6.79}$$

This gives the operator norm $\|[\mathcal{D}, \pi^N(a_s)]\|_{op} = 1$ for each N, $2 \leq N \leq \infty$:

$$\|[\mathcal{D}, \pi(a_s)\|_{op} \equiv \lim_{N\to\infty} \|[\mathcal{D}, \pi^N(a_s)]\|_{op} \equiv \|[\mathcal{D}, \pi^N(a_s)]\|_{op} = 1 . \tag{6.80}$$

Note that the basis (6.3) can be obtained naturally from a different orthonormal and complete basis

$$\left\{ |n, \uparrow\rangle\rangle = |n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix} \ ; \ |n, \downarrow\rangle\rangle = |n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix} \right\} , \tag{6.81}$$

by first leaving out $\begin{pmatrix} |0\rangle \\ 0 \end{pmatrix} \equiv |0\rangle\rangle$ separately and then pairing $|n, \uparrow\rangle\rangle$ and $|n-1, \downarrow\rangle\rangle$ as

$$|n\rangle\rangle_\pm = \frac{1}{\sqrt{2}} \left(|n, \uparrow\rangle\rangle \pm |n-1, \downarrow\rangle\rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} |n\rangle \\ \pm|n-1\rangle \end{pmatrix} ; \ n = 1, 2, 3, \ldots . \tag{6.82}$$

The projector $\mathbb{P}_N$ constructed from the basis (6.3), rather than from (6.81) for its natural association with the Dirac operator.

Let $\langle\langle . \mid . \rangle\rangle$ denotes an inner product between a pair of elements $A_1, A_2 \in \mathcal{H}_q \otimes M_2(\mathbb{C})$,

$$\langle\langle A_1 \mid A_2 \rangle\rangle \;=\; \mathrm{Tr}_{\mathcal{H}_M}(A_1^\dagger A_2)\;. \tag{6.83}$$

Note that $\mathcal{H}_M \equiv \mathcal{H}_c \otimes \mathbb{C}^2$ and the above inner product (6.83) is the counter part of the inner product (2.4) on $\mathcal{H}_c$. Since for a given pair of operators $a_1, a_2 \in \mathcal{H}_q$, we can define its representations $A_1 = \pi(a_1), A_2 = \pi(a_2) \in \mathcal{H}_q \otimes M_2(\mathbb{C})$ such that the inner product (2.4) between a_1, a_2 can be related to the inner product (6.83) of $\pi(a_1), \pi(a_2)$ as

$$(a_1, a_2) \;=\; \frac{1}{2}\langle\langle \pi(a_1) \mid \pi(a_2) \rangle\rangle\;. \tag{6.84}$$

Thus, we now have

$$(d\rho, a_s) \;=\; \frac{1}{2}\big|\langle\langle \pi(d\rho) \mid \pi(a_s) \rangle\rangle\big| \;=\; \sqrt{2\theta}\,|dz|. \tag{6.85}$$

Here one can easily see that $\nexists$ any $a \in \mathcal{H}_q$ s.t. $\pi(a) = \mathbb{P}_2\pi(a_s)\mathbb{P}_2$ and one can not simply relate (6.83) with any inner products $(.,.)$ of $\mathcal{H}_q$. Indeed, if it were to exist, we could have identified this 'a', using (6.80) and (6.85), to be the optimal element itself, which by definition has to belong to $\mathcal{H}_q = \mathcal{A}$, or at best to the multiplier algebra. In fact, this will be a persistent feature with any finite $(2N+1)$-dimensional projection $\pi^N(a_s) = \mathbb{P}_N\pi(a_s)\mathbb{P}_N$ (6.75) as

$$\frac{1}{2}\big|\langle\langle \pi(d\rho) \mid \mathbb{P}_N\pi(a_s)\mathbb{P}_N \rangle\rangle\big| = \sqrt{2\theta}|dz|\;, \tag{6.86}$$

is independent of N if $N \geq 2$. One can note at this stage, however, that one can keep on increasing the rank of the projection operator $\mathbb{P}_N$ indefinitely. That is, even in the limit $N \to \infty$, $\mathbb{P}_N \to \mathbb{1}_{\mathcal{H}_q \otimes M_2(\mathbb{C})}$ (6.5) where $\mathbb{P}_N\pi(a_s)\mathbb{P}_N \to \pi(a_s)$, we can interprete with (6.80)

$$(d\rho, a_s) = \frac{1}{2}\big|\langle\langle \pi(d\rho) \mid \pi(a_s) \rangle\rangle\big| \equiv \lim_{N\to\infty}\frac{1}{2}\big|\langle\langle \pi(d\rho) \mid \mathbb{P}_N\pi(a_s)\mathbb{P}_N \rangle\rangle\big| = \sqrt{2\theta}\,|dz|\;. \tag{6.87}$$

Thus, instead of inserting a Gaussian factor, as in (6.73), we have a sequence $\{\mathbb{P}_N\pi(a_s)\mathbb{P}_N\}$ of trace-class operators living in $\mathcal{H}_q \otimes M_2(\mathbb{C})$ (note that $\mathcal{H}_q \otimes M_2(\mathbb{C})$ can be regarded as Hilbert-Schmidt operators acting on $\mathcal{H}_c \otimes \mathbb{C}^2$) and each of them satisfy the ball condition (6.42), (6.80).

It is finally clear from the above analysis that the upper bound (6.70) is saturated in the infinitesimal case through the sequence $\{\mathbb{P}_N\pi(a_s)\mathbb{P}_N\}$ in the limit $N \to \infty$, allowing one to identify

$$d(\rho_0, \rho_{dz}) = d(|0\rangle\langle 0|, |dz\rangle\langle dz|) = \sqrt{2\theta}\,|dz|\;, \tag{6.88}$$

with the optimal element now belonging to the multiplier algebra. Invoking translational symmetry (6.61) it is clear that

$$d(\rho_0, \rho_{dz}) = d(\rho_z, \rho_{z+dz}) = \sqrt{2\theta}\,|dz| \;\; \forall \;\; z \in \{zt, t \in [0,1]\} \tag{6.89}$$

and one concludes that for finitely separated states, one can write

$$d(\rho_0, \rho_z) = \sqrt{2\theta}\,|z|, \tag{6.90}$$

reproducing the result (6.74) and identify the straight line joining $z = 0$ to z to be geodesic of the Moyal plane enabling one to integrate the infinitesimal distance (6.89) along this geodesic to compute finite distance. As we shall subsequently see this feature will not persist for other generic non-commutative spaces and we will demonstrate this through the example of the fuzzy sphere later. In fact, one can easily see at this stage that the distance (6.90) can be written as the sum of distances $d(\rho_0, \rho_{zt})$ and $d(\rho_0, \rho_{zt})$ as,

$$d(\rho_0, \rho_z) = d(\rho_0, \rho_{zt}) + d(\rho_{zt}, \rho_z), \tag{6.91}$$

where ρ_{zt} is an arbitrary intermediate pure state from the one-parameter family of pure states, introduced in (6.63), so that the respective triangle inequality becomes an equality.

6.4 SPECTRAL TRIPLE ON $\mathcal{H}_q$ AND THE CONNECTION OF GEOMETRY WITH STATISTICS

Here we provide a brief review of the computation of infinitesimal distance on $\mathcal{H}_q$ [79] which on exploiting the presence of the additional degrees of freedom reveals a deep connection between geometry and statistics [79]. We shall carry out a similar analysis on fuzzy sphere $\mathbb{S}^2_\star$ in the next chapter.

On the quantum Hilbert space $\mathcal{H}_q$ of Moyal, the following spectral triple is defined in [79]:

$$\mathcal{A}_M^q = \mathcal{L}^2(\mathcal{H}_q)^3 \; ; \quad \mathcal{H}_M^q = \mathcal{H}_q \otimes \mathbb{C}^2 \; ; \quad \mathcal{D}_M^q = \sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & \hat{B}^\ddagger \\ \hat{B} & 0 \end{bmatrix} , \tag{6.92}$$

where $\hat{B}$ and $\hat{B}^\ddagger$ are the ones introduced in (2.8). Here too we consider the usual action of $\mathcal{A}_M^q$ on $\mathcal{H}_M^q$ through the diagonal representation analogous to the action of $\mathcal{A}_M$ (6.7) on the configuration space $\mathcal{H}_c$. Here, the density matrices on $\mathcal{H}_q$ are of the form $\rho_{\psi,\phi} \equiv |\psi,\phi)(\psi,\phi| \in \mathcal{L}^2(\mathcal{H}_q)$ where $|\psi,\phi) \equiv |\psi\rangle\langle\phi| \in \mathcal{H}_q$ with $|\psi\rangle, |\phi\rangle \in \mathcal{H}_c$.

We know that for the infinitesimal distance between a pair of discrete states $\rho_{m+1,\phi'} \equiv |m+1,\phi')(m+1,\phi'|$ and $\rho_{m,\phi} \equiv |m,\phi)(m,\phi|$ where $\langle\phi|\phi'\rangle = \delta_{\phi,\phi'}$, we can use the lower bound formula (6.28) as it gives the exact infinitesimal distance. For simplicity, we can further

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simplify the distance computation by computing the trace norm instead of operator norm in the denominator of (6.28) as these two norms differ only by a numerical factor [79]. So, we can define a modified infinitesimal spectral distance on the quantum Hilbert space as

$$\tilde{d}(\omega_{\rho_{m+1,\phi'}}, \omega_{\rho_{m,\phi}}) = \frac{\mathrm{tr}_q\{(d\rho)^2\}}{\|[\mathcal{D}_M^q, \pi(d\rho)]\|_{\mathrm{tr}}} \; ; \quad d\rho = \rho_{m+1,\phi'} - \rho_{m,\phi} \; . \tag{6.93}$$

It is found that the distance between a pair of discrete pure states given by $\rho_{m+1,\phi}$ and $\rho_{m,\phi}$ is always less than the one between $\rho_{m+1,\phi'}$ and $\rho_{m,\phi}$. That is,

$$d(\omega_{\rho_{m+1,\phi'}}, \omega_{\rho_{m,\phi}}) = \begin{cases} \sqrt{\frac{\theta}{2(m+1)}}\,, & \text{if } \phi = \phi'; \\ \sqrt{\frac{2\theta}{m+2}}\,, & \text{if } \phi \neq \phi'. \end{cases} \tag{6.94}$$

This reveals the importance of right-hand sector $|\phi\rangle$ for the states $|m,\phi) \equiv |m\rangle\langle\phi| \in \mathcal{H}_q$ [81] and hence the additional degrees of freedom (reviewed in section 1.1.2) present in non-commutative space like Moyal plane. In [79], a more general situation has been considered by taking the right-hand sector to be a statistical mixture that changes from point to point. That is, the following density matrix on $\mathcal{H}_q$ is considered:

$$\rho_q(n) = \sum_k p_k(n)|n,k)(n,k| \; \in \mathcal{L}^2(\mathcal{H}_q)\,, \quad \sum_k p_k(n) = 1, \;\; \forall n. \tag{6.95}$$

The physical meaning of such states has been understood by computing the average of the radial operator in $\rho_q(n)$ which yields $\mathrm{tr}_q(\hat{B}_L^\ddagger \hat{B}_L \, \rho_q(n)) = n$. That is, these states are localized at a fixed radial distance n as the fluctuations vanish. Then the infinitesimal distance between a pair of mixed states $\rho_q(n+1)$ and $\rho_q(n)$ can be computed using the lower bound formula (6.28) as mentioned earlier for infinitesimally separated discrete states (6.28) gives the exact infinitesimal distance. In case $d\rho = \rho_q' - \rho_q \equiv \rho_q(n+1) - \rho_q(n)$, the following distance is obtained:

$$\tilde{d}(\omega_{n+1}, \omega_n) = \frac{\sqrt{\theta}}{2} \frac{\sum_k \{p_k^2(n+1) + p_k^2(n)\}}{\sqrt{\sum_k \{(2n+3)p_k^2(n+1) + (2n+1)p_k^2(n) + 2(n+1)p_k(n+1)p_k(n)\}}}. \tag{6.96}$$

This depends on the probabilities $p_k(n+1)$ and $p_k(n)$. Two particular choices of probabilities $p_k(n)$ were considered in [79] which we can briefly review here. However, the details of similiar computations on quantum Hilbert space of fuzzy sphere is given in chapter 7.

1. Choosing p_k's such that the distance between a pair of points is minimized, it is found that

$$\tilde{d}(\omega_{n+1}, \omega_n) = \frac{\sqrt{\theta}}{6\Omega} \frac{1}{\sqrt{n+1}}; \quad \Omega \to \text{a cut-off for } \sum_k . \tag{6.97}$$

2. Choosing p_k's that maximize the local entropy $S(n) = -\sum_k p_k(n) \log p_k(n)$; with $\sum_k p_k(n) E_k = E(n)$ held fixed, yielding a local Boltzmann distribution:

$$p_k(n) = \frac{e^{-\beta(n)E_k}}{Z(\beta(n))} \quad ; \quad Z(\beta(n)) = \sum_k e^{-\beta(n)E_k} \ , \tag{6.98}$$

where $\beta(n)$ is the local inverse temperature. From this, the following distance has been obtained

$$\tilde{d}(\omega_{n+1}, \omega_n) = \frac{\theta}{6} \frac{\sqrt{Z(2\beta)}}{Z(\beta)} \frac{1}{\sqrt{n+1}}, \quad Z(\beta) = \sum_k e^{-\beta E_k} \tag{6.99}$$

This clearly shows the connection between the distance and partition function describing the statistical properties of the 2D Moyal plane with quantum states in thermal equilibrium. However, in this case the distance decreases as the temperature T increases since the value of the factor $\frac{\sqrt{Z(2\beta)}}{Z(\beta)}$ lies within 0 to 1 and goes to 0 for $T \to \infty$ [79].

7

SPECTRAL DISTANCES ON FUZZY SPHERE

As mentioned earlier, fuzzy sphere $\mathbb{S}^2_\star$ can be considered as a quantized 2-sphere $\mathbb{S}^2$. We know the metric properties of $\mathbb{S}^2$ from usual Riemannian geometry. Here, we study the metric properties of its quantized version $\mathbb{S}^2_\star$. Using the generalized Perelomov's $SU(2)$ coherent states (5.52), we try to make comparison between the metric properties of $\mathbb{S}^2$ and the non-commutative analogue of homogeneous space of $\mathbb{S}^2_\star$ i.e. the space of Perelomov's $SU(2)$ coherent states [87]. As we have seen it is the Dirac operator, one of the most important ingredients of spectral triple, that gives the metric properties of a generalized space associated with a given spectral triple. For a review of the construction of Dirac operator on fuzzy sphere given in [88] see appexndix D.

7.1 SPECTRAL TRIPLE ON THE CONFIGURATION SPACE $\mathcal{F}_j$ OF FUZZY SPHERE

Let us denote by $\mathbb{S}^2_j$, a single fuzzy sphere with a fixed radius $r_j = \theta_f\sqrt{j(j+1)}$. Then, the spectral triple of a fuzzy sphere $\mathbb{S}^2_j$ can be constructed [82] in the same way as done for Moyal plane [79]:

- The Algebra $\mathcal{A}_f = \mathcal{H}_j = \text{Span}\{|j,m\rangle\langle j,m'|, -j \leq m, m' \leq j\}$.
- The Hilbert space $\mathcal{H}_f = \mathcal{F}_j \otimes \mathbb{C}^2 = \left\{ \begin{pmatrix} |j,m\rangle \\ |j,m'\rangle \end{pmatrix} \right\}$.
- Dirac operator $\mathcal{D}_f = \frac{1}{r_j}\hat{\vec{J}} \otimes \vec{\sigma}$, obtained in (D.35).

This triple is also introduced in [89]. Here, the algebra $\mathcal{A}_f$ acts on the Hilbert sapce $\mathcal{H}_f$ through the diagonal representation

$$\pi(a)\Phi = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \;;\quad a \in \mathcal{A}\,,\quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \mathcal{H}_f\,. \tag{7.1}$$

This spectral triple is a legitimate spectral triple as the Dirac operator $\mathcal{D}_f$ satisfy the conditions that $[\mathcal{D}_f, \pi(a)]$ are bounded for all $a \in \mathcal{A}_f$ and the resolvent $(\mathcal{D}_f - \mu)^{-1}$ is compact for all $\mu \notin \mathbb{R}$.

For a given j, we have $-j \leq m \leq j$ so the configuration space $\mathcal{F}_j$ (2.67) and the quantum Hilbert space $\mathcal{H}_j$ (2.69) of a given fuzzy sphere with radius r_j are all finite dimensional. For an operator A acting on a Hilbert space of dimension d, the operator norm and trace norms are equivalent, in the sense that,

$$\|A\|_{\text{op}} \leq \|A\|_{\text{tr}} \leq \sqrt{d}\|A\|_{\text{op}} . \tag{7.2}$$

Note that the Dirac operator $\mathcal{D}_f$ can be effectively written as

$$\mathcal{D}_f = \frac{1}{r_j}\begin{pmatrix} \hat{J}_3 & \hat{J}_1 - i\hat{J}_2 \\ \hat{J}_1 + i\hat{J}_2 & -\hat{J}_3 \end{pmatrix} . \tag{7.3}$$

By a simple computation, we can see that

$$\|[\mathcal{D}_f, \pi(a)]\|_{\text{tr}}^2 = \frac{1}{r_j^2}\left(2\|[\hat{J}_3, a]\|_{\text{tr}}^2 + \|[\hat{J}_+, a]\|_{\text{tr}} + \|[\hat{J}_-, a]\|_{\text{tr}}\right) . \tag{7.4}$$

We know that $\|A+B\|_{\text{tr}} \leq \|A\|_{\text{tr}} + \|B\|_{\text{tr}}$ and $\|AB\|_{\text{tr}} \leq \|A\|_{\text{tr}}\|B\|_{\text{tr}}$, which are, of course, true for all norms. Moreover, we have $\|\hat{J}_3\|_{\text{tr}}^2 = \frac{1}{2}\|\hat{J}_+\|_{\text{tr}}^2 = \frac{1}{2}\|\hat{J}_-\|_{\text{tr}}^2 = \frac{1}{3}j(j+1)(2j+1)$. With these, we can conclude that

$$\|[\mathcal{D}_f, \pi(a)]\|_{\text{op}} \leq \|[\mathcal{D}_f, \pi(a)]\|_{\text{tr}} \leq \frac{2}{\theta_f}\sqrt{2(2j+1)}\|\pi(a)\|_{\text{tr}} \leq \frac{2\sqrt{2}}{\theta_f}(2j+1)\|\pi(a)\|_{\text{op}} < \infty. \tag{7.5}$$

The reslovent of $\mathcal{D}_f$ can be explicitly computed as

$$(\mathcal{D}_f - \mu)^{-1} = r_j^2\{j(j+1) - \mu r_j(\mu r_j + 1)\}^{-1}\left(\mathcal{D}_f + \frac{1}{r_j} + \mu\right), \tag{7.6}$$

the only singularities occur at $\mu = \frac{j}{r_j} \in \mathbb{R}$ and $\mu = \frac{-(j+1)}{r_j} \in \mathbb{R}$. For $\mu \notin \mathbb{R}$, $(\mathcal{D}_f - \mu)^{-1}$ is non-singular and bounded which maps $\mathcal{H}_f$ into itself and thus it is a finite rank operator and hence a compact operator. Note that these properties are valid only for a particular fuzzy sphere $\mathbb{S}_j^2$ with finite j but these cannot be extended to the union of all the fuzzy spheres $\bigoplus_{j\in\mathbb{Z}^+/2} \mathbb{S}_j^2 \equiv \mathbb{R}_{\theta_f}^3$. In fact our distance computation will be restricted to a particular fuzzy sphere $\mathbb{S}_j^2$ with a fixed radius $r_j = \theta_f\sqrt{j(j+1)}$.

7.1.1 *Eigenspinors of Dirac operator $\mathcal{D}_f$ [89]*

The eigenvalues of $\mathcal{D}_f$ with the corresponding eigenspinors are obtained in [89] as

$$\mathcal{D}_f|m\rangle\rangle_+ = \frac{j}{r_j}|m\rangle\rangle_+ \,; \quad \mathcal{D}_f|m'\rangle\rangle_- = -\frac{j+1}{r_j}|m'\rangle\rangle_- \,, \tag{7.7}$$

$$\text{where,} \quad |m\rangle\rangle_+ = \frac{1}{\sqrt{2j+1}}\begin{bmatrix} \sqrt{j+m+1}|j,m\rangle \\ \sqrt{j-m}|j,m+1\rangle \end{bmatrix}, \; m = -j-1, ..., j \tag{7.8}$$

$$|m'\rangle\rangle_- = \frac{1}{\sqrt{2j+1}}\begin{bmatrix} -\sqrt{j-m'}|j,m'\rangle \\ \sqrt{j+m'+1}|j,m'+1\rangle \end{bmatrix}, \; m' = -j,, ..., j-1. \tag{7.9}$$

Note that the eigenvalues for a particular j is independent of m or m' and is responsible for a $(2j+2)$-fold degeneracy in the positive eigenvalue sector and a $2j$-fold degeneracy in the negative eigenvalue sector. This can be understood from the tensor product structure of Dirac operator (D.35) and the Clebsch-Gordon decomposition of a tensor product of a pair of $SU(2)$ representations. For example, if $\hat{\vec{J}}$ in (D.35) corresponds to the $j = 1/2$ representation i.e. $\hat{\vec{J}} = \vec{\sigma}/2$, then it will split into the direct sum of $j = 1$ (triplet) and $j = 0$ (singlet) representations of three and one dimension, respectively. Lastly, we would like to mention that the positive eigenvalue of $\mathcal{D}_f$ obtained in [89] is $\frac{j+1}{r_j}$ and the negative is $-\frac{j}{r_j}$, unlike the above (7.7). This is because of the fact that we ignore the identity term (responsible of chiral symmetry) in the Dirac operator of [89], also the one obtained in [88], as this term does not survive in the commutator $[\mathcal{D}_f, \pi(a)]$.

7.1.2 *Ball condition with Dirac eigenspinors*

As in the case of Moyal plane, the computation of operator norm $\|[\mathcal{D}_f, \pi(a)]\|_{\text{op}}$ for fuzzy sphere is simplified drastically using the eigenspinors (7.7) basis. A straightforward computation yields,

$${}_+\langle\langle j,m|[\mathcal{D}_f,\pi(a)]|j,m'\rangle\rangle_- = \frac{1}{r_j}A_{(2j+2)\times 2j} \,; \quad {}_+\langle\langle j,m|[\mathcal{D}_f,\pi(a)]|j,n\rangle\rangle_+ = 0 \,; \tag{7.10}$$

$${}_-\langle\langle j,m'|[\mathcal{D}_f,\pi(a)]|j,m\rangle\rangle_+ = -\frac{1}{r_j}A^\dagger_{2j\times(2j+2)} \,; \quad {}_-\langle\langle j,m'|[\mathcal{D}_f,\pi(a)]|j,n'\rangle\rangle_- = 0 \,, \tag{7.11}$$

where $m,n = -j-1,-j,\cdots,j$ and $m',n' = -j,-j-1,\cdots,j-1$ with A's, given by

$$\begin{aligned} A_{(2j+2)\times 2j} &= (2j+1)_+\langle\langle j,m|\pi(a)|j,m'\rangle\rangle_- \\ &= \left\{\sqrt{(j-m)(j+m'+1)}\, a_{m+1,m'+1} - \sqrt{(j-m')(j+m+1)}\, a_{m,m'}\right\} ; \end{aligned} \tag{7.12}$$

$$\begin{aligned} A^\dagger_{2j\times(2j+2)} &= (2j+1)_-\langle\langle m'|\pi(a)|m\rangle\rangle_+ \\ &= \left\{\sqrt{(j-m)(j+m'+1)}\, a_{m'+1,m+1} - \sqrt{(j-m')(j+m+1)}\, a_{m',m}\right\} . \end{aligned} \tag{7.13}$$

That is, in the matrix form we get

$$[\mathcal{D}_f, \pi(a)] = \frac{1}{r_j} \left(\begin{array}{c|c} 0_{(2j+2)\times(2j+2)} & A_{(2j+2)\times 2j} \\ \hline -\ A^\dagger_{2j\times(2j+2)} & 0_{(2j)\times(2j)} \end{array} \right) , \tag{7.14}$$

which yields

$$[\mathcal{D}_f, \pi(a)]^\dagger [\mathcal{D}_f, \pi(a)] = \frac{1}{r_j^2} \left(\begin{array}{c|c} (AA^\dagger)_{(2j+2)\times(2j+2)} & 0_{(2j+2)\times 2j} \\ \hline 0_{2j\times(2j+2)} & (A^\dagger A)_{2j\times 2j} \end{array} \right) . \tag{7.15}$$

With this, we get the operator norm as

$$\|[\mathcal{D}_f, \pi(a)]\|^2_{\text{op}} = \|[\mathcal{D}_f, \pi(a)]^\dagger [\mathcal{D}_f, \pi(a)]\|_{\text{op}} = \frac{1}{r_j^2}\|AA^\dagger\|_{\text{op}} = \frac{1}{r_j^2}\|A^\dagger A\|_{\text{op}} = \frac{1}{r_j^2}\|A\|^2_{\text{op}} . \tag{7.16}$$

Thus, the computation of operator norm $\|[\mathcal{D}_f, \pi(a)]\|_{\text{op}}$ reduces to the computation of $\|A\|_{\text{op}}$ and that too by computing $\|A^\dagger A\|_{\text{op}}$, rather than $\|AA^\dagger\|_{\text{op}}$ as the former involves $A^\dagger A$ which is a matrix of smaller rank. With this, the ball condition reduces to

$$\|[\mathcal{D}_f, \pi(a)]\|_{\text{op}} \ \leq \ 1 \quad \Rightarrow \quad \|A\|_{\text{op}} \ \leq \ r_j . \tag{7.17}$$

7.1.3 *Spectral distance between discrete states*

The role of generalized points on $\mathcal{F}_j$ will be played by the pure states ω_ρ where ρ is the pure density matrix. We can take $\rho = |j,m\rangle\langle j,m|$, giving us the discrete pure states and we can also take $\rho = |z\rangle\langle z|$ where $|z\rangle$ is the Perelomov's coherent state (5.52).

7.1.3.1 *Infinitesimal distance*

Let us define the Connes' spectral distance between two infinitesimally separated discrete pure states $\omega_{\rho_{m+1}} \equiv \omega_{m+1}$ and $\omega_{\rho_m} \equiv \omega_m$, where $\rho_m = |j,m\rangle\langle j,m| \equiv |m\rangle\langle m|$ (suppressing j, as the value of j is fixed). We have

$$d_j(\omega_{m+1}, \omega_m) = \sup_{a\in B_s} |\text{tr}(\rho_{m+1}a) - \text{tr}(\rho_m a)| = \sup_{a\in B_s} |\langle m+1|a|m+1\rangle - \langle m|a|m\rangle| \ ; \tag{7.18}$$

where $B_s = \{a \in \mathcal{A}_f \ : \ a^\dagger = a = c^\dagger c,\ 0 < \|[\mathcal{D}_f, \pi(a)]\|_{\text{op}} \leq 1\}$. In the same way as done for Moyal case, we get

$$d_j(\omega_{m+1}, \omega_m) = \sup_{a\in B_s} \left|\langle m| \frac{\hat{J}_- a \hat{J}_+}{\sqrt{j(j+1) - m(m+1)}} - a|m\rangle\right| = \sup_{a\in B_s} \frac{|\langle m|[\hat{J}_-, a]|m+1\rangle|}{\sqrt{j(j+1) - m(m+1)}} . \tag{7.19}$$

Using again the Bessel's inequality (6.40), we get the distance to be bounded above by

$$d_j(\omega_{m+1},\omega_m) \leq \frac{\|[\hat{J}_-,a]\|_{op}}{\sqrt{j(j+1)-m(m+1)}} = \frac{\|[\hat{J}_+,a]\|_{op}}{\sqrt{j(j+1)-m(m+1)}} \leq \frac{r_j}{\sqrt{j(j+1)-m(m+1)}} \,. \tag{7.20}$$

The last inequality follows from the ball condition and (7.4) to get

$$\frac{1}{r_j}\|[\hat{J}_-,a]\|_{\text{op}} \leq \|[\mathcal{D}_f,\pi(a)]\|_{\text{op}} \leq 1 \;\Rightarrow\; \|[\hat{J}_-,a]\|_{\text{op}} \leq r_j \,. \tag{7.21}$$

It is now quite straightforward to check that this upper bound in (7.20) coincides exactly with the lower bound (6.28) i.e, the optimal element $a_s \in B_s$ saturating the above inequality (7.20) is given by

$$a_s = \frac{d\rho}{\|[\mathcal{D}_f,\ \pi(d\rho)]\|_{\text{op}}}\,, \quad \text{where} \quad d\rho = \rho_{m+1}-\rho_m = |m+1\rangle\langle m+1| - |m\rangle\langle m| \;. \tag{7.22}$$

All that we need to do is to compute the operator norm $\|[\mathcal{D},\pi(d\rho)]\|_{\text{op}}$ the using Dirac operator's eigen-spinor basis (7.7). To that end, note that

$$[\mathcal{D}_f,\pi(d\rho)] = \frac{1}{r_j}\left(\begin{array}{c|c} 0 & A \\ \hline -A^\dagger & 0 \end{array}\right), \quad \text{where} \;\; A = (2j+1)_+\langle\langle j,m|\pi(d\rho)|j,m'\rangle\rangle_- \tag{7.23}$$

$$\text{i.e.}\;\; A = \left\{\sqrt{(j-m)(j+m'+1)}\,(d\rho)_{m+1,m'+1} - \sqrt{(j-m')(j+m+1)}\,(d\rho)_{m,m'}\right\}$$

$$= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} ; \quad (\text{note that} \;\; d\rho = |m+1\rangle\langle m+1| - |m\rangle\langle m|) \tag{7.24}$$

$$\begin{aligned} \text{with,} \quad a_{11} &= -\sqrt{j(j+1)-m(m-1)} \\ a_{22} &= 2\sqrt{j(j+1)-m(m+1)} \\ a_{33} &= -\sqrt{j(j+1)-(m+1)(m+2)}\,. \end{aligned} \tag{7.25}$$

Here, the rows and columns of the matrix A are labeled from top to bottom and left to right respectively by $|j,m-1\rangle\rangle_+, |j,m\rangle\rangle_+, |j,m+1\rangle\rangle_+$ and $|j,m-1\rangle\rangle_-, |j,m\rangle\rangle_-, |j,m+1\rangle\rangle_-$. This readily yields $\|[\mathcal{D}_f,\pi(d\rho)]\|_{\text{op}} = \frac{2}{r_j}\sqrt{j(j+1)-m(m+1)}$. The lower bound formula (6.28) then gives, on using $\text{tr}\{(d\rho)^2\} = 2$

$$d_j(\omega_{m+1},\omega_m) \geq \frac{\|d\rho\|^2_{\text{tr}}}{\|[\mathcal{D}_f,\pi(d\rho)]\|_{\text{op}}} = \frac{\theta_f\sqrt{j(j+1)}}{\sqrt{j(j+1)-m(m+1)}}\,. \tag{7.26}$$

Thus, the distance is given by

$$d_j(\omega_{m+1}, \omega_m) = \frac{\theta_f \sqrt{j(j+1)}}{\sqrt{j(j+1) - m(m+1)}} . \tag{7.27}$$

Finally, we show that this a_s is not unique and there exists another optimal element a'_s:

$$a'_s = \frac{r_j}{\sqrt{j(j+1) - m(m+1)}} |m+1\rangle\langle m+1| \ , \tag{7.28}$$

which too saturates inequality (7.20). This reproduces the result of [89].

7.1.3.2 *A heuristic derivation of the above distance formula* (7.27)

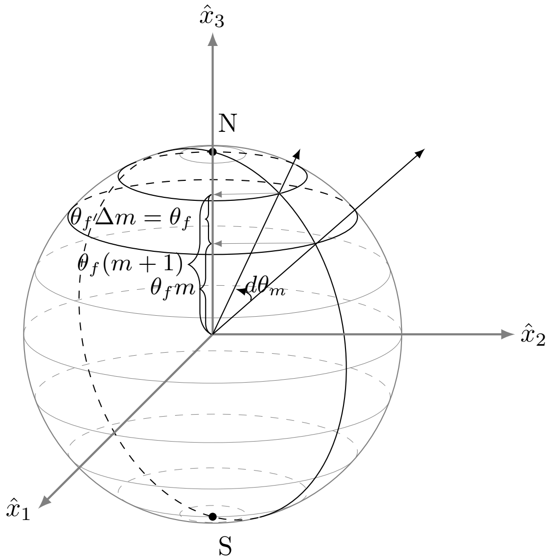

Figure 7.1: Infinitesimal change on the surface of sphere with respect to the change in m

To clarify the adjective "infinitesimal" in this context, let us recall from the theory of angular momentum that the state $|j, m\rangle$ can be visualized as the vector $\vec{x}$ precessing the x_3- axis along a cone, in such a manner that the tip of the vector $\vec{x}$ lies on the circle of latitude on a sphere of radius $\theta_f \sqrt{j(j+1)}$, maintaining a fixed x_3- component $\theta_f m$ with m varying in the interval $-j \leq m \leq j (j \in \mathbb{Z}/2)$ in the steps of unity (see fig.1). The associated polar angles are therefore quantized as,

$$\theta_m = \sin^{-1}\left(\frac{m}{\sqrt{j(j+1)}}\right). \tag{7.29}$$

Now, let us treat m to be a continuous variable for a moment. This yields

$$d\theta_m = \frac{dm}{\sqrt{j(j+1) - m^2}}. \tag{7.30}$$

The distance, which is identified with arc length in figure 1, is then obtained by multiplying with the quantized radius to get

$$ds(m) = \frac{\theta_f \sqrt{j(j+1)}\, dm}{\sqrt{j(j+1) - m^2}}. \tag{7.31}$$

This almost matches with the distance expression (7.27); in fact with the formal replacement $dm \to \Delta m = 1$ and $m^2 \to m(m+1)$ one reproduces (7.27) exactly. Further, (7.31) can be shown to follow, albeit somewhat heuristically, from (7.27) by taking the average of the m dependence in (7.27), which gives $d(m, m+1)$ and that of the m -dependence occurring in $d(m-1, m)$: $\frac{1}{2}[m(m+1) + m(m-1)] = m^2$.

7.1.3.3 *Finite Distance*

Let us consider a pair of finitely separated discrete pure states ω_m and ω_n with $\rho_m = |m\rangle\langle m|$ and $\rho_n = |n\rangle\langle n|$ respectively, such that $k = m - n \geq 2$ where $-j \leq m, n \leq j$. We follow here the same technique as adopted in Moyal plane, i.e., we can write

$$|\mathrm{tr}\{(\rho_m - \rho_n)a\}| = \left|\sum_{i=1}^{k} \mathrm{tr}\{(\rho_{n+i} - \rho_{n+i-1})a\}\right| = \left|\sum_{p=n}^{m-1} \mathrm{tr}\{(\rho_{p+1} - \rho_p)a\}\right| ;\; p = m + i - 1. \tag{7.32}$$

From the infinitesimal case, we found

$$|\mathrm{tr}\{(\rho_{p+1} - \rho_p)a\}| \leq \frac{r_j}{\sqrt{j(j+1) - p(p+1)}}\,, \tag{7.33}$$

so that the finite distance is bounded by

$$d_j(\omega_m, \omega_n) = \sup_{a\in B_s} |\mathrm{tr}\{(\rho_m - \rho_n)a\}| \leq \sum_{p=n}^{m-1} \frac{r_j}{\sqrt{j(j+1) - p(p+1)}} \tag{7.34}$$

$$= \sum_{i=1}^{k} \frac{r_j}{\sqrt{j(j+1) - (n+i)(n+i-1)}}\,. \tag{7.35}$$

We can easily see that the optimal element $a_s \in B_s$ saturating this inequality is of the form:

$$a_s = \theta_f \sum_{p=n}^{m-1} \left(\sum_{i=1}^{m-p} \frac{\sqrt{j(j+1)}}{\sqrt{j(j+1) - (p+i)(p+i-1)}} |p\rangle\langle p| \right)\,. \tag{7.36}$$

This gives the finite distance between a pair of discrete pure states ω_m and ω_n as

$$d_j(\omega_m, \omega_n) = |\mathrm{tr}\{(\rho_m - \rho_n)\, a_s\}| = \sum_{i=1}^{k} \frac{r_j}{\sqrt{j(j+1) - (n+i)(n+i-1)}}\,. \tag{7.37}$$

We can easily check that this distance is additive, i.e, it saturates the triangle inequality [89]:

$$d_j(\omega_m, \omega_n) = d_j(\omega_m, \omega_l) + d_j(\omega_l, \omega_n)\,, \quad \text{for any } l \in \quad n \leq l \leq m\,. \tag{7.38}$$

In particular, we can check the distance between north pole (N), represented by the pure state $\omega_j = |j\rangle\langle j|$ and south pole (S), by $\omega_{-j} = |-j\rangle\langle -j|$ is given by

$$d_j(N,S) \;=\; d_j(\omega_j,\omega_{-j}) \;=\; \theta_f \sum_{k=1}^{2j} \frac{\sqrt{j(j+1)}}{\sqrt{k(2j+1-k)}}\,, \quad \forall\, j\,. \tag{7.39}$$

Note that both the discrete pure states ω_j and ω_{-j} representing north and south poles are also coherent states. This implies that we should be able to reproduce the same results from the computation of finite distance between coherent pure states which will be discussed in the next subsection. Let us see for $j = \frac{1}{2}$, which corresponds to a fuzzy sphere with maximal

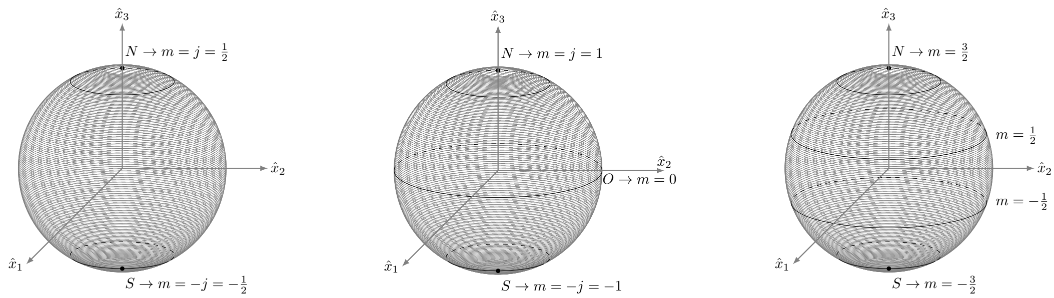

Figure 7.2: Fuzzy spheres, $\mathbb{S}^2_{\frac{1}{2}}$, $\mathbb{S}^2_{1}$, $\mathbb{S}^2_{\frac{3}{2}}$

non-commutativity, the distance between north (N) and south poles (S); also for $j = 1$ and $j = \frac{3}{2}$ respectively can be obtained as

$$d_{\frac{1}{2}}(\mathrm{N,S}) = r_{\frac{1}{2}}\;;\;\; d_1(\mathrm{N,S}) = \sqrt{2}\,r_1 = 1.4142\,r_1\;;\;\; d_{\frac{3}{2}}(\mathrm{N,S}) = \left(\frac{1}{2} + \frac{2\sqrt{3}}{3}\right) r_{\frac{3}{2}} = 1.6547\, r_{\frac{3}{2}}\,. \tag{7.40}$$

This indicates that the fuzzy spheres: $\mathbb{S}^2_{\frac{1}{2}}$, $\mathbb{S}^2_{1}$, $\mathbb{S}^2_{\frac{3}{2}}$ are highly deformed spaces as the corresponding distances (7.40) are way below the corresponding commutative spheres $\pi r_{\frac{1}{2}}$, πr_1 and $\pi r_{\frac{3}{2}}$ respectively. However, we can expect the commutative results can be obtained in the limit $j \to \infty$, i.e., $\mathbb{S}^2_j \xrightarrow{j\to\infty} \mathbb{S}^2$. In the limit $j \to \infty$, we can take $\frac{k}{j} = x_k$ and $\Delta x = x_k - x_{k-1} = \frac{1}{j}$ as the increment such that $\Delta x \to 0$ as $j \to \infty$.

That is, in the limit $j \to \infty$, we can write the ratio of distance d_j with radius r_j as

$$\lim_{j\to\infty} \frac{d_j(N,S)}{r_j} \;=\; \lim_{j\to\infty} \sum_{k=1}^{2j} \frac{1}{\sqrt{k(2j+1-k)}} = \lim_{j\to\infty} \sum_{k=1}^{2j} \frac{\frac{1}{j}}{\sqrt{\frac{k}{j}(2+\frac{1}{j}-\frac{k}{j})}} \tag{7.41}$$

$$= \lim_{\Delta x\to 0} \sum_{x=x_1}^{x_{2j}=2} \frac{\Delta x}{\sqrt{x_k(2+\Delta x - x_k)}} \tag{7.42}$$

$$= \int_0^2 \frac{dx}{\sqrt{x(2-x)}} = 2\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \pi\,, \quad (x = 1-t)\,. \tag{7.43}$$

This is exactly the ratio between the distance of north pole and south pole of $\mathbb{S}^2$ with its radius [89] which implies that indeed $\mathbb{S}^2_j \xrightarrow{j\to\infty} \mathbb{S}^2$.

7.1.4 *Spectral distance between coherent states*

The space of Perelomov's $SU(2)$ coherent states is constructed in [87] on fuzzy sphere $\mathbb{S}_j^2$ which serves as a non-commutative analogue of homogeneous space of $\mathbb{S}_j^2$. Such coherent states along the meridien i.e. the great circle connecting north and south poles are given by (5.52)

$$|z\rangle \;=\; U_f(z)|j,j\rangle \;\equiv\; e^{-i\hat{J}_2\theta}|j\rangle \;=\; e^{\frac{\theta}{2}(\hat{J}_- - \hat{J}_+)}|j\rangle \;;\;\text{ where }\;\frac{\theta}{2} = \tan^{-1}(|z|)\;,\;\;\phi = 0. \quad (7.44)$$

The corresponding pure coherent states on $\mathcal{F}_j$ are given by

$$\omega_z(a) = \text{tr}(\rho_z a) = \langle j|U_f^\dagger(z) a U_f(z)|j\rangle\;,\quad\text{where}\quad \rho_z = |z\rangle\langle z| = U_f(z)|j\rangle\langle j|U_f^\dagger(z)\;. \quad (7.45)$$

By using the Hadamard identity and putting $G(\theta) = \frac{\theta}{2}(\hat{J}_+ - \hat{J}_-)$, we get

$$U_f^\dagger(z) a U_f(z) = e^{G(\theta)} a\, e^{-G(\theta)} = a + [G(\theta), a] + \frac{1}{2!}[G(\theta), [G(\theta), a]] + \cdots\;. \quad (7.46)$$

7.1.4.1 *An upper bound on $d_j(\omega_z, \omega_0)$ which is unsaturated for finite j*

The difference between the pure states ω_z and ω_0 (note that $\omega_0 \equiv \omega_{\rho_0}$ with $\rho_0 = |j\rangle\langle j|$ as we take $|j\rangle \equiv |z = 0\rangle$) as the highest weight state can be written as

$$|\omega_z(a) - \omega_0(a)| = \big|\langle j|U_f^\dagger(z) a U_f(z) - a|j\rangle\big| = \big|\langle j|[G(\theta), a] + \frac{1}{2!}[G(\theta), [G(\theta), a]] + \cdots|j\rangle\big|. \quad (7.47)$$

Like in the case of Moyal plane here also, we can introduce a one-parameter family of pure states:

$$\rho_{zt} = |zt\rangle\langle zt| = U_f(zt)|j\rangle\langle j|U_f^\dagger(zt) \quad (\text{with}\quad t \in [0,1])\;, \quad (7.48)$$

and define the function

$$W(t) \;=\; \omega_{zt}(a) \;=\; \text{tr}(\rho_{zt} a) \;=\; \langle j|U_f^\dagger(zt)\, a\, U_f(zt)|j\rangle\;. \quad (7.49)$$

Note that we have $U_f(zt)$ as

$$U_f(zt) = e^{-G(\theta_t)}\;,\quad G(\theta_t) = \frac{\theta_t}{2}(\hat{J}_+ - \hat{J}_-)\;,\quad\text{where}\quad \theta_t = 2\tan^{-1}(|z|t)\;. \quad (7.50)$$

We can rewrite (7.47) in terms of this function $W(t)$ (7.49) to get the following inequality

$$|\omega_z(a) - \omega_0(a)| \;=\; \left|\int_0^1 \frac{dW(t)}{dt} dt\right| \;\leq\; \int_0^1 \left|\frac{dW(t)}{dt}\right| dt\;. \quad (7.51)$$

From the computation, we get

$$\left|\frac{dW(t)}{dt}\right| = \frac{|z|}{1+|z|^2t^2}\left|\left\{[\hat{J}_+ - \hat{J}_-, a] + [G(\theta_t), [\hat{J}_+ - \hat{J}_-, a]] + \frac{1}{2!}[G(\theta_t), [G(\theta_t), [\hat{J}_+ - \hat{J}_-, a]]] + \cdots\right\}\right|$$
$$= \frac{|z|}{1+|z|^2t^2}\,\omega_{zt}([\hat{J}_+ - \hat{J}_-, a]) = \frac{|z|}{1+|z|^2t^2}\left|\omega_{zt}([\hat{J}_-, a]) + \omega_{zt}([\hat{J}_-, a]^\dagger)\right| . \quad (7.52)$$

Using Cauchy-Schwartz inequality, we get

$$\left|\omega_{zt}([\hat{J}_-, a]) + \omega_{zt}([\hat{J}_-, a]^\dagger)\right| \leq \sqrt{2}\sqrt{\left|\omega_{zt}([\hat{J}_-, a])\right|^2 + \left|\omega_{zt}([\hat{J}_-, a]^\dagger)\right|^2}$$
$$\leq 2\|[\hat{J}_-, a]\|_{op} \leq 2r_j , \quad (7.53)$$

since every state is bounded as $\omega_{zt}([\hat{J}_-, a]) \leq \|[\hat{J}_-, a]\|_{op}$ and $\|[\hat{J}_-, a]\|_{op} = \|[\hat{J}_-, a]^\dagger\|_{op}$.

Thus, from (7.51), (7.52) and (7.53) , we get

$$|\omega_z(a) - \omega_0(a)| \leq (2r_j)\int_0^1 \left(\frac{|z|dt}{1+|z|^2t^2}\right) = 2r_j \tan^{-1}(|z|) = r_j\theta \quad (7.54)$$

That is, the Connes' spectral distance between two pure states ω_z and ω_0 is bounded above by the geodesic distance on $\mathbb{S}^2$ as

$$d_j(\omega_z, \omega_0) = \sup_{a\in B_s} |\omega_z(a) - \omega_0(a)| \leq r_j\,\theta . \quad (7.55)$$

This Connes' spectral distance would have corresponded to geodesic distance on $\mathbb{S}^2$ if we were able to find an optimal element $a_s \in \mathcal{A}_f \equiv \mathcal{H}_j$ saturating this upper bound. However, it has already been indicated in the previous subsection that spectral distance between north and south poles on $\mathbb{S}^2_j$ is always less than the geodesic distance πr_j for any finite j (see for example (7.40)) and reduces to it only in the limit $j \to \infty$ (7.43). Thus, there simply does not exist any optimal element a_s saturating the above inequality (7.55) for any finite j. This feature displays a stark difference between Moyal plane $\mathbb{R}^2_\star$ and fuzzy sphere $\mathbb{S}^2_\star$. For the latter case, we shall see later, the notion of conventional geodesics do not exist; we need to consider interpolating the extremal pure states $\omega_0 \equiv \rho_0$ and $\omega_z \equiv \rho_z$ by a one-parameter family of suitable mixed states, rather than pure states ρ_{zt} (7.48) as we show in the sequel.

7.1.5 *Lower bound to the spectral distance between infinitesimally separated continuous pure states*

From above computation of the spectral distance between finitely separated coherent states, we cannot obtain the exact distance function, in absence of any optimal element a_s saturating (7.55). Moreover, computation using the exact distance formula (6.24) obtained in chapter 6 involves computation of operator norm i.e. the eigenvalues of matrices of the form: $(A^\dagger A)_{2j\times 2j}$ (7.16). This is, however, virtually impossible to carry out for any $\mathbb{S}^2_j$ for large j. However, we can easily compute the lower bound (6.28) of the distance for a pair of infinitesimally separated coherent states for any j. Expectedly, this lower bound distance will differ from the exact one by just a numerical factor, as in case of Moyal plane.

To that end, let us consider a pair of infinitesimally separated coherent states $\omega_{dz} \equiv \omega_{\rho_{dz}}$ ($\rho_{dz} = |dz\rangle\langle dz|$) and $\omega_0 \equiv \omega_{\rho_0}$ ($\rho_0 = |j\rangle\langle j|$). The latter can be taken to correspond to the north pole. Here,

$$|dz\rangle \;=\; U_f(dz)|j\rangle \;=\; e^{\frac{d\theta}{2}(\hat{J}_- - \hat{J}_+)}|j\rangle \;=\; |j\rangle + \frac{d\theta}{2}\sqrt{2j}|j-1\rangle \;. \tag{7.56}$$

We then introduce the following difference:

$$d\rho \;=\; \rho_{dz} - \rho_0 \;=\; |dz\rangle\langle dz| - |j\rangle\langle j| \;=\; \frac{d\theta}{2}\sqrt{2j}(|j\rangle\langle j-1| + |j-1\rangle\langle j|)\;, \tag{7.57}$$

We can easily see that $\|d\rho\|_{\text{tr}} = \sqrt{j}\,d\theta$ and we are left with the computation of $\|[\mathcal{D}_f, \pi(d\rho)]\|_{\text{op}}$. We know that the commutator $[\mathcal{D}_f, \pi(d\rho)]$ will take the following form in its eigenbasis (7.7,7.8,7.9) just like (7.14) where the matrix A, given by (7.12). Substituting $d\rho$ (7.57) , we get

$$[\mathcal{D}_f, \pi(d\rho)] \equiv \frac{1}{r_j}\left(\begin{array}{c|c} 0 & A \\ \hline -A^\dagger & 0 \end{array}\right) \;;\quad A = d\theta\begin{pmatrix} \sqrt{j(j+\frac{1}{2})} & 0 & 0 \\ 0 & -\sqrt{j(j-\frac{1}{2})} & 0 \\ -j\sqrt{2} & 0 & 0 \end{pmatrix} . \tag{7.58}$$

Here, rows (columns) of A are labeled from up to down (left to right) respectively by $|j\rangle\rangle_+, |j-1\rangle\rangle_+, |j-2\rangle\rangle_+ \left(|j-1\rangle\rangle_-, |j-2\rangle\rangle_-, |-3\rangle\rangle_-\right)$. We then get

$$A^\dagger A = j(d\theta)^2\begin{pmatrix} 3j+\frac{1}{2} & 0 & 0 \\ 0 & j-\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\Rightarrow\; \|A\|_{\text{op}} = d\theta\sqrt{j(3j+\frac{1}{2})} \tag{7.59}$$

Thus, we get

$$\|[\mathcal{D}_f, \pi(d\rho)]\|_{\text{op}} = \frac{1}{r_j}\|A\|_{\text{op}} = \frac{1}{r_j}\,d\theta\sqrt{j(3j+\frac{1}{2})} \tag{7.60}$$

With this, we get the lower bound to the spectral distance as

$$d_j(\omega_{dz}, \omega_0) \geq \frac{\|d\rho\|_{\text{tr}}^2}{\|[\mathcal{D}, \pi(d\rho)]\|_{\text{op}}} = r_j\,d\theta\sqrt{\frac{2j}{6j+1}}\;. \tag{7.61}$$

For the specific cases of first three smallest fuzzy spheres : $\mathbb{S}^2_{\frac{1}{2}}$, $\mathbb{S}^2_1$, $\mathbb{S}^2_{\frac{3}{2}}$, we get

$$r_{\frac{1}{2}}\frac{d\theta}{2} \leq d_{\frac{1}{2}}(\omega_{dz},\omega_0) < r_{\frac{1}{2}}\, d\theta \; ; \tag{7.62}$$

$$r_1\, d\theta\sqrt{\frac{2}{7}} \leq d_1(\omega_{dz},\omega_0) < r_1\, d\theta \; ; \tag{7.63}$$

$$r_{\frac{3}{2}}\, d\theta\sqrt{\frac{3}{10}} \leq d_{\frac{3}{2}}(\omega_{dz},\omega_0) < r_{\frac{3}{2}}\, d\theta \; . \tag{7.64}$$

In general, we have the following estimation for spectral distance between a pair of infinitesimally separated coherent states, given by ρ_{dz} and ρ_0:

$$r_j\, d\theta\sqrt{\frac{2j}{6j+1}} \leq d_j(\omega_{dz},\omega_0) < r_j\, d\theta \; . \tag{7.65}$$

However, we shall show that it is possible to compute the exact finite and infinitesimal spectral distance for $j = \frac{1}{2}$ coinciding with the lower bound in (7.62) and we can also find an almost exact distance for $j = 1$. At the level of infinitesimal distance itself, however, we get an improvement over and above the lower bound given in (7.63), indicating a non-trivial role played by the transverse component $\Delta\rho_\perp$ in (6.24). Needless to say that this feature with persist at the level of finite distance computation as well.

7.1.6 *Spectral distance on the configuration space of fuzzy sphere* $\mathbb{S}^2_{\frac{1}{2}}$

Clearly, for $\mathbb{S}^2_{\frac{1}{2}}$, the configuration space is $\mathcal{F}_{\frac{1}{2}} = \text{Span}\{|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle\}$ and the quantum Hilbert space $\mathcal{H}_{\frac{1}{2}}$ consists of elements which are complex 2×2 matrices. Since we know that the spectral distance gets saturated by positive elements of the algebra [110], we can consider only the Hermitian 2×2 matrices. The elements of $\mathcal{H}_{\frac{1}{2}}$ have the following form:

$$a = a_0\mathbb{I}_2 + a_i\sigma_i \; ; \;\; a_\mu \in \mathbb{R}\, , \mu = 0,1,2,3\, , \;\; \mathbb{I}_2 \rightarrow \text{Identity matrix}\, , \; \sigma_i \rightarrow \text{Pauli matrices}\, . \tag{7.66}$$

Since the first term including a_0 does not survive in the commutator $[\mathcal{D}_f, \pi(a)]$, we can dis regard this term and simply take $a_0 = 0$, without loss of generality. Thus, we have the algebra elements

$$a = \vec{a}.\vec{\sigma} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \in su(2)\, , \quad \vec{a} \in \mathbb{R}^3\, . \tag{7.67}$$

We can compute the commutator (7.14) for this a (7.67) as follows

$$[\mathcal{D}_f, \pi(a)] \equiv \frac{1}{r_{\frac{1}{2}}}\left(\begin{array}{c|c} 0_{3\times 3} & A_{3\times 1} \\ \hline -A^\dagger_{1\times 3} & 0 \end{array}\right)\, , \quad \text{where} \quad A_{3\times 1} = \begin{pmatrix} \sqrt{2}(a_1 + ia_2) \\ 2a_3 \\ -\sqrt{2}(a_1 - ia_2) \end{pmatrix} . \tag{7.68}$$

Note that $A_{3\times 1} = 2\ _{+}\langle\langle m|\pi(a)|m'\rangle\rangle_{-}$, where $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ and $m' = -\frac{1}{2}$ and cealrly, we labeled the rows of A from top to bottom by $|-\frac{3}{2}\rangle\rangle_{+}, |-\frac{1}{2}\rangle\rangle_{+}, |\frac{1}{2}\rangle\rangle_{+}$ respectively and the single column by $|-\frac{1}{2}\rangle\rangle_{-}$. Clearly, we get $A^{\dagger}A = 4(a_1^2 + a_2^2 + a_3^2) = 4|\vec{a}|^2$, just a number. We had (7.16) which can be written as

$$\|[\mathcal{D}_f, \pi(a)]\|_{op} = \frac{1}{r_{\frac{1}{2}}}\|A\|_{op} = \frac{2}{r_{\frac{1}{2}}}|\vec{a}| \leq 1 \ \Rightarrow\ |\vec{a}| \leq \frac{1}{2}\, r_{\frac{1}{2}} \,. \tag{7.69}$$

Interestingly, this reduces to usual ball of radius $\frac{1}{2}\, r_{\frac{1}{2}}$ in $\mathbb{R}^3$. Now, we want to compute the spectral distance between a pair of coherent states which are of the form: $\omega_\theta(a) = \mathrm{tr}(\rho_\theta\ a)$. For $j = \frac{1}{2}$, we know that all pure states are coherent states [89]. Let us consider the following pair of pure density matrices (see figure 7.3 for the points N and P):

1. $\rho_N = \rho_{\theta=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

2. $\rho_P = \rho_{\theta_0} = U(\theta_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} U^{\dagger}(\theta_0) = \frac{1}{2} \begin{pmatrix} 1+\cos\theta_0 & \sin\theta_0 \\ \sin\theta_0 & 1-\cos\theta_0 \end{pmatrix}$,

   where $U(\theta_0) = \begin{pmatrix} \cos\frac{\theta_0}{2} & -\sin\frac{\theta_0}{2} \\ \sin\frac{\theta_0}{2} & \cos\frac{\theta_0}{2} \end{pmatrix} \in SU(2)$.

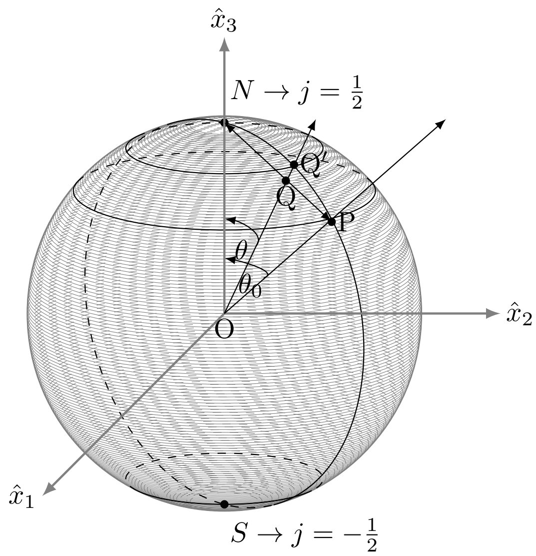

Figure 7.3: Space of Perelomov's $SU(2)$ coherent states for $j = \frac{1}{2}$.

Introducing $\Delta\rho = \rho_{\theta_0} - \rho_0$ which is a traceless Hermitian matrix, we can also expand it as follows (7.67):

$$\Delta\rho \ = \ \vec{\Delta\rho}.\vec{\sigma}\ , \quad \text{where} \ \ (\Delta\rho)_1 = \frac{\sin\theta_0}{2}\ ; \ \ (\Delta\rho)_2 = 0\ ; \ \ (\Delta\rho)_3 = \frac{-1+\cos\theta_0}{2}. \tag{7.70}$$

With this, we can write

$$|\omega_{\theta_0}(a) - \omega_0(a)| = |\mathrm{tr}(\Delta\rho\ a)| = 2|\vec{a}.\vec{\Delta\rho}|. \tag{7.71}$$

Since both $\vec{a}$ and $\vec{\Delta\rho} \in \mathbb{R}^3$, the supremum of $|\vec{a}.\vec{\Delta\rho}|$ will be attained when $\vec{a}$ and $\vec{\Delta\rho}$ are parallel or anti-parallel to each other. Thus, with the ball condition (7.69) and (7.70), we get the spectral distance between ω_{θ_0} and ω_0 as

$$d_{\frac{1}{2}}(\omega_{\theta_0}, \omega_0) = \sup_{|\vec{a}| \leq \frac{r_{1/2}}{2}} |\omega_{\theta_0}(a) - \omega_0(a)| = r_{\frac{1}{2}} \sqrt{(\Delta\rho)_1^2 + (\Delta\rho)_3^2} = r_{\frac{1}{2}} \sin\frac{\theta_0}{2}. \tag{7.72}$$

This is just half of the chordal distance connecting NP (see figure 7.3), reproducing the result of [89]. The corresponding infinitesimal $d(\omega_{d\theta}, \omega_0) = r_{\frac{1}{2}} \frac{d\theta}{2}$ which is exactly the lower bound obtained (7.62). It is important to point out in this context that the optimal element a_s in this case is associated with the 3-vector $\vec{a}_s \in \mathbb{R}^3$ is proportional to $\vec{\Delta\rho}$, as we have observed. Consequently, the lower bound (6.28) itself yields the exact finite distance, with $\Delta\rho_\perp$ playing no role.

Let us now consider a one-parameter family of mixed states $\{\rho_t\}$, $0 \leq t \leq 1$ (as shown in the figure 7.3). A generic mixed state ρ_Q, represented by the point Q on the chord NP inside the sphere), obtained by the following convex sum:

$$\rho_t = (1-t)\rho_N + t\rho_P = \frac{1}{2}\begin{pmatrix} 2 - t(1-\cos\theta_0) & t\sin\theta_0 \\ t\sin\theta_0 & t(1-\cos\theta_0) \end{pmatrix}; \tag{7.73}$$

Clearly, $\rho_0 = \rho_N$ and $\rho_1 = \rho_P$. Introducing $(\Delta\rho)_{QN} = \rho_t - \rho_N$ and $(\Delta\rho)_{PN} = \rho_P - \rho_t$ such that

$$\begin{aligned} ((\Delta\rho)_{QN})_1 &= \frac{t\ \sin\theta_0}{2}, \quad ((\Delta\rho)_{QN})_2 = 0, \quad ((\Delta\rho)_{QN})_3 = -\frac{t(1-\cos\theta_0)}{2} \\ ((\Delta\rho)_{PN})_1 &= \frac{(1-t)\sin\theta_0}{2}, \quad ((\Delta\rho)_{PN})_2 = 0, \quad ((\Delta\rho)_{PN})_3 = -\frac{(1-t)(1-\cos\theta_0)}{2}, \end{aligned}$$

we get the spectral distances between the mixed state ω_t and ω_N and ω_P respectively as

$$d(\omega_t, \omega_N) = t\ r_{\frac{1}{2}} \sin\frac{\theta_0}{2} \quad \text{and} \quad d(\omega_P, \omega_t) = (1-t)\ r_{\frac{1}{2}} \sin\frac{\theta_0}{2}. \tag{7.74}$$

The fact that the distance of this mixed state ω_t from the extremal, pure states ω_N and ω_P are proportional to the parameters t and $(1-t)$ respectively indicates that we can identify a unique pure state Q' (see fig 1) nearest to a given mixed state Q just by extending the straight line OQ from the center O to the surface of the sphere. This distance can therefore be used alternatively to characterize the 'mixedness' of a spin-1/2 system. Further, we have

$$d(\omega_{\theta_0}, \omega_0) = d(\omega_P, \omega_N) = d(\omega_P, \omega_t) + d(\omega_t, \omega_N). \tag{7.75}$$

This is just analogous to (6.91), except that the intermediate state ω_t is not pure. This chord cannot therefore be identified as a conventional geodesic. This same family can be parametrised alternatively [89] as

$$\rho_\theta = \frac{1}{2}(\mathbb{1}_2 + \vec{\sigma}.\vec{n}_\theta) = \frac{1}{2}\begin{pmatrix} 1+|\vec{n}_\theta|\cos\theta & |\vec{n}_\theta|\sin\theta \\ |\vec{n}_\theta|\sin\theta & 1-|\vec{n}_\theta|\cos\theta \end{pmatrix} \tag{7.76}$$

where, $|\vec{n}_\theta|$ is the magnitude the vector $\vec{n}_\theta$ parametrising each of the mixed states between the two extremal pure states and is given by

$$\vec{n}_\theta = |\vec{n}_\theta| \left(\sin\theta,\ 0,\ \cos\theta \right) \tag{7.77}$$

Clearly, $|\vec{n}_\theta|$ is strictly less than 1 : $|\vec{n}_\theta| < 1$ except for the extremal pure states at $\theta = 0$ and $\theta = \theta_0$. Further the mixed state ρ_θ for the open interval $(0, \theta_0)$ represents a point Q in the chord connecting the north pole $N(\theta = 0)$ and point $P(\theta = \theta_0)$, and therefore lies in the interior of the sphere. Indeed, these two different parameters t and θ for the same state can be related by setting $\rho_\theta = \rho_t$, to get

$$t = \frac{1-|\vec{n}_\theta|\cos\theta}{1-\cos\theta_0} = \frac{|\vec{n}_\theta|\sin\theta}{\sin\theta_0} \Rightarrow |\vec{n}_\theta| = \frac{\cos\left(\frac{\theta_0}{2}\right)}{\cos\left(\theta - \frac{\theta_0}{2}\right)} \equiv \frac{OQ}{r_{1/2}}, \quad ON = OP = r_{1/2}. \tag{7.78}$$

Thus, we can recast the spectral distance between mixed state represented by ρ_θ and the pure states ρ_N and ρ_P (7.74) respectively as

$$d(\omega_\theta, \omega_N) = \frac{r_{\frac{1}{2}}\sin\theta}{2\cos\left(\theta - \frac{\theta_0}{2}\right)} \quad \text{and} \quad d(\omega_P, \omega_\theta) = \frac{r_{\frac{1}{2}}\sin(\theta_0 - \theta)}{2\cos\left(\theta - \frac{\theta_0}{2}\right)}. \tag{7.79}$$

For the case $\theta = \theta_0$, we get $d(\omega_{\theta_0}, \omega_N) = r_{\frac{1}{2}} \sin\frac{\theta_0}{2}$ and $d(\omega_P, \omega_{\theta_0}) = 0$.

Finally, we can also obtain the distance between pure states represented by $\rho_\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (South pole S) and ρ_{θ_0} (P) as

$$d(\rho_\pi, \rho_{\theta_0}) = r_{\frac{1}{2}} \cos\frac{\theta_0}{2}. \tag{7.80}$$

This implies that

$$[d(\rho_0, \rho_{\theta_0})]^2 + [d(\rho_\pi, \rho_{\theta_0})]^2 = r_{1/2}^2. \tag{7.81}$$

That is, the Pythagoras identity $(NP^2 + SP^2 = NS^2)$ is obeyed.

All these features, however, will not persist for higher 'n', as we shall see.

7.1.6.1 *Analogy with $\mathbb{C}P^1$ model and mixed states*

For the spectral triple (5.30), the space of vector states is CP^1 [110] which we have reviewed in section 5.6 of chapter 5. We had parametrized a pair of CP^1-doublets, associated to the pair of points lying in the same latitude (i.e. the same polar angle θ) [82] as

$$\chi = \begin{pmatrix} \sin\frac{\theta}{2}e^{i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} \longrightarrow \rho = \chi\chi^\dagger; \ \ \chi' = \begin{pmatrix} \sin\frac{\theta}{2}e^{i\phi'} \\ \cos\frac{\theta}{2} \end{pmatrix} \longrightarrow \rho' = \chi'\chi'^\dagger. \tag{7.82}$$

The spectral distance between these two points is obtained as (5.41)

$$d(\omega_{\rho'},\omega_\rho) = \frac{2\sin\theta}{|D_1 - D_2|}\left|\sin\left(\frac{\phi-\phi'}{2}\right)\right|. \tag{7.83}$$

which corresponds to the distance measured along the chord connecting the pair of points (θ,ϕ) and (θ,ϕ'), at the same latitude θ. Recall that the distance between a pair of states belonging to different lattitudes ($\theta \neq \theta'$) diverges.

Again, let us define a one-parameter family of mixed states out of this pair of pure states ρ and ρ', in an analogous way as defined for $n = 1/2$ representation (7.73), as

$$\rho_t = (1-t)\rho + t\rho'. \tag{7.84}$$

Similarly, we obtain the distances between the mixed state ρ_t and the corresponding pure states ρ representing the point (θ,ϕ) and ρ' representing (θ,ϕ') as

$$d(\omega_{\rho_t},\omega_\rho) = t\frac{2\sin\theta}{|D_1 - D_2|}\left|\sin\left(\frac{\phi-\phi'}{2}\right)\right|; \ \text{and} \ d(\omega_{\rho'},\omega_{\rho_t}) = (1-t)\frac{2\sin\theta}{|D_1 - D_2|}\left|\sin\left(\frac{\phi-\phi'}{2}\right)\right|. \tag{7.85}$$

Clearly, we have

$$d(\omega_{\rho'},\omega_\rho) = d(\omega_{\rho'},\omega_{\rho_t}) + d(\omega_{\rho_t},\omega_\rho). \tag{7.86}$$

7.1.7 *Spectral distance on the configuration space of fuzzy sphere $\mathbb{S}^2_1$*

The computation of spectral distance on $\mathbb{S}^2_1$ $j = 1$ is expected to be much more complicated than the computation on $\mathbb{S}^2_{\frac{1}{2}}$ because here the algebra $\mathcal{H}_1$ consists of complex 3×3 matrices. Here, just like for the case of $j = \frac{1}{2}$, we will consider only the Hermitian traceless complex 3×3 as the identity matrix $\mathbb{I}_3$ commutes with the Dirac operator and hence will give no contribution in the operator norm $\|[\mathcal{D},\pi(a)]\|_{op}$. The algebra element with some extra restrictions provide us with a simple expression of the distance using (6.9), which we then corroborate with a more rigorous calculation using (6.24). The role of $\Delta\rho_\perp$ turns out to be very important in (6.24) for $j = 1$ fuzzy sphere and we employ the most general form of

$\Delta\rho_\perp$ possible to improve the estimate of the spectral distance as best as we can from the lower bound (6.28). The determination of an exact value, even with the help of *Mathematica*, remains a daunting task.

7.1.7.1 *Ball condition using Dirac eigenspinors*

For $j = 1$, the positive and negative eigenspinors (7.7) of Dirac operator $\mathcal{D}_f$ have the ranges: $-2 \leq m \leq 1$ and $-1 \leq m' \leq 0$ respectively such that the commutator (7.14) takes following off-block diagonal form:

$$[\mathcal{D}, \pi(a)] = \frac{1}{r_1} \left(\begin{array}{c|c} 0_{4\times 4} & A_{4\times 2} \\ \hline -A^\dagger_{2\times 4} & 0_{2\times 2} \end{array} \right), \tag{7.87}$$

where $A_{4\times 2} = 3\ {}_+\langle\langle j, m|\pi(a)|j, m'\rangle\rangle_-$ (7.12) and $A^\dagger_{2\times 4} = 3\ {}_-\langle\langle j, m'|\pi(a)|j, m\rangle\rangle_+$ (7.13) take the following forms:

$$A_{4\times 2} = \begin{pmatrix} -\sqrt{3}a_{1,0} & -\sqrt{6}a_{1,-1} \\ \sqrt{2}(a_{1,1} - a_{0,0}) & (a_{1,0} - 2a_{0,-1}) \\ (2a_{0,1} - a_{-1,0}) & \sqrt{2}(a_{0,0} - a_{-1,-1}) \\ \sqrt{6}a_{-1,1} & \sqrt{3}a_{-1,0} \end{pmatrix}; \tag{7.88}$$

$$A^\dagger_{2\times 4} = \begin{pmatrix} -\sqrt{3}a_{0,1} & \sqrt{2}(a_{1,1} - a_{0,0}) & (2a_{1,0} - a_{0,-1}) & \sqrt{6}a_{1,-1} \\ -\sqrt{6}a_{-1,1} & (a_{0,1} - 2a_{-1,0}) & \sqrt{2}(a_{0,0} - a_{-1,-1}) & \sqrt{3}a_{0,-1} \end{pmatrix}. \tag{7.89}$$

Note that here $a_{m,n}$ are the usual matrix elements $\langle m| a |n\rangle$, with $m, n \in \{1, 0, -1\}$. Using above result for $[\mathcal{D}, \pi(a)]$ we readily obtain:

$$[\mathcal{D}, \pi(d\rho)]^\dagger [\mathcal{D}, \pi(d\rho)] = \frac{1}{r_1^2} \left(\begin{array}{c|c} (AA^\dagger)_{4\times 4} & 0_{4\times 2} \\ \hline 0_{2\times 4} & (A^\dagger A)_{2\times 2} \end{array} \right). \tag{7.90}$$

By exploiting the property of operator norm one has the freedom to choose between the two block-diagonal square matrices as $\frac{1}{r_1^2}\|AA^\dagger\|_{op} = \frac{1}{r_1^2}\|A^\dagger A\|_{op} = \|[\mathcal{D}, \pi(a)]^\dagger [\mathcal{D}, \pi(a)]\|_{op} = \|[\mathcal{D}, \pi(a)]\|^2_{op}$. We choose to work with the more convenient one i,e the 2×2 matrix $(A^\dagger A)_{2\times 2}$ which turns out to be:

$$M := (A^\dagger A)_{2\times 2} = \begin{pmatrix} M_{11} & M_{12} \\ M^*_{12} & M_{22} \end{pmatrix} \tag{7.91}$$

where,

$$\begin{aligned}
M_{11} &= 3|a_{0,1}|^2 + 2(a_{0,0} - a_{1,1})^2 + |a_{0,-1} - 2a_{1,0}|^2 + 6|a_{1,-1}|^2 \\
M_{22} &= 3|a_{0,-1}|^2 + 2(a_{0,0} - a_{-1,-1})^2 + |a_{1,0} - 2a_{0,-1}|^2 + 6|a_{1,-1}|^2 \\
M_{12} &= \sqrt{2}\,\{3a_{1,-1}(a_{0,1} + a_{-1,0}) + (a_{0,0} - a_{1,1})(2a_{0,-1} - a_{1,0}) + (a_{0,0} - a_{-1,-1})(2a_{1,0} - a_{0,-1})\}
\end{aligned} .$$

This matrix M has the following two eigenvalues $E_\pm$ which can be obtained by solving a quadratic equation to get,

$$E_\pm := \frac{1}{2}\left(P \pm \sqrt{Q}\right) = \frac{1}{2}\left((M_{11} + M_{22}) \pm \sqrt{(M_{11} - M_{22})^2 + 4|M_{12}|^2}\right) . \tag{7.92}$$

Here, both P and Q can be written as a sum of several whole square terms and thus they are both positive definite for any algebra elements a. Clearly,

$$E_+ \geq E_- \quad \forall \;\; a \in B \tag{7.93}$$

yielding, for a particular $a \in B$,

$$\|[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{1}{r_1}\sqrt{E_+}. \tag{7.94}$$

Now the corresponding infimum $\inf_{a\in B}\|[\mathcal{D}, \pi(a)]\|_{\text{op}}$ is computed by varying the entries in the algebra elements, within the admissible ranges and obtaining the global minimum of E_+. This gives

$$\inf_{a\in B}\|[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{1}{r_1}\min\left(\sqrt{E_+}\right) = \frac{1}{r_1}\sqrt{\min(E_+)} \tag{7.95}$$

Now the eigenvalue E_+ will always have a “concave-up" structure in the parametric space as it can be written as the sum of square terms only (7.91,7.92). There can be points in the parametric space where E_+ and E_- are equal, namely points where Q becomes 0, but since E_+ can not become less than E_-, determining the minimum of E_+ will alone suffice in calculating the infimum of the operator norm as is clear from (7.94). So we work with E_+ alone and take the help of *Mathematica* again to get the desired result.

7.1.7.2 *General form of* $\Delta\rho$

The pure states corresponding to points on the fuzzy sphere $\mathbb{S}^2_1$ can be obtained by the action of the $SU(2)$ group element:

$$\hat{U} = e^{i\theta\hat{J}_2} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}\sin\theta & \sin^2\frac{\theta}{2} \\ -\frac{1}{\sqrt{2}}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \sin^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta & \cos^2\frac{\theta}{2} \end{pmatrix} \tag{7.96}$$

on the pure state $|1\rangle\langle 1|$ corresponding to the north pole (N) of $\mathbb{S}^2_1 : \rho_\theta = \hat{U}\,|1\rangle\langle 1|\,\hat{U}^\dagger$. Note that we have taken for convenience the azimuthal angle $\phi = 0$. This can be done without loss of generality. Correspondingly,

$$\begin{aligned} \Delta\rho = \rho_\theta - \rho_0 &= \hat{U}|1\rangle\langle 1|\hat{U}^\dagger - |1\rangle\langle 1| \\ &= \begin{pmatrix} \cos^4\frac{\theta}{2} - 1 & -\frac{1}{\sqrt{2}}\sin\theta\cos^2\frac{\theta}{2} & \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \\ -\frac{1}{\sqrt{2}}\sin\theta\cos^2\frac{\theta}{2} & \frac{1}{2}\sin^2\theta & -\frac{1}{\sqrt{2}}\sin\theta\sin^2\frac{\theta}{2} \\ \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} & -\frac{1}{\sqrt{2}}\sin\theta\sin^2\frac{\theta}{2} & \sin^4\frac{\theta}{2} \end{pmatrix}. \end{aligned} \tag{7.97}$$

Clearly, all entries are real here just like the case of $j = \frac{1}{2}$ (7.70); indeed by writing $\Delta\rho = (\Delta\rho)_i\lambda_i$ (λ_i's are the Gell-Mann matrices) the coefficients of λ_2, λ_5 and λ_7 vanishes. This $\Delta\rho$ however only provides us with a lower bound (6.28) of the distance in Connes' formula and the actual distance is reached by some optimal algebra element (a_S) of the form

$$a_s = \Delta\rho + \kappa\Delta\rho_\perp\,; \quad \mathrm{tr}(\Delta\rho\ \Delta\rho_\perp) = 0, \tag{7.98}$$

for which the infimum is reached in (say in (6.24)). This should be contrasted with the optimal element, for which the supremum is reached in (7.55). In any case, let us first try to have an improved estimate of the upper bound of the distance. This will be followed by the computation involving $\Delta\rho_\perp$.

7.1.7.3 *An improved but realistic estimate using Gell-mann matrices*

The upper bound for the spectral distance, obtained previously in (7.55) corresponded to that of a commutative sphere $\mathbb{S}^2$, but that lies much above the realistic distance for any fuzzy sphere $\mathbb{S}^2_j$ associated to the j-representation of $SU(2)$, as discussed in section 6.2. It is therefore quite imperative that we try to have a more realistic estimate of this where this upper bound will be lowered considerably. At this stage, we can recall the simple example of H_2-atom, where the energy gap between the ground state ($n = 1$) and first excited state ($n = 2$) is the largest one and the corresponding gaps in the successive energy levels go on decreasing and virtually become continuous for very large n ($n \gg 1$). One can therefore expect a similar situation here too. Indeed, a preliminary look into the distance between north and south poles (7.40) already support this in the sense that $\frac{\left(d_{3/2}(\mathrm{N,S})/r_{3/2}\right)}{\left(d_1(\mathrm{N,S})/r_1\right)} < \frac{\left(d_1(\mathrm{N,S})/r_1\right)}{\left(d_{1/2}(\mathrm{N,S})/r_{1/2}\right)}$. One can therefore expect the distance function $d_1(\rho_0, \rho_\theta)$ to be essentially of the same form as that of $d_{\frac{1}{2}}(\rho_0, \rho_\theta)$, except to be scaled up by a $\sqrt{2}$-factor (7.40) and a miniscule deformation in the functional form. For large-values of j, the corresponding ratios $\frac{\left(d_j(\mathrm{N,S})/r_j\right)}{\left(d_{j-1}(\mathrm{N,S})/r_{j-1}\right)} \to 1$, and the functional deformations are expected to be pronounced over a large variation in j. However, the exact determination of this form is extremely difficult and we will have to be contended with a somewhat heuristic analysis in this sub-section and a more careful analysis, using (6.27) in the next subsection. To that end, we start here with the most general form of algebra element a, as a linear combination of all the 8 Gell-Mann matrices (λ_i) as follows and look for an optimal element form $a \in B$ giving sup $|(\Delta\rho, a)|$, with some additional restrictions which are to be discussed later. We write

$$a = x_i\lambda_i = \begin{pmatrix} x_3 + \frac{x_8}{\sqrt{3}} & x_1 - ix_2 & x_4 - ix_5 \\ x_1 + ix_2 & -x_3 + \frac{x_8}{\sqrt{3}} & x_6 - ix_7 \\ x_4 + ix_5 & x_6 + ix_7 & -\frac{2x_8}{\sqrt{3}} \end{pmatrix}, \tag{7.99}$$

in analogy with (7.67) in $j = \frac{1}{2}$, $\mathbb{S}^2_{\frac{1}{2}}$. Again the rows/columns are labeled from top to bottom/left to right by $\big(\langle 1|, \langle 0|, \langle -1|\big) / \big(|1\rangle, |0\rangle, |-1\rangle\big)$. Now we calculate $\mathrm{tr}(\Delta\rho a)$ using the $\Delta\rho$ matrix (7.97) and the above algebra element (7.99) to get

$$\begin{aligned} \mathrm{tr}(\Delta\rho\ a) &= \left(x_3 + \frac{x_8}{\sqrt{3}}\right)\left(\cos^4\frac{\theta}{2} - 1\right) - \frac{2x_8}{\sqrt{3}}\sin^4\frac{\theta}{2} + \frac{x_4}{2}\sin^2\theta + \frac{1}{2}\sin^2\theta\left(\frac{x_8}{\sqrt{3}} - x_3\right) \\ &\quad -\sqrt{2}\sin\theta\left(x_1\cos^2\frac{\theta}{2} + x_6\sin^2\frac{\theta}{2}\right). \end{aligned} \tag{7.100}$$

This clearly demonstrates the expected independence of imaginary components viz. x_2, x_5 and x_7. We therefore set $x_2 = x_5 = x_7 = 0$ to begin with. This simplifies the matrix elements of M (7.91), using (7.99), as,

$$M_{11} = 3x_1^2 + 6x_4^2 + 8x_3^2 + (x_6 - 2x_1)^2 \tag{7.101}$$

$$M_{22} = 3x_6^2 + 6x_4^2 + (x_1 - 2x_6)^2 + 2\left(\sqrt{3}x_8 - x_3\right)^2 \tag{7.102}$$

$$M_{12} = \sqrt{2}\left(3x_4(x_1 + x_6) + 2x_3(x_1 - 2x_6) - (2x_1 - x_6)\left(x_3 - \sqrt{3}x_8\right)\right) \tag{7.103}$$

Like-wise the above expression (7.100) simplifies as

$$\begin{aligned} |\mathrm{tr}(\Delta\rho\ a)| &= \Bigg|\Bigg[\sin\left(\frac{\theta}{2}\right)\Bigg\{\sin\left(\frac{\theta}{2}\right)\left\{x_3 + 3x_3\cos^2\left(\frac{\theta}{2}\right) + \sqrt{3}x_8\sin^2\left(\frac{\theta}{2}\right)\right\} + \cos\left(\frac{\theta}{2}\right)\Big\{2\sqrt{2}x_1\cos^2\left(\frac{\theta}{2}\right) \\ &\quad +2\sqrt{2}x_6\sin^2\left(\frac{\theta}{2}\right) - 2x_4\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\Big\}\Bigg\}\Bigg]\Bigg|. \end{aligned} \tag{7.104}$$

Since our aim is to obtain a simple form of the "Ball" condition and eventually that of Connes spectral distance (6.24) so that we might obtain an improved estimate for the spectral distance over the lower bound (6.28), obtained by making use of $\Delta\rho$ (7.97) and more realistic than (7.63). We shall see shortly that with few more additional restrictions, apart from the ones (like vanishing of x_2, x_5 and x_7 imposed already) it is possible to simplify the analysis to a great extent, which in turn yields a distance estimate which has a same mathematical structure as that of the exact distance (7.72) for $j = 1/2$ case upto an overall factor. To that end, we impose the following new constraints:

$$x_1 = x_6;\quad x_3 = \frac{x_8}{\sqrt{3}};\quad x_4 = 0, \tag{7.105}$$

as a simple observation of (7.104) suggests that it simplifies it even further to the following form:

$$|\text{tr}(\Delta\rho\, a)| = \left|\left[\sin\left(\frac{\theta}{2}\right)\left\{4x_3\sin\left(\frac{\theta}{2}\right) + 2\sqrt{2}x_1\cos\left(\frac{\theta}{2}\right)\right\}\right]\right| = 2\sqrt{2}\sqrt{x_1^2+2x_3^2}\left|\sin\left(\frac{\theta}{2}\right)\cos\left(\zeta-\frac{\theta}{2}\right)\right|, \tag{7.106}$$

where $\cos\zeta = \frac{x_1}{\sqrt{x_1^2+2x_3^2}}$ and $\sin\zeta = \frac{\sqrt{2}x_3}{\sqrt{x_1^2+2x_3^2}}$. Moreover, putting the above constraints (7.105) in the equation (7.101)- (7.103), we get

$$M_{11} = M_{22} = 4x_1^2 + 8x_3^2 \;\; ; \quad M_{12} = 0. \tag{7.107}$$

With this, the eigenvalue E_+ (7.92) and the corresponding ball condition (7.94) can be obtained as

$$E_+ = 4x_1^2 + 8x_3^2 \;\; \Rightarrow \;\; \|[\mathcal{D},\pi(a)]\|_{op} = \frac{1}{r_1}2\sqrt{x_1^2+2x_3^2} \leq 1. \tag{7.108}$$

Putting this ball condition on the above (7.106), we get

$$|\text{tr}(\Delta\rho\, a)| \leq \sqrt{2}r_1\sin\left(\frac{\theta}{2}\right)\cos\left(\zeta-\frac{\theta}{2}\right). \tag{7.109}$$

Hence, a suggestive form of the spectral distance between a pair of pure states $\rho_0 = |1\rangle\langle 1|$ and $\rho_\theta = U|1\rangle\langle 1|U^\dagger$ for $j=1$ representation can be easily obtained by identifying the optimal value of the last free parameter ζ to be given by $\zeta = \frac{\theta}{2}$. This yields

$$d_1^a = \sup_{a\in B}\left\{|\text{tr}(\Delta\rho a)|\right\} = \sqrt{2}r_1\sin\left(\frac{\theta}{2}\right). \tag{7.110}$$

The corresponding form of the optimal algebra element a_s is obtained after a straightforward computation to get

$$\hat{a}_s = \frac{r_1}{2}\begin{pmatrix} \sqrt{2}\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) & 0 \\ \cos\left(\frac{\theta}{2}\right) & 0 & \cos\left(\frac{\theta}{2}\right) \\ 0 & \cos\left(\frac{\theta}{2}\right) & -\sqrt{2}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{7.111}$$

When $\theta=\pi$, the distance is exactly the same between the two pure coherent states $|1\rangle\langle 1|$ and $|-1\rangle\langle -1|$ and the above distance (7.110) gives $d_1^a(\rho_0,\rho_\pi) = \sqrt{2}r_1$ which exactly matches with the one (7.40), computed using the discrete formula (7.39).

Note that we have made use of all the restrictions $x_2 = x_4 = x_5 = x_1 - x_6 = x_3 - \frac{x_8}{\sqrt{3}} = 0$ and $\zeta = \frac{\theta}{2}$, imposed at various stages. Finally, we would like to mention that this simple form (7.110) was obtained by imposing the above ad-hoc constraints resulting in $M_{11} - M_{22} = M_{12} = 0$. Consequently, one cannot expect, a priori this to reflect the realistic distance either. At best, this can be expected to be closer to the realistic one, compared to (7.63). The only merit in (7.110) being that it has essentially the same structure as that of (7.72) for $j = \frac{1}{2}$ case. Nevertheless, as we shall show below that the computation involving $\Delta\rho_\perp$, using (6.24),(6.27) matches with (7.110) to a great degree of accuracy.

Note that we are denoting the analytical distanceas as (d^a), to distinguish it from other distances to be calculated in the next section. Also note that since the analytical distance (7.110) has the same form (7.72), it therefore corresponds to $\sqrt{2}$-times the half of chordal distance. Further, it satisfies the Pythagoras equality (7.81) just like the $n = \frac{1}{2}$ case.

Before we conclude this subsection, we would like to point out that we could have perhaps reversed our derivation by simply requiring the matrix M (7.91) to be diagonal: $M_{12} = 0$. But in that case $E_+ = \max\{M_{11}(a), M_{22}(a)\}$ for a particular choice of algebra element, satisfying the aforementioned conditions viz $x_2 = x_5 = x_7 = M_{12} = 0$. Now given the structures of M_{11} (7.101) and M_{22} (7.102) they will have shapes which are concave upwards, when the hyper-surfaces are plotted against the set of independent parameters occurring in 'a' $\in \mathcal{R}$, where $\mathcal{R}$ represents the sub-region in the parameter space, defined by these conditions. Now it may happen that $M_{11}(a) \neq M_{22}(a), \ \ \forall\, a \in \mathcal{R}$, in which case one of them, say $M_{11}(a)$, exceeds the other: $M_{11} > M_{22}$. Then clearly

$$\inf_{a\in\mathcal{R}} \|[\mathcal{D}, \pi(a)]\|_{op} = \frac{1}{r_1}\sqrt{\min_{a\in\mathcal{R}}(M_{11})}. \tag{7.112}$$

Otherwise, the hyper-surfaces given by $M_{11}(a)$ and $M_{22}(a)$ will definitely intersect and (7.112) will reduce to

$$\inf_{a\in\bar{\mathcal{R}}} \|[\mathcal{D}, \pi(a)]\|_{op} = \frac{1}{r_1}\sqrt{\min_{a\in\bar{\mathcal{R}}}(M_{11})} = \frac{1}{r_1}\sqrt{\min_{a\in\bar{\mathcal{R}}}(M_{22})}. \tag{7.113}$$

where $\bar{\mathcal{R}} \subset \mathcal{R}$ represents the sub-region where $M_{11}(a) = M_{22}(a)$. In fact, this is a scenario, which is more likely in this context, as suggested by our analysis of infinitesimal distance presented in the next subsection (see also Fig. 2). We therefore set also $M_{11} = M_{22}$. With the additional condition like $x_1 = x_6$ (7.105), we can easily see that one gets, apart from (7.105), another set of solutions like,

$$x_1 = x_6, \quad x_3 = -\sqrt{3}x_8, \quad x_4 = -\frac{2}{\sqrt{3}}x_8. \tag{7.114}$$

This, however, yields the following ball condition

$$\sqrt{x_1^2 + 8x_8^2} \leq \frac{r_1}{2}, \tag{7.115}$$

the counterpart of (7.108), and in contrast to (7.106), cannot in anyway be related to its counterpart here, given by

$$|\mathrm{tr}(\Delta\rho a)| = \sqrt{\frac{2}{3}}\sin\theta\big(\sqrt{2}x_8\sin\theta - \sqrt{3}x_1\big). \tag{7.116}$$

We therefore reject (7.114) from our consideration, as it will not serve our purpose.

7.1.7.4 *Spectral distance using* $\Delta\rho_\perp$

In this section we will employ the modified distance formula (6.24) by constructing a most general form of $\Delta\rho_\perp$ for both finite as well as infinitesimal distances. We show that in both of the cases the distance calculated using this more general (numerical) method matches with the corresponding result given by d_a (7.110) to a very high degree of accuracy suggesting that d_a should be the almost correct distance for arbitrary θ.

INFINITESIMAL DISTANCE: In this case the $d\rho$ takes a simpler form by expanding (7.97) and keeping only upto the first order in the infinitesimal angle $d\theta$, we get

$$d\rho = \rho_{d\theta} - \rho_0 = -\frac{d\theta}{\sqrt{2}}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) . \tag{7.117}$$

The most general structure of the transverse part $d\rho_\perp$ here is obtained by taking all possible linear combinations of generic state like $|i\rangle\langle j|$ i.e.

$$d\rho_\perp = \sum_{i,j} C_{ij}|i\rangle\langle j| \;\; ; \;\; i,j \in \{-1,0,+1\} , \tag{7.118}$$

where $C_{ij} = C^*_{ji}$ because of the hermiticity of $d\rho_\perp$. Clearly, the complex parameters C_{ij} are exact analogues of suitable combinations of x_i's in (7.99). The orthogonality condition (7.98) here requires the coefficient C_{10} to be purely imaginary. Moreover we can demand that the matrix representation of $\Delta\rho_\perp$ should be traceless as discussed in section 7.1.7. We thus impose $C_{11} + C_{22} + C_{33} = 0$. To better understand the significance of each term we write (7.118) in matrix form as follows:

$$d\rho_\perp = \begin{pmatrix} \mu_1 & i\alpha_1 & \gamma \\ -i\alpha_1 & \mu_0 & \beta \\ \gamma^* & \beta^* & -(\mu_1+\mu_0) \end{pmatrix} \;\; ; \;\; \mu_1, \mu_0, \mu_{-1}, \alpha_1 \in \mathbb{R} \text{ and } \beta, \gamma \in \mathbb{C}. \tag{7.119}$$

With this our optimal algebra element a_S (7.98) would become:

$$a_S = \begin{pmatrix} \mu_1 & -\frac{d\theta}{\sqrt{2}} + i\alpha_1 & \gamma \\ -\frac{d\theta}{\sqrt{2}} - i\alpha_1 & \mu_0 & \beta \\ \gamma^* & \beta^* & -(\mu_1+\mu_0) \end{pmatrix} \tag{7.120}$$

where we have absorbed κ inside the coeffiecients of $d\rho_\perp$ (7.119). With 7 independent parameters, it is extremely difficult to vary all the parameters simultaneously to compute the infimum analytically. However, as far as infinitesimal distances are concerned, it may be quite adequate to take each parameter to be non-vanishing one at a time. Thus, by keeping one of these diagonal/complex conjugate pairs like β and γ to be non-zero one at a time and computing the eigen values of the 2×2 matrix (7.91) using (7.98), (7.117) and (7.119), it is found that only the real part of β to be contributing non-trivially to the infimum of the operator norm $\|[\mathcal{D}, \pi(a)]\|_{\text{op}}$ in the sense that the operator norm of this object with $a_S = \begin{pmatrix} 0 & -\frac{d\theta}{\sqrt{2}} & 0 \\ -\frac{d\theta}{\sqrt{2}} & 0 & \beta \\ 0 & \beta^* & 0 \end{pmatrix}$ in (7.98) is a monotonically increasing function of $Im(\beta)$ but yields a non-trivial value for the infimum which is less than the one with vanishing β i.e. $d\rho$ itself. Like-wise, both real and imaginary parts of γ and α_1 in (7.119) do not contribute to the infimum. On computation, we get

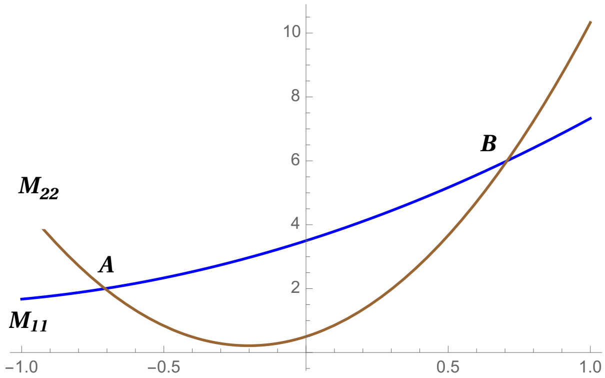

Figure 7.4: Infimum corresponding to the plots of $M_{11}/(d\theta)^2$ and $M_{22}/(d\theta)^2$ vs $\beta_1/d\theta$

$$\|[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{1}{r_1}\sqrt{\max\{M_{11}, M_{22}\}}\,, \tag{7.121}$$
$$\text{where } M_{11} = \beta_1^2 + 2\sqrt{2}\beta_1 d\theta + \frac{7}{2}d\theta^2 \text{ and } M_{22} = 7\beta_1^2 + 2\sqrt{2}\beta_1 d\theta + \frac{1}{2}d\theta^2;$$

where $\beta_1 = Re(\beta)$ and the 2×2 matrix (7.91) takes a diagonal form $diag\{M_{11}, M_{22}\}$ thus yielding eigenvalues trivially (7.121). From the plot of these eigen values(see figure 7.4) it is clear that the infimum of the operator norm over the full range of β_1 is given by the minimum value of the two intersections at A ($\beta_1 = -\frac{d\theta}{\sqrt{2}}$) and B ($\beta_1 = +\frac{d\theta}{\sqrt{2}}$) which comes out to be $\frac{\sqrt{2}d\theta}{r_1}$ i.e. $\|[\mathcal{D}, \pi(a)]\|_{\text{op}} = \frac{\sqrt{2}d\theta}{r_1}$. Note that in this context

$$E_+ = \begin{cases} M_{22}, & \text{for } \beta_1 < -\frac{d\theta}{\sqrt{2}} \text{ and } \beta_1 > +\frac{d\theta}{\sqrt{2}} \textit{ i.e.} \text{ left of A and right of B} \\ M_{11}, & \text{for } -\frac{d\theta}{\sqrt{2}} < \beta_1 < +\frac{d\theta}{\sqrt{2}} \textit{ i.e.} \text{ in between A and B.} \end{cases}.$$

Also since $\|d\rho\|_{tr}^2 = d\theta^2$, as follows from (7.117), the infinitesimal spectral distance (6.24) is given by

$$d_1(\rho_{d\theta}, \rho_0) = r_1 \frac{d\theta}{\sqrt{2}}\,. \tag{7.122}$$

We now corroborate the same result by varying all the 7 parameters simultaneously. Of course we shall have to employ *Mathematica* now. To that end, first note that P, Q (7.91, 7.92) are now given as:

$$P = 2\{1+\mu_1^2+2\beta_1^2+3\alpha^2+4\mu_0^2+6|\gamma|^2+(\mu_1+\mu_0)^2+(\sqrt{2}\beta_1+1)^2+(2\beta_2-\alpha)^2\} \tag{7.123}$$

$$\begin{aligned} Q &= 9\{-1+4\mu_0\mu_1+2(|\beta|^2+\mu_0^2-\alpha^2)\}^2+4\{3(\mu_0+\mu_1+\gamma_1)+4.242(\beta_1\mu_1-\beta_2\gamma_2-\alpha\gamma_2-\beta_1\gamma_1)\}^2 \\ &\quad +4\{3\gamma_2+4.242(\beta_2\mu_1+\beta_2\gamma_1+\alpha\gamma_1-\beta_1\gamma_2-\alpha\mu_0-\alpha\mu_1)\}^2, \end{aligned} \tag{7.124}$$

where $\beta = \beta_1 + i\beta_2$ and $\gamma = \gamma_1 + i\gamma_2$. Interestingly enough we get the same infimum i.e. $2d\theta^2$ by finding the minimum of the eigenvalue E_+ as discussed in section 7.1.7.1 with the 7 parameter eigenvalue (7.92) where P and Q are given by (7.123,7.124). We therefore recover the distance (7.122) which also matches with (7.110) for $\theta \to d\theta$.

FINITE DISTANCE For any finite angle θ, the $\Delta\rho$ matrix (7.97) can be directly used to compute the square of the trace norm $\|\Delta\rho\|_{tr}^2$. Moreover, this $\Delta\rho$ can be used as the algebra element a to compute the eigenvalues of (7.92) and then the operator norm $\|[\mathcal{D}, \pi(a)]\|_{op}$, yielding the lower bound for the spectral distance using (6.28). More specifically for $\theta = \frac{\pi}{2}$ (i.e. the distance between the north pole N and any point E on equator) the lower bound is found to be

$$d_1(N, E) \geq 0.699 r_1 \tag{7.125}$$

Contrasting with the corresponding value (7.110) of $d_1^a = \sqrt{2} r_1 \sin\left(\frac{\pi}{4}\right) = r_1$, we see that the transverse component $\Delta\rho_\perp$ must play a vital role here. On the other hand for $\theta = \pi$ (i.e. distance between north and south pole) we find $d(N, S) = \sqrt{2}\, r_1$ which matches exactly with the result of (7.40) for the distance between discrete states $|1\rangle\langle 1|$ and $|-1\rangle\langle -1|$. This means that there is no contribution of $\Delta\rho_\perp$ to this distance for $\theta = \pi$. There are however non-trivial contribution from $\Delta\rho_\perp$ in the distance for any general value of angle $0 < \theta < \pi$ as we have illustrated above through the example of $\theta = \frac{\pi}{2}$. Now the most general $\Delta\rho_\perp$ in this case can be constructed as follows:

$$\begin{aligned} \Delta\rho_\perp = [&\mu_1 |1\rangle\langle 1| + \mu_0 |0\rangle\langle 0| + \mu_{-1} |-1\rangle\langle -1| + \alpha |1\rangle\langle 0| + \alpha^* |0\rangle\langle 1| \\ &+ \gamma |1\rangle\langle -1| + \gamma^* |-1\rangle\langle 1| + \beta |0\rangle\langle -1| \,| + \beta^* |-1\rangle\langle 0|] \end{aligned} \tag{7.126}$$

where $\mu_1, \mu_0, \mu_{-1} \in \mathbb{R}$. Again we can take a_S (7.98) to be traceless as before, which implies $\mu_1 + \mu_0 + \mu_{-1} = 0$ and we can eliminate one of them, say μ_{-1}. Further imposing the orthogonality condition (7.98) we have a relation between all the remaining 8 parameters and one of them can be eliminated from that. In this case, using the $\Delta\rho$ (7.97) and $\Delta\rho_\perp$ (7.126) in (7.98) while absorbing κ inside the coefficients of $\Delta\rho_\perp$, we have

$$\mathrm{tr}\Big(\Delta\rho\,\Delta\rho_\perp\Big) = 0 \implies \mu_1 = -\frac{1}{2}\mu_0\sin^2\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}}\beta_1\sin\theta + \cos^2\left(\frac{\theta}{2}\right)\left(\gamma_1 + \mu_0 - \sqrt{2}\alpha_1\cot\frac{\theta}{2}\right) \tag{7.127}$$

where α_1, β_1 and γ_1 are the real components and α_2, β_2 and γ_2 are the imaginary components of α, β and γ respectively. With these substitutions the eigenvalues of the matrix (7.91) become a function of 7 parameters. We now calculate P,Q (7.92) for $\theta = \frac{\pi}{2}$ case to get

$$\begin{aligned} P &= 5 + \frac{45}{4}\mu_0^2 + 8(|\alpha|^2 + |\beta|^2) + 2(\alpha_1^2 + \beta_1^2) + \gamma_1^2 + 12|\gamma|^2 - 4\gamma_1 + 6\mu_0 \\ &- 3\sqrt{2}\mu_0(\beta_1 + \alpha_1) - 4\alpha_1\beta_1 + 3\gamma_1\mu_0 - 2\sqrt{2}\gamma_1(\beta_1 + \alpha_1) - 8\alpha_2\beta_2\,; \end{aligned} \tag{7.128}$$

$$\begin{aligned} Q &= 9\Big[\{1 + 2|\alpha|^2 - 3\mu_0^2 - 2|\beta|^2 - \gamma_1 + \frac{1}{2}\mu_0 + 2\sqrt{2}\beta_1 - 2\mu_0\gamma_1 + 2\sqrt{2}\mu_0(\alpha_1 + \beta_1)\}^2 \\ &+ \frac{1}{2}\Big\{\big(\sqrt{2} + 2\sqrt{2}\alpha_1^2 - 2\sqrt{2}\beta_1^2 + 2\sqrt{2}\gamma_1 - 4\beta_1 + \sqrt{2}\mu_0 + \mu_0\beta_1 - 4\gamma_2\beta_2 - 6\gamma_1\alpha_1 - 4\gamma_2\alpha_2 - 5\alpha_1\mu_0\big)^2 \\ &+ \big(2\beta_2 - 2\sqrt{2}\gamma_2 - 2\alpha_2 - 6\gamma_1\beta_2 - 2\sqrt{2}\alpha_1\alpha_2 + 4\alpha_1\gamma_2 + 2\sqrt{2}\alpha_1\beta_2 + \beta_2\mu_0 + 4\gamma_2\beta_1 \\ &+ 2\sqrt{2}\beta_1\beta_2 - 2\gamma_1\alpha_2 + 5\alpha_2\mu_0 - 2\sqrt{2}\beta_1\alpha_2\big)^2\Big\}\Big]\,. \end{aligned} \tag{7.129}$$

We now minimize E_+ as before to get

$$d(N,E) = r_1 \tag{7.130}$$

which exactly matches with the result of (7.110), which is quite remarkable. As for $\theta = \pi$, the terms P,Q (7.92) of eigen values $E_\pm$ comes out to be

$$P = 4 + 9\mu_0^2 + 8(|\alpha|^2 + |\beta|^2) + 12|\gamma|^2 - 8(\alpha_1\beta_1 + \alpha_2\beta_2) \tag{7.131}$$

$$\begin{aligned} Q &= 36\big(|\beta|^2 - |\alpha|^2 - 2\mu_0\big)^2 + 18\Big[\Big\{(\beta_1 + \alpha_1)(2\gamma_1 + \mu_0) + 2(\beta_1 - \alpha_1) + 2\gamma_2(\alpha_2 + \beta_2)\Big\}^2 \\ &+ \Big\{(\beta_2 + \alpha_2)(2\gamma_1 - \mu_0) - 2(\beta_2 - \alpha_2) - 2\gamma_2(\alpha_1 + \beta_1)\Big\}^2\Big] \end{aligned} \tag{7.132}$$

Again we need to calculate the spectral distance by minimizing E_+ which gives $d_1(\rho_{\theta=\pi}, \rho_0) = \sqrt{2}r_1$. This is precisely the lower bound result $d_1(N,S)$ (7.63) as discussed previously and hence support our claim that $\Delta\rho_\perp$ will not contribute here at all. For arbitrary angle θ, we present here [see table 1] both the distances i.e. the one calculated using the formula (7.110) and another one calculated using the global minima of eigenvalue E_+ (say d_1) for various angles between 0 to π.

Angle (degree)	d_1^a/r_1	d_1/r_1
10	0.1232568334	0.1232518539
20	0.2455756079	0.2455736891
30	0.3660254038	0.3660254011
40	0.4836895253	0.4836894308
50	0.5976724775	0.5976724773
60	0.7071067812	0.7071067811
70	0.8111595753	0.8111595752
80	0.9090389553	0.9090389553
90	1	0.9999999998
100	1.0833504408	1.0833504407
110	1.1584559307	1.1584559306
120	1.2247448714	1.2247448713
130	1.2817127641	1.2817127640
140	1.3289260488	1.3289260487
150	1.3660254038	1.3660254037
160	1.3927284806	1.3927284806
170	1.4088320528	1.4088320527

Table 7.1: Data set for various distances corresponding to different angles

It is very striking that the distance d_1^a (7.110) matches almost exactly with d_1 for all these angles as one see from the table 1. This strongly suggests that (7.110) is indeed very very close to the exact distance! In fact, for larger angles like $50°$ and above the results agree upto 9 decimal places, whereas for smaller angles ($< 50°$) they agree upto 5 decimal places and show some miniscule deviations from 6 decimal point onwards. One can expect to see more pronounced deformations in the functional form, away from the like of (7.110), when the overall scale of magnification of size will start reducing monotonically with $n \to \infty$ and eventually merge with commutative results.

7.2 SPECTRAL TRIPLE AND DISTANCE ON QUANTUM HILBERT SPACE OF FUZZY SPHERE

Here we present our final computation of the infinitesimal distance on the quantum Hilbert space $\mathcal{H}_q$. This is the counterpart of the computation, presented in section 6.4 for Moyal plane and here too the presence of additional degrees of freedom will be exploited to uncover an intriguing connection between geometry and statistics. The quantum Hilbert space is spanned by the Hilbert-Schmidt operators acting on the fuzzy sphere described by $\mathcal{F}_j$. That is,

$$\mathcal{H}_j = Span\{|j,m\rangle\langle j,m'| \equiv |m,m') : \ \ \mathrm{tr}_c(\Psi^\dagger\Psi) < \infty\}. \tag{7.133}$$

Since we consider a particular fuzzy sphere indexed by j, we have suppressed the index j above and by taking analogy with the case of Moyal plane, we can construct the spectral triple for this case as:

1. The Algebra $\mathcal{A} = \mathrm{Span}\Big\{|m,m')(l,l'| \ : \ -j \leq m,m,l,l' \leq j\ , \ \text{ with } j \text{ being fixed } \Big\}$.
2. The Hilbert space $\mathcal{H} = \mathcal{H}_j \otimes \mathbb{C}^2 = \left\{ \begin{pmatrix} |m,m') \\ |l,l') \end{pmatrix} \right\}$.
3. Dirac operator $\mathcal{D} = \frac{1}{r_j}\,\hat{\vec{J}} \otimes \vec{\sigma}$,

where the action of $\hat{J}_i$ on $\mathcal{H}_q$ is given in (D.39).

In this case, we can define spectral distance between both pure and mixed states of the algebra. First consider the pure states of the algebra $\mathcal{A}$ corresponding to the density matrices $\rho_q(m,m') = |m,m')(m,m'|$ and $\rho_q(m+1,l') = |m+1,l')(m+1,l'|$. Then we have the operator $d\rho_q = |m+1,l')(m+1,l'| - |m,m')(m,m'|$, which should reproduce the infinitesimal distance between states computed earlier in (7.27) when we take $m' = l'$. To this end, let us begin by computing

$$[\mathcal{D}_f, \pi(d\rho_q)] = \frac{1}{r_j}\begin{bmatrix} 0 & \frac{1}{\theta_f}[\hat{X}_-, d\rho_q] \\ \frac{1}{\theta_f}[\hat{X}_+, d\rho_q] & 0 \end{bmatrix}, \tag{7.134}$$

so that

$$[\mathcal{D}_f, \pi(d\rho_q)]^\ddagger[\mathcal{D}_f, \pi(d\rho_q)] = \frac{1}{r_j^2}\begin{bmatrix} \frac{1}{\theta_f^2}[\hat{X}_-, d\rho_q]^\ddagger[\hat{X}_-, d\rho_q] & 0 \\ 0 & \frac{1}{\theta_f^2}[\hat{X}_+, d\rho_q]^\ddagger[\hat{X}_+, d\rho_q] \end{bmatrix}. \tag{7.135}$$

Here, $\hat{X}_i$ and $\hat{X}_\pm$ (2.65) are the position operators and the corresponding ladder operators acting on $\mathcal{H}_q$.

After computation, we get

$$\|[\mathcal{D}_f, \pi(d\rho_q)]\|_{\mathrm{op}} = \begin{cases} \frac{2\sqrt{[j(j+1)-m(m+1)]}}{\theta_f\sqrt{j(j+1)}}, & \text{if } \ m' = l'. \\ \frac{\sqrt{[j(j+1)-m^2+|m|]}}{\theta_f\sqrt{j(j+1)}}, & \text{otherwise.} \end{cases} \tag{7.136}$$

Since here we also have $\mathrm{tr}_q(d\rho_q)^2 = 2$, we get the infinitesimal distance on the quantum Hilbert space by using a formula with the same form as that of (**??**):

$$d(\rho_q(m+1,l'),\rho_q(m,m')) = \begin{cases} \frac{\theta_f\sqrt{j(j+1)}}{\sqrt{[j(j+1)-m(m+1)]}}, & \text{if } m'=l'. \\ \frac{2\theta_f\sqrt{j(j+1)}}{\sqrt{[j(j+1)-m^2+|m|]}}, & \text{otherwise.} \end{cases} \tag{7.137}$$

This shows that just like in the Moyal case [79] the distance on quantum Hilbert space $\mathcal{H}_j$ of the fuzzy sphere depends on the right hand sectors and it increases when the right hand sectors are taken differently, although the Dirac operator acts only on the left hand sector.

This motivates us to consider a more general situation where the density matrices are of the mixed form, given by

$$\rho_q(m) = \sum_l P_l(m)\ |m,l)(m,l|, \quad \sum_l P_l = 1, \ \ \forall\, m. \tag{7.138}$$

Clearly P_l are probabilities that are position-dependent. As mentioned in [79], the distance formula (6.28) will yield the true Connes' distance between the mixed states for which the probabilities P_l are position independent.

However, instead of using the operator norm to calculate the infinitesimal distance, one can use trace norm which will give the closely related distance function

$$\tilde{d}(\rho_q(m+1,\rho_q(m)) = \frac{\mathrm{tr}_c(d\rho_q)^2}{\|[\mathcal{D},\pi(d\rho_q)]\|_{\mathrm{tr}}}. \tag{7.139}$$

This distance given by (7.139) will expectedly be different from the Connes infinitesimal distance given by (6.28) by a numerical factor only so we can employ (7.139) instead of (6.28) for computational simplicity.

Now, introducing $d\rho_q(m+1,m) = \rho_q(m+1) - \rho_q(m)$, we can compute the closely related distance function between the mixed states on the subspace $\mathcal{H}_j$ of quantum Hilbert space using the formula (7.139). After the straightforward computation, we get

$$\mathrm{tr}_q(d\rho_q(m+1,m))^2 = \sum_l \left[P_l^2(m+1) + P_l^2(m)\right], \tag{7.140}$$

$$\|[\mathcal{D},\pi(d\rho_q(m+1,m))]\|_{\mathrm{tr}} = \frac{2}{r_j}\sqrt{\sum_l [P_l^2(m+1)s_1 + P_l^2(m)s_2 + P_l(m+1)P_l(m)s_3]}, \tag{7.141}$$

$$\text{where,}\quad s_1 = \{j(j+1)-(m+1)^2\}\ ;$$
$$s_2 = \{j(j+1)-m^2\}\ ;$$
$$s_3 = \{j(j+1)-m(m+1)\}\ . \tag{7.142}$$

so that we obtain the distance function as

$$\tilde{d}(m+1,m) = \frac{r_j}{2}\frac{\sum_l[P_l^2(m+1)+P_l^2(m)]}{\sqrt{\sum_l[P_l^2(m+1)s_1 + P_l^2(m)s_2 + P_l(m+1)P_l(m)s_3]}}\ , \tag{7.143}$$

where $s_1, s_2.s_3$ are given in the above equation (7.142) Clearly, the distance depends upon the probabilities which shows the connection between geometry and statistics. Proceeding in the same way as [79], we can take two choices of probability distribution: one that minimize the distance between two generalized points and another that maximize the local entropy, while fixing the local average energy.

Let us consider the first choice. Since we have the infinitesimal distance between the mixed states, we can define the distance between two generalized points n_i and n_f on $\mathcal{H}_j$ as

$$\tilde{d}(n_f, n_i) = \sum_{m=n_i}^{n_f-1} \tilde{d}(m+1, m). \tag{7.144}$$

After long computation, we obtain that the probabilities that minimize the distance must satisfy

$$\Delta\, P_l = 2\alpha, \qquad \forall\;\, l, \tag{7.145}$$

where

$$\Delta = \begin{pmatrix} a(n_i) & b(n_i) & 0 & . & . \\ b(n_i) & a(n_i+1) & b(n_i+1) & 0 & . \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & 0 & b(n_f-2) & a(n_f-1) & b(n_f-1) \\ . & . & 0 & b(n_f-1) & a(n_f) \end{pmatrix}; \tag{7.146}$$

$$P_l = \begin{pmatrix} P_l(n_i) \\ P_l(n_i+1) \\ . \\ . \\ P_l(n_f-1) \\ P_l(n_f) \end{pmatrix} ; \quad \alpha = \begin{pmatrix} \alpha(n_i) \\ \alpha(n_i+1) \\ . \\ . \\ \alpha(n_f-1) \\ \alpha(n_f) \end{pmatrix}. \tag{7.147}$$

Here, $\alpha(m)$ (m taking value from n_i to $n_f - 1$) are the Lagrange multipliers imposing the constraints that the probabilities sum to 1 and the matrix elements of Δ are given by

$$\begin{aligned} a(m) &= r_j[g(m) + g(m-1) - \{j(j+1) - m^2\}\{f(m) + f(m-1)\}], \\ b(m) &= -r_j[\{j(j+1) - m(m+1)\}f(m)], \end{aligned}$$

with $f(m)$ and $g(m)$ given by

$$f(m) = \frac{1}{2}\frac{\sum_l [P_l^2(m+1) + P_l^2(m)]}{[\sum_l [P_l^2(m+1)\ s_1 + P_l^2(m)\ s_2 + P_l(m+1)P_l(m)\ s_3]]^{\frac{3}{2}}}\ , \tag{7.148}$$

$$g(m) = \frac{1}{\sqrt{\sum_l [P_l^2(m+1)\ s_1 + P_l^2(m)\ s_2 + P_l(m+1)P_l(m)\ s_3]}}\ . \tag{7.149}$$

From equation (7.145), we see that P_l is independent of l since both Δ and α are independent of l so that we get

$$\sum_{l=-n}^{n} P_l(m) = 1 \Rightarrow P_l(m) = \frac{1}{(2j+1)}. \tag{7.150}$$

Substituting this in the equation (7.143), we get the distance function as

$$\tilde{d}(m+1,m) = \frac{1}{\sqrt{(2j+1)}}\frac{r_j}{\sqrt{3\{j(j+1) - m(m+1) - \frac{1}{3}\}}}. \tag{7.151}$$

This distance differs from the true Connes infinitesimal distance just by a numerical factor resulting from the use of the trace instead of operator norm.

Let us consider the second choice where we introduce a local entropy as

$$S(m) = \sum_l P_l(m)\ \log P_l(m), \tag{7.152}$$

with the further condition that

$$\sum_l P_l(m)E_l = E(m) \quad \text{in addition to} \quad \sum_l P_l(m) = 1\ . \tag{7.153}$$

After maximizing the local entropy, we get the following Maxwell-Boltzmann form:

$$P_l(m) = \frac{e^{-\beta(m)E_l}}{\sum_l\ e^{-\beta(m)E_l}} = \frac{e^{-\beta(m)E_l}}{Z(\beta(m))}, \tag{7.154}$$

where $\beta(m)$ is the local inverse temperature introduced as a Lagrange multiplier imposing the local energy constraint (7.153) and $Z(\beta(m)) = \sum_l\ e^{-\beta(m)E_l}$ is the partition function.

If we take the local average energy and so the temperature to be independent of m, then putting (7.154) in (7.143), we get the distance function as

$$\tilde{d}(m+1,m) = \frac{\sqrt{Z(2\beta)}}{Z(\beta)}\frac{r_j}{\sqrt{3\{j(j+1) - m(m+1) - \frac{1}{3}\}}}. \tag{7.155}$$

This clearly shows the connection between the distance and partition function describing the statistical properties of a system with quantum states given by (7.138) in thermal equilibrium.

However, in this case the distance decreases as the temperature increases since the value of the factor $\frac{\sqrt{Z(2\beta)}}{Z(\beta)}$ lies within 0 to 1 and goes to 0 for $T = \frac{1}{\beta} \to \infty$.

8

CONCLUSION

The issue of maximal localization of the physical Voros position basis on 3D Moyal space has been studied in the first part of the thesis. Although, the Voros basis states are maximally localized states on the 6D phase space, they are not maximally localized states on the 3D configuration space unlike its 2D counterpart [38]. Besides, the Voros basis has an isotropic structure, in the sense that the symplectic eigenvalues of the corresponding commutative variance matrix yield the same pair of eigenvalues for three independent "modes", which are now essentially decoupled from each other.

We have also re-visited the issue of twisted symmetry on 3D Noncommutative Moyal space in a completely operatorial framework using Hilbert-Schmidt operators to investigate whether the twisted bosons/fermions [26] necessarily occurs in conjunction with the twisted deformed coproduct on Moyal space [23], where the twisted fermions were shown to violate Pauli principle [27]. Further, even within this scheme, we have shown that there exists a basis in the multi-particle sector called "quasi-commutative basis" which satisfies orthonormality and completeness relation and is symmetric/antisymmetric under the usual i.e. un-deformed exchange operator so that one has usual bosons/fermions and can avoid introducing twisted bosons/fermions. The correlation functions and the associated thermal effective potential is then shown to conform to Pauli principle, apart from preserving the $SO(3)$ symmetry, both in Moyal and Voros basis, in contrast to the case of twisted bosons/fermions, where there is a $SO(3) \rightarrow SO(2)$ symmetry breaking. In all these cases, the resulting expressions in Moyal and Voros bases exhibit the same structural form, except that in the Voros case, one gets a θ-deformed thermal wavelength ensuring that it has a non-vanishing lower bound, which is in conformity with the requirement that wavelengths $\lesssim \sqrt{\theta}$ are suppressed exponentially. Thus, in Voros basis one gets a non-commutative deformation even in the quasi-commutative basis and this Voros basis should be regarded as physical as one can talk sensibly about the inter-particle separation, as one can introduce spectral distance a la Connes, unlike its " Moyalian" counterpart [79].

In the second part of the thesis, we have provided a general algorithm to compute the finite spectral distance on non-commutative spaces viz Moyal plane and fuzzy sphere, adaptable with the Hilbert-Schmidt operator formulation of noncommutative quantum mechanics. We have extensively studied the geometry of the Moyal plane ($\mathbb{R}^2_*$) and that of the Fuzzy sphere ($\mathbb{S}^2_*$) using both the mentioned general algorithm and also emulated the method of [77] to compute the upper bound and then look for an optimal element saturating this upper bound. In the case of the Moyal plane, we succeed in identifying such optimal element 'a_s' belonging to the multiplier algebra. We then constructed a sequence of projection operators $\pi^N(a_s)$ in the finite dimensional subspace spanned by eigen-spinors of the Dirac operator that converge to $\pi(a_s)$ and saturates the upper bound, allowing us to identify the upper bound itself with the distance. Eventually, this enables us to relate the one parameter family of pure states to the geodesic of the Moyal plane which is nothing but the straight line. In contrast, on the fuzzy sphere, although an analogous upper bound can be constructed for any finite j-representation of $su(2)$, there simply does not exist an optimal element a_s saturating the inequality. Indeed, for the case of extremal non-commutativity $j = \frac{1}{2}$, the finite distance turns out to be half the chordial distance. Here, except for the extremal points, the interpolating "points" correspond to mixed states. This in turn helps us to find the distance between a given mixed state and a uniquely defined nearest pure state lying on the "surface" of $\mathbb{S}^2_*$. The corresponding distance can then be taken as an alternative characterization of the "mixedness" of a state. This exercise shows that in Connes' framework no discrimination is made between pure and mixed states; it scans through the entire set of pure and mixed states to compute the supremum in (1.25).

All these calculations are enormously simplified by working in the eigen-spinor basis of the respective Dirac operator so that we are able to compute the distance in the '$j = 1$' fuzzy sphere, using this revised algorithm. Since this algorithm involves also the transverse $\Delta\rho_\perp$ components in addition to the longitudinal $\Delta\rho$ component, this becomes somewhat less user-friendly. For the '$j = 1$' case, for example, it involves a minimization in seven parameters. Needless to say that we have to make use of *Mathematica* after solving the quadratic characteristic equation. For higher j's, the corresponding characteristic equations will not only involve higher degree polynomials, it will also involve a very large number of independent parameters to be varied. Consequently, the computation for the $j > 1$ fuzzy sphere, even with the help of *Mathematica*, remains virtually intractable and for the Moyal plane the number of parameters is simply infinite! To put our findings in a nutshell, we observe that the finite distance for $j = 1$ and that of $j = \frac{1}{2}$ have almost the same functional form except for an overall scaling by a factor of $\sqrt{2}$ and a miniscule deformation at small 'θ' and that too only from the sixth decimal onwards.

Moreover, we have computed the infinitesimal spectral distance between a pair of discrete mixed states on the quantum Hilbert space of fuzzy sphere using the lower bound formula (6.28). Just like the Moyal plane case [79], we find a deep connection between the geometry of the quantum Hilbert space and the statistical properties of the quantum system associated with the fuzzy sphere.

Part III

Appendices

A

BASIC THEORY OF HOPF ALGEBRA

A.1 ALGEBRA [115]

A unital algebra A is a vector space over a field $\mathbb{K}$ with the following maps:

1. the multiplication map $m : A \otimes A \to A$ (the product) which is associative in the sense $(ab)c = a(bc), \;\; \forall\, a, b, c \in A$. Note that $m(a \otimes b) = ab$.

2. the map $\eta : \mathbb{K} \to A$ (the unit) such that $\eta(1) = \mathbb{I}_A$, where $\mathbb{I}_A$ is the unit element of A which satisfies $a\mathbb{I}_A = a\mathbb{I}_A a, \;\; \forall\, a \in A$.

A.2 CO-ALGEBRA [115]

A co-algebra C is a vector space over the field $\mathbb{K}$ with the following maps:

1. the map $\Delta : C \to C \otimes C$ (the co-product) which is co-associative in the sense

$$\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}, \;\; \forall\, c \in C. \tag{A.1}$$

   Note that we have used the notation $\Delta c \equiv \sum c_{(1)} \otimes c_{(2)}$ in the above equation (A.1).

2. the map $\epsilon : C \to \mathbb{K}$ (the co-unit) obeying $\sum \epsilon(c_{(1)})c_{(2)} = c = \sum c_{(1)}\epsilon(c_{(2)}), \;\; \forall\, c \in C$.

A.3 BI-ALGEBRA [115]

A bi-algebra H is both a unital algebra H with the maps m and η and a co-algebra H with the co-product Δ and the co-unit ϵ where $H \otimes H$ has the tensor product algebra structure $(h \otimes g)(h' \otimes g') = hh' \otimes gg', \;\; h, h', g, g' \in H$.

A.4 HOPF ALGEBRA [115]

A Hopf algebra H is a bi-algebra H i.e., it satisfies the properties of both algebra and co-algebra with the algebra maps: product m and co-product Δ; and the unit map η and the co-unit ϵ with an additional map $S : H \to H$, called the antipode such that $\sum(Sh_{(1)})h_{(2)} = \epsilon(h) = \sum h_{(1)}Sh_{(2)}, \quad \forall\, h \in H$.

The definition of a Hopf algebra can be summarized by the commutativity of the diagram:

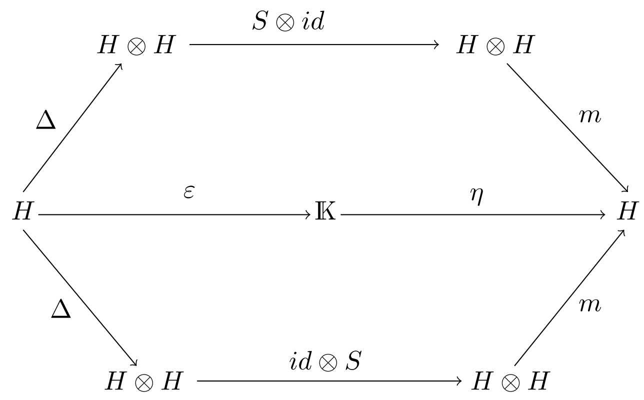

Figure A.1: Commutative diagram showing the axioms on the bi-algebra H that make it Hopf algebra.

A.4.1 *Co-commutative*

A Hopf algebra H is commutative if it is commutative as an algebra and co-commutative if it is co-commutative as a co-algebra i.e. if $\Sigma \circ \Delta = \Delta$ where $\Sigma : H \otimes H \to H \otimes H$ is the flip map, $\Sigma(h \otimes g) = g \otimes h, \;\; \forall\, h, g \in H$.

A.4.2 *Almost Co-commutative*

A Hopf algebra H is said to be almost cocommutative if there exists an invertible element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$ such that $\Sigma \circ \Delta = \mathcal{R}\Delta\mathcal{R}^{-1}$.

A.4.3 *Quasi-triangular*

An almost cocommutative Hopf algebra H is said to be quasi-triangular if

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \quad \text{and} \quad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{A.2}$$

where the product is in $H \otimes H \otimes H$; and $\mathcal{R}_{12} = \mathcal{R}^{\alpha} \otimes \mathcal{R}_{\alpha} \otimes \mathbf{1}$, $\mathcal{R}_{13} = \mathcal{R}^{\alpha} \otimes \mathbf{1} \otimes \mathcal{R}_{\alpha}$ and $\mathcal{R}_{23} = \mathbf{1} \otimes \mathcal{R}^{\alpha} \otimes \mathcal{R}_{\alpha}$ (denoting $\mathcal{R} = \sum_{\alpha} \mathcal{R}^{\alpha} \otimes \mathcal{R}_{\alpha} \in H \otimes H$) are the embedding of $\mathcal{R}$ in higher tensor powers of H. This element $\mathcal{R}$ is called **quasi-triangular structure** or **universal R-matrix**.

Additionally, if $\mathcal{R}_{21} = \mathcal{R}^{-1}$, H is said to be **triangular**. Every co-commutative Hopf algebra is trivially triangular with $\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$.

A.4.4 *Twisted Hopf algebra*

Given a co-commutative Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$, an invertible element $\mathcal{F} \in H \otimes H$ is said to be a **twist** (**Drinfel'd twist**) if it satisfies the following conditions:

$$\begin{aligned} (\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} &= (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F} \quad \rightarrow \quad \text{co-cycle condition;} && \text{(A.3)} \\ (\varepsilon \otimes id)\mathcal{F} &= \mathbf{1} = (id \otimes \varepsilon)\mathcal{F}. && \text{(A.4)} \end{aligned}$$

Further, if we have an invertible element of H as

$$\chi = m(id \otimes S)\mathcal{F} \quad \text{with} \quad \chi^{-1} = m(S \otimes id)\mathcal{F}^{-1}, \tag{A.5}$$

then the twist element $\mathcal{F}$ determines a new Hopf algebra structure on $\mathcal{H}$ with twisted co-product $\Delta_{\mathcal{F}}$ and twisted antipode $S_{\mathcal{F}}$ defined respectively as

$$\Delta_{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1} \quad \text{and} \quad S_{\mathcal{F}} = \chi S \chi^{-1}.$$

This resulting Hopf algebra is called a **twisted Hopf algebra**, $H_{\mathcal{F}}$ with the same underlying algebra structure $\mu_{\mathcal{F}} = \mu$ and counit $\varepsilon_{\mathcal{F}} = \varepsilon$ as H but with twisted co-product $\Delta_{\mathcal{F}}$ and twisted anti-pode $S_{\mathcal{F}}$. It can be shown that $H_{\mathcal{F}}$ or $(H_{\mathcal{F}}, m, \eta, \Delta_{\mathcal{F}}, \varepsilon, S_{\mathcal{F}})$ is a triangular Hopf algebra with universal $\mathcal{R}$-matrix given by $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ which satisfies the quantum Yang-Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{A.6}$$

This twisted or deformed Hopf algebra $\mathcal{H}_{\mathcal{F}}$ is usually called the **Quantum group**. The concept of quantum group is important in non-commutative spaces because it serves as a proper mathematical structure to capture the properties of symmetry in noncommutative spaces.

B

ROBERTSON AND SCHRÖDINGER UNCERTAINTY RELATIONS

The Schrödinger Uncertainty relation represents the most generalized form of uncertainty relation. It can be shown in the following:

The variance $\Delta\hat{A}^2$ of any Hermitian operator $\hat{A}$ in a state $|\Psi\rangle$ can be written as

$$\Delta\hat{A}^2 = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle = \langle f_A|f_A\rangle \ , \ \ \text{where} \ \ |f_A\rangle = |(\hat{A} - \langle\hat{A}\rangle)\Psi\rangle. \tag{B.1}$$

Using the Schwarz inequality for a pair of such observables $\hat{A}$ and $\hat{B}$, we have

$$\Delta\hat{A}^2\Delta\hat{B}^2 = \langle f_A|f_A\rangle\langle f_B|f_B\rangle \geq |\langle f_A|f_B\rangle|^2. \tag{B.2}$$

We can then split $\langle f_A|f_A\rangle\langle f_B|f_B\rangle \geq |\langle f_A|f_B\rangle|^2$ into real and imaginary terms as

$$|\langle f_A|f_B\rangle|^2 = \left(\frac{\langle f_A|f_B\rangle + \langle f_B|f_A\rangle}{2}\right)^2 + \left(\frac{\langle f_A|f_B\rangle - \langle f_B|f_A\rangle}{2i}\right)^2. \tag{B.3}$$

Using the fact that $\langle f_A|f_B\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$ we see that

$$\langle f_A|f_B\rangle - \langle f_B|f_A\rangle = \langle[\hat{A},\hat{B}]\rangle \ \ \text{and} \ \ \langle f_A|f_B\rangle + \langle f_B|f_A\rangle = \langle\{\hat{A},\hat{B}\}\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{B.4}$$

Robertson Uncertainty Relation

Ignoring the square of real part in (B.3), i.e.

$$|\langle f_A|f_B\rangle|^2 \geq (Im.\langle f_A|f_B\rangle)^2 = \left(\frac{\langle f_A|f_B\rangle - \langle f_B|f_A\rangle}{2i}\right)^2, \tag{B.5}$$

we get the Robertson Uncertainty Relation:

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle. \tag{B.6}$$

Schrödinger Uncertainty Relation

If we retain both the squares of real and imaginary parts of (B.3), i.e.

$$|\langle f_A|f_B\rangle|^2 = \left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2 \;\; ; \;\; \text{using B.3 and B.4} \tag{B.7}$$

we finally get the Schrödinger Uncertainty Relation:

$$\Delta\hat{A}\Delta\hat{B} \geq \sqrt{\left(\frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2 + \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2}. \tag{B.8}$$

C

SOME DEFINITIONS

C.1 C^*-ALGEBRA [113, 116]

A complex vector space $\mathcal{A}$ equipped with a multiplication map $f : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\forall\ f,\ g \in \mathcal{A}$, $f \times g \in \mathcal{A}$ is called an **algebra**. Suppose we can define an anti-linear mapping $* : \mathcal{A} \to \mathcal{A}$ called an involution or adjoint operation such that the following properties are satisfied:

1. $\forall\ f \in \mathcal{A}, \quad f^{**} = f;$

2. $\forall\ f, g \in \mathcal{A}, \quad (f \times g)^* = g^* \times f^*;$

3. $\forall\ f, g \in \mathcal{A}$ and $a, b \in \mathbb{C}, \quad (af + bg)^* = \bar{a}f^* + \bar{b}g^*;$

then $\mathcal{A}$ is called a **$*$-algebra or an involutive algebra**. If $\forall\ f \in \mathcal{A}$, we can associate a real number $\|f\|$ called the norm of f such that $\forall\ f, g \in X$ and $c \in \mathbb{C}$

1. $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f = 0$,

2. $\|cf\| = |c|\|f\|$,

3. $\|f + g\| \leq \|f\| + \|g\|$, (triangle inequality)

4. $\|f \times g\| \leq \|f\|\|g\|$, (product inequality or Cauchy-Schwarz inequality),

then $\mathcal{A}$ is called a **normed or Banach algebra**. An involutive as well as normed algebra is called a **Banach $*$-algebra**.

A C^***-algebra** is a Banach $*$-algebra with the property, called C^*-identity:

$$\|f^* \times f\| = \|f\|^2, \quad \forall\ f \in \mathcal{A}. \tag{C.1}$$

For example, given a Hilbert space $\mathcal{H}$, the set $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is a C^*-algebra with the adjoint operation and operator norm:

$$\|A\| = \sup\{\|A\psi\|\ ;\ \psi \in \mathcal{H}, \quad \|\psi\| = 1\}, \tag{C.2}$$

which satisfies the C^*-norm property: $\quad \|A^* \times A\| = \|A\|^2$.

A C^*-algebra $\mathcal{A}$ is a **unital** if it has an identity element $\mathbb{1}$, i.e. $f = \mathbb{1} \times f = f \times \mathbb{1}, \ \forall f \in \mathcal{A}$. Otherwise, it is called **non-unital** C^***-algebra**. A **commutative unital** C^***-algebra** is the one with $f \times g = g \times f, \ \forall f, g \in \mathcal{A}$. An element f of a unital C^*-algebra $\mathcal{A}$ is said to be **normal** iff $f \times f^* = f^* \times f$; **self-adjoint** iff $f = f^*$; **unitary** iff $f \times f^* = f^* \times f = \mathbb{1}$; **projection** iff $f = f^* = f^2$; **positive** iff $f = g \times g^*$ for some $g \in \mathcal{A}$; and **invertible** iff $\exists \ g \in \mathcal{A}$ such that $f \times g = g \times f = \mathbb{1}$. The **resolvent set** $R_{\mathcal{A}}(f)$ of an element $f \in \mathcal{A}$ is defined as the set of $r \in \mathbb{C}$ such that $r\mathbb{1} - f$ is invertible. The **spectrum** $\Sigma_{\mathcal{A}}(f)$ of f is defined as the complement of $R_{\mathcal{A}}(f)$ in $\mathbb{C}$. And the inverse $(r\mathbb{1} - f)^{-1}$ where $r \in R_{\mathcal{A}}(f)$ is called the **resolvent** of f at r.

C.2 $*$-HOMOMORPHISM AND FUNCTIONAL OF C^*-ALGEBRA

Given a pair of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is called a **$*$-homomorphism** if $\phi(f^*) = \phi(f)^*, \ \forall f \in \mathcal{A}$. A linear functional on $\mathcal{A}$ is a linear mapping $\omega : \mathcal{A} \to \mathbb{C}$ and the space of all continuous linear functionals is called the **dual** $\mathcal{A}'$ of $\mathcal{A}$.

1. **Character**: A non-zero $*$-homomorphism $\mu : \mathcal{A} \to \mathbb{C}$, i.e., $\mu(f)\mu(g) = \mu(f \times g), \ \forall f, g \in \mathcal{A}; \ \mu(f^*) = \mu(f)^* \ \mu(f) \neq 0, \ \forall f \in \mathcal{A}$. The set of all characters of $\mathcal{A}$ is called the character space of $\mathcal{A}$ and is a subset of $\mathcal{A}'$, denoted by $\mathcal{M}(\mathcal{A})$. The characters of commutative unital C^*-algebra $\mathcal{A} = C(X)$, the algebra of all continuous functions on a compact set X are the evaluations $\mu_x(f) = f(x), \ \forall x \in X$. The characters of the algebra $\mathcal{A}_n^D$ of $n \times n$ diagonal matrices are the maps $\mu_m : a \to a_{mm}, \ 1 \leq m \leq n, \ \forall a \in \mathcal{A}_n^D$.

2. **State**: A positive and normalized linear functional, i.e. $\omega : \mathcal{A} \to \mathbb{C}$ with $\omega(f^* \times f) \geq 0 \ , \ \forall f \in \mathcal{A}$ and $\|\omega\| = 1$. If ω_1 and ω_2 are the states over $\mathcal{A}$ and $0 < t < 1$, then $\omega = t\omega_1 + (1-t)\omega_2$ is also a state. Thus, the space of all states $\mathcal{S}(\mathcal{A})$ over $\mathcal{A}$ form a convex subset of the dual $\mathcal{A}'$. The extremal elements of $\mathcal{S}(\mathcal{A})$ are called the **pure states** of $\mathcal{A}$ and so they cannot be expressed in terms of states ω_1 and ω_2. The set of pure states is denoted by $\mathcal{P}(\mathcal{A})$.

Since the characters of a C^*-algebra $\mathcal{A}$ are the algebraic $*$-homomorphisms, the existence of $(r\mathbb{1} - f)^{-1}$ implies that of $(r\mathbb{1} - \mu(f))^{-1}, \ \forall f \in \mathcal{A}$ and hence for all $\mu \in \mathcal{M}(\mathcal{A}), \ \mu(f) \in \Sigma(f)$ with $|\mu(f)| \leq \|f\|$. For a commutative unital C^*-algebra $\mathcal{A}$, the character space $\mathcal{M}(\mathcal{A})$ is identified with the spectrum $\Sigma(\mathcal{A})$. Moreover, $\mu(f^* \times f) = \mu(f^*)\mu(f) = |\mu(f)|^2 \geq 0$ and $\mu(\mathbb{1}) = 1$ so that every character is a state. Suppose $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ $(0 \leq \alpha \leq 1)$ with $\mu_1(f) > \mu_2(f)$ for some $f > 0$ such that $\mu(f) = c\mu_1(f)$ with $c < 1$, then we have

$$\alpha\mu_1(f^n) \leq \mu(f^n) = \mu(f)^n = c^n(\mu_1(f))^n \leq c^n\mu_1(f^n), \ \forall n \quad \Rightarrow \quad \alpha = 0. \tag{C.3}$$

Thus, $\mathcal{M}(\mathcal{A})$ contains all pure states. As a subset of the dual $\mathcal{A}'$ of the commutative unital C^*-algebra $\mathcal{A}$, the character space can be topologized through restriction of any of the topologies of $\mathcal{A}'$.

C.3 REPRESENTATIONS OF A C^*-ALGEBRA

A **representation** π of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. That is, $\forall\ a_i \in \mathcal{A},\ \lambda_i \in \mathbb{C}$, we have

$$\pi(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 \pi(a_1) + \lambda_2 \pi(a_2); \pi(a_1 a_2) = \pi(a_1)\pi(a_2); \pi(a_i^*) = \pi(a_i)^*. \quad \text{(C.4)}$$

If $\pi(a) \neq 0$ whenever $a \neq 0$ or, $\ker\pi = \{0\}$ then the representation π is called **faithful**. Each representation π a C^*-algebra $\mathcal{A}$ defines a faithful representation on the quotient algebra $\mathcal{A}_\pi = \mathcal{A}/\ker\pi$. For any C^*-algebra $\mathcal{A}$, the range $\mathcal{B}_\pi = \{\pi(a);\ a \in \mathcal{A}\}$ of representation π on $\mathcal{H}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{T}$ be a subspace of $\mathcal{H}$ such that $\mathcal{M}\mathcal{T} \subset \mathcal{T}$, then $\mathcal{T}$ is called an invariant subspace of $\mathcal{H}$. Let $\mathcal{M} = \mathcal{B}_\pi$ such that $\mathcal{H}$ has no non-trivial invariant sunbspaces, then the representation π is called an **irreducible**.

In any representation π of $\mathcal{A}$ on $\mathcal{H}$, every vector $\psi \in \mathcal{H}$ with $\|\psi\| = \sqrt{\langle\psi|\psi\rangle} = 1$ produces a state $\omega_\psi(a) = \langle\psi|\pi(a)|\psi\rangle,\ \forall\ a \in \mathcal{A}$ called **vector state** of $\mathcal{A}$. The vector state corresponding to an irreducible representation is always a pure state.

C.4 HILBERT-SCHIMDT OPERATORS

For a separable infinite dimensional Hilbert space $\mathcal{H}$, if $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ denotes a space of operators of finite rank i.e. an $a \in \mathcal{L}$ maps $\mathcal{H}$ to a finite-dimensional space, then the completion of $\mathcal{L}$ in the norms:

1. $\|a\|_1 = \mathrm{Tr}(|a|)\ ,\quad |a| = \sqrt{a^* a}$, is denoted by $\mathcal{L}^1$ and such operators are called trace-class operators;
2. $\|a\|_2^2 = \mathrm{Tr}(a^* a)$ is denoted by $\mathcal{L}^2$ and such operators are called Hilbert-Schimdt operators;
3. $\|a\|_\infty = \|a\|$ is denoted by $\mathcal{K}$ and such operators are called compact operators.

For example, diagonal matrices with eigenvalues α_i belong to

1. $\mathcal{L}^1$ provided that $\sum_i |\alpha_i| < \infty$;
2. $\mathcal{L}^2$ provided that $\sum_i |\alpha_i|^2 < \infty$;
3. $\mathcal{K}$ provided that $\lim_{i\to\infty} \alpha_i = 0$.

With this, we have the following inclusions:

$$\mathcal{L} \subset \mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K} \subset \mathcal{B}(\mathcal{H}). \tag{C.5}$$

If the dimension of the underlying Hilbert space $\mathcal{H}$ is finite, say n i.e., $\mathcal{B}(\mathcal{H}) \simeq \mathcal{B}(\mathbb{C}^n)$, then all linear functionals on $\mathcal{B}(\mathbb{C}^n)$ are of the form:

$$\omega(a) = \mathrm{Tr}(\rho a) \equiv (\rho|a) \ , \forall\ a \in \mathcal{B}(\mathbb{C}^n) \ \text{ where } \ \rho \in \big(\mathcal{B}(\mathbb{C}^n)\big)' \equiv \mathcal{B}(\mathbb{C}^n). \tag{C.6}$$

C.5 NORMAL STATE

If the dimension of the underlying Hilbert space $\mathcal{H}$ is infinite, then the dual of $\mathcal{L}$ is $\mathcal{L}^1$ and that of $\mathcal{L}^1$ is $\mathcal{B}(\mathcal{H})$ such that all trace-class operators provide linear functionals on the bounded operators by $a \mapsto \mathrm{Tr}(\rho a)$, $a \in \mathcal{B}(\mathcal{H})$, $\rho \in \mathcal{L}^1$ [113]. Note that a Hermitian trace-class operator $\rho = \rho^*$ such that $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$ is called a **density operator**. The positive and normalized linear functionals provided by density operators $\rho \in \mathcal{L}^1$ on $\mathcal{B}(\mathcal{H})$ are called **normal states** and they are given by

$$\omega_\rho(a) = \mathrm{Tr}(\rho a)\ , \ \ a \in \mathcal{B}(\mathcal{H}). \tag{C.7}$$

If the density matrix ρ is one-dimensional projection such that $\rho^2 = \rho$, then the corresponding normal state is pure. Otherwise, it is a mixed state.

C.6 MODULE

A **left module** M over an algebra A (or, left A-module) is an abelian group M together with the scalar multiplication $A \times M \to M$ i.e. $(a, u) \mapsto au$ such that

$$a(u+v) = au + av \ ; \ (a+b)u = au + av \ ; \ a(bu) = (ab)u \ , \ \ \forall\ a, b \in A,\ u, v \in M. \tag{C.8}$$

Similarly, we can define the **right module** M over B (or, right B-module) as the abelian group M with the scalar multiplication $M \times B \to M$ i.e. $(u, a) \mapsto ua$ satisfying the similar conditions:

$$(u+v)a = ua + va \ ; \ u(a+b) = ua + va \ ; \ (ua)b = u(ab)\ , \ \ \forall\ a, b \in B,\ u, v \in M. \tag{C.9}$$

Given two algebras A and B, an abelian group M with the scalar multplications $A \times M \to M$ and $M \times B \to M$ satisfying (C.8) and (C.9) with the compatibility condition: $a(ub) = (au)b = aub$ is called a (A, B)-module. If $A = B$, then the (A, A)-module is called a **bi-module**.

A module M over A is a **free module** if there exists a linearly independent generating set E which form a basis for M. That is, $\forall\ m \in M,\ \exists\ e_i \in E \subseteq M,\ a_i \in A$ such that $m = \sum_{i=1}^{n} a_i e_i$ where $\sum_{i=1}^{n} a_i e_i = 0_m \Rightarrow a_1 = a_2 = .. = a_n = 0_A$. Every vector space is a free module.

A module M over A is a **finitely generated** if there exists a generating set $\{f_1, f_2, .., f_n\},\ f_i \in M$ such that $\forall\ m \in M,\ \exists\ a_1, a_2, .., a_n \in A$ with $m = \sum_{i=1}^{n} a_i f_i$. Here, $\{f_1, f_2, .., f_n\},\ f_i \in M$ may not be linearly independent.

A module M over A is a **projective** if there exists a set $\{m_1, m_2, .., m_n\},\ m_i \in M$ and another set $\{f_i = \in Hom(M, A)\}$ such that $\forall\ m \in M,\ f_i(m)$ is only nonzero for finitely many $i \in \mathcal{I}$ and $m = \sum f_i(m) a_i,\ a_i \in A$. Or, a module M over A is a projective if and only if there is a free module F and another module N such that $F = M \oplus N$.

If A is a unital C^*-algebra and p is a projection in $End_A(A^n)$, i.e. $p^2 = p = \begin{pmatrix} p_{11} & .... & p_{1s} \\ .. & .... & .. \\ .. & .... & .. \\ p_{s1} & .... & p_{ss} \end{pmatrix}$, $p_{ij} \in A$, then $\mathcal{E} = pA^n$ is a **finitely generated projective right** A**-module**. Further, $\mathcal{E}$ is a pre-Hilbert A-module with A-valued inner product: $\langle u|v\rangle = \sum_{i=1}^{n} u_i^* v_i,\ u, v \in \mathcal{E}$.

If M is a differentiable manifold and we have a vector bundle $E \xrightarrow{\pi} M$ with the space of sections $\Gamma(M, E) \equiv \Gamma(E) = \{s : M \to E\ :\ \pi \circ s = \mathrm{Id}_M\}$, then $\Gamma(E)$ is finitely generated projective right module over $A = C^\infty(M)$ with $s \circ f \in \Gamma(E),\ \forall\ f \in A$ defined as $(s \circ f)(x) = s(x) f(x),\ \forall\ x \in M$.

D

CONSTRUCTION OF DIRAC OPERATORS

Here we will review the construction of Dirac operator $\mathcal{D}_M$ on Moyal plane and $\mathcal{D}_f$ on fuzzy sphere. Dirac operator is one of the major ingredients of a spectral triple and plays the most important role in the computation of spectral distance.

D.1 MOYAL PLANE

The Moyal plane $\mathbb{R}^2_\star$ is the quantization of Euclidean plane $\mathbb{R}^2$ and we know that for the latter $\mathbb{R}^2$ the Dirac operator is

$$\mathcal{D} = -i(\sigma_1\partial_1 + \sigma_2\partial_2)\ , \quad \text{where} \ \ \sigma_1, \sigma_2 \ \text{ are the first two Pauli matrices.} \tag{D.1}$$

For Moyal plane where $[\hat{x}_\alpha, \hat{x}_\beta] = i\theta\epsilon_{\alpha\beta}\ ;\ \alpha, \beta = 1, 2$, we can define an analogous operator $\partial_\alpha \to \hat{\partial}_\alpha$'s only through the momentum operator $\hat{P}_\alpha$ acting adjointly on $\mathcal{H}_q$ with the non-commutative Heisenberg algebra (2.5). Since it is only the quantum Hilbert space $\mathcal{H}_q$ that furnish a complete representation of the entire Heisenberg algebra we need to construct our Dirac operator on this space. Thus, we can construct the Dirac operator for Moyal plane as

$$\mathcal{D}_M \equiv \sigma_\alpha\hat{P}_\alpha = \sigma_1\hat{P}_1 + \sigma_2\hat{P}_2\ , \quad \text{acting adjointly on} \quad \Phi = \begin{pmatrix} |\phi_1) \\ |\phi_2) \end{pmatrix} \in \mathcal{H}_q \otimes \mathbb{C}^2\ . \tag{D.2}$$

Using (2.10), we get $\mathcal{D}_M\Phi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} [i\hat{b}^\dagger, \phi_2] \\ [-i\hat{b}, \phi_1] \end{pmatrix}$. Let us define a diagonal representation π of $a \in \mathcal{H}_q$ as $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and its action on the same $\Phi \in \mathcal{H}_q \otimes \mathbb{C}^2$ as

$$\pi(a)\Phi = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} |\phi_1) \\ |\phi_2) \end{pmatrix} = \begin{pmatrix} a|\phi_1) \\ a|\phi_2) \end{pmatrix}, \tag{D.3}$$

such that the action of the commutator $[\mathcal{D}_M, \pi(a)]$ on Φ gives

$$[\mathcal{D}_M, \pi(a)]\Phi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [i\hat{b}^\dagger, a] \\ [-i\hat{b}, a] & 0 \end{pmatrix} \Phi \;\Rightarrow\; [\mathcal{D}_M, \pi(a)] \equiv \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [i\hat{b}^\dagger, a] \\ [-i\hat{b}, a] & 0 \end{pmatrix}. \quad \text{(D.4)}$$

Since $\hat{b}/\hat{b}^\dagger$ and $a \in \mathcal{H}_q$ are the operators acting on $\mathcal{H}_c$ and the diagonal action $\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ on $\mathcal{H}_q \otimes \mathbb{C}^2$ can be understood as a diagonal representation $\pi(a)$ acting on $\mathcal{H}_c \otimes \mathbb{C}^2$, the commutator $[\mathcal{D}_M, \pi(a)]$ thus yields the following form of Dirac operator acting on $\mathcal{H}_c \otimes \mathbb{C}^2$:

$$\mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & i\hat{b}^\dagger \\ -i\hat{b} & 0 \end{pmatrix}. \quad \text{(D.5)}$$

This is further simplified by considering the transformations $\hat{b} \to i\hat{b}$ and $\hat{b}^\dagger \to -i\hat{b}^\dagger$, which just corresponds to a $SO(2)$ rotation by an angle $\frac{\pi}{2}$ in $\hat{x}_1, \hat{x}_2$ space. With this transformation the Dirac operator takes the following hermitian form :

$$\mathcal{D}_M = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & \hat{b}^\dagger \\ \hat{b} & 0 \end{pmatrix}. \quad \text{(D.6)}$$

Clearly, this Dirac operator $\mathcal{D}_M$ has a well-defined left action on $\Psi = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \in \mathcal{H}_c \otimes \mathbb{C}^2$:

$$\mathcal{D}\Psi = \sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix} \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \sqrt{\frac{2}{\theta}} \begin{pmatrix} b^\dagger |\psi_2\rangle \\ b\,|\psi_1\rangle \end{pmatrix}. \quad \text{(D.7)}$$

D.2 FUZZY SPHERE [88]

The construction of Dirac operator for fuzzy sphere can be understood by first understanding the construction of Dirac operator for 3-sphere and hence for 2-sphere. Here, we provide a review of the same by essentially following [88].

D.2.1 *Dirac operator on $\mathbb{S}^3$ and $\mathbb{S}^2$*

Let us consider a flat $\mathbb{C}_0^2$ manifold, isomorphic to $\mathbb{R}^4 - \{0\}$ which can be defined as

$$\mathbb{C}_0^2 = \left\{ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathbb{C}^2 \;:\; \chi^\dagger \chi > 0 \right\}. \quad \text{(D.8)}$$

With the imposition of the constraint $\chi^\dagger\chi = |\chi_1|^2 + |\chi_2|^2 = 1$ on (D.8), we get the 3-sphere $\mathbb{S}^3$ which is the $SU(2)$ group manifold [117]. Note that Hopf has shown that $\mathbb{S}^3$ is a $U(1)$ bundle over $\mathbb{S}^2$ [118]. This can be easily seen by making use of the $U(1)$ freedom $\chi \to e^{i\alpha}\chi$ (for arbitrary α) in (D.8). That is, we can choose α such that either $\chi^* = \chi_1$ or $\chi_2^* = \chi_2$ in local neighborhoods giving either $\chi^\dagger\chi = \chi_1^2 + |\chi_2|^2 = 1$ or, $\chi^\dagger\chi = |\chi_1|^2 + \chi_2^2 = 1$ which is clearly $\mathbb{S}^2$. Now, we know that a $\mathbb{CP}^1$ manifold is defined as

$$\mathbb{CP}^1 = \left\{ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \approx \lambda\chi \in \mathbb{C}^2 \;:\; \lambda \neq 0 \in \mathbb{C} \right\}. \tag{D.9}$$

This implies that by first normalising i.e. setting $\chi^\dagger\chi = 1$ and then choosing a suitable section from the $U(1)$ bundle over $\mathbb{S}^2$, we can identify $SU(2)/U(1) \simeq \mathbb{S}^2 \simeq \mathbb{CP}^1$. This can be obtained in two ways which we can be explained clearly by choosing a standard parametrization of $\mathbb{S}^3$.

Let θ, ϕ, ψ be the three Euler angles on $\mathbb{S}^3$ which have the following identification with the complex doublets of $\mathbb{C}_0^2$ manifold (D.8):

$$\mathbb{S}^3 = \left\{ \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{\frac{i}{2}(\phi+\psi)} \\ \sin\left(\frac{\theta}{2}\right) e^{-\frac{i}{2}(\phi-\psi)} \end{pmatrix} \in \mathbb{C}^2 \;:\; \chi^\dagger\chi = |\chi_1|^2 + |\chi_2|^2 = 1 \right\}. \tag{D.10}$$

Let us choose the following two charts:

1. $U_+ = \{\chi \in \mathbb{S}^3 \;:\; \chi_1^* = \chi_1\}$ in the neighbourhood $\chi_1 \neq 0$ by setting $\psi = -\phi$

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \chi' = \begin{pmatrix} \chi_1' \\ \chi_2' \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}; \tag{D.11}$$

2. $U_- = \{\chi \in \mathbb{S}^3 \;:\; \chi_2^* = \chi_2\}$ in the neighbourhood $\chi_2 \neq 0$ by setting $\psi = \phi$

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \chi'' = \begin{pmatrix} \chi_1'' \\ \chi_2'' \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{D.12}$$

Clearly, χ' and χ'' can be interpreted as the local coordinates of $\mathbb{S}^2$ on the northern hemisphere V_+ and southern hemisphere V_- respectively (which can be expressed as familiar coordinates $x_1 = \sin\left(\frac{\theta}{2}\right)\cos\phi$, $x_2 = \sin\left(\frac{\theta}{2}\right)\sin\phi$ and $x_3 = \cos\left(\frac{\theta}{2}\right)$ of $\mathbb{S}^2$). On the overlap region $V_+ \bigcap V_-$, we have the transition function as

$$\chi' = e^{i\varphi}\chi''. \tag{D.13}$$

Let us now consider the spinor bundle $S(C_0^2)$ over $\mathbb{C}_0^2$ manifold, which is actually a trivial one. The sections of $S(\mathbb{C}_0^2)$ have the following form in terms of homogeneous coordinates χ_1 and χ_2:

$$\Psi(\chi,\chi^*) = \begin{pmatrix} \Psi_1(\chi,\chi^*) \\ \Psi_2(\chi,\chi^*) \end{pmatrix} \; ; \; \Psi_\alpha = \sum_{n_1,n_2,m_1,m_2} a^\alpha_{n_1,n_2,m_1,m_2} \chi_1^{*n_1} \chi_2^{*n_2} \chi_1^{m_1} \chi_2^{m_2} \; ; \; \alpha = 1,2. \quad \text{(D.14)}$$

Here, $n_1, n_2, m_1, m_2 \in \mathbb{Z}$. The sections of the spinor bundle $S(\mathbb{S}^3)$ over $\mathbb{S}^3$ can be easily obtained from the sections of $S(\mathbb{C}_0^2)$ (D.14) by putting the constraint $\chi^\dagger \chi = 1$. The spinor bundle $S(\mathbb{S}^3)$ persists to be a trivial one. In terms of the Euler angles (D.10), we can write the sections (D.14) as

$$\Psi_\alpha = \sum_{n_1,n_2,m_1,m_2} a^\alpha_{n_1,n_2,m_1,m_2} \left(\cos\frac{\theta}{2}\right)^{n_1+m_1} \left(\sin\frac{\theta}{2}\right)^{n_2+m_2} e^{-i\frac{\phi}{2}(m_1-m_2-n_1+n_2)} e^{-ik\frac{\psi}{2}} \, , \quad \text{(D.15)}$$

where

$$k = (m_1 + m_2 - n_1 - n_2) \in \mathbb{Z} \, . \quad \text{(D.16)}$$

The sections of $S(\mathbb{C}_0^2)$ and $S(\mathbb{S}^3)$ have a natural grading where the subbundles $S_k(C_0^2)$ and $S_k(\mathbb{S}^3)$ are formed by sections of the form (D.15) with fixed k.

The differential operators acting on sections of $S(\mathbb{C}_0^2)$ are

$$J_i = \frac{1}{2}(\chi_\alpha \sigma_i^{\beta\alpha} \partial_{\chi_\beta} - \chi_\alpha^* (\sigma^*)_i^{\beta\alpha} \partial_{\chi_\beta^*}) \;\; \text{and} \;\; K = \frac{1}{2}(\chi_\alpha \partial_{\chi_\alpha} - \chi_\alpha^* \partial_{\chi_\alpha^*}) \, . \quad \text{(D.17)}$$

The actions of differential operators (D.17) on sections of $\mathbb{S}^3$ remains unchanged. The differential operators J_i satisfy the $su(2)$ algebra: $[J_i, J_j] = i\epsilon_{ijk} J_k$ and can be identified with the orbital angular momenta. The sections of $S_k(\mathbb{C}_0^2)$ or, $S_k(\mathbb{S}^3)$ are eigenstates of the dilatation operator K:

$$K\Psi = k\Psi \, , \quad \Psi \in S_k(\mathbb{C}_0^2) \;\; \text{or,} \;\; S_k(\mathbb{S}^3). \quad \text{(D.18)}$$

In terms of the Euler angles (D.10), these differential operators (D.17) have the following form:

$$\begin{aligned} J_1 &= i \sin\varphi \frac{\partial}{\partial\theta} + i \cos\phi \cot\theta \frac{\partial}{\partial\phi} - i \frac{\cos\phi}{\sin\theta} \frac{\partial}{\partial\psi}, \\ J_2 &= -i \cos\phi \frac{\partial}{\partial\theta} + i \cot\theta \sin\phi \frac{\partial}{\partial\phi} - i \frac{\sin\phi}{\sin\theta} \frac{\partial}{\partial\psi}, \\ J_3 &= -i \frac{\partial}{\partial\phi}, \\ K &= i \frac{\partial}{\partial\psi}. \end{aligned} \quad \text{(D.19)}$$

The sections of $S_k(\mathbb{C}_0^2)$ or, $S_k(\mathbb{S}^3)$ are modules with respect to multiplication by elements of the algebra $\mathcal{A}_0$, formed by functions of the form:

$$f(\chi,\chi^*) = \sum_{n_1,n_2,m_1,m_2} a^\alpha_{n_1,n_2,m_1,m_2} \chi_1^{*n_1}\chi_2^{*n_2}\chi_1^{m_1}\chi_2^{m_2} \text{ with } k=0 \Rightarrow n_1+n_2 = m_1+m_2 \ . \quad \text{(D.20)}$$

Since $\chi^\dagger\chi = 1$ and we have the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$:

$$x_i = \chi^\dagger\sigma_i\chi \ ; \quad i = 1,2,3 \ , \quad \text{(D.21)}$$

satisfying $\vec{x}^2 = x_i x_i = 1$. Note that the algebra of $\mathcal{A}_0$ is isomorphic to the algebra of functions $C(\mathbb{S}^2)$ on $\mathbb{S}^2$. Further, note that the Jordan-Schwinger map (2.59) is the operatorial version of this Hopf map (D.21).

The sections of spinor bundle $S(\mathbb{S}^2)$ have the following form:

$$\Psi'(\chi',\chi'^*) = \begin{pmatrix} \psi_1'(\chi',\chi'^*) \\ \psi_2'(\chi',\chi'^*) \end{pmatrix} \text{ on } V_+ \ ; \ \Psi''(\chi'',\chi''^*) = \begin{pmatrix} \psi_1''(\chi'',\chi''^*) \\ \psi_2''(\chi'',\chi''^*) \end{pmatrix} \text{ on } V_- \ . \quad \text{(D.22)}$$

On the overlap region $V_+ \bigcap V_-$, we have the relation:

$$\Psi'(\chi',\chi'^*) \ = \ e^{ik\phi}\, \Psi''(\chi'',\chi''^*) \ , \quad \text{(D.23)}$$

up to a global phase factor $e^{i\delta}$. This k, which can be identified with the topological index (Chern class), classifies the spinor bundles $S_k(\mathbb{S}^2)$ over $\mathbb{S}^2$ and the angle ϕ is given by (D.13). Two sections Ψ_0 and Ψ_1 of $S_k(\mathbb{S}^2)$ are said to be equivalent if $\Psi_0' = \Psi_1'$, $\Psi_0'' = e^{i\delta}\Psi_1''$. The equivalence class $\tilde{\Psi}$ of a given section Ψ of $S_k(\mathbb{S}^2)$ can be represented as

$$\begin{aligned} \tilde{\Psi}'_\alpha(\chi',\chi'^*) &= \sum_{n_1,n_2,m_1,m_2} a^\alpha_{n_1,n_2,m_1,m_2} {\chi'}_1^{*n_1}{\chi'}_2^{*n_2}{\chi'}_1^{m_1}{\chi'}_2^{m_2} \quad \text{on} \quad V_+ \ , \\ \tilde{\Psi}''_\alpha(\chi'',\chi''^*) &= \sum_{n_1,n_2,m_1,m_2} a^\alpha_{n_1,n_2,m_1,m_2} {\chi''}_1^{*n_1}{\chi''}_2^{*n_2}{\chi''}_1^{m_1}{\chi''}_2^{m_2} \quad \text{on} \quad V_- \ . \end{aligned} \quad \text{(D.24)}$$

Note that the coefficients $a^\alpha_{n_1,n_2,m_1,m_2}$ are the same in both charts V_+ and V_- with $k = n_1 + n_2 - m_1 - m_2$ such that the following transition rule is satisfied:

$$\tilde{\Psi}'_\alpha(\chi',\chi'^*) \ = \ e^{-ik\phi}\, \tilde{\Psi}''_\alpha(\chi'',\chi''^*). \quad \text{(D.25)}$$

Denoting the bundle formed by the sections of the form (D.24) as $\tilde{S}_k(\mathbb{S}^2)$, we can see a one-to-one mapping between the sections of $\tilde{S}_k(\mathbb{S}^2)$ (D.24) and the sections of $S_k(\mathbb{C}_0^2)$ or, $S_k(\mathbb{S}^3)$ (D.14) with the same coefficients $a^\alpha_{n_1,n_2,m_1,m_2}$ with $k = n_1 + n_2 - m_1 - m_2$. This gives the following mapping of the section $\Psi \in S(\mathbb{S}^3)$ to the section $\tilde{\Psi} \in \tilde{S}(\mathbb{S}^2)$:

$$\begin{aligned} \tilde{\Psi}'(\chi',\chi'^*) &= e^{\frac{i}{2}k(\phi+\psi)}\Psi(\chi,\chi^*) \quad \text{on} \quad V_+ \ , \\ \tilde{\Psi}''(\chi'',\chi''^*) &= e^{-\frac{i}{2}k(\phi-\psi)}\tilde{\Psi}_\alpha(\chi,\chi^*) \quad \text{on} \quad V_- \ , \end{aligned} \quad \text{(D.26)}$$

where the bundle $\tilde{S}(\mathbb{S}^2)$ is defined as the direct Whitney sum : $\tilde{S}(\mathbb{S}^2) = \oplus\tilde{S}_k(\mathbb{S}^2)$.

The free Dirac operator $\tilde{D}_k : \tilde{S}(\mathbb{S}^2) \to \tilde{S}(\mathbb{S}^2)$ is defined by

$$\tilde{D}'_k = [i\sigma'^{\alpha}(\partial'_\alpha + iA'_\alpha)] \;\; on \;\; V_+ \;\; ; \;\; \tilde{D}''_k = [i\sigma''^{\alpha}(\partial''_\alpha + iA''_\alpha)] \;\; on \;\; V_- \; ; \;\; \alpha = 1,2 \; > \tag{D.27}$$

Here, ∂_α denotes the derivative ∂_θ, ∂_ϕ in the local coordiantes θ and ϕ in $V_+ \bigcap V_-$ and $(\partial_\alpha + iA_\alpha)$ represents covariant derivative. The σ's (σ^θ and σ^ϕ) satisfies the following Clifford algebra:

$$\{\sigma^\alpha, \sigma^\beta\} = 2g^{\alpha\beta} \; , \quad \text{on} \quad V_+ \bigcap V_- \tag{D.28}$$

where $\{g^{\alpha\beta}\} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$ is the inverse of the metric tensor of the unit sphere $S^2(\vec{x}^2 = 1)$ and A_μ is the k- monopole field given by [119]:

$$A'_\alpha = ik\chi'^{\dagger}\partial_\alpha\chi' \quad \text{on} \quad V_+ \;\; ; \;\; A''_\alpha = ik\chi''^{\dagger}\partial_\alpha\chi'' \quad \text{on} \quad V_-. \tag{D.29}$$

The field A'_α and A''_α in $V_+ \bigcap V_-$ are related by the gauge transformation

$$A'_\alpha = A''_\alpha + k\partial_\alpha\phi \;\Rightarrow\; A'_\theta = A''_\theta = 0 \; ; \;\; A'_\phi = \frac{k}{2}(\cos\theta - 1) \; ; \;\; A''_\phi = \frac{k}{2}(\cos\theta + 1) \; , \tag{D.30}$$

since on $V_+ \bigcap V_-$, we have $\sigma^\theta = \begin{pmatrix} 1 & -\cot\theta e^{-i\phi} \\ -\cot\theta e^{i\phi} & -1 \end{pmatrix}$ and $\sigma^\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$. The eigenvalue problem of the Dirac operator in $\tilde{S}(\mathbb{S}^2)$:

$$\tilde{D}'_k\tilde{\Psi}'(\chi', \chi'^*) = \lambda\tilde{\Psi}'(\chi', \chi'^*) \;\; \text{on} \;\; V_+ \; ; \;\; \tilde{D}''_k\tilde{\Psi}''(\chi'', \chi''^*) = \lambda\tilde{\Psi}''(\chi'', \chi''^*) \;\; \text{on} \;\; V_- \; , \tag{D.31}$$

is switched to a problem in $S_k(S^3)$, $D_k\Psi = \lambda\Psi$, $\Psi \in S_k(S^3)$ where we use

$$\Psi = e^{-\frac{i}{2}k(\phi+\psi)}\tilde{\Psi}' \;\; \text{on} \;\; V_+ \; ; \quad \Psi = e^{\frac{i}{2}k(\phi-\psi)}\tilde{\Psi}'' \;\; \text{on} \;\; V_- \; ; \tag{D.32}$$

$$\text{and} \;\; D_k = e^{-\frac{i}{2}k(\phi+\psi)}\tilde{D}_k e^{\frac{i}{2}k(\phi+\psi)} \;\; \text{on} \;\; V_+ \; ; \;\; D_k = e^{\frac{i}{2}k(\phi-\psi)}\tilde{D}_k e^{-\frac{i}{2}k(\phi-\psi)} \;\; \text{on} \;\; V_-. \tag{D.33}$$

On both V_+ and V_-, a straightforward computation shows that the Dirac operator on $\mathbb{S}^3$ with radius r is given by

$$D_k = \frac{1}{r}\,\sigma_j(J_j - \frac{k}{2}\frac{x_j}{r}). \, . \tag{D.34}$$

The Dirac operator on $\mathbb{S}^2$ with radius r can be obtained by putting $k = 0$ as

$$D_0 = \frac{1}{r}\,\sigma_j J_j = \frac{1}{r}\,\vec{J}\otimes\vec{\sigma} \; . \tag{D.35}$$

D.2.2 *Dirac operator on fuzzy sphere* $\mathbb{S}^2_\star$

We have the algebra $\mathcal{A}_0$ of functions on $\mathbb{S}^2$ which have the form:

$$\mathcal{A}_0 = \left\{ \Phi = \sum_{n_1,n_2,m_1,m_2} a^{\alpha}_{n_1,n_2,m_1,m_2} \chi_1^{*n_1} \chi_2^{*n_2} \chi_1^{m_1} \chi_2^{m_2} \quad : \quad k = n_1 + n_2 - m_1 - m_2 = 0 \right\}. \quad \text{(D.36)}$$

Note that this algebra $\mathcal{A}_0$ is analogous to the quantum Hilbert space $\mathcal{H}_j$ (2.69) of fuzzy sphere with radius $r_j = \theta_f \sqrt{j(j+1)}$ which form an algebra:

$$H_j = \left\{ \hat{\Phi} = \sum_{n_1,n_2,m_1,m_2} a^{\alpha}_{n_1,n_2,m_1,m_2} \hat{\chi}_1^{\dagger n_1} \hat{\chi}_2^{\dagger n_2} \hat{\chi}_1^{m_1} \hat{\chi}_2^{m_2} \quad : \quad k = n_1 + n_2 - m_1 - m_2 = 0 \right\}. \quad \text{(D.37)}$$

The actions of angular mometum operators $\hat{J}_i$ and the dilatation operator $\hat{K}$ on fuzzy sphere can be obtained by replacing the derivatives in (D.17) with the following commutators:

$$\partial_{\chi_\alpha} \Psi \sim [\chi_\alpha^\dagger, \Psi] \;\; ; \;\; \partial_{\chi_\alpha^\dagger} \Psi \sim [\chi_\alpha, \Psi]. \quad \text{(D.38)}$$

This gives the adjoint action of $\hat{J}_i$ and $\hat{K}$ on the quantum Hilbert space $\mathcal{H}_j$ as

$$\hat{J}_i \Phi = \frac{1}{\theta_f} [\hat{x}_i, \Phi] \quad ; \quad \hat{K}\Phi = [\hat{N}, \Phi] = 0 \,, \quad \Phi \in \mathcal{H}_j \,, \quad \text{(D.39)}$$

since all elements of $\mathcal{H}_j$ are of the form $\Phi \sim |j,m)(m',j|$. The action of $\hat{J}_i$ on the configuation space $\mathcal{F}_j$ is given by

$$\hat{J}_i |j,m\rangle = \frac{1}{\theta_f} \hat{x}_i |j,m\rangle \,, \quad |j,m\rangle \in \mathcal{F}_j \,. \quad \text{(D.40)}$$

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