Interlinking Fundamental Quantum Concepts With Information Theoretic Resources

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To Amiya Samanta and Avijit Lahiri who motivated me into research in physics.

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ABSTRACT

In Quantum theory there are several points of departure from classical theory for describing nature. The most well known non-classical concept about quantum mechanics is the uncertainty principle. Uncertainty relation and related randomness are associated with the probabilistic structure of quantum theory, which is not like classical probability theory where any kind of randomness arises due to subjective ignorance. To reduce quantum theory to classical probability theory with some additional variables is the program of so called hidden variable theory. There are three no-go theorems arising from quantum correlations. No local-realist model pertaining to spatial correlation, no non-contextual model and no macro-realist model for quantum theory pertaining to temporal correlation. These foundational studies have many information theoretic applications such as quantum cryptography, factorisation problem, computation, genuine random number generation etc.

This thesis contains some foundational issues and applications as well. Generalised form of Heisenberg's uncertainty relation is turned into witness of purity or mixedness of quantum system by choosing observables suitably. A new uncertainty relation in the presence of quantum memory is derived which is optimal in the context of experimental verification. Then problem of sharing of nonlocality by multiple observers is addressed. Violation of macrorealism (MR) is a promising ground for studying quantum-classical transition. We show how to obtain optimal violation of Leggett-Garg inequality and a necessary condition of MR, dubbed Wigner form of LGI is proposed. Quantum-classical transition is addressed considering coarse-grained measurements in cases of large spin systems in uniform magnetic field and simple harmonic oscillator with increasing mass. Finally LGI is linked with device independent randomness generation by deriving it from a new set of assumptions, no signalling in time and predictability.

LIST OF PUBLICATIONS

Publications relevant to the Thesis:

- **S. Mal**, T. Pramanik, A. S. Majumdar: "*Detecting mixedness of qutrit systems using the uncertainty relation*"; Phys. Rev. A **87**, 012105 (2013).
- T. Pramanik, **S. Mal**, A. S. Majumdar: "*Lower bound of quantum uncertainty from extractable classical information*"; Quantum Inf Process **15**, 981 (2016).
- **S. Mal**, A. S. Majumdar: "*Optimal violation of Leggett-Garg inequality for arbitrary spin and emergence of classicality through unsharp measurement*"; Phys. Lett. A **380**, 2265 (2016).
- D. Saha, **S. Mal**, P. Panigrahi, D. Home: "*Wigner's form of the Leggett-Garg inequality, No-Signalling in Time, and Unsharp Measurements*"; Phys. Rev. A **91**, 032117 (2015).
- **S. Mal**, D. Das, D. Home: "*Quantum mechanical violation of macrorealism for large spin and its robustness against coarse-grained measurements*"; Phys. Rev. A **94**, 062117 (2016).
- S. Bose, D. Home, **S. Mal**: "*Uncovering a Nonclassicality of the Schrödinger Coherent State up to the Macro-Domain*"; arXiv:1509.00196(2015).
- **S. Mal**, A. S. Majumdar, D. Home: "*Sharing of Nonlocality of a single member of an Entangled Pair of Qubit Is Not Possible by More Than Two Unbiased Observers on the other wing*"; Mathematics, **4**, 48 (2016).
- **S. Mal**, M. Banik, S. K. Choudhury: "*Temporal correlations and device-independent randomness*"; Quantum Inf Process **15**, 2993 (2016).

Additional Publications during the Ph.D. thesis but not forming Part of it:

- **S. Mal**, A. S. Majumdar, D. Home: "*Probing hierarchy of temporal correlation requires either generalised measurement or nonunitary evolution*"; arXiv:1408.0526 (2015) (submitted).

- D. Das, A. Gayen, R. Das, and **S. Mal**: "*Exploring the role of 'biasedness' parameter of a generalized measurement in the context of quantum violation of macrorealism for arbitrary spin systems*"; arXiv: (2017) (submitted).

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CHAPTER 1

INTRODUCTION

Birth of quantum theory is marked by the year 1900 due to Max Planck. After then it got huge success in explaining newly explored natural phenomena which were not possible to comprehend from the then existing theories (classical physics). Apart from gravity, it provides completely correct description for all natural phenomena in microscopic domain. Various counter classical quantum phenomena were discovered and analysed throughout the last century and till now exploration is going on. Conceptual revolution always facilitates technological revolution. It is indeed with the quantum mechanical understanding of the structure and properties of matter that physicists and engineers were able to invent and develop transistor and laser.

The most well known non-classical concept about quantum mechanics is uncertainty principle of Heisenberg [1]. Uncertainty relations and related randomness are associated with the probabilistic structure of quantum theory, which is not like classical probability theory where any kind of randomness arises due to subjective ignorance. To reduce quantum theory to classical probability theory with some additional variables is the program of so called hidden variable theory(HVT). Quantum entanglement which lies at the heart of EPR paradox indicates one of the famous conflicts between classical and quantum description of nature. Bell's no-go theorem asserts that one cannot construct a local realist model for quantum theory[2]. Another no-go theorem is known as contextuality [3], which states that non-contextual hidden variable model cannot explain some temporal correlations emerging from sequential compatible quantum measurements. The latest no-go theorem in this direction is due to Leggett-Garg[4], which imply macro-realist theories, compatible with classical physics, are untenable with quantum theory.

Our every day experience with macroscopic world does not manifest quantum features. Quantum mechanics allow superposition of states and well describe micro-world phenomena. But it leads to Schrödinger cat paradox when macro-world comes into the picture. Formalism of quantum measurement requires classical apparatus which is to be entangled with quantum system to be measured leads to notorious measurement problem in quantum mechanics. Quantum superposition of macroscopic system inevitably arises through such description, which is not observed. There are three major approaches to this problem. One is objective collapse models[5, 6] which put a limit beyond which quantum superposition disappears. Decoherence program[7] considers interaction between system and environment for resolving this issue. Third approach limits power of observability for describing emergence of classicality out of quantum features. Based on the idea of Peres [8], Kofler and Brukner established the approach of emergence of classicality through coarse-grained measurements. First two approaches do not yield fully satisfactory answer to the already settled experimental facts and third approach does not provide a sharp boundary of quantum-classical transition. Hence, quantum to classical transition is one of the most fundamental and interesting area of study not only due to its prior importance for the future development towards macroscopic superposition and entanglement but also necessary for a consistent description of nature.

These foundational studies have many applications as several no-go results lead to various quantum information processing tasks outperforming their classical counter parts such as quantum cryptography [9, 10, 11], search algorithm, factorisation problem, computation, genuine random number generation [12, 13]. Therefore it is important to identify proper resources for the information processing tasks. Recently non-locality has been proven to be resource for device independent tasks. Contextuality is linked with computational tasks.

Outline of the thesis: This thesis contains some foundational issues and applications as well. New application of one of generalised forms of Hiesenberg's uncertainty relation is found. A new uncertainty relation in the presence of quantum memory is derived. How non-local correlation can be shared between multiple observers is addressed. At the later part, this thesis mainly deals with issues of macro-realism. How the no-go theorem in this case differs from that of scenario of local-realism is emphasised. Quantum-classical transition is addressed considering coarse-grained measurement in greater detail. A novel formalism is introduced using simple harmonic oscillator, which is well described in classical and quantum theory as well, to explore macroscopic superposition. Finally Leggett-Garg inequality, a necessary condition for macro-realism, is utilised in the context of device independent randomness generation.

In the remaining part of the introductory chapter, the mathematical framework which is relevant for comprehending different results in later chapters is discussed. It begins with Postulates of quantum theory. Then mathematical representation of single and bipartite quantum systems ranging from simplest two level system to system of any dimension are discussed. Derivation of generalised uncertainty relation and entropic uncertainty relation are discussed briefly. Three no-go theorem and some issues regarding these are discussed. We end with briefly stating *ontological model* framework for the operational quantum theory.

In Chapter-2 we demonstrate an application of Robertson-Schrödinger generalized uncertainty relation(GUR) in the context of detecting mixedness or purity of a quantum system. Advantages of purity detection scheme using GUR over state tomography approach in terms of number of measurements is addressed. Then a new uncertainty relation is proposed in the presence of quantum memory. Lower bound of this uncertainty relation is optimal in the experimental conditions. We also identify the proper resource dubbed extractable classical information responsible for the reduction of lower bound in this scenario.

In Chapter-3 we provide a brief discussion on the quantum theory of measurement and positive operator valued measure. Then using this formalism we show that unsharp observables characterized by a single unsharpness parameter saturate the optimal pointer condition with respect to the trade-off between disturbance and information gain. Then we consider the problem of sharing of nonlocality by multiple observers. Specifically we prove nonlocality pertaining to a single member of an entangled pair of particles can be shared with two independent observers who sequentially perform measurements on the other member of the entangled pair but not more than two.

In Chapter-4 we discuss macrorealism and its violation probed through violation of Leggett-Garg inequality. We show how to obtain optimal violation of LGI involving dichotomic measurements for arbitrary spin system and then how classicality emerges with unsharp measurements. Then we derive a new necessary condition of macrorealism dubbed Wigner form of LGI and show its robustness with compare to conventional LGI with respect to unsharp measurement. We also consider another necessary condition of MR, namely no-signalling in time(NSIT) and demonstrate its maximal robustness among other necessary conditions of MR with respect to unsharp measurement.

In Chapter-5 we discuss Quantum-classical transition considering two type of systems. Firstly we consider arbitrary spin system in uniform magnetic field. Invoking general kind of coarse-grained measurement i.e., measurement with varying degree of coarseness in conjunction with fuzziness we discuss issues of quantum-classical transition. Then we consider oscillator system with dichotomic position measurement and investigated quantum-classical transition with increasing mass.

In Chapter-6 we propose an important application of violation of LGI in the context of certification of randomness generation. This is done by deriving LGI from different set of assumptions: no signalling in time and predictability. This derivation of LGI allows us to conclude that in a situation, when NSIT is satisfied, the violation of LGI imply the presence of certifiable randomness.

1.1 A brief introduction to quantum mechanics

To present preliminary ideas the postulates of quantum mechanics are listed below.

1.1.1 Postulates of Quantum mechanics

The 1st postulates deals with suitable space where quantum phenomena occur at the level of theory

> **P1. State space of system:** *Every quantum mechanical system S, is associated with a separable Hilbert space $\mathcal{H}_S$ over complex field, known as the state space of the system. The dimension of the associated Hilbert space depends on the multiplicity of degree of freedom being considered for the system.*

This association of state space to a particular system is not given by quantum mechanics and rather a different problem of physics. Through some reasonable assumptions a particular Hilbert space is chosen for a particular system of interest. For example if only the spin degree of freedom of a spin–1/2 particle(two level system also called a qubit) is considered, the corresponding Hilbert space is $\mathbb{C}^2$, a two dimensional complex Hilbert space. An arbitrary qubit state can be written as $|\psi\rangle = a|0\rangle + b|1\rangle$, where $|0\rangle$ and $|1\rangle$ are orthonormal basis states for $\mathbb{C}^2$ and $|a|^2 + |b|^2 = 1$. The Hilbert space associated to a simple harmonic oscillator is the infinite dimensional complex separable Hilbert space $\mathcal{L}^2(-\infty, +\infty)$ of all complex valued functions. Each of which is square integrable over the entire real line. The system S is completely described by its density operator ρ which is a positive semidefinite trace class operator acting on the state space $\mathcal{H}_S$ of the system. Collection of all density operators $\mathcal{T}(\mathcal{H}_S)$, acting on the state space $\mathcal{H}_S$, forms a convex compact subset of set of all bounded hermitian operators acting on $\mathcal{H}_S$. The density operators corresponding to the extreme point of the convex set $\mathcal{T}(\mathcal{H}_S)$ are called pure state, otherwise they are called mixed state. Mathematically pure states are characterized as Tr$(\rho^2) = 1$ and the mixed states satisfy Tr$(\rho^2) < 1$. The set of pure density operators are isomorphic to the projective Hilbert space $\mathcal{P}(\mathcal{H}_S)$ and in such case density operators have one-one correspondence with the ray vectors $|\psi\rangle \in \mathcal{H}_S$, as considered in normal text books.

> **P2. Observable:** *Observables, which are measurable quantities like position, momentum, energy, spin are associated with self adjoint operators on the Hilbert space $\mathcal{H}_S$.*

As observables are self adjoint operators, it have real eigenvalues which appear as measurement outcomes. Any such operator A has spectral representation $A = \sum_i a_i P_i$. Where a_is are eigenvalues and P_is are associated projectors.

> **P3. Dynamics:** *The evolution of a closed quantum system is described by a unitary transformation. That is, the state ρ_{t_1} of the system at time t_1 is transformed to the state ρ_{t_2} of the system at later time t_2 by a unitary operator U which depends only on time interval, i.e.,*
>
> $$\rho_{t_1} \to \rho_{t_2} = U(t_1, t_2)\rho_{t_1}U^\dagger(t_1, t_2) \tag{1.1}$$

A more refined version of this postulate can be given which describes the evolution of a quantum system in continuous time. Considering the system is in the pure state $|\psi\rangle$, the time evolution of the state of a closed quantum system can also be described by the well known Schrödinger equation which reads as:

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle, \tag{1.2}$$

where H is a Hermitian operator known as the Hamiltonian of the closed system. Hamiltonian picture of dynamics and unitary operator picture are connected by their relation,

$$U(t_1, t_2) = \exp^{-iH(t_2-t_1)/\hbar}. \tag{1.3}$$

P4. Measurement: *Quantum measurements are described by a collection $\{M_k\}$ of positive operators. These operators acting on the state space of the system being measured. The index k denotes measurement outcomes that may occur in the experiment. The measurement operators satisfy the completeness relation*

$$\sum_k M_k^\dagger M_k = \mathbf{1},$$

where $\mathbf{1}$ denotes the identity operator acting on $\mathcal{H}_S$.

If the state of the quantum system is ρ immediately before the measurement then the probability that result k occurs is given by *generalized* **Born** rule, i.e.,

$$p(k) = \text{Tr}(M_k^\dagger M_k \rho), \tag{1.4}$$

and the state of the system ρ_k, conditioned that the result k is obtained in the measurement, is given by

$$\rho \to \rho_k = \frac{M_k \rho M_k^\dagger}{\text{Tr}(M_k^\dagger M_k \rho)}. \tag{1.5}$$

Evolution of the quantum state after the measurement process can not be described by a continuous unitary dynamics in orthodox interpretation. The state transformed into another state conditioned on the result of measurement outcome. This process is called *measurement induced collapse*.

Projective measurement: A special class of measurement frequently used in quantum theory is projective measurements. A projective measurement is described by an observable, R, a Hermitian operator on the state space of the system being observed. Spectral decomposition of the observable is written as,

$$R = \sum_r r P_r, \tag{1.6}$$

where P_r is the projector onto the eigenspace of R having eigenvalue r and $P_r P_q = \delta_{r,q} P_r$. Projective measurements are repeatable in the sense that if a projective measurement is performed once, and outcome m is obtained then repeating the same measurement gives

the outcome m again not changing the state further.
The average value of the observable for the state $|\psi\rangle$ is $\langle\psi|R|\psi\rangle$. Standard deviation associated to observation of R is $\Delta(R) = \langle R^2\rangle - \langle R\rangle^2$. This formulation of measurement and standard deviation gives rise to *Heisenberg uncertainty principle*, which is discussed later.

Positive operator valued measure: In reality not every measurements are repeatable. A general kind of measurement known as positive operator valued measure or POVM. Suppose a measurement described by measurement operator M_m is performed on a quantum system $|\psi\rangle$. Then the probability of outcome is given by following Born's rule, $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$. Let us define $E_m = M_m^\dagger M_m$. The set of positive operators E_m satisfying normalisation condition $\sum_m M_m = \mathbb{I}$ are known as POVM elements. The corresponding state update rule is given by generalised Lüders transformation

$$\rho \to \frac{M_m \rho M_m^\dagger}{Tr[M_m \rho M_m^\dagger]}. \tag{1.7}$$

Projective measurement is an example of POVM, where POVM elements are projectors satisfying $E_m = P_m^\dagger P_m = P_m$.

The following postulate describes the state space of a composite system consisting of several subsystems.

P5. Composite system: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems S_i, i.e.,

$$\mathcal{H}_{1,2,...,n} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n.$$

If an composite state $\rho_{1,2,...,n} \in \mathcal{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$ can be expressed as $\rho_{1,2,...,n} = \rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$, with $\rho_i \in \mathcal{T}(\mathcal{H}_i)$, then the state is called product state. States which are convex combination of product states are called separable state $\rho^{sep}_{1,2,...,n} = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes ... \otimes \rho_n^i$. Let us denote the collection of all separable states as $\mathcal{S}\text{ep}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n) \subset \mathcal{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$. States belonging in $\mathcal{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$, but not belonging in $\mathcal{S}\text{ep}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$ are called entangled, i.e., $\rho^{ent}_{1,2,...,n} \in \mathcal{T}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$, but $\rho^{ent}_{1,2,...,n} \notin \mathcal{S}\text{ep}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_n)$.

1.1.2 Simplest quantum system: Qubit

Qubits or quantum bits are the simplest quantum system with minimal dimension. They provide a mathematically simple framework in which the basic concepts of quantum physics can be easily understood. Qubits are 2-level quantum system and the Hilbert space associated with a system is $\mathbb{C}^2$. A pure state of an 2-level quantum system is a vector $|\psi\rangle \in \mathbb{C}^2$ which is normalised, i. e., $|\langle\psi|\psi\rangle|^2 = 1$. Thus $|\psi\rangle$ as a unit vector. Since the global phase factor $e^{i\phi}$ $(\phi \in \mathbb{R})$ is insignificant, vectors $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ correspond to the same physical state.

Bloch sphere representation: As discussed above, the global phase is physically irrelevant. Thus without the loss of generality a pure state $|\psi\rangle \in \mathbb{C}^2$ can be expressed as,

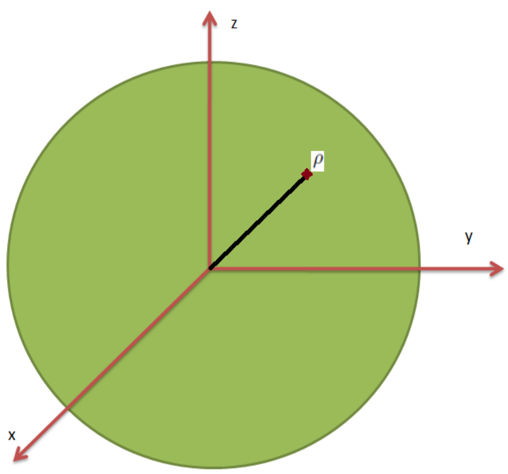

Fig. 1.1: Bloch sphere representation for qubit. The points on the surface of the sphere correspond to pure states and the points inside the surface correspond to mixed states.

$$|\psi\rangle \equiv \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\varphi}\sin(\frac{\theta}{2}) \end{pmatrix},$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. There is a one-to-one correspondence between pure qubit states and the points on a unit sphere $\mathcal{S}^2$ in $\mathcal{R}^3$ (see Fig.1.1). The Bloch vector for state $|\psi\rangle$ is $\hat{n} = (x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin, \cos\theta)$, which lies on the surface of the sphere. The density matrix for the state $|\psi\rangle$ is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

Any density operator ρ can also be written in terms of operator basis $\{\mathbf{1}, \sigma_x, \sigma_y, \sigma_z\}$, as, $\rho = \frac{1}{2}(\mathbf{1} + \hat{n}.\vec{\sigma})$. Here $\sigma_x, \sigma_y, \sigma_z$ are the well known Pauli matrices and $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$.

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From the positivity and trace conditions norm of $\vec{n}$ should be bounded by unity, i.e., $0 \leq |\vec{n}| \leq 1$. For pure states we have $|\vec{n}| = 1$, for the mixed states we have $0 \leq |\vec{n}| < 1$. As for example $|\vec{n}| = 0$ corresponds to the completely mixed state $\mathbf{1}/2$.

1.1.3 Three level quantum system: Qutrit

The structure of the state space of the generalised Bloch sphere (Ω_d), is much richer for $d \geq 3$ [14, 15]. Qutrit states can be expressed in terms of Gellmann matrices that are familiar generators of the unimodular unitary group $SU(3)$ in its defining representation with eight Hermitian, traceless and orthogonal matrices $\lambda_j, j = 1, ..., 8$ satisfying $tr(\lambda_k \lambda_l) = 2\delta_{kl}$, and $\lambda_j \lambda_k = (2/3)\delta_{jk} + d_{jkl}\lambda_l + i f_{jkl}\lambda_l$. The expansion coefficients f_{jkl}, the structure constants of the Lie algebra of $SU(3)$, are totally anti-symmetric, while d_{jkl} are totally symmetric. Explicitly d_{jkl} are

$$\begin{aligned} d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}, d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \\ d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}. \end{aligned} \tag{1.8}$$

Single-qutrit states can be expressed as

$$\rho(\vec{n}) = \frac{I + \sqrt{3}\vec{n}.\vec{\lambda}}{3}, \vec{n} \in R^{8.} \tag{1.9}$$

Eight Gellmann matrices are the following.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The set of all extremals (pure states) of Ω_3 constitute also CP^2, and can be written as $\Omega_3^{ext} = CP^2 = \{\vec{n} \in R^8 | \vec{n}.\vec{n} = 1, \vec{n} * \vec{n} = \vec{n}\}$, with $\vec{n} * \vec{n} = \sqrt{3} d_{jkl} n_k n_l \hat{e_j}$. Here $\hat{e_j}$ is the unit vector belongs to R^8. Non-negativity of ρ demands that $\vec{n}$ should satisfy the additional inequality $|\vec{n}|^2 \leqslant 1$. The boundary $\partial\Omega_3$ of Ω_3 is characterised by $\partial\Omega_3 = \{\vec{n} \in R^8 | 3\vec{n}.\vec{n} - 2\vec{n} * \vec{n}.\vec{n} = 1, \vec{n}.\vec{n} \leqslant 1\}$, and the state space Ω_3 is given by $\Omega_3 = \{\vec{n} \in R^8 | 3\vec{n}.\vec{n} - 2\vec{n} * \vec{n}.\vec{n} \leqslant 1, \vec{n}.\vec{n} \leqslant 1\}$. For two-level systems the whole boundary of the state space represents pure states, i.e., $\Omega_2^{ext} = \partial\Omega_2$, while for three-level systems $\Omega_3^{ext} \subset \partial\Omega_3$.

1.1.4 Multilevel quantum system: Qudit

State of a qudit system is represented by a density operator in the Hilbert-Schmidt space acting on the d-dimensional Hilbert space $\mathcal{H}_d$ that can be written as a matrix called density matrix in the standard basis $\{|k\rangle\}$ with $k = 0, 1, 2, ..., d-1$. For practical purpose Bloch vector decomposition of qudit is expressed in a convenient basis system including identity matrix and $d^2 - 1$ traceless matrices $\{\Gamma_i\}$

$$\rho = \frac{\mathbf{1}}{d} + \vec{b}.\vec{\Gamma}. \tag{1.10}$$

Where Γs are the higher dimension extension of Pauli matrices (for qubits) and Gellmann matrices (for qutrits) and are called generalised Gellmann matrices(GGM) which are standard $SU(N)$ generators. There are $d^2 - 1$ Hermitian, traceless, orthogonal GGM and defined as three different types of matrices. In operator notation they have the following form

(i)$d(d-1)/2$ symmetric GGM

$$\Lambda_s^{jk} = |j\rangle\langle k| + |k\rangle\langle j|, \quad 1 \le j < k \le d; \tag{1.11}$$

(ii)$d(d-1)/2$ antisymmetric GGM

$$\Lambda_a^{jk} = -|j\rangle\langle k| + |k\rangle\langle j|, \quad 1 \le j < k \le d; \tag{1.12}$$

(iii)(d-1) diagonal GGM

$$\Lambda^l = \sqrt{\frac{2}{l(l+1)}}(\sum_{j=1}^{l} |j\rangle\langle j| - l|l+1\rangle\langle l+1|), \quad i \leq l \leq d-1. \tag{1.13}$$

Qubit observables: Two outcome projective measurement performed of a qubit system is represented by the Hermitian operator $\hat{m}.\vec{\sigma}$ with outcomes denoted by ± 1. The eigenstates corresponding to eigenvalues ± 1 are $\frac{1}{2}(\mathbf{1} \pm \hat{m}.\vec{\sigma})$. Generally the eigenstates of σ_z observable are denoted as $|0\rangle$ and $|1\rangle$, which form an orthonormal basis for the Hilbert space $\mathbb{C}^2$. Projectors corresponding to the outcome 1 and -1 are respectively $|0\rangle\langle 0| = \frac{1}{2}(\mathbf{1} + \sigma_z)$ and $|1\rangle\langle 1| = \frac{1}{2}(\mathbf{1} - \sigma_z)$. The eigenstates of σ_x observable are $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and that of σ_y are $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. If the measurement $\hat{m}.\vec{\sigma}$ is performed on a qubit prepared in the state $\rho_{\vec{n}}$, the probability $p(\pm|\rho_{\vec{n}}, \hat{m})$ of obtaining the the outcome $\pm$ turns out to be

$$p(\pm|\rho_{\vec{n}}, \hat{m}) = \text{Tr}\left(\rho_{\vec{n}}\frac{1}{2}(\mathbf{1} \pm \hat{m}.\vec{\sigma})\right) = \frac{1}{2}(1 \pm \vec{n}.\hat{m}). \tag{1.14}$$

Qubit POVM: Any linear operator acting on $\mathbf{C}^2$ can be written in terms of identity matrix and Pauli matrices. The most general form of two outcome POVM are given by qubit effect operators. These effect operators are characterised by two parameters and given by

$$\begin{aligned} E^{+} &= \frac{1}{2}[(1+\gamma)\mathbf{1} + \lambda\hat{n}.\sigma] \\ E^{-} &= \frac{1}{2}[(1-\gamma)\mathbf{1} - \lambda\hat{n}.\sigma] \end{aligned} \tag{1.15}$$

λ is known as sharpness parameter and γ called biasedness of measurement. Positivity and normalisation conditions of POVM elements demands $|\gamma| + |\lambda| \leq 1$. These effect operators reduce to projectors in the limit of $\lambda = 1$ and $\gamma = 0$, i.e., unbiased sharp effects.

1.1.5 Composite system

Let us now discuss on composite quantum system. We consider here only bipartite quantum states.

Two qubit: Assume that we have two quantum systems each of which are qubit system. According to the postulate of *composite system* (postulate **P5**) the Hilbert space associated with two qubit system is $\mathbb{C}^2 \otimes \mathbb{C}^2$. Suppose eigenstates of σ_z are $|0_i\rangle$ and $|1_i\rangle$, which form an orthonormal basis for the i^{th} system ($i = 1, 2$), the set of composite states $\{|0_1\rangle \otimes |0_2\rangle, |0_1\rangle \otimes |1_2\rangle, |1_1\rangle \otimes |0_2\rangle, |1_1\rangle \otimes |1_2\rangle\}$ form an orthonormal basis for the composite Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Tensor product between two arbitrary states $|\phi_1\rangle \equiv (a_1, b_1)^{\mathcal{T}}$ of first system and $|\phi_2\rangle \equiv (a_2, b_2)^{\mathcal{T}}$ of second system is defined as (here the superscript $\mathcal{T}$ denotes transposition):

$$|\phi_1\rangle \otimes |\phi_2\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \equiv \begin{pmatrix} a_1a_2 \\ a_1b_2 \\ b_1a_2 \\ b_1b_2 \end{pmatrix}.$$

For economy of symbols we will denote $|\phi_1\rangle \otimes |\phi_2\rangle$ as $|\phi_1\phi_2\rangle$. Any composite state which can be expressed as tensor product of pure states of the corresponding sub systems is called pure product state. However, there are pure state which can not be written as tensor product of pure states of two sub systems. Such states are called entangled states. Example of 2-qubit entangled states are the well known Bell states $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, where $|\psi^{-}\rangle$ is called singlet states and rest three are called triplet states. These are maximally entangled states in $2 \otimes 2$ dimension also.

Generic form of any two qubit state: Quantum systems can be mixture of pure states also. Then an arbitrary state of the $\mathbb{C}^2 \otimes \mathbb{C}^2$ system can be represented as:

$$\rho_{12} = \frac{1}{4}\left(\mathbf{1} \otimes \mathbf{1} + \vec{r}.\vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{s}.\vec{\sigma} + \sum_{n,m=1}^{3} t_{nm}\sigma_n \otimes \sigma_m\right), \tag{1.16}$$

where $\vec{r}, \vec{s} \in \mathbb{R}^3$, with $0 \leq |\vec{r}|, |\vec{s}| \leq 1$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$ and all other notations having usual meaning. The coefficients $t_{nm} = \text{Tr}(\rho_{12}\sigma_n \otimes \sigma_m)$ form a real matrix denoted by T called correlation matrix. Vectors $\vec{r}$ and $\vec{s}$ are local parameters and they determine density operator of the subsystems and given by,

$$\rho_1 \equiv \text{Tr}_2\rho_{12} = \frac{1}{2}(\mathbf{1} + \vec{r}.\vec{\sigma}), \quad \rho_2 \equiv \text{Tr}_1\rho_{12} = \frac{1}{2}(\mathbf{1} + \vec{s}.\vec{\sigma}). \tag{1.17}$$

Here Tr_i denotes partial trace over the i^{th} sub system. If an density matrix can be expressed as convex combination of pure product states, i.e., $\rho_{12} = \sum_j p_j \rho_1^j \otimes \rho_2^j$, with $\{p_j\}$ being a probability distribution, then the state is called a separable state. States which are not separable are called entangled. Entanglement of a 2-qubit state is determined by Peres-Horodecki *positive partial transposition* (PPT) criteria [16, 17]. Let us denote partial transposition of the state ρ_{12} as $\rho_{12}^{T_i}$ (here transposition is taken on i^{th} system). If

$\rho_{12}^{T_i}$ is a positive operator then ρ_{12} is called a PPT state, otherwise it a negative-PT (NPT) state. A 2-qubit state is entangled if and only if it is NPT. For 2×2 and 2×3 system PPT criterion is a necessary and sufficient condition for an composite density operator to be separable. However for higher dimensional system this is only a necessary condition. In higher dimensional system there exists entangled state which are PPT. Considering the unextendible product basis (UPB) one can easily construct such PPT entangled states [18].

Schmidt decomposition and state of $d \otimes d$ system: Schmidt decomposition provides an useful representation of the pure states of any bipartite quantum systems, i.e., systems which are composed of two sub systems. A bipartite pure state $|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim(\mathcal{H}_1) = d_1$ and $\dim(\mathcal{H}_2) = d_2 \geq d_1$, with Schmidt rank r is written as $|\psi_{12}\rangle = \sum_{j=1}^{r} \alpha_j |e_1^j\rangle \otimes |f_2^j\rangle$, where $r \leq d_1$, $\sum_{j=1}^{r} \alpha_j^2 = 1$, $\alpha_j > 0 \; \forall \; j$, $\{|e_1^j\rangle\}_{j=1}^{r}$ is an orthonormal set of vectors in $\mathcal{H}_1$ and $\{|f_1^j\rangle\}_{j=1}^{r}$ is an orthonormal set of vectors in $\mathcal{H}_2$. Number of non vanishing terms in mixed decomposition is known as Schmidt rank.

1.2 Uncertainty principle and relations:

Now we discuss uncertainty principle, which is the very first principle known about quantum theory and different formulations of uncertainty relation. It prohibits certain properties of quantum systems from being simultaneously well-defined. Originally Heisenberg[1] proposed uncertainty principle by demonstrating no precise measurement of two conjugate variables position and momentum simultaneously. A generalised form of uncertainty relation was proposed by Robertson[19] and Schrödinger[20] and since then, several other versions of the uncertainty relations have been suggested. The consideration of state-independence has lead to the formulation of entropic versions of the uncertainty relation [21]. We first demonstrate derivation of generalised uncertainty relation due to Robertson-Schrödinger and then entropic uncertainty relation.

1.2.1 Derivation of generalised uncertainty relation

Let us assume an ensemble of identical noninteracting quantum system, each in state $|\psi\rangle$. Derivation for mixed mixed state is straight forward application of this. On half of ensemble observable A is measured and on another half B is measured. with $(\Delta A)^2$ and $(\Delta B)^2$ representing the variances of the observables, A and B, respectively, given by $(\Delta A)^2 = (\langle A^2 \rangle) - (\langle A \rangle)^2$, $(\Delta B)^2 = (\langle B^2 \rangle) - (\langle B \rangle)^2$, and the square (curly) brackets representing the standard commutators (anti-commutators) of the corresponding variables.

Suppose, $[A,B]=iC$ and $\alpha=A-\langle A\rangle$, $\beta=B-\langle B\rangle$.
With this choice it one can find that $[\alpha,\beta]=iC$, $(\Delta\alpha)^2=(\Delta A)^2=\langle\alpha^2\rangle$ and $(\Delta\beta)^2=(\Delta B)^2=\langle\beta^2\rangle$.
In this scenario we have to find lower bound of $(\Delta A)^2(\Delta B)^2=\langle\psi|\alpha^2|\psi\rangle\langle\psi|\beta^2|\psi\rangle$.
Now for vectors $|\phi\rangle$ and $|\chi\rangle$, Schwartz inequality is given by

$$|\langle\phi|\chi\rangle|^2\leq\langle\phi|\phi\rangle\langle\chi|\chi\rangle. \tag{1.18}$$

Equality sign holds iff $\phi=c\chi$, where c is a constant. Now put $|\chi\rangle=\beta|\psi\rangle$ and $|\phi\rangle=\alpha|\psi\rangle$. Then

$$\langle\psi|\alpha^2|\psi\rangle\langle\psi|\beta^2|\psi\rangle\geq|\langle\psi|\alpha\beta|\psi\rangle|^2. \tag{1.19}$$

Now

$$\alpha\beta=\frac{\alpha\beta+\beta\alpha}{2}+\frac{\alpha\beta-\beta\alpha}{2}=\frac{\alpha\beta+\beta\alpha}{2}+\frac{i}{2}C. \tag{1.20}$$

Hence,

$$(\Delta A)^2(\Delta B)^2\geq\frac{1}{4}|\langle\alpha\beta+\beta\alpha\rangle+iC|^2. \tag{1.21}$$

After some algebra this becomes

$$(\Delta A)^2(\Delta B)^2\geq\frac{1}{4}|\langle\{A,B\}\rangle-2\langle A\rangle\langle B\rangle|^2+\frac{1}{4}|\langle[A,B]\rangle|^2. \tag{1.22}$$

This is Robertson-Schrödinger uncertainty relation which we call generalised uncertainty relation (GUR) in the subsequent text.

1.2.2 Entropic uncertainty relation:

In information theoretic purpose the uncertainty is measured by Shannon entropy of the probability distribution of measurement outcome. For a probability distribution $\{p_i\}$, Shannon entropy is given by

$$\mathcal{H}=-\sum_i p_i\log p_i. \tag{1.23}$$

The Shannon information entropy later has been generalized by Renyi[22]. The Renyi entropy is a one-parameter family of entropic measures that share with the Shannon entropy many important properties. It is defined as

$$\mathcal{H}_{\alpha} = \frac{1}{1-\alpha}\log[\sum_k p_k^{\alpha}]. \tag{1.24}$$

p_k is a set of probability distribution and α is positive number. In the limit of $\alpha \to 1$, Renyi entropy becomes Shannon entropy. The entropic uncertainty relation for two measurement was, first, introduced by Deutsch [23]. For two probability distribution $\{p_i\}$ and $\{q_j\}$, it is given by

$$\mathcal{H}^{(A)} + \mathcal{H}^{(B)} \geq -2\log[\frac{1+C}{2}]. \tag{1.25}$$

Here, $C = max_{i,j}\langle a_i|b_j\rangle$ and $|a_i\rangle, |b_j\rangle$ are eigenstate of A and b respectively.
This inequality was improved in the version conjectured in Ref.[24] and then proved in Ref.[25]. The form of improved entropic uncertainty relation for the measurement of two observables (R and S) on a quantum system, A (in the state ρ_A) is given by

$$\mathcal{H}_{\rho_A}(R) + \mathcal{H}_{\rho_A}(S) \geq \log_2\frac{1}{c}, \tag{1.26}$$

where, $\mathcal{H}_{\rho_A}(\alpha)$ is the Shannon entropy of the probability distribution of measurement outcome of observable α ($\in \{R, S\}$) on the quantum system (A) and $\frac{1}{c}$ quantifies the complementarity of the observables. Eq.1.26 is known as Maassen-Uffink inequality. We sketch here a brief derivation of this inequality. For more one can see[26, 27]

Derivation of Maassen-Uffink inequality: We present here a brief derivation following Ref.[26]. Every uncertainty relation is based on some mathematical theorem. In the case of the Maassen-Uffink relation this role is played by the Riesz theorem which states that for every N-dimensional complex vector X and a unitary transformation matrix $\hat{T}$ with coefficients t_{ji}, the following inequality between the norms holds

$$c^{1/\mu} \parallel X \parallel^{\mu} \leq c^{1/\nu} \parallel \hat{T}X \parallel^{\nu}. \tag{1.27}$$

With constant $c = sup_{i,j}|t_{ji}|$ and μ, ν obey the relation

$$\frac{1}{\mu} + \frac{1}{\nu} = 2. \tag{1.28}$$

Where, $1 \leq \nu \leq 2$ and norms are defined as $\| X \| = [\sum_k |x_k|^{\mu}]^{1/\mu}$.
Now take $x_i = \langle a_i|\psi\rangle, t_{ji} = \langle b_j|a_i\rangle$ so that

$$\sum_{i=1}^{N} t_{ji}x_i = \langle b_j|\psi\rangle. \tag{1.29}$$

Suppose $q_j = \langle a_j|\psi\rangle, p_i = \langle b_i|\psi\rangle$, then above theorem gives

$$c^{1/\mu}[\sum_j q_j^{\mu/2}]^{1/\mu} \leq c^{1/\nu}[\sum_i p_i^{\nu/2}]^{1/\nu}. \tag{1.30}$$

Now take $\mu = 2\alpha, \nu = 2\beta$. Using these parameters and taking logarithm of both side of above inequality we obtained uncertainty relation for Renyi entropy

$$\mathcal{H}_{\alpha}^{A} + \mathcal{H}_{\beta}^{B} \geq -2\log c. \tag{1.31}$$

In the limit $\alpha \to 1, \beta \to 1$ this yields Maassen-Uffink uncertainty relation.

1.3 Correlations and no-go theorems

Natural events occur in the background of space-time. Measurement outcomes obtained from spatially separated systems give rise to spatial correlation. Issue of quantum non-locality is associated with spatial correlation. On the other hand measurements done on a single system at different times give rise to temporal correlation. Measurement done on a single system with time ordering is also known as sequential measurement. Again sequential measurements can be commutative or non-commutative. First kind of temporal correlation associated with contextuality of quantum theory whereas second kind of temporal correlation considered in the context of macro-realism. Quantum correlations are incompatible with classical theory. For different kind of correlations there are different no-go theorems which reflects the incompatibility between quantum and classical description of nature.

1.3.1 No local realist model for spatial correlation

Einstein, Podolsky and Rosen(EPR) in their famous 1935 paper [28], used a peculiar feature of quantum entanglement to establish the incompleteness of quantum mechanics. EPR have shown that quantum theory does not satisfy a necessary condition of completeness for any physical theory. Nearly thirty years after EPR work, John Bell, in 1966, provided an empirically testable criterion which is always satisfied by a *local realistic* theory [2, 29]. Surprisingly, quantum correlation violates this criterion and results to one of the most counterintuitive conclusion that quantum theory is not compatible with *local realism*. This is famously known as Bell's no-go theorem.

A. EPR paradox and Bell's no-go theorem

Quantum theory is probabilistic by nature. This probability is not due to subjective ignorance about the pre-assigned value of a dynamical variable, rather it is objective in nature. On the other hand, according to Copenhagen interpretation, quantum system is completely described by its wave function. This intrinsic probabilistic nature of quantum theory was not accepted by Einstein. He believed that the fundamental theory of nature should be *deterministic* in nature. In [28], they designed an *gedanken* experiment to establish the incompleteness of wave function as the description of physical systems. Their argument is based on the following assumptions:

Necessary condition for completeness: A necessary condition for the completeness of any physical theory is that "every element of the physical reality must have a counterpart in the physical theory".

Sufficient condition for reality: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity".

Locality principle: "Elements of reality belonging to one system can not be affected (instantaneously) by measurements performed on another system which is spatially separated from the former".

EPR originally considered predictions from measurements of position and momentum on quantum systems for formulating their argument. Later D. Bohm formulated this argument for two qubit system [30].

Suppose two observers, Alice and Bob, interacted in the past and then perform measurements on their respective spin-1/2 particles. Let the observers share singlet state:

$$|\psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \otimes |1_B\rangle - |1_A\rangle \otimes |0_B\rangle). \tag{1.32}$$

An interesting property of this state is that it is invariant under the same rotations of observables in the two labs, i.e., the state is symmetric under $U \otimes U$, where U is any arbitrary unitary operator. For instance, in x-basis (eigen states of spin observable σ_x) it takes the same form:

$$|\psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|+_A\rangle \otimes |-_B\rangle - |-_A\rangle \otimes |+_B\rangle), \tag{1.33}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Measurement outcomes of Alice and Bob are perfectly anti-correlated. If Alice measures σ_z then she can predict with certainty the outcome of Bob's σ_z measurement. Thus, according to EPR-assumptions there exists an element of physical reality associated with the σ_z measurement. Similarly Alice could also measure σ_x and predict with certainty, without in any way perturbing the system, the outcome of a possible σ_x measurement by Bob. Again, seemingly there exists an element of reality associated with the σ_x measurement. Locality is assumed here by considering that the physical reality at Bob's site is independent of anything that occurs at Alice's site. Since due to uncertainty relation, quantum mechanics does not allow simultaneous knowledge of both σ_z and σ_x, it lacks some concepts which are necessary for the theory to be complete.

Consequently EPR paper naturally raised the question whether a complete theory can be constructed (at least in principle) underlying quantum mechanics. Bell motivated by the work of Bohm [31, 32] considered whether there is possibility of any completion of quantum theory. For quantum systems composed of more than one spatially separated subsystems, Bell investigated whether any local realistic theory can reproduce all the statistical results of such systems? He succeeded to provide certain constraint (in form of inequalities) which is satisfied by all local realist theories [2] and famously known as Bell's inequality.

Consider a joint system consisting of two subsystems shared between Alice and Bob. Alice performs measurements, randomly chosen from $\{A_1, A_2\}$, on her subsystem while Bob chooses his measurement from the set $\{B_1, B_2\}$. Let the corresponding measurement results are $a, b \in \{+1, -1\}$. Let $\lambda \in \Lambda$ is local-realistic complete state associated with this joint system distributed according to a distribution $p(\lambda) : \ p(\lambda) \geq 0 \ \forall \ \lambda$ and $\int_{\lambda\in\Lambda} p(\lambda) = 1$. For this state, values of every observables are definite locally, i.e., the measurement results of each of the distant (space-like separated) observers (here Alice and Bob) are independent of the choice of observable of the other observer. This assumption reflects the locality condition inherent in the arguments of EPR. For ontic state $\lambda \in \Lambda$ expectation value of the joint observables $\langle A_i B_j \rangle_\lambda$ ($i, j \in \{1, 2\}$) is calculated as:

$$\langle A_i B_j \rangle_\lambda = \sum_{a,b \in \{+1,-1\}} abp(a, b|A_i, B_j, \lambda),$$

where $p(a, b|A_i, B_j, \lambda)$ denotes the probability of obtaining outcome 'a' and 'b' by Alice and Bob for measurements A_i and B_j performed by them respectively. Due to realistic nature of the theory, $\langle A_i B_j \rangle_\lambda \in \{+1, -1\}$. Consider now the expression $\mathcal{B}_{CHSH}$ defined as,

$$\mathcal{B}_{CHSH} = \langle A_1 B_1 \rangle_\lambda + \langle A_1 B_2 \rangle_\lambda + \langle A_2 B_1 \rangle_\lambda - \langle A_2 B_2 \rangle_\lambda.$$

It is straight forward to see that for any fixed $\lambda \in \Lambda$, $\mathcal{B}_{CHSH} = \pm 2$, which in turns implies that the average of $\langle \mathcal{B}_{CHSH} \rangle$ over some distribution $p(\lambda)$ of hidden variables is

$$-2 \leq \langle \mathcal{B}_{CHSH} \rangle = \int_{\lambda \in \Lambda} d\lambda p(\lambda) \mathcal{B}_{CHSH} \leq 2.$$

Thus we obtain the following Bell-CHSH inequality in terms of experimentally observable correlation functions $\langle A_i B_j \rangle$,

$$|\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2. \tag{1.34}$$

It is observed that correlations of entangled quantum particles violates this inequality which implies that quantum theory is not compatible with local realistic framework. For more on this issue see [33].

B. Quantum theory violates Bell's inequality

Consider Alice and Bob share an EPR pair of Eq.(1.32) and can only operate locally on their respective subsystem in two distant laboratories. If Alice and Bob perform spin measurements along $\hat{m}_A$ and $\hat{n}_B$ direction respectively, then it can be shown that the expectation value of the local joint observable becomes:

$$\langle \psi_{AB}^- | \hat{m}_A . \vec{\sigma} \otimes \hat{n}_B . \vec{\sigma} | \psi_{AB}^- \rangle = -\hat{m}_A . \hat{n}_B. \tag{1.35}$$

Let us now choose $A_1 = \frac{\sigma_z + \sigma_x}{\sqrt{2}}$, $A_2 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$, $B_1 = \sigma_z$, and $B_2 = \sigma_x$. Using Eq.(1.35), the value for the left hand side of Eq.(1.34) turns out to be

$$|\langle A_1 B_1 \rangle_{\psi_{AB}^-} + \langle A_1 B_2 \rangle_{\psi_{AB}^-} + \langle A_2 B_1 \rangle_{\psi_{AB}^-} - \langle A_2 B_2 \rangle_{\psi_{AB}^-}| = 2\sqrt{2}. \tag{1.36}$$

Hence, we see violation of Bell-CHSH inequality in quantum mechanics. The experimental tests performed so far show this violation upto some loopholes. These technical loopholes are gradually being closed and are now believed not to have any fundamental impact on confirmation of Bell's inequality violation. Therefore, contrary to the intuition envisaged by EPR, there can be no underlying local-realistic hidden variable description for correlations from which quantum mechanical predictions can be always derived.

Cirel'son bound : It is demonstrated that quantum correlations violate Bell's inequality (1.34). The maximum algebraic value of the left hand side if (1.34) is 4. Now the question is what is the maximum value obtained by spatial quantum correlation? B.S. Cirel'son showed that the maximum quantum violation of the Bell-CHSH inequality is limited to $2\sqrt{2}$, which is known as Cirel'son's bound [34]. In the following we sketch Cirel'son's proof. The Bell operator corresponding to Bell-CHSH expression can be written as

$$\mathbb{B}_{CHSH} := \mathbb{A}_1 \otimes \mathbb{B}_1 + \mathbb{A}_1 \otimes \mathbb{B}_2 + \mathbb{A}_2 \otimes \mathbb{B}_1 - \mathbb{A}_2 \otimes \mathbb{B}_2. \tag{1.37}$$

For any pure quantum state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ shared between Alice and Bob the value for the Bell-CHSH expression can be calculated as$\langle\psi_{AB}|\mathbb{B}_{CHSH}|\psi_{AB}\rangle$. Consideration of only pure states is sufficient here as mixed states being statistical mixture of pure states must also satisfy the derived upper bound. Actually it is only needed to derive a bound for sup-norm $||.||_{sup}$ of the Bell-CHSH operator and the result easily follows (the sup-norm of a bounded linear operator $\mathbb{O}$ is defined as $||\mathbb{O}||_{sup} = Sup_{|\psi\rangle}\frac{||\mathbb{O}|\psi\rangle||}{|||\psi\rangle||}$). According to quantum mechanics, Alice and Bob's dichotomic observables producing outcomes $\{+1,-1\}$ must obey following relations:

$$\mathbb{A}_1^2 = \mathbb{A}_2^2 = \mathbb{B}_1^2 = \mathbb{B}_2^2 = \mathbf{1},\ [\mathbb{A}_1, \mathbb{B}_1] = [\mathbb{A}_1, \mathbb{B}_2] = [\mathbb{A}_2, \mathbb{B}_1] = [\mathbb{A}_2, \mathbb{B}_2] = \ \ 0. \tag{1.38}$$

where $[\mathbb{A}_i, \mathbb{B}_j] = \mathbb{A}_i\mathbb{B}_j - \mathbb{A}_j\mathbb{B}_i$ are commutators of Alice and Bob's observables. Under these conditions one can find an identity

$$\mathbb{B}_{CHSH}^2 = 4\mathbf{1} + [\mathbb{A}_1, \mathbb{A}_2][\mathbb{B}_1, \mathbb{B}_2].$$

Also, the following inequality holds for two bounded hermitian operators $\mathbb{T}_1$ and $\mathbb{T}_2$

$$||[\mathbb{T}_1, \mathbb{T}_2]||_{sup} \leq 2||\mathbb{T}_1||_{sup}||\mathbb{T}_2||_{sup}.$$

Then on applying this inequality we get

$$||\mathbb{B}^2_{CHSH}||_{sup} \leq 8 \Rightarrow ||\mathbb{B}_{CHSH}||_{sup} \leq 2\sqrt{2} \Rightarrow \langle \mathbb{B}_{CHSH} \rangle_{|\psi_{AB}\rangle} \leq 2\sqrt{2} \text{ for any state } |\psi_{AB}\rangle.$$

We also seen that the Cirel'son's bound can be achieved within quantum mechanics.
Next we discuss another class of no-go theorem due to Kochen-Specker [3]. Although the present thesis does not deal with contextuality, for the sake of self consistency and some motivation for the later part we discuss it very shortly.

1.3.2 Temporal correlation and contextuality

In the preceding discussion on Bell's theorem, it was shown that for nonfactorable state i.e., entangled state it is possible to find pairs of observables whose correlations violate Bell's inequality. Bells theorem strongly constraints the interpretation of measurements as revealing preexisting properties of physical systems. A natural question is whether such a behaviour of quantum correlations appears also in more general measurement scenario, where measurements are not necessarily performed on separated systems.
In quantum mechanics commuting or compatible observables can be jointly measured and their measurement statistics can be described by classical probability theory. Moreover commuting measurements can be performed in sequence of any order and repeated many times, and the outcomes of each measurement are confirmed by the subsequent ones. This phenomenon suggests the idea that compatible measurements do not disturb each other and that each measurement apparatus should behave in the same way, independently of which other compatible measurements are performed together. From Bell's no-go theorem, we already know that despite such properties a description in terms of noncontextual hidden variable is, in general, impossible (if measurement done at one site defines the context of measurement done on other site, then 'no local realist model' for quantum theory can be described as a special case of 'no noncontextual hidden variable theory' for that). However, such an approach allows to investigate new phenomena arising from single systems, with potential new applications[35, 36].
From the assumptions of *realism* (Observables represent well defined properties of the system, which are just revealed by the measurement process) and *noncontextuality* (The value of an observable is independent of the measurement context, compatible measurements cannot be in a relation of causal influence), the following inequality can be derived [37]. There is also free will assumption i.e., experimenter is able to choose measurement

settings freely.

$$\langle A_0 A_1\rangle + \langle A_1 A_2\rangle + \langle A_2 A_3\rangle + \langle A_3 A_4\rangle + \langle A_4 A_0\rangle \geq -3 \tag{1.39}$$

where A_is are dichotomic measurements. Noncontextual hidden variable (NCHV) i.e., classical model does not violates the bound of the above inequality. Whereas by performing measurements on three level system it is found that the inequality can be violated. For system $|\psi\rangle = (1, 0, 0)$ and measurement settings $A_i = 2|v_i\rangle\langle v_i| - \mathfrak{I}$ with $|v_i\rangle = (\cos\theta, \sin\theta\cos(i4\pi/5), \sin\theta\sin(i4\pi/5)), \cos^2\theta = \frac{\cos\pi/5}{1+\cos\pi/5}$, the above inequality becomes -3.94.

Kochen and Speckers original approach[3] focused on a more strict notion of NCHV, i.e., state independent cotextuality. More precisely, it focused on reproducing also the state-independent predictions of QM, namely, those given by functional relations between commuting quantum observables. For more details on this topic one can see[8, 38, 39].

1.3.3 MACRO-REALISM

Another class of no-go theorem is introduced by Leggett and Garg [4]. This asserts that quantum mechanics is incompatible with macro-realist hidden variable theory. The notion of macrorealism is characterized by the following assumptions -

Macroscopic realism per se: At any given instant, a macroscopic object is in a definite one of the states available to it.

Non-invasive measurability: It is possible, in principle, to determine which of the states the system is in, without affecting the state itself or the system's subsequent behaviour.

There is an another assumption implicit in this context is that measurement result at a time would not be affected by past or future measurements.

A. Derivation of LGI

We begin with a short derivation of LGI following the ontological framework discussed in[40, 41]. In this framework any Heisenberg picture operator in quantum mechanics can be written as an average over a set of hidden variables λ. The role of the initial state is to provide a probability distribution on the set of hidden variables, which we denote as $\rho(\lambda)$, called the ontic state. The average of an observable can be written as

$$< \hat{A}(t) >= \int d\lambda A(\lambda, t)\rho(\lambda), \tag{1.40}$$

where $A(\lambda, t)$ is the value taken by the observable on the hidden variable λ. The correlation between two observables is given by

$$< \hat{B}(t_2)\hat{A}(t_1) >= \int d\lambda B(\lambda, t_2)A(\lambda, t_1)\rho(\lambda|A, t_1). \tag{1.41}$$

Non-invasive measurability (NIM) can be defined as $\rho(\lambda|A, t_1, B, t_2...) = \rho(\lambda)$, i.e., a measurement performed does not change the distribution of λ (like the locality condition in Bell's theorem). Let us take A, B as observables measured on a single system at different times denoted by $Q(t_1), Q(t_2)$. Now, following similar steps as in the derivation of the Bell inequality, one obtains

$$\begin{aligned} < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) >= & \int d\lambda[Q(\lambda, t_2)Q(\lambda, t_1) - Q(\lambda, t_4)Q(\lambda, t_1)]\rho(\lambda|Q, t_1) \\ = & \int d\lambda Q(\lambda, t_2)Q(\lambda, t_1)[1 \pm Q(\lambda, t_4)Q(\lambda, t_3)]\rho(\lambda|Q, t_1) \\ - & \int d\lambda Q(\lambda, t_4)Q(\lambda, t_1)[1 \pm Q(\lambda, t_3)Q(\lambda, t_2)]\rho(\lambda|Q, t_1) \end{aligned} \tag{1.42}$$

Now,

$$\begin{aligned} | < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) > | \leq 2 \pm [& \int d\lambda Q(\lambda, t_4)Q(\lambda, t_3)\rho(\lambda|Q, t_1) \\ + & \int d\lambda Q(\lambda, t_3)Q(\lambda, t_2)\rho(\lambda|Q, t_1)]. \end{aligned} \tag{1.43}$$

Invoking NIM, we have,

$$| < \hat{Q}(t_2)\hat{Q}(t_1) > - < \hat{Q}(t_4)\hat{Q}(t_1) > | \mp [< \hat{Q}(t_3)\hat{Q}(t_2) > + < \hat{Q}(t_4)\hat{Q}(t_3) >] \leq 2. \tag{1.44}$$

This is four term Leggett-Garg inequality.

B. Quantum theory violates macro-realism

In an actual experiment, $Q(t)$, a dichotomic observable measured at time t, is found to take a value $+1(-1)$ depending on whether the system is in the state $1(2)$. We consider series of measurements with the same initial conditions such that in the first series Q is measured at times t_1 and t_2, in the second at t_2 and t_3, in the third at t_3 and t_4, and in the fourth at t_1 and t_4 (here $t_1 < t_2 < t_3 < t_4$). From such measurements one obtains the temporal correlations $C_{ij} = \langle Q_i Q_j \rangle = p^{++}(Q_i, Q_j) - p^{-+}(Q_i, Q_j) - p^{+-}(Q_i, Q_j) + p^{--}(Q_i, Q_j)$, where $p^{++}(Q_i, Q_j)$ is the joint probability of getting '+' outcomes at both times t_i and t_j. Experimentally, these joint probabilities are determined from the Bayes' rule $p^{++}(Q_i, Q_j) = p^{+}(Q_i)p^{+|+}(Q_j|Q_i)$, where $p^{+|+}(Q_j|Q_i)$ is the conditional probability of getting '+' outcome at t_j given that '+' outcome occurs at t_i.

Let us now briefly describe how quantum violation of the LGI was obtained in[42]. Consider precession of a spin $1/2$ particle under the unitary evolution $U_t = e^{-i\omega t\sigma_x/2}$, where ω is the angular precession frequency. Measurement of σ_z at times t_1 and t_2 yields the temporal correlation $C_{12} = \cos\omega(t_2 - t_1)$. Here the state transformation rule is given by $\rho \to P_\pm \rho P_\pm / Tr[P_\pm \rho P_\pm]$. Choosing equidistant measurement times with time difference $\Delta t = t_2 - t_1 = \pi/4\omega$, the maximum value taken by the l.h.s of Eq.(1.44) is given by $2\sqrt{2}$. For a spin j system with a maximally mixed initial state $\frac{1}{2j+1}\sum_{m=-j}^{m=+j}|m\rangle\langle m|$, evolving unitarily under $U_t = e^{-i\omega t \hat{J}_x}$, measurement of the dichotomic parity operator $\sum_{m=-j}^{m=+j}(-1)^{j-m}|m\rangle\langle m|$, leads to the two-time correlation function given by

$$C_{12} = \sin[(2j+1)\omega\Delta t]/(2j+1)\sin[\omega\Delta t]. \tag{1.45}$$

With these correlations the LGI expressed as $K = C_{12} + C_{23} + C_{34} - C_{14} \leq 2$ becomes

$$K = \frac{3\sin x}{x} - \frac{\sin 3x}{3x} \leq 2, \tag{1.46}$$

where $x = (2j+1)\omega\Delta t$. For $x \approx 1.054$, the maximal violation in this case is obtained for infinitely large j, with the value 2.481, i.e., 42 percent short of the largest violation of $2\sqrt{2}$ allowed by quantum theory.

We end introduction chapter by discussing framework for ontological model introduced by Harrigan and Spekkens [43] as this is related to the last chapter of the thesis where we propose a new derivation of LGI and show how device independent randomness can be certified through violation of LGI. For more study see [44]

1.4 Ontological Model for Quantum Theory

In this section we briefly discuss the ontological models framework, introduced by Harrigan and Spekkens [43] which is formulated mainly with a view to deal with the issue of status of quantum state. The nature of quantum state has been debated since the inception of quantum theory [28, 45, 46, 47]. When a quantum state $|\psi\rangle$ is assigned to a physical system, does this mean that there is some independently existing property of that individual system which is in one-to-one correspondence with $|\psi\rangle$, or is $|\psi\rangle$ simply a mathematical tool for determining probabilities? In the ontological models framework, introduced by Harrigan and Spekkens [43], this kind of discussion has been made much more precise.

While an operational theory is *epistemic* by nature and does predict the outcome probabilities of certain experiments performed in a laboratory it does not tell anything about *ontic* state (a state of reality) of the system. On the other hand, in an ontological model of an operational theory, the primitives of description are the properties of microscopic systems. A preparation procedure is assumed to prepare a system with certain properties and a measurement procedure is assumed to reveal something about those properties. A complete specification of the properties of a system is referred to as the ontic state of that system.

1.4.1 Basic mathematical structure

We, in the following, briefly describe the ontological framework of an operational theory (for details of this framework, we refer to [48]), as this will subsequently be used in our derivation of LGI.

The primitive elements of an operational theory are preparation procedures $P \in \mathcal{P}$, transformations $T \in \mathcal{T}$, and measurement procedures $M \in \mathcal{M}$, where $\mathcal{P}, \mathcal{T}$ and $\mathcal{M}$ denote collection of all permissible preparations, transformations and measurements respectively. An operational theory specifies the probabilities of different outcomes of a measurement performed on a system prepared according to some definite procedure. Let $p(k|P, M) \in [0, 1]$ denote the probability of outcome k when a measurement M is performed on a system prepared according to some procedure P. Clearly We have $\sum_{k\in\mathcal{K}_M} p(k|P, M) = 1, \;\; \forall\, P, M$, where $\mathcal{K}_M$ denotes the outcome set of the measurement M.

In an ontological model for quantum theory, a particular preparation method P_ψ which prepares the quantum state $|\psi\rangle$, actually puts the system into some ontic state $\lambda \in \Lambda$, Λ

denotes the ontic state space. An observer who knows the preparation P_ψ may nonetheless have incomplete knowledge of λ. Thus, in general, an ontological model associates a probability distribution $\mu(\lambda|P_\psi)$ with preparation P_ψ of $|\psi\rangle$. $\mu(\lambda|P_\psi)$ is called the *epistemic state* as it encodes observer's *epistemic ignorance* about the state of the system. It must satisfy

$$\int_\Lambda \mu(\lambda|P_\psi)d\lambda = 1 \;\; \forall \; |\psi\rangle \text{ and } P_\psi.$$

Similarly, the model may be such that the ontic state λ determines only the probability $\xi(k|\lambda, M)$, of different outcomes k for the measurement method M. However, in a deterministic model $\xi(k|\lambda, M) \in \{0, 1\}$. The response functions $\xi(k|\lambda, M) \in [0, 1]$, should satisfy

$$\sum_{k \in \mathcal{K}_M} \xi(k|\lambda, M) = 1 \;\; \forall \;\; \lambda, \;\;\; M.$$

Thus, in the ontological model, the probability $p(k|M, P)$ is specified as

$$p(k|M, P) = \int_\Lambda \xi(k|M, \lambda)\mu(\lambda|P)d\lambda.$$

As the model is required to reproduce the observed frequencies (quantum predictions) hence the following must also be satisfied

$$\int_\Lambda \xi(\phi|M, \lambda)\mu(\lambda|P_\psi)d\lambda = |\langle\phi|\psi\rangle|^2.$$

The transformation processes T are represented by stochastic maps from ontic states to ontic states. $\mathcal{T}(\lambda'|\lambda)$ represents the probability distribution over subsequent ontic states given that the earlier ontic state one started with was λ.

CHAPTER 2

APPLICATIONS OF UNCERTAINTY RELATIONS

The uncertainty principle being most known of quantum mechanics, provides one of the first and foremost point of departure from classical concepts. As originally formulated by Heisenberg [1], it prohibits certain properties of quantum systems from being simultaneously well-defined. A generalized form of the uncertainty relation was proposed by Robertson [19] and Schrödinger [20], and since then, several other versions of the uncertainty relations have been suggested. The consideration of state-independence has lead to the formulation of entropic versions of the uncertainty principle [21]. A modification of the entropic uncertainty relation occurs in the presence of quantum memory associated with quantum correlations [49]. Another version provides a fine-grained distinction between the uncertainties inherent in obtaining possible different outcomes of measurements [50]. Uncertainty relations have many areas of important applications. To mention a few it has been used for discrimination between separable and entangled quantum states[51, 52, 53], and the Robertson-Schrödinger generalized uncertainty relation (GUR) has also been applied in this context of detecting multipartite and bound entanglement as well [54]. The fine-grained uncertainty relation in conjunction with steering can be used to determine the nonlocality of the underlying physical system [50, 55] and detection of steerability as well[56].

This chapter is based on two works [57, 58]. First we demonstrate an application of

Robertson-Schrödinger generalized uncertainty relation(GUR) in the context of detecting mixedness/purity of a quantum system. This application considers single qubit system and classes of two-qubit, single qutrit and two-qutrit system. We also discuss advantages of purity detection scheme using GUR over state tomography approach in terms of number of measurements. In the second work we derive a new uncertainty relation in the presence of quantum memory. Lower bound of this uncertainty relation is optimal in the experimental conditions. We also identify the proper resource dubbed extractable classical information responsible for the reduction of lower bound in this scenario.

2.1 Detection of mixedness or purity

We define a quantity $Q(A, B, \rho)$ by taking all the terms on the left hand side of GUR. Then GUR for any pair of observables A, B and for any quantum state represented by the density operator ρ becomes

$$Q(A, B, \rho) \geq 0, \tag{2.1}$$

where,

$$Q(A, B, \rho) = (\Delta A)^2(\Delta B)^2 - |\frac{\langle [A, B] \rangle}{2}|^2 - |(\frac{\langle \{A, B\} \rangle}{2} - \langle A \rangle \langle B \rangle)|^2 \tag{2.2}$$

with conventional notations discussed in the introduction chapter. The quantity $Q(A, B, \rho)$ involves the measurable quantities, i.e., the expectation values and variances of the relevant observables in the state ρ. Pure states correspond to the condition $\rho^2 = \rho$ which is equivalent to the scalar condition $tr[\rho^2] = 1$. Hence, complement of the trace condition can be taken as a measure of mixedness given by the linear entropy defined for a d-level system as $S_l(\rho) = (d/(d-1))(1 - tr(\rho^2))$. Hence to detect purity of a system one has to determine ρ experimentally i.e., through state tomography. Now we show how the quantity $Q(A, B, \rho)$ can act as an experimentally realizable measure of mixedness of a system without knowing ρ. Explicitly we show that for a pair of suitably chosen spin observables, GUR is satisfied as an equality for the states extremal, i.e., the pure states, and as an inequality for points other than extremals, i.e., for the mixed states. This characterization is shown for all single qubit states and class of two qubit and single and two qutrit states.

2.1.1 Single qubit system

We first briefly describe the status of GUR with regard to the purity of qubit states. The density operator for two-level systems can be expressed in terms of the Pauli matrices. The state of a single qubit can be written as

$$\rho(\vec{n}) = \frac{(I + \vec{n}.\vec{\sigma})}{2}, \quad \vec{n} \in \mathbb{R}^3 \tag{2.3}$$

Positivity of this Hermitian unit trace matrix demands $|\vec{n}|^2 \leqslant 1$. It follows that single qubit states are in one to one correspondence with the points on or inside the closed unit ball centred at the origin of $\mathbb{R}^3$. Points on the boundary correspond to pure states. The linear entropy of the state ρ can be written as $S_l(\rho) = (1 - \vec{n}^2)$. If we choose spin observables along two different directions, i.e., $A = \hat{r}.\vec{\sigma}$ and $B = \hat{t}.\vec{\sigma}$, then Q becomes

$$Q(A, B, \rho) = (1 - (\Sigma r_i t_i)^2) S_l(\rho) \tag{2.4}$$

It thus follows that for $\hat{r}.\hat{t} = 0$, Q coincides with the linear entropy. For orthogonal spin measurements, the uncertainty quantified by GUR, Q and the linear entropy S_l are exactly same for single qubit systems. Thus, it turns out that $Q = 0$ is both a necessary and sufficient condition for any single qubit system to be pure when the pair of observables are qubit spins along two different directions.

2.1.2 Two qubit system

For the treatment of composite systems the states considered are taken to be polarized along a specific known direction, say, the z- axis forming the Schmidt decomposition basis. The choice of A and B, in order to enable $Q(A, B, \rho)$ as a mixedness measure, for the two-qubit case, are given by

$$A = (\hat{m}.\vec{\sigma}^1) \otimes (\hat{n}.\vec{\sigma}^2) \qquad B = (\hat{p}.\vec{\sigma}^1) \otimes (\hat{q}.\vec{\sigma}^2) \tag{2.5}$$

where $\hat{m}, \hat{n}, \hat{p}, \hat{q}$ are unit vectors. For enabling $Q(A, B, \rho)$ to be used for discerning the purity/mixedness of given two qubit state specified, say, z-axis, the appropriate choice of observables A and B is found to be that of lying on the two dimensional $x - y$ plane (i.e.,$\hat{m}, \hat{n}, \hat{p}, \hat{q}$ are all taken to be on the $x - y$ plane), normal to the z-axis pertaining to the relevant Schmidt decomposition basis. Then, $Q(A, B, \rho) = 0$ (i.e., GUR is satisfied as an equality) necessarily holds good for pure two-qubit states whose individual spin orientations are all along a given direction (say, the z-axis) normal to which lies the plane on which the observables A and B are defined. On the other hand, $Q(A, B, \rho) > 0$ holds good for most settings of A and B for two qubit isotropic states, Werner states and one parameter two-qubit states which comprise of pure states whose Schmidt basis is orthogonal to the plane on which the observables A and B are defined.

2.1.3 Single qutrit system

We demonstrate detection of mixedness scheme elaborately for single qutrit system as it is more involved than the previous two examples. From the introduction chapter we know that any single qutrit state can be written in terms of identity and eight Gelmann matrices as

$$\rho(\vec{n}) = \frac{I + \sqrt{3}\vec{n}.\vec{\lambda}}{3}, \vec{n} \in R^8. \tag{2.6}$$

For qutrit the most general type of observables can be written as $A = \hat{a}.\vec{\lambda} = a_i\lambda_i$, $B = \hat{b}.\vec{\lambda} = b_i\lambda_i$, where, $\Sigma a_i^2 = 1$ and $\Sigma b_i^2 = 1$. The measurement of qutrit observables composed of the various λ_i's, can be recast in terms of qutrit spin observables [59], e.g., $\lambda_1 = (1/\sqrt{2})(S_x + 2\{S_z, S_x\})$, and similarly for the other λ_i's. Where the qutrit spins are given by

$$\sqrt{2}S_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \sqrt{2}S_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.7}$$

Note that with the choice of $A = \hat{A}.\hat{\lambda}$ and $B = \hat{B}.\hat{\lambda}$, Q becomes

$$\begin{aligned} Q = (4/9)(1 - (\hat{A}.\hat{B})^2) + (4/9)(((\hat{A} * \hat{A}).\vec{n}) + ((\hat{B} * \hat{B}).\vec{n}) \\ -2(\hat{A}.\hat{B})((\hat{A} * \hat{B}).\vec{n})) + (4/9)(((\hat{A} * \hat{A}).\vec{n})((\hat{B} * \hat{B}).\vec{n}) - ((\hat{A} * \hat{B}).\vec{n})^2 \\ +4(\hat{A}.\hat{B})(\hat{A}.\vec{n})(\hat{B}.\vec{n}) - 2(\hat{A}.\vec{n})^2 - 2(\hat{B}.\vec{n})^2 - 3((\hat{A} \wedge \hat{B}).\vec{n})^2) \\ -(4/9)(2((\hat{A} * \hat{A}).\vec{n}))(\hat{B}.\vec{n})^2 + 2(\hat{A}.\vec{n})^2((\hat{B} * \hat{B}).\vec{n})) - 4((\hat{A} * \hat{B}).\vec{n})(\hat{A}.\vec{n})(\hat{B}.\vec{n})) \end{aligned} \tag{2.8}$$

where $(\hat{A} * \hat{B})_k = \sqrt{3}d_{ijk}A_iB_j$ and $(\hat{A} \wedge \hat{B})_k = f_{ijk}A_iB_j$. From the expression of Q it is clear that it changes if ρ is changed by some unitary transformation. For such change of states the norm of $\vec{n}$ does not change. Purity/mixedness property of a state does not change under unitary operations on the state. Hence, it is desirable for any mixedness measure to remain invariant under unitary operation. This would be possible if Q becomes some function of only $|\vec{n}|^2$ for suitable choice of observables. However, unlike the case of the single qubit, for the single qutrit Q becomes independent of the linear and cubic terms of $|\vec{n}|$ only for the trivial choice of observables, i.e., $\hat{A} = \hat{B}$, in which case Q becomes zero, whatever be the state, pure or mixed. Here we employ suitably chosen observables and a sequence of measurements to turn Q to a detector of mixedness, i.e., $Q = 0$ for pure, and $Q > 0$ for mixed states.

Note further, that under a basis transformation $\lambda_i' = U\lambda_i U^\dagger$, the state becomes $\rho' = (1/3)(I + \sqrt{3}\vec{n}'.\vec{\lambda}') = U(1/3)(I + \sqrt{3}\vec{n}'.\vec{\lambda})U^\dagger$. Now, for any observable χ' in the prime basis, one has $Tr[\chi'\rho'] = Tr[\chi(1/3)(I + \sqrt{3}\vec{n}'.\vec{\lambda})]$. Thus, any non-vanishing expectation value in the primed basis cannot vanish in the unprimed one, and vice-versa. Hence, in order to measure in another basis one has to simply choose observables which are unitary conjugates to the observables written in terms of standard λ basis. Such observables would again yield $Q = 0$ for pure, and $Q > 0$ for mixed states in the new basis. Hence, though we have specified our scheme based on the single qutrit state in terms of the standard λ basis [14, 15], our scheme remains invariant with regard to the choice of the basis as long as the knowledge of the specific basis chosen is available to the experimenter. This means that the experiment shall involve not only the observables A and B but also a possibility for simultaneous unitary rotations of these observables.

In what follows we take up to three-parameter family of states (means coefficient at most any three λ's can be non-zero) from the state space of qutrit Ω_3 [15], and find that there exist observable pairs which for pure states exhibit minimum uncertainty, *viz.* $Q = 0$. Our scheme runs as follows. Economizing on the number of measurements required, we take λ_3 as A and sequentially, the members of any one of the pairs $(\lambda_7, \lambda_6), (\lambda_5, \lambda_4), (\lambda_1, \lambda_2)$ as B. The significance of such pairing will be clear later. To be precise in this case what we show is that if two successive measurements taking B from any of the above pairs yield $Q = 0$, the state concerned is pure. In contrast, if B taken from all the above pairs sequentially, yields $Q > 0$, the state is found to be mixed. (See, Fig. 2.1 for an illustration of the scheme).

Let us first consider the one-parameter family of single-qutrit states for which only one of the eight parameters $(n_i, i = 1, ..., 8)$ is non-zero. The linear entropy of this class of states is given by $S_l(\rho) = 1 - n_i^2$. There exist many pairs of observables which can detect

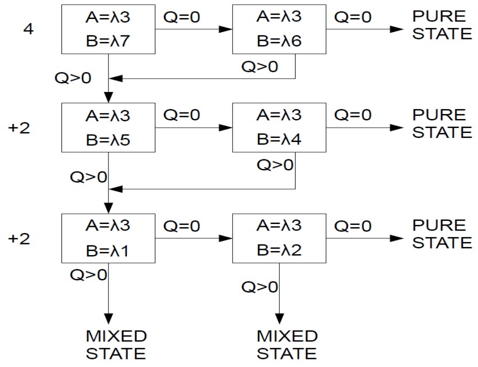

FIG. 2.1: Detection scheme for purity of single qutrit states of up to three parameters. The numbers to the left of the boxes indicate the number of measurements required corresponding to each of the horizontal levels.

mixedness of this class of states unambiguously. For example, when $i = 8$, the only pure state of this class is given by $n_8 = -1$ [15]. Here

$$Q(\lambda_3, \lambda_7) = Q(\lambda_3, \lambda_6) = (4/9)(2 - n_8)(1 + n_8) \tag{2.9}$$

Hence, $Q = 0$ only for $n_8 = -1$, but $Q > 0$ otherwise. Next, for example when $i = 1$, one has

$$Q(\lambda_3, \lambda_7) = Q(\lambda_3, \lambda_6) = Q(\lambda_3, \lambda_5) = Q(\lambda_3, \lambda_4) = 4/9 \tag{2.10}$$

It turns out that there is no choice of B from both the sequential pairs (as depicted in Fig. 2.1) for which $Q = 0$ as there is no pure state for this case. Similar considerations are valid also for other single parameter qutrit states.

Moving to the two-parameter family of density matrices, (two of the eight parameters $n_1 ... n_8$ are non zero, while remaining six vanish), note that in this case there are twenty-eight combinations of different pairs of non-zero parameters, and these classes belongs to one of the four different types of unitary equivalence classes, *viz.*, circular, parabolic, elliptical and triangular [15]. In this case, for example, for states belonging to the parabolic class, by choosing n_3 and n_4 to be non-vanishing, Q takes the forms

$$\begin{aligned} Q(\lambda_3, \lambda_5) &= (2/9)(2 + \sqrt{3}n_3)(1 - 2n_3^2) - n_4^2/3 \\ Q(\lambda_3, \lambda_4) &= (1/9)(4 - 8n_3^2 - 4\sqrt{3}n_3^3 - 11n_4^2 + 2\sqrt{3}n_3(1 + 4n_4^2)) \end{aligned} \tag{2.11}$$

Here pure states occur for $(n_3, n_4) = (1/\sqrt{3}, \pm\sqrt{2/3})$, leading to $Q = 0$, while $Q > 0$ corresponding to all mixed states, as is also evident from the expression for the linear entropy given by $S_l(\rho) = (1 - n_3^2 - n_4^2)$. Similar considerations apply to other single qutrit states of the two parameter family, enabling the detection of pure states when two successive measurements with B taken from sequential pairs (Fig. 2.1) lead to $Q = 0$.

Next consider the three-parameter family of qutrit states where there are seven geometrically distinct and ten unitary equivalent types of three-sections out of fifty-six standard three-sections. Considering an example of states belonging to the parabolic geometric shape, Q has the forms

$$\begin{aligned} Q(\lambda_3, \lambda_5) &= (1/9)(4 - 8n_3^2 - 4\sqrt{3}n_3^3 - 3n_4^2 - 11n_5^2 + 2\sqrt{3}n_3(1 + 4n_5^2)) \\ Q(\lambda_3, \lambda_4) &= (1/9)(4 - 8n_3^2 - 4\sqrt{3}n_3^3 - 3n_5^2 - 11n_4^2 + 2\sqrt{3}n_3(1 + 4n_4^2)). \end{aligned} \tag{2.12}$$

The linear entropy of this class of states is given by $S_l(\rho) = 1 - n_3^2 - n_4^2 - n_5^2$.

When B is chosen from the (λ_4, λ_5) pair as above, Q turns out to be zero for pure states given by $n_3 = 1/\sqrt{3}$ and $n_4^2 + n_5^2 = 2/3$, and Q is greater than zero for all mixed states. It can be checked that the purity of all three parameter family of single qutrit states can be determined by the scheme depicted in Fig. 2.1.

2.1.4 Two qutrit

Let us now discuss the case of two-qutrit state discrimination. Here we assume that the states considered are taken to be polarized along a specific known direction, say, the z-axis forming the Schmidt decomposition basis.

A two-qutrit pure state in the Schmidt form can be written as $|\psi\rangle = k_1|11\rangle + k_2|22\rangle + k_3|33\rangle$ where, k_1, k_2, k_3 are real with $k_1^2 + k_2^2 + k_3^2 = 1$, and $|1\rangle$, $|2\rangle$ and $|3\rangle$ are orthonormal unit vectors in $\mathbb{C}^3$. In our purpose a general form of observables acting on the two-qutrit system is given by $A = \hat{r}_1.\vec{\lambda} \otimes \hat{r}_2.\vec{\lambda}$, and $B = \hat{t}_1.\vec{\lambda} \otimes \hat{t}_2.\vec{\lambda}$, where $\hat{r}_1, \hat{t}_1, \hat{r}_2, \hat{t}_2$ are unit vectors in $\mathbb{R}^8$. For our purpose it is sufficient to take observables of the form

$$A = \lambda_i \otimes (\cos\theta_2\lambda_i + \sin\theta_2\lambda_j), \;\; B = (\cos\theta_3\lambda_i + \sin\theta_3\lambda_j) \otimes (\cos\theta_4\lambda_i + \sin\theta_4\lambda_j) \tag{2.13}$$

where (i, j) are taken from the pair $(1, 2)$,$(3, 8)$, $(4, 5)$, $(6, 7)$, and $\theta_2, \theta_3, \theta_4$ are angles between $\hat{r}_1$ and $\hat{r}_2, \hat{t}_1, \hat{t}_2$, respectively. With the choice of observables ($i = 1, j = 2$), the uncertainty becomes $Q(A, B, \rho_{pure}) = 4k_1^2k_2^2k_3^2 \sin(\theta_2 - \theta_3 - \theta_4)$. Hence, choosing $\theta_2 - \theta_3 = \theta_4$, we can make $Q = 0$ for every pure state.

Now consider a one-parameter class of two-qutrit mixed states expressed as

$$\rho_m = p\rho_1 + (1-p)\rho_2 \tag{2.14}$$

where ρ_1 and ρ_2 are arbitrary pure states parametrized as $\rho_1 = |\psi_1\rangle\langle\psi_1|$ with $|\psi_1\rangle = k_1|11\rangle + k_2|22\rangle + k_3|33\rangle$, and $\rho_2 = |\psi_2\rangle\langle\psi_2|$ with $|\psi_2\rangle = k_4|11\rangle + k_5|22\rangle + k_6|33\rangle$. For such states the linear entropy is given by $S_l(\rho_m) = \frac{3}{2}p(1-p)$.
The expression for Q under the condition $\theta_2 - \theta_3 = \theta_4$ is given by $Q(A, B, \rho_m) = 4k_1^2 p(1-p)(1 - k_6^2 - 4k_4^2 k_5^2 (1-p)\cos^2(\theta_3 + \theta_4))\sin^2(\theta_3)$ which when maximized over all observables in the selected region ($i = 1, j = 2$) leads to $Q = 4k_1^2(1-k_6^2)p(1-p)$.
We observe that the expression for the uncertainty may coincide with the value of linear entropy for certain choices of the state parameters. In general, Q always vanishes for pure states, and remains positive for mixed ones, for $k_1 \neq 0$, and $k_6 \neq 1$.
As another example of two-qutrit states, we consider the popular class of isotropic states that are invariant under the action of local unitary operations of the form $U \otimes U^*$. Two-qutrit isotropic states can be written as

$$\rho = p\rho_i + \frac{1-p}{9} I \otimes I \tag{2.15}$$

where, $0 \leq p \leq 1$, and $\rho_i = |\phi_i\rangle\langle\phi_i|$, with $|\phi_i\rangle = (1/\sqrt{3})(|11\rangle + |22\rangle + |33\rangle)$. The linear entropy of this state is given by $S_l(\rho) = \frac{2}{3}(1-p^2)$. Our choice of observables leads to $Q = (8/81)(-1+p)(-3-3p+2p^2+(-1+p)\cos(2\theta_3)+2p^2\cos(2(\theta_3+\theta_4)))^2\sin\theta_3$. Maximizing over all observables in the selected region we get $Q = \frac{16}{81}(1-p)(1+2p)$. which is quadratic in the parameter p similar to the linear entropy, and is able to distinguish mixed states from the pure state ($p = 1$). It may be noted that for the Werner class of states that are invariant under the local unitary operations of the form $U \otimes U$, and which differ from the Isotropic class for qutrits, there exists no pure state for qutrits, a fact that is reflected in the corresponding expression for Q that turns out to be $Q > 0$ always. Class of states considered are shown schematically in the following figure.
Realistic experimental scenario: It may be noted here that the limitation of instrumental precision could make the observed value of Q for pure states to be a small number instead of exactly zero. In order to take into account the experimental inaccuracy, a parameter ε may be introduced in the analysis. For a single-qubit system, by choosing the measurement

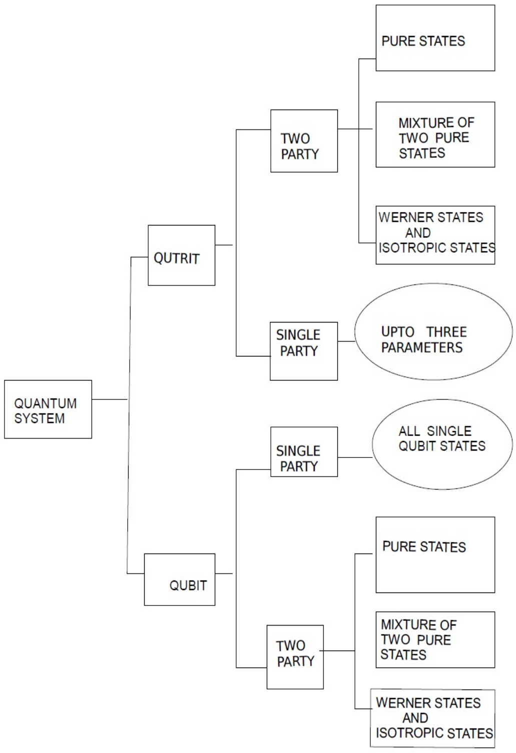

FIG. 2.2: Family of states that can be distinguished using the uncertainty relation.

settings for A and B as qubit spins along z and x directions, respectively, the measured value of the uncertainty obtained as $Q \geq \varepsilon$ leads to the conclusion that the given state is mixed. This prescription of determining mixedness holds for all single-qubit states $\rho(\vec{n}) = \frac{(I+\vec{n}.\vec{\sigma})}{2}$, except those lying in the narrow range $1 \geq n \geq \sqrt{1-2\varepsilon/3}$, as determined by putting $Q < \varepsilon$ in Eq.(2.4). A somewhat more elaborate procedure is required for qutrits, as may be expected from the richer structure of their state space. For the case of single qutrits belonging to the one, two or three-parameter family of states, one has to find Q taking $A = \lambda_3$ and B from the $(\lambda_6, \lambda_7), (\lambda_4, \lambda_5), (\lambda_1, \lambda_2)$ pairs in succession as depicted in the Fig. 1. If $Q < \varepsilon$ for the settings B corresponding to both members of a same pair measured in succession, then the state is pure within the limitations of experimental accuracy. Whenever $Q \geq \varepsilon$, B is chosen from the pair vertically below. If there exists no such pair for which $Q < \varepsilon$, then the state is mixed. In order to maximize the uncertainty measured by the variable Q, such that $Q \geq \varepsilon$ for the maximum number of mixed states, the observables need to chosen so as to avoid $|a_i/b_i| \approx 1$.

For the case of two-qutrit states, the measurement of the observables given by Eq.(2.13)

System	in tomography	using GUR
Single qubit	3	3
two qubit	15	3-5
Single qutrit	8	4-8
Two Qutrit	80	4-8

TAB. 2.1: A comparison between the number of measurements required in tomographic method and in our method is shown for the categories of states considered. Number of measurements for detecting mixedness/purity for bipartite system is much less and for single party system this method is becoming advantageous with increasing dimension.

with the λ_i's chosen from the regions spanned by (λ_1, λ_2), together with the restriction on the angle $\theta_3 \neq \pi$, suffices to distinguish pure and mixed states. Such a procedure is able to detect all mixed states within the margin of experimental accuracy. For example, for the case of the two-qutrit isotropic states the method would fail only for states lying in the parameter range $\sqrt{1 - 3\varepsilon/2} < p < 1$.

2.1.5 ADVANTAGE OVER STATE TOMOGRAPHY

The determination of mixedness using GUR may require in certain cases a considerably lesser number of measurements compared to tomography. In the case of single qutrit states, full tomography involves the estimation of eight parameters, while in our prescription sometimes four measurements may suffice for detecting purity of a single qutrit state. For instance, the number 4 besides the top box in Fig. 2.1, means that the four measurements ($\langle\lambda_3\rangle$, $\langle\lambda_7\rangle$,$\langle\lambda_8\rangle$ and$\langle\lambda_6\rangle$) are all that is required for the first horizontal level. This follows from the algebra $\langle\{\lambda_3, \lambda_7\}\rangle = -\langle\lambda_7\rangle/2$, $\langle[\lambda_3, \lambda_7]\rangle = \langle\lambda_6\rangle/2$, $\langle\lambda_3^2\rangle = \frac{2}{3}I + \frac{1}{\sqrt{3}}\langle\lambda_8\rangle$, $\langle\lambda_7^2\rangle = \frac{2}{3}I - \frac{1}{2\sqrt{3}}\langle\lambda_8\rangle - \frac{1}{2}\langle\lambda_3\rangle$, $\langle[\lambda_3, \lambda_6]\rangle = -\langle\lambda_7\rangle/2$, $\langle\{\lambda_3, \lambda_6\}\rangle = -\langle\lambda_6\rangle/2$, $\langle\lambda_6^2\rangle = \frac{2}{3}I - \frac{1}{2\sqrt{3}})\langle\lambda_8\rangle - \frac{1}{2}\langle\lambda_3\rangle$, which are required to determine $Q(A, B, \rho)$. To proceed vertically down to the next level in Fig.1, the number of extra measurements are indicated besides the boxes. It may be mentioned that in our scheme it does not matter if any horizontal pair of boxes are interchanged with another pair at a different level. A maximum of eight measurements thus suffices to distinguish between pure and mixed states of single qutrit up to three-parameter families. Advantage over tomography in detecting mixedness becomes prominent for higher dimension as evident from Tab. 2.1.

2.2 Uncertainty in the presence of quantum memeory

A form of the entropic uncertainty relation (EUR) introduced by Deutsch [23], was later improved in the version conjectured in Ref.[24] and then proved in Ref.[25], given by

$$\mathcal{H}(R) + \mathcal{H}(S) \geq \log_2 \frac{1}{c}, \tag{2.16}$$

where $\mathcal{H}(k)$ represents the Shannon entropy for the measurement of observable $k \in \{R, S\}$ and the complementarity of the observables R and S is quantified by the quantity c ($= \max_{i,j} c_{i,j} = \max_{i,j} |\langle r_i|s_j\rangle|^2$, with $|r_i\rangle$ and $|s_j\rangle$ are the eigenvectors of R and S, respectively).

Considering the correlation of the observed system with another system called quantum memory, Berta et al. [49] modified the lower bound of entropic uncertainty given by inequality (2.16). Recently, Coles and Piani [60] have made the bound tighter. Here, Bob is able to reduce his uncertainty about Alice's measurement outcome with the help of communication from Alice regarding the choice of her measurement performed, but not its outcome. The modified form of EUR in the presence of quantum memory is given by [60]

$$\mathcal{S}(R_A|B) + \mathcal{S}(S_A|B) \geq c'(\rho_A) + \mathcal{S}(A|B) \tag{2.17}$$

where $c'(\rho_A) = \max\{c'(\rho_A, R_A, S_A), c'(\rho_A, S_A, R_A)\}$. $c'(\rho_A, R_A, S_A)$ and $c'(\rho_A, S_A, R_A)$ are defined by

$$c'(\rho_A, R_A, S_A) = \sum_i p_i^r \log_2 \frac{1}{\max_j c_{ij}}, \quad c'(\rho_A, S_A, R_A) = \sum_j p_j^s \log_2 \frac{1}{\max_i c_{ij}}, \tag{2.18}$$

where $p_i^r = \langle r|\rho_A|r\rangle$ with $\sum_i p_i^r = 1$ and $p_j^s = \langle s|\rho_A|s\rangle$ with $\sum_j p_j^s = 1$. Here, the uncertainty for the measurement of the observable R_A (S_A) on Alice's system by accessing the information stored in the quantum memory with Bob is measured by $\mathcal{S}(R_A|B)$ ($\mathcal{S}(S_A|B)$) which is the conditional von Neumann entropy of the state given by

$$\begin{aligned} \rho_{R_A(S_A)B} &= \sum_j (|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I) \\ &= \sum_j p_j^{R_A(S_A)} \Pi_j^{R_A(S_A)} \otimes \rho_{B|j}^{R_A(S_A)}, \end{aligned} \tag{2.19}$$

where $\Pi_j^{R_A(S_A)}$'s are the orthogonal projectors on the eigenstate $|\psi_j\rangle_{R_A(S_A)}$ of observable $R_A(S_A)$, $p_j^{R_A(S_A)} = Tr[(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)]$,
$\rho_{B|j}^{R_A(S_A)} = Tr_A[(|\psi_j\rangle_{R_A(S_A)}\langle\psi_j| \otimes I)\rho_{AB}(|\psi_j\rangle_{R(S)}\langle\psi_j| \otimes I)]/p_j^{R_A(S_A)}$ and ρ_{AB} is the state of joint system 'A' and 'B'. EUR in presence of quantum memory is modified by the quantity $\mathcal{S}(A|B)$ $(= \mathcal{S}(\rho_{AB}) - \mathcal{S}(\rho_B)$, where $\rho_B = Tr_A[\rho_{AB}])$ which measures the amount of one-way distillable entanglement [61]. For shared maximal entanglement (i.e., $S(A|B) = -1$) between the system and the memory, there is no uncertainty in the measurement of incompatible observables. EUR in the presence of quantum memory has been brought out in two recent experiments using respectively, pure [62] and mixed states [63]. For experimental purposes [63], one can obtain the uncertainty relation from the inequality (2.17) with the help of Fano's inequality [64] and it is given by

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq c'(\rho_A) + \mathcal{S}(A|B), \tag{2.20}$$

where p_d^R (p_d^S) is the probability of getting different outcomes when Alice and Bob measure the same observables R (S) on their respective system. Here lower bound of the uncertainty relation is $\mathcal{L}_1(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(A|B)$.
Later Pati et al. [65] have derived a tighter lower bound of the uncertainty relation which is

$$\mathcal{S}(R_A|B) + \mathcal{S}(S_A|B) \geq c'(\rho_A) + \mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB}) - C_A^M(\rho_{AB})\}, \tag{2.21}$$

where the quantum discord $D_A(\rho_{AB})$ is given by [66, 67], $D_A(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - C_A^M(\rho_{AB})$, with $\mathcal{I}(\rho_{AB})$ $(= \mathcal{S}(\rho_A) + \mathcal{S}(\rho_B) - \mathcal{S}(\rho_{\mathcal{AB}}))$ being the mutual information of the state ρ_{AB} which contains the total correlation present in the state ρ_{AB}, and the classical information $C_A^M(\rho_{AB})$ for the shared state ρ_{AB} (when Alice measures on her system) is given by, $C_A^M(\rho_{AB}) = \max_{\Pi^{R_A}}[\mathcal{S}(\rho_B) - \sum_{j=0}^{1} p_j^{R_A}\mathcal{S}(\rho_{B|j}^{R_A})]$. In this case, the lower bound of the above uncertainty relation is given by

$$\mathcal{L}_2(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB}) - C_A^M(\rho_{AB})\}, \tag{2.22}$$

$\mathcal{L}_2$ is tighter than $\mathcal{L}_1$ for those state whose quantum discord is larger than the classical information. This is true for a class of states including Werner states and isotropic states.

2.2.1 Optimal bound and classical information

In a recent work [68], it was shown that the lower bound of the uncertainty relation given by Eqs. (2.17) and (2.21) are not *optimal* as illustrated by the analysis of an experiment using mixed states [63]. In [68] the *optimal* lower bound of entropic uncertainty relation was obtained using fine-grained uncertainty relation [50]. Considering a situation [63] where Alice and Bob both measure the same observable on their system, a new uncertainty relation was derived [68] which is

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S), \tag{2.23}$$

where the lower bound given by $\mathcal{L}_3(\rho_{AB}) = \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S)$.
From 2.23 it is not clear what is the actual resource in terms of correlation between parties, which is responsible for the modification of the lower bound optimally. Whereas for 2.17 and 2.21 resources are identified with conditional entropy $\mathcal{S}(A|B)$ and discord $D_A(\rho_{AB})$ respectively. In the present section we show that the sought after resource is identified with *extractable classical information* by deriving a new uncertainty relation in terms of it. Interestingly it is also found that in the experimental scenario considered, classical correlation in the absence of quantum correlation can also be useful for reduction of the uncertainty-lower bound.

2.2.2 Uncertainty relation using extractable classical information

To derive the sum of uncertainties for the measurement of two incompatible observables R and S, we consider the following memory game [49]. In this game Bob prepares a particle (labeled by 'A') in a particular state, say, ρ_A and sends it to Alice who measures an observable chosen from the non-commuting set $\{R_A, S_A\}$ and communicates only the choice of the observable to Bob. Bob's task is to reduce his uncertainty about the Alice's measurement outcome. To win the game, Bob chooses one of the following two strategies – (i) *classical strategy*; (ii) *quantum strategy*.

Classical strategy : Here, Bob prepares two particles in the identical state, $\rho = \rho_A = \rho_B$. The combined state of two particles is given by

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{2.24}$$

Note here that Bob keeps full information of the state of Alice since he himself has prepared it. Here, the uncertainty relation prevents Bob to know with arbitrary precision the measurement outcomes of two non-commuting observables. The EUR which gives the lower bound for the measurement of the above two non-commuting observables follows from Eq.(2.17) for product states and is given by [60]

$$\mathcal{H}(R_B) + \mathcal{H}(S_B) \geq c'(\rho_B) + \mathcal{S}(\rho_B), \tag{2.25}$$

where the subscript B labels Bob's measurement. The inequality (2.25) is tighter than the entropic uncertainty relation given by inequality (2.21), and hence, Bob can not reduce his uncertainty about Alice's measurement outcome below the lower bound $\mathcal{L}_0(\rho_{AB}) = c'(\rho_B) + \mathcal{S}(\rho_B)$.

Note that, the state given by Eq.(2.24) has zero classical correlation (i.e., $C_A^M = 0$) and zero quantum correlation (i.e., $D_A = 0$) [69]. The inequality (2.25) represents the entropic uncertainty relation for Bob's measurements of two non-commuting observables R_B and S_B on his system, and pertains to the situation when there is either no correlation with the other system called quantum memory, or the correlation with the quantum memory is not considered.

Quantum strategy : In this strategy, Bob prepares two particles in a correlated state. To reduce his uncertainty further from the bound $c'(\rho_B)+\mathcal{S}(\rho_B)$ (which is the lower bound of uncertainty corresponding to the *classical strategy*), Bob uses the correlations (quantum and/or classical) present in the state ρ_{AB}. According to our considered game, when Alice communicates her choice, say, R_A (where 'A' labels Alice's choice), Bob measures same the observable $R_B = R_A$ on his particle (labeled by 'B'). Due to Bob's measurement, the reduced uncertainty measured by the conditional von-Neumann entropy of the state, $\rho_{AR_B}(= \sum_j p_j^{R_B(S_B)} \rho_{A|j}^{R_B(S_B)} \otimes \Pi_j^{R_B(S_B)})$ now becomes $\mathcal{S}(A|R_B) = \mathcal{S}(\rho_A) - C_B^R(\rho_{AB})$. After Bob's measurement, Alice measures the observable R_A on her particle. Now, Alice's reduced uncertainty for the measurement of observable R_A is given by

$$\mathcal{H}(R_A|R_B) = \mathcal{H}(R_A) - C_{A,B}^{R,R}(\rho_{AB}), \tag{2.26}$$

with

$$C_{A,B}^{R,R}(\rho_{AB}) = \mathcal{H}(R_A) - \sum_i p_i^{R_B} \mathcal{H}(q_i^{R_A}), \tag{2.27}$$

where $\mathcal{H}(R_A)$ is the Shannon entropy of the probability distribution $\{q_k^{R_B}\}$ corresponding to different measurement outcomes $\{k\}$ for the measurement of observable R_A on Alice's particle and $\mathcal{H}(q_i^{R_A})$ is the Shannon entropy of the conditional probability distribution $\{q_{k|i}^{R_A}\}$ for the measurement of observable R_A on Alice's particle, given that Bob gets ith outcome for the measurement of the same observable (R_B) on his particle. We define the quantity $C_{A,B}^{R,R}(\rho_{AB})$ as the "extractable classical information".

Similarly, when both Alice and Bob measures the observable S, the conditional entropy becomes $\mathcal{H}(S_A|S_B) = \mathcal{H}(S_A) - C_{A,B}^{S,S}(\rho_{AB})$, where $C_{A,B}^{S,S}(\rho_{AB})$ is the extractable classical information for the measurement of the observable S on the both particles. Now, combining the above results

$$\mathcal{H}(R_S|R_B) + \mathcal{H}(S_A|S_B) = \mathcal{H}(R_A) + \mathcal{H}(S_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}) \tag{2.28}$$

The sum of the first two terms on the r.h.s. of the above equation (2.28) represents the entropy of a single system (system A) when there is either no correlation with the other system called quantum memory, or the correlation with the quantum memory is not considered. Hence, the sum $\mathcal{H}(R_A)+\mathcal{H}(S_A)$ can be constrained through the inequality (2.25), using which we obtain

$$\mathcal{H}(R_A|R_B) + \mathcal{H}(S_A|S_B) \geq c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}), \tag{2.29}$$

where ρ_A is the density state of Alice's particle. Now, using Fano's inequality [64], Eq.(2.29) becomes

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}), \tag{2.30}$$

where $\mathcal{H}(p_d^\alpha)$ is the Shannon entropy of the probability distribution $\{p_d^\alpha\}$ when Alice and Bob measure same observable $\alpha \in \{R, S\}$ and get different outcomes. Eq.(2.30) represents our new uncertainty relation when both Alice and Bob measure two incompatible observables R and S. Hence, the lower bound of Bob's uncertainty about Alice's measurement outcomes is given by

$$\mathcal{L}_4(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}). \tag{2.31}$$

2.2.3 Examples

In the following analysis we compare the bound $\mathcal{L}_4(\rho_{AB})$ with the lower bounds obtained earlier in the literature, *viz.*, $\mathcal{L}_1(\rho_{AB})$ [49, 63], the bound $\mathcal{L}_2(\rho_{AB})$ [65], the bound $\mathcal{L}_3(\rho_{AB})$ [68], as well as the bound $\mathcal{L}_0(\rho_{AB})$ for various classes of pure and mixed entangled and separable states. We show that the lower bound given by Eq.(2.31) is optimal as obtained through fine-graining [68] for all the cases considered here.

Pure entangled state : Here we consider a pure entangled state ρ_{AB}^{PE}, given by

$$\rho_{AB}^{PE} = \sqrt{\alpha}|01\rangle_{AB} - \sqrt{1-\alpha}|10\rangle_{AB}, \tag{2.32}$$

where α lies between 0 and 1, and the state ρ_{AB}^{PE} is maximally entangled for $\alpha = \frac{1}{2}$. Now lower bound for different uncertainty relations are given as

$$\begin{aligned}\mathcal{L}_0(\rho_{AB}^{PE}) = 1 + \mathcal{H}(\alpha),\;\; \mathcal{L}_1(\rho_{AB}^{PE}) = \mathcal{L}_2(\rho_{AB}^{PE}) = 1 - \mathcal{H}(\alpha),\\ \mathcal{L}_3(\rho_{AB}^{PE}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}).\end{aligned} \tag{2.33}$$

However, in practice Bob is unable to reduce his uncertainty upto the above level, since $\mathcal{L}_1(\rho_{AB}^{PE})$ is not the optimal lower bound. The main reason is that Bob only extracts the information $C_{A,B}^{\sigma_z,\sigma_z}(\rho_{AB}^{PE})$ ($C_{A,B}^{\sigma_x,\sigma_x}(\rho_{AB}^{PE})$) given by $\mathcal{H}(\alpha)$ $(1 - \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}))$ when both of them measure the same spin observables $R = \sigma_z$ $(S = \sigma_x)$ on their respective particle. Hence, the lower bound (given by Eq.(2.31)) of Bob's uncertainty is given by

$$\mathcal{L}_4(\rho_{AB}^{PE}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}). \tag{2.34}$$

which coincides with $\mathcal{L}_3$ as expected.

Werner State : For the class of Werner State $\rho_{AB}^{W}(= \frac{1-p}{4} I \otimes I + p|\psi^-\rangle\langle\psi^-|)$, these bounds are given by

$$\mathcal{L}_0(\rho_{AB}^{W}) = 2,\;\; \mathcal{L}_1(\rho_{AB}^{W}) = 2 - \mathcal{I}(\rho_{AB}^{W}),\;\; \mathcal{L}_2(\rho_{AB}^{W}) = 2 - 2C_B^M(\rho_{AB}^{W}) = 2\mathcal{H}(\frac{1-p}{2}), \tag{2.35}$$

$$\mathcal{L}_3(\rho_{AB}^{W}) = 2 - 2C_B^M(\rho_{AB}^{W}) = 2\mathcal{H}(\frac{1-p}{2}),\;\; \mathcal{L}_4(\rho_{AB}^{W}) = 2\mathcal{H}(\frac{1-p}{2}). \tag{2.36}$$

Thus, for the Werner class of states, Bob can actually minimize his uncertainty about Al-

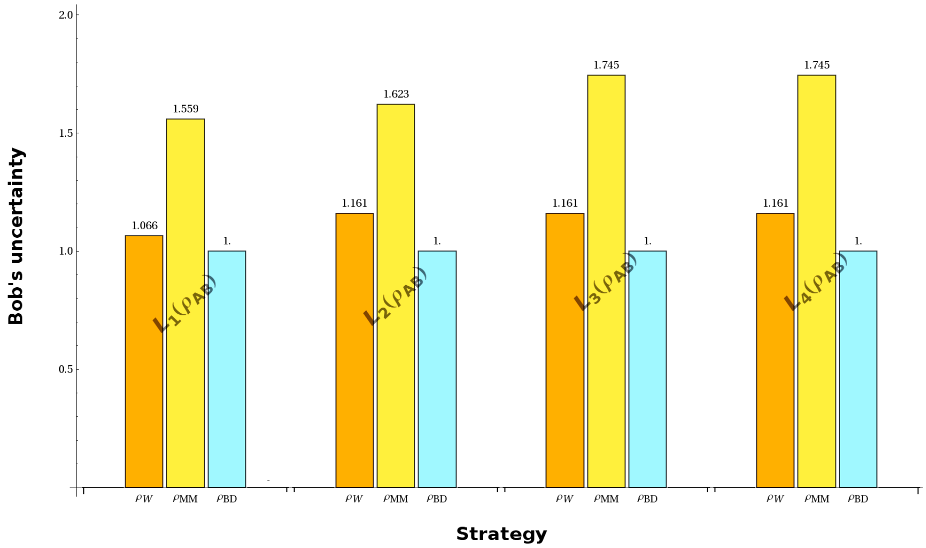

FIG. 2.3: A comparison of the different lower bounds for the (i) Werner state with $p = 0.723$, (ii) the state with maximally mixed marginals with the ci's given by $c_x = 0.5,\ c_y = -0.2$, and $c_z = -0.3$, and (iii) the Bell diagonal state with $p = 0.5$.

ice's measurement outcome upto $2\mathcal{H}(\frac{1-p}{2})$. The lower bound $\mathcal{L}_1(\rho^W_{AB})$ ($\leq \mathcal{L}_3(\rho^W_{AB})$) is not experimentally reachable.

Bell diagonal state : Bell diagonal state, $\rho^{BD}_{AB} = p\rho_2 + (1-p)\rho_S$, used in Ref.[63], where ρ_2 is the density matrix of the state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$. For this class of states lower bounds are given by

$$\mathcal{L}_0(\rho^{BD}_{AB}) = 2,\ \ \mathcal{L}_1(\rho^{BD}_{AB}) = \mathcal{L}_2(\rho^{BD}_{AB}) = \mathcal{H}(p),\ \ \mathcal{L}_3(\rho^{BD}_{AB}) = \mathcal{H}(p),\ \ \mathcal{L}_4(\rho^{BD}_{AB}) = \mathcal{H}(p). \tag{2.37}$$

Here the lower bound predicted by [49, 65] is optimal. From the expression of $\mathcal{L}_3$ and $\mathcal{L}_4$ it is clear that the extractable classical information $C^{\sigma_z,\sigma_z}_{A,B}(\rho^{BD}_{AB}) = 1-\mathcal{H}(p)$, $C^{\sigma_y,\sigma_y}_{A,B}(\rho^{BD}_{AB}) = 1$ is responsible for reducing Bob's uncertainty optimally.

Maximally mixed marginal state : The maximally mixed marginal state ρ^{MM}_{AB} is given by, $\rho^{MM}_{AB} = \frac{1}{4}(\mathcal{I} + \sum_{i=x,y,z} c_i\sigma_i \otimes \sigma_i)$. where the coefficients c_i's ($i \in \{x, y, z\}$) are constrained by the eigenvalues $\lambda_i \in [0, 1]$ of ρ^{MM}_{AB} given by

$$\lambda_0 = \frac{1-c_x-c_y-c_z}{4},\ \lambda_1 = \frac{1-c_x+c_y+c_z}{4},\ \lambda_2 = \frac{1+c_x-c_y+c_z}{4},\ \lambda_3 = \frac{1+c_x+c_y-c_z}{4} \tag{2.38}$$

Here we consider $c_x = 0.5,\ c_y = -0.2$, and $c_z = -0.3$, and for this choices, the observable $R = \sigma_z$ and $S = \sigma_x$ minimizes Bob's uncertainty [68]. For the above choices lower bounds are

$$\mathcal{L}_0(\rho_{AB}^{MM}) = 2, \mathcal{L}_1(\rho_{AB}^{MM}) \approx 1.558, \mathcal{L}_2(\rho_{AB}^{MM}) \approx 1.622, \mathcal{L}_3(\rho_{AB}^{MM}) \approx 1.745, \mathcal{L}_4(\rho_{AB}^{MM}) \approx 1.745. \tag{2.39}$$

For the three classes of the states depicted, one sees that $\mathcal{L}_1 \leq \mathcal{L}_2 \leq (\mathcal{L}_3 = \mathcal{L}_4)$ holds.

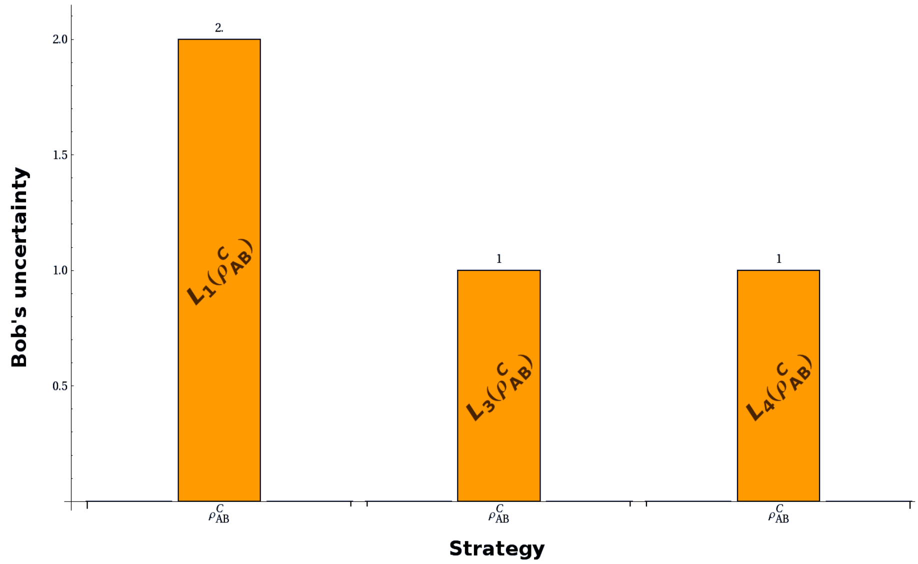

FIG. 2.4: A comparison of the different lower bounds for the shared classical state choosing p=0.5.

Classical state : Now, we consider classical state f the form $\rho^C_{AB} = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|$. For this zero discord state lower bounds are

$$\mathcal{L}_0(\rho^C_{AB}) = 1 + \mathcal{H}(p), \ \mathcal{L}_1(\rho^C_{AB}) = \mathcal{L}_2(\rho^C_{AB}) = \mathcal{L}_3\rho^C_{AB} = \mathcal{L}_4\rho^C_{AB} = 1. \tag{2.40}$$

$$\tag{2.41}$$

Hence in this case, $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_4 = 1 < \mathcal{L}_0$. We thus observe that even purely classical correlations can play a role in reducing the uncertainty using a shared bipartite state when the quantum strategy is employed. This result is displayed in Fig.2.4.

2.3 Concluding remarks and future perspective

The generalized uncertainty corresponding to the measurement of suitable observables vanishes for pure states and is positive definite for mixed states. Using this feature we have proposed a scheme to distinguish pure and mixed states belonging to the classes of all single-qubit, single-qutrit states up to three parameters, as well as several classes of two-qubit and two-qutrit states, when prior knowledge of the basis is available. The procedure suggested here could be helpful also for the detection of entanglement, since purity of subsystems is related to the entanglement of the joint system. The method of detecting mixedness using the uncertainty relation is advantageous over tomography in terms of the number of measurements required, significantly for bipartite qutrit systems, which may have applications in information processing protocols such as distributed computing [70] and security enhancement of quantum cryptography [71].

In the later work we have reformulated the uncertainty relation in the presence of quantum memory. The lower bound of uncertainty is derived here using an approach that is different from the fine-graining employed earlier in the context of memory[68] and steering[56]. Though it turns out that for several important and widely considered examples of two-qubit states the bound derived here and that using fine-graining turn out to be numerically equivalent, there is as yet no proof of formal equivalence between the two derived bounds using two a priori different concepts of classical information and fine-graining, respectively. Further investigation into this issue is called for in order to clarify whether the connection between fine-graining and extractable classical information would hold true for other classes of two-qubit states, or could even be extended to the case of higher dimensional systems.

CHAPTER 3

SHARING OF NONLOCALITY IN QUANTUM THEORY

Correlations in quantum theory is discussed in introduction for three different scenarios. Nonlocality pertaining to spatial correlation is one of the well known counter classical features of quantum mechanics. In classical theory correlation obtained through measurements on space like separated systems is always explained in *local realistic* framework. However, there exists quantum correlations emerging from entangled states which cannot be explained by *local realist* model. Due to monogamy nature of nonlocal correlation, which is no-signalling also, it cannot be shared by more than two observers. In this chapter we consider the problem of sharing of nonlocality in a scenario where no-signalling constraint is not valid. Specifically we prove nonlocality pertaining to a single member of an entangled pair of particles can be shared with two independent observers who sequentially perform measurements on the other member of the entangled pair but not more than two. This chapter is based on *Sharing of Nonlocality of a single member of an Entangled Pair Is Not Possible by More Than Two Unbiased Observers on the other wing*, S Mal, A S Majumdar, D Home, Mathematics 4 48(2016) [72].

3.1 MONOGAMY OF NONLOCAL CORRELATION

Monogamy of nonlocal correlation can be demonstrated by considering CHSH inequality in tripartite scenario. Let three observer Alice, Bob and Charlie can perform two possible dichotomic measurements, say, A_x, B_y, C_z with $x, y, z \in \{0, 1\}$. CHSH expression between Alice and Bob is denoted by $\mathcal{B}_{AB}$ and Alice and Charlie is $\mathcal{B}_{AC}$. In [73], Scarani and Gisin shown for three qubit state if $\langle \mathcal{B}_{AB} \rangle > 2$ then $\mathcal{B}_{AC} \leq 2$. In other words if Alice and Bob demonstrate nonlocality then Alice and Charlie cannot. Later Toner and verstraete [74] shown for any arbitrary quantum state shared by three parties, we have $\langle \mathcal{B}_{AB} \rangle^2 + \langle \mathcal{B}_{AC} \rangle^2 \leq 8$. It is to be mentioned here that no-signalling correlation follow monogamy constraint also. So in this way nonlocality cannot be shared by more than two observers. if no-signaling constraint is dropped then monogamy constraint should not be obeyed. Now let us see in what scenario nonlocality can be shared between more than two observers and how far this can go.

3.2 SHARING OF NONLOCALITY

A new fundamental question on the sharing of nonlocality by multiple observers was posed recently [75]: Can the nonlocality pertaining to a single member of an entangled pair of particles be shared among more than two independent observers who sequentially perform measurements on the other member of the entangled pair? Note that the monogamy constraints [73, 76] on entanglement and nonlocality do not apply in this scenario since the condition of no-signalling is violated. Though the observers who perform consecutive measurements are independent of one another, the observer(s) who perform the prior measurement(s) implicitly signal to the latter one(s) through the choice of their measurement(s).

Specification of the problem: The experimental scenario considered here [75] is as follows. One of the particles of an entangled pair is measured by a single observer Alice who performs projective measurements on one side. There exist multiple observers (Bobs) on the other side who act sequentially and independently. Using weak measurements with optimized pointer settings it was shown [75] that Bell-CHSH inequalities between Alice and an arbitrary number of sequential Bobs can be consecutively violated in case of biased observation settings used by the various Bobs. In other words, the protocol works when one of the inputs to the various independent observers occurs a lot more often than the other input. Though, in the unbiased input scenario numerical evidence indicated that violation of the CHSH inequality is impossible by more than two Bobs, it was left as an open problem to show this analytically [75]. next let us describe how this result can be proved analytically.

3.3 Formalism

In this section we concentrate on the formalism which is required for proving the above stated problem analytically. To this end we utilise the framework of unsharp measurements [77] or POVMs with a single parameter, based upon the notion of generalised observables beyond the usual framework projective valued measures (PVM) or sharp measurements. In the measurement process after interaction of the physical system with the apparatus the latter indicates a particular value corresponding to the former. This indication is modelled by means of pointer observable assuming an actual value corresponding to a value of the physical quantity of interest. Actual measurements in which the apparatus are represented by broad meter states, are seldom compatible with PVMs. On the other hand, the generalised notion of POVMs turns out to be very helpful not only in explaining some of the conceptual problems of quantum theory, such as joint measurability of noncommutative observables [78, 79], but in also performing tasks such as probing non-locality in the case of mixed states. There are non-separable mixed states for which the Bell-CHSH inequalities are violated not for the usual pair of sharp spins but only for suitable families of POVMs. For two outcome measurements only projective measurements are sufficient for Bell violation whereas advantages of POVMs are discernible for measurements with more outcomes [80]. This is an illustration of the fact that optimisation of information gain in measurements can under certain conditions only be achieved with POVMs but not with PVMs. A comprehensive introduction to the topic of POVMs and their application in quantum foundations and experiments can be found in the monographs [77, 81] and references therein.

Now we provide a brief discussion on the quantum theory of measurement and POVMs. Then using the formalism of POVMs we show that unsharp observables characterized by a single unsharpness parameter saturate the optimal pointer condition with respect to the trade-off between disturbance and information gain, a condition that was earlier obtained using numerical optimization [75].

3.3.1 Quantum measurements

The minimal content of the notion of measurement in quantum mechanics [82] is given by the probability reproducibility requirement, according to which a particular measurement scheme qualifies as a measurement of a given observable E if for all initial states of the system the associated probability distributions of E are reproduced in the resulting statistics of pointer readings. The information available by a given measurement depends on the statistical dependencies established by the interaction between the system and the apparatus. Let S be the system with associated Hilbert space $\mathcal{H}_{\mathcal{S}}$, and $\mathcal{A}$ be the measuring apparatus with Hilbert space $\mathcal{H}_{\mathcal{A}}$. The initial joint state of system and the apparatus is transformed unitarily during *pre-measurement*, and is given by $\mathcal{U}(\rho_S \otimes \rho_A) \mapsto \mathcal{U}\rho_S \otimes \rho_A \mathcal{U}^*$. An explicit construction of pre-measurement for discrete sharp observables has been known since the work of von Neumann [83]. Any pre-measurement of an observable determines a state transformer on a measurable space $(\Omega, \mathcal{F})$, $\mathcal{I} : \mathcal{F} \mapsto \mathcal{L}(T(H_s))$ through the relation

$$\mathcal{I}_M(X)(\rho) := tr[\mathbb{I} \otimes Z(\mathcal{U}\rho \otimes \rho_A \mathcal{U}^*)\mathbb{I} \otimes Z] \tag{3.1}$$

where, $X \in \mathcal{F}$, and $\mathcal{L}(T(H_s))$ is the set of operators acting on a set of density states. The state transformer summarizes all the features of the pre-measurement. It recovers the observable via the relation

$$tr[E(X)\rho] = tr[\mathcal{I}_M(X)\rho] \forall X \in \mathcal{F}, \rho \in T(H_S) \tag{3.2}$$

The state transformer for projective measurement of a discrete observable A with eigenvalues a_is is given by $\mathcal{I}_M(\rho) = \sum_{a_i \in X} P_i \rho P_i$.

For an observable $A = \sum a_i P_i$ with eigenvalue a_i and eigenprojectors P_i, pre-measurement is given by

$$\mathcal{U}(\varphi \otimes \phi) = \sum P_i \varphi \otimes \phi_i, \varphi \in \mathcal{H}_{\mathcal{S}}, \phi \in \mathcal{H}_{\mathcal{A}} \tag{3.3}$$

Let $\mathcal{Z} = \sum z_i \mathcal{Z}$ be an observable of apparatus $\mathcal{A}$, known as pointer observable. The reduced state of the apparatus is given by $W(\varphi) = \sum_{a_i} p^A_{\varphi}(a_i) P[\phi_i]$ (all the $P[\phi_i]$ are not necessarily mutually orthogonal) with the probability reproducibility condition given by

$$p^A_{\varphi}(a_i) = p^Z_{W(\varphi)}(z_i), \tag{3.4}$$

where $p_{\varphi}^{A}(a_i)$ is the probability distribution of outcomes of the observable A and $p_{W(\varphi)}^{Z}(z_i)$ is that of the pointer observable. Equation (3.4) implies that the outcome probabilities for observable A are recovered as the distribution of the pointer values in the final apparatus state. The emerging observable out of this measurement scheme is given by $E_i = \sum p_{W(\varphi)}^{Z}(z_i)P_j$.

Now, following [81] let us see how POVM emerges quite naturally in an actual measurement on a two level system. Let us take the system-apparatus coupling as $\mathcal{U} = \exp^{i\lambda\sigma.\hat{n}\otimes\mathcal{P}}$. where $\mathcal{P}$ is the momentum operator and the pre-measurement is given by

$$|\Psi\rangle = \mathcal{U}(\varphi \otimes \phi) = \sum P_i\varphi \otimes \exp^{i\lambda a_i \mathcal{P}} \phi = C_+\varphi_+ \otimes \phi_+ + C_-\varphi_- \otimes \phi_- \tag{3.5}$$

Vectors $\exp^{i\lambda a_i \mathcal{P}} \phi$ or $\phi_\pm$ need not be mutually orthogonal. Next, to describe registration of spots on the screen, the pointer observable is modelled by $P_\pm$, projectors onto the upper and lower half of the screen. For unsharp observables the state transformer is given by the generalised Lüder transformer as

$$\mathcal{I}_M(\rho) = \sum_{a_i \in X} \sqrt{E_i}\rho\sqrt{E_i} \tag{3.6}$$

This measurement scheme yields

$$< \Psi|\mathbb{I} \otimes P_\pm\Psi >= |C_+|^2\langle\phi_+|P_+\phi_+\rangle + |C_-|^2\langle\phi_-|P_+\phi_-\rangle := \langle\varphi|E_\pm\varphi\rangle \tag{3.7}$$

where the effects $E_\pm$ constitute the unsharp spin observables actually measured in this experiment, given by

$$E_\pm = \langle\phi_+|P_+\phi_+\rangle P[\varphi_+] + \langle\phi_-|P_-\phi_-\rangle P[\varphi_-] \tag{3.8}$$

with $E_+ + E_- = \mathbb{I}$, and $E_\pm^2 \neq E_\pm$, i.e., $\langle\varphi_+|E_+\varphi_+\rangle, \langle\varphi_-|E_-\varphi_-\rangle \neq 0$ or 1. If the center of mass of the wave-packets $\phi_\pm$ were well separated and localized in the appropriate half planes, i.e., if $\langle\phi_\pm|P_\pm\phi_\pm\rangle = 1$, then $\langle\phi_\pm|P_\mp\phi_\pm\rangle = 0$, in which case $E_\pm$ coincides with $P[\varphi_\pm]$. However, due to spreading of this wave-packet this coincidence is achieved only approximately. This provides a possible source of inaccuracy due to quantum indeterminism inherent in the center of mass wave-function. Even if spin is prepared sharply *a priori*, its value can only be ascribed with some uncertainty.

3.3.2 Optimality of Information Gain Versus Disturbance Trade-off

Following the work of [75], let us consider a spin $1/2$ particle whose initial state is described by the state $|\psi\rangle(=\alpha|0\rangle+\beta|1\rangle)$. Considering von Neumann type measurements, after interaction with a meter with the state $\phi(q)$, the joint state of system and apparatus goes to $\alpha|0\rangle\otimes\phi(q-1)+\beta|1\rangle\otimes\phi(q+1)$. On tracing out the pointer state the reduced state of the system is given by

$$\rho' = F\rho + (1-F)(\pi^{+}\rho\pi^{+} + \pi^{-}\rho\pi^{-}) \tag{3.9}$$

where $\rho=|\psi\rangle\langle\psi|$, and $\pi^{\pm}$ are projectors onto states $|0\rangle,|1\rangle$, and $F(\phi)=\int_{-\infty}^{\infty}\langle\phi(q+1)|\phi(q-1)\rangle dq$, is called the quality factor of the measurement. The probability of getting outcomes '$\pm$' is given by

$$p(\pm) = G\langle\psi|\pi^{\pm}|\psi\rangle + (1-G)\frac{1}{2} \tag{3.10}$$

Here, $G=\int_{-1}^{1}\phi^{2}(q)dq$, which quantifies the precision of the measurement. It is independent of the state of the spin and depends on the width of the pointer compared to the distance between the eigenvalues. These two parameters F and G characterize a weak measurement (the case with $F=0$ and $G=1$ corresponds to a strong measurement). It was found in [75] that a square pointer yields the relation $G=1-F$. However, such a pointer is not optimal. An optimal pointer is defined as the one which gives the best trade-off, i.e., for a given quality factor F, it provides the largest precision G. It was shown that the optimal pointers satisfy the information-disturbance trade-off condition given by $F^2+G^2=1$.

For two outcome measurements the notion of unsharp measurement discussed in previous section is captured by the effect operator, $E^{\lambda}=(\mathbb{I}+\lambda n_i\sigma_i)/2, i=1,2,3.$, with $\lambda\in(0,1]$. Thus, the set of effects can be written as a linear combination of sharp projectors with white noise, $E^{\lambda}\equiv\{E^{\lambda}_{+},E^{\lambda}_{-}|E^{\lambda}_{+}+E^{\lambda}_{-}=\mathbb{I}\}$, given by $E^{\lambda}_{\pm}=\lambda P_{\pm}+\frac{1-\lambda}{2}\mathbb{I}$.

In the unsharp formalism the non-selective un-normalised state of the system after pre-measurement according to the Lüder transformation rule (3.6) is given by $\rho'=\sqrt{E^{\lambda}_{+}}\rho\sqrt{E^{\lambda}_{+}}+$

$\sqrt{E_-^\lambda}\rho\sqrt{E_-^\lambda}$. From this we get

$$\rho' = \sqrt{1-\lambda^2}\rho + (1-\sqrt{1-\lambda^2})(P_+\rho P_+ + P_-\rho P_-) \tag{3.11}$$

The probabilities of getting the outcomes $\pm$ are given by

$$p(\pm) = tr[E_\pm^\lambda \rho] = \lambda tr[P_\pm \rho] + \frac{1-\lambda}{2} \tag{3.12}$$

Comparing the two formalisms, i.e., comparing Equation (3.9) with Equation (3.11) and Equation (3.10) with Equation (3.12), one sees that $G = \lambda$ and $F = \sqrt{1-\lambda^2}$. Hence, λ characterises the precision of the measurement. For $G = \lambda = 1$, F becomes zero, this being the limit of sharp measurement. We thus find that the optimal pointer state condition, $F^2 + G^2 = 1$, derived through an optimization in [75] emerges explicitly within the formalism of unsharp measurements. In other words, unsharp measurement yields the maximum information about the system while disturbing the original state minimally.

3.4 Alice cannot share nonlocality with more than two Bobs

We show here analytically that using a pair of entangled spin $1/2$ particles Alice cannot share non-locality with more than two Bobs. To address this question let us consider the following Bell-CHSH scenario. All the observers have two measurement choices which they perform one at a time. Tsirelson's bound is achieved when Alice measures in the direction $\hat{X}$ and $\hat{Z}$ and Bob measures in directions $\frac{-(\hat{Z}+\hat{X})}{\sqrt{2}}, \frac{-\hat{Z}+\hat{X}}{\sqrt{2}}$. As we want to see how many Bobs can have measurement statistics violating the CHSH inequality with a single Alice, the 1st Bob cannot measure sharply. This would destroy the entanglement between the particles shared by Alice and Bob and there would be no possibility of violation of the CHSH inequality for the 2nd Bob. Hence, in order to share nonlocality among n Bobs, $n-1$ of them have to measure weakly. Here, it is important to note that each Bob measures independently of the previous Bobs on the particle of his possession. Hence, any Bob has to consider the average effect of possible choices of measurements done by previous Bobs [75]. Here, we do not consider multiple Alice's, and thus it is not required to consider unsharp measurement for Alice as it reduces the violation of the CHSH inequality thereby reducing the possibility of sharing nonlocality for multiple Bobs. However, there may be a possibility of getting an advantage if Alice performs sharp nonorthogonal measurements. We comment on this issue later.

The joint probability of getting the outcome 'a' and 'b_n' by Alice and Bob^n (the n-th Bob) respectively, is given by

$$p(a,b_n) = p(a)p(b_n|a) = \frac{1}{2}Tr[\frac{\mathbb{I} + \lambda_n b_n \hat{y_n}.\vec{\sigma}}{2}\rho_{n|y_1 \cdots y_{n-1}}] \tag{3.13}$$

where $\rho_{n|y_1 \cdots y_{n-1}}$ is the state of the pair of spin-$\frac{1}{2}$ particles before the measurements of Alice and Bob^n, and y_i is the measurement done by the i-th Bob. For two Bob measuring in succession, the joint probability is given by

$$p(a,b_2) = \frac{\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{y_2}.\hat{x}}{2} + \frac{1-\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}}{2}. \tag{3.14}$$

The measurement directions chosen for Alice are $\hat{X}$, $\hat{Z}$, and those for Bob are $\frac{-(\hat{Z}+\hat{X})}{\sqrt{2}}$, $\frac{-\hat{Z}+\hat{X}}{\sqrt{2}}$. For the first Bob measuring weakly and the second Bob measuring sharply, the CHSH values are given by $CHSH_{AB_1} = 2\sqrt{2}\lambda_1$, and $CHSH_{AB_2} = \sqrt{2}(1+\sqrt{1-\lambda_1^2})$ respectively. This result coincides with that obtained in [75]. In this case both Bobs obtain violation of the Bell-CHSH inequality when the precision of the 1st Bob remains within the range $1/\sqrt{2}$ and $\sqrt{2(\sqrt{2}-1)}$.

Now consider the case of three Bobs with a single Alice. In this case the 1st and 2nd Bobs both measure weakly, while the last Bob measures sharply. We get

$$p(a,b_3) = \tfrac{1}{2}[\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}\tfrac{1-ab_3\lambda_3\hat{y_3}.\hat{x}}{2} + (1-\sqrt{1-\lambda_1^2})\sqrt{1-\lambda_2^2}\tfrac{1-ab_3\lambda_3\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_3}}{2} +$$
$$\sqrt{1-\lambda_1^2}(1-\sqrt{1-\lambda_2^2})\tfrac{1-ab_3\lambda_3\hat{x}.\hat{y_2}\hat{y_2}.\hat{y_3}}{2} + (1-\sqrt{1-\lambda_1^2})(1-\sqrt{1-\lambda_2^2})\tfrac{1-ab_3\lambda_3\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}\hat{y_2}.\hat{y_3}}{2}]. \tag{3.15}$$

Here λ_2 is precision of measurement by the 2nd Bob. The correlation between Alice and Bob^3 is given by

$$C_3 = \lambda_3[\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}\hat{y_3}.\hat{x} + (1-\sqrt{1-\lambda_1^2})\sqrt{1-\lambda_2^2}\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_3}+$$
$$\sqrt{1-\lambda_1^2}(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y_2}\hat{y_2}.\hat{y_3} + (1-\sqrt{1-\lambda_1^2})(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}\hat{y_2}.\hat{y_3}]. \tag{3.16}$$

As any Bob is ignorant about inputs of previous Bobs, this correlation has to be averaged over all possible earlier inputs. Hence, one has $\bar{C_3} = \sum_{y_1 y_2} C_3 P(y_1)P(y_2)$.

With this average correlation we find the CHSH sum between Alice and Bob^3 given by

$$\mathcal{I}^3 = \frac{(1+\sqrt{1-\lambda_1^2})(1+\sqrt{1-\lambda_2^2})}{\sqrt{2}}. \tag{3.17}$$

For the 1st and 2nd Bobs the corresponding CHSH values are given by $CHSH_{AB_1} = 2\sqrt{2}\lambda_1$ and $CHSH_{AB_2} = \lambda_2\sqrt{2}(1+\sqrt{1-\lambda^2})$, respectively. In order for the 1st Bob to obtain violation of the Bell-CHSH inequality, his measurement precision λ_1 has to be greater than $1/\sqrt{2}$. For the 2nd Bob to get the violation, it is required that $\lambda_2 > \frac{2}{\sqrt{2}+1}$. Thus, it follows from Equation (3.17) that if the first two Bobs obtain violation, the subsequent CHSH value corresponding to Bob^3 cannot be greater than 2. In the worst case scenario of violation of CHSH by Bob^1 and Bob^2, i.e., when both of them obtain minimal violation, $\mathcal{I}^3$ can not becomes greater than 1.88.

The above arguments were based on the assumption of orthogonality for the measurements performed by the Bobs. Let us now consider the scenario when the orthogonality condition for the Bobs' measurements is relaxed. In this case the meausurements performed by the n-th Bob may be denoted as

$$y_n^0 = \cos\theta_{n0}\hat{Z} + \sin\theta_{n0}\hat{X}, y_n^1 = \cos\theta_{n1}\hat{Z} + \sin\theta_{n1}\hat{X}. \tag{3.18}$$

Alice's measurements are the same as before, i.e., $\hat{X}, \hat{Z}$. In the case of non-orthogonal measurements done by all the Bobs CHSH violation is not possible by more than two of them. With the non-orthogonal measurements defined above, we have $\mathcal{I}^1(xy_1) = \lambda_1\tilde{\mathcal{I}}^1(xy_1)$. where $\tilde{\mathcal{I}}^1(xy_1) = (\cos[\theta_{10}] - \cos[\theta_{11}] + \sin[\theta_{10}] + \sin[\theta_{11}])$. After the 1st and 2nd Bob measuring weakly the CHSH expression between Alice and 2nd Bob is given by

$$\mathcal{I}^2(x(y_1)y_2) = \lambda_2[(1-F_1)\tilde{\mathcal{I}}^2(x(y_1)y_2) + F_1\tilde{\mathcal{I}}^1(xy_2)]. \tag{3.19}$$

In the above expression $\tilde{\mathcal{I}}^2(x(y_1)y_2) = \frac{1}{2}\sum_{i,j=0}^{1}((-1)^j\cos\theta_{1i} + \sin\theta_{1i})\cos(\theta_{1i} - \theta_{2j})$, and in the argument of $\tilde{\mathcal{I}}^2$, (y_1) implies averaging over the inputs of Bob^1. Now denoting $F_i = \sqrt{1-\lambda_i^2}$, and with the 3rd Bob measuring sharply, we have

$$\begin{aligned}\mathcal{I}^3(x(y_1y_2)y_3) &= \tfrac{F_1+F_2}{2}\tilde{\mathcal{I}}^1(xy_3) + \tfrac{(1-F_1)F_2}{2}(\tilde{\mathcal{I}}^2(x(y_1)y_3) - \tilde{\mathcal{I}}^2(x(y_1+\tfrac{\pi}{2})y_3)) + \\ &\tfrac{(1-F_2)F_1}{2}(\tilde{\mathcal{I}}^2(x(y_2)y_3) - \tilde{\mathcal{I}}^2(x(y_2+\tfrac{\pi}{2})y_3)) + \tfrac{(1-F_1)(1-F_2)}{16}\tilde{\mathcal{I}}^3(x(y_1y_2)y_3).\end{aligned} \tag{3.20}$$

Here $(y_i + \frac{\pi}{2})$ means averaging over the measurement directions of Bob^i after rotation by $\pi/2$ and $\tilde{\mathcal{I}}^3(x(y_1y_2)y_3) = \sum_{k=0}^{1}[\sum_{i=1,2,j=0,1} 2\sin(2\theta_{ij} - \theta_{3k}) + 4\sin\theta_{3k} + \sum_{i,j=0,1}\sin(2\theta_{1i} - 2\theta_{2j} + \theta_{3k})] + \sum_{k=0}^{1}(-1)^k[\sum_{i=1,2,j=0,1} 2\cos(2\theta_{ij} - \theta_{3k}) + 4\cos\theta_{3k} + \sum_{i,j=0,1}\cos(2\theta_{1i} - 2\theta_{2j} + \theta_{3k})]$.

With the above expressions for most general settings we find that when $\mathcal{I}^1 = 2\mathcal{I}^2 = 2$, then one obtains $\mathcal{I}^3 = 2$. When both the 1st and 2nd Bobs get 5 percent violation, i.e., having CHSH expression equalling 2.1, then $\mathcal{I}^3 \to 1.89$ at most, with the settings $y_1^0 \approx 0.19\hat{Z} + 0.98\hat{X}, y_1^1 \approx -0.19\hat{Z} + 0.98\hat{X}$, $y_2^0 \approx 0.19\hat{Z} + 0.98\hat{X}, y_2^1 \approx -0.19\hat{Z} + 0.98\hat{X}$ and $y_3^0 \approx 0.04\hat{Z} + 0.99\hat{X}, y_3^1 \approx -0.04\hat{Z} + 0.99\hat{X}$.
It should be noted here that even if we consider nonorthogonal measurements by Alice along with all the Bobs we get similar results with expressions involving more variables. There exists settings for which $\mathcal{I}^3$ becomes at most 2 when $\mathcal{I}^1 = \mathcal{I}^2 = 2$. Again, when the 1st and 2nd Bobs get 5 percent violation, $\mathcal{I}^3 \to 1.89$ at most, and for these settings we find that Alice's measurements are orthogonal. It is thus clear that more than two Bobs can never share the nonlocality of a pair of spin 1/2 particles with a single Alice, a result that was numerically conjectured in [75]. One may note that the sequence of the particular Bobs is not important in this scenario. For example, Bob^3 may obtain violation if the sharpness of the 2nd Bob's measurement is sufficiently weak for the latter not to get a violation. There exists a range of unsharpness parameters for each Bob so that any one pair of Bobs in the combinations (Bob^1, Bob^2), (Bob^1, Bob^3), or (Bob^2, Bob^3) can simultaneously demonstrate non-locality.

3.5 Concluding remarks and future perspective

Generalised notion of quantum measurements modelled by POVM has various kind of advantages over notion of sharp von Neumann type measurements in explaining many quantum features. In the context of sharing nonlocality of a single member of an entangled pair of qubits by multiple observers on the other wing it is shown analytically that more than two observers can not share nonlocality by considering unsharp measurements which is one parameter POVM. Next it will be interesting to study what if there are several observers in both wings also.

CHAPTER 4

INCOMPATIBILITY BETWEEN MACROREALISM AND QUANTUM THEORY

Complementing the long pursued exploration of the nonclassical features of the microphysical world, the probing of the implications and validity of QM at the macroscopic level has been gradually gaining momentum over the past decade [84, 85, 86, 87]. Relevant to these studies, a crucial ingredient is provided by the Leggett-Garg form of macrorealist inequality [4] involving testable temporal correlation functions, and its validity can be regarded as a necessary condition for what is known as macrorealism. The notion of macrorealism is characterized by the following assumptions -

Macroscopic realism per se: At any given instant, a macroscopic object is in a definite one of the states available to it.

Non-invasive measurability: It is possible, in principle, to determine which of the states the system is in, without affecting the state itself or the system's subsequent behavior.

There is an another assumption implicit in this context is that measurement result at a time would not be affected by past or future measurements.

Quantum mechanics is incompatible with macrorealism. This chapter is based on two works [88, 89],

i) *Optimal violation of Leggett-Garg inequality for arbitrary spin and emergence of classi-*

cality through unsharp measurement, S Mal, A S Majumdar, Phys. Lett. A **380** pp 2265 2270(2016)

ii) *Wigner's form of the Leggett-Garg inequality, No-Signalling in Time, and Unsharp Measurements*, D Saha, S Mal, P Panigrahi, D Home, Phys. Rev. A **91**, 032117 (2015).

In the first work [88] we show how to obtain optimal violation of LGI involving dichotomic measurements for arbitrary spin system and then how classicality emerges with unsharp measurement. In the second work [89] we derive a new necessary condition of macrorealism dubbed Wigner form of LGI and then show its robustness with compare to conventional LGI with respect to unsharp measurement. We also consider another necessary condition of MR, namely no-signalling in time(NSIT) and demonstrate its maximal robustness among other necessary conditions of MR with respect to unsharp measurement.

4.1 Optimal violation of LGI for arbitrary spin system and emergence of classicality

In probing violation of MR majority of the experiments done with micro-systems. It is desired to study with large systems e. g., systems with large mass or large quantum number such as spin or with large number of constituent subsystems. In this direction Kofler and Brukner studied system with arbitrary large spin[42]. Inspired by the earlier ideas of Peres [8] on the classical limit of quantum mechanics, they have presented a different theoretical approach to emergence of classical physics within quantum theory. They showed that if consecutive eigenvalues of the spin component can be sufficiently resolved, the LGI will be violated for arbitrary large spin. On the other hand, with sufficiently coarse grained measurements, classical laws would emerge for a macroscopic system with very large dimension. This approach is rather different from the decoherence program[7]. However, the violation they obtained for large spin is not maximal. It remains unclear as to why the violation is lesser than the value $2\sqrt{2}$ which is achieved for spin $1/2$ particles and remains constant asymptotically for large spin. The choice of observables may indeed have a role to play in the quantum violation of the LGI. In this section we show there exists measurement scheme so that optimal violation for arbitrary spin is obtained.

4.1.1 Optimal violation of LGI for arbitrary spin

In order to obtain optimal violation of the LGI for arbitrary spin, we employ a variant of the measurement scheme suggested earlier by Gisin and Peres [90] in the context of testing local realism. Interestingly, for the case of spatial correlations the above measurement scheme [90] yields maximal violation of a local realist inequality only for half-integral spin systems. For integral spin systems the amount of violation drops, and the value of $2\sqrt{2}$ is achieved only when the spin becomes infinitely large.

We now show how the maximum correlation up to $2\sqrt{2}$ which is the upper bound of quantum theory for dichotomic measurements, can be achieved not only for spin $1/2$ particles, but for systems having arbitrary spin too.

***Lemma**:* If a dichotomic observable Q is measured successively at times t_i and t_j on any state ρ of a two dimensional system evolving unitarily, then the two-time correlation function is given by $C_{ij} = \frac{1}{2}tr[Q(t_i)Q(t_j)]$. Here $Q(t_i) = U^\dagger(t_i)QU(t_i)$ and $Q(t_j) = U^\dagger(t_j)QU(t_j)$ are time evolved observables in the Heisenberg picture.

***Proof**:* The initial state ρ is evolved to $U(t_i)\rho U^\dagger(t_i)$. At t_i, Q is measured and according to the outcome '$\pm$', the post-measurement state becomes $P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm/tr[P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm]$, where $P_\pm$ are the two orthogonal projectors of the observable Q, and $P_\pm U(t_i)\rho U^\dagger(t_i)P_\pm/p_\pm = P_\pm$. Here, $p_\pm = tr[P_\pm U(t_i)\rho U^\dagger(t_i)]$, are probability of getting outcomes '$\pm$'. Again, this post-measurement state is evolved to time t_j and becomes $U(\Delta t)P_\pm U^\dagger(\Delta t)$, with $\Delta t = t_j - t_i$. Now, the conditional probabilities are given by $p_{k|l} = tr[P_k U(\Delta t)P_l U^\dagger(\Delta t)$, where $p_{k|l}$ denotes the probability of getting an outcome k at time t_j when the outcome l occurs at time t_i. Hence, the two-time correlation is given by

$$\begin{aligned} C_{ij} &= p_+(p_{+|+} - p_{-|+}) + p_-(p_{-|-} - p_{+|-}) \\ &= p_+(tr[(P_+ - P_-)U(\Delta t)P_+U^\dagger(\Delta t)]) + p_-(tr[(P_- - P_+)U(\Delta t)P_-U^\dagger(\Delta t)]) \end{aligned} \tag{4.1}$$

Now, using $P_+ - P_- = Q$, $tr[Q] = 0$, $p_+ + p_- = 1$, and the cyclic property of the trace, we have $C_{ij} = tr[P_+U^\dagger(\Delta t)QU(\Delta t)]$. Since, $P_+ = \frac{\mathbb{I}+Q}{2}$ and $U(\Delta t) = U(t_2)U^\dagger(t_1)$, we finally have

$$C_{ij} = tr[QU^\dagger(\Delta t)QU(\Delta t)]/2 = tr[U^\dagger(t_1)QU(t_1)U^\dagger(t_2)QU(t_2)]/2. \tag{4.2}$$

This completes the proof of the lemma.

Theorem: For any state of a single quantum system of arbitrary spin there exists observables with eigenvalues ± 1 and a measurement scheme such that the Leggett-Garg inequality can be maximally violated.

Proof: Let $\Gamma_x, \Gamma_y, \Gamma_z$ be block-diagonal matrices, in which each block is an ordinary Pauli matrix, σ_x, σ_y and σ_z respectively. The only nonvanishing elements are given by

$$(\Gamma_x)_{2n-1,2n} = (\Gamma_x)_{2n,2n-1} = 1, (\Gamma_y)_{2n-1,2n} = i, (\Gamma_y)_{2n,2n-1} = -i, (\Gamma_z)_{n,n} = (-1)^{n-1}. \tag{4.3}$$

Suppose mixed states of spin j particles coming from a source are in diagonal form in some basis $\{|k\rangle\}$, i.e.,

$$\sum_{k=-j}^{k=j} p_k|k\rangle\langle k| = \sum_{k=1/2(0)}^{k=j} (p_k| + k\rangle\langle +k| + p_{-k}| - k\rangle\langle -k|) \tag{4.4}$$

where, $\sum_{k=-j}^{k=j} p_k = 1$. We define an observable Q following [90] in the way given below:

$$Q = \frac{\Gamma_z + \Pi}{\sqrt{2j+1}} = (\sigma_z^1 \oplus \sigma_z^2 + ... \oplus \sigma_z^j + \Pi)/\sqrt{2j+1} \tag{4.5}$$

where, Π is the null matrix when $N(=2j+1)$ is even, and for odd N the only nonvanishing element of Π is $(\Pi)_{N,N} = \frac{1}{\sqrt{2}}$. In order to maintain optimal violation of the four-term LGI, we require our time-evolved observables to remain in the block diagonal form mentioned above. This can not be ensured by arbitrary rotations of the SG apparatus in space, except for two dimensional systems. However, this is achieved if each block is evolved separately [90]. As $\oplus_j \exp^{i\theta_j\sigma_x} \sigma_z \exp^{-i\theta_j\sigma_x} = \oplus_j \exp^{i\theta_j\sigma_x} \oplus_j \sigma_z^j \oplus_j \exp^{-i\theta_j\sigma_x}$, time evolution of the system is affected by

$$U(t) = \exp^{-i\theta_1\sigma_x} \oplus \exp^{-i\theta_2\sigma_x} \oplus ... \oplus \exp^{-i\theta_j\sigma_x} \tag{4.6}$$

We explain in next paragraph explicitly how this kind of evolution and measurements are realised experimentally. First, the system is evolved to time t_1 and Q is measured. The post-measurement state is further evolved to time t_2, and Q is measured again. This scheme can be recast into the Heisenberg picture. Taking all $\theta_j = \omega t/2 = \alpha$, we have

$$U^{\dagger}(t)QU(t) = (\cos\alpha\Gamma_z + \sin\alpha\Gamma_y + \Pi)/\sqrt{2j+1}. \tag{4.7}$$

The two-time correlation function C_{12} using lemma 1, for even N is given by

$$C_{12} = Tr[U^{\dagger}(t_1)QU(t_1)U^{\dagger}(t_2)QU(t_2)] = [\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2]. \tag{4.8}$$

For odd N, one gets

$$C_{12} = \frac{2j(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) + 1/\sqrt{2}}{(2j+1)}. \tag{4.9}$$

Similarly, C_{23} and C_{34} are also obtained. For obtaining the correlation function C_{14}, the operator Q is taken to be $(\Gamma_z - \Pi)/\sqrt{2j+1}$. Now, in order to obtain the maximal violation of the LGI, we choose the time intervals such that $\alpha_1 = 0, \alpha_2 = \pi/4, \alpha_3 = \pi/2, \alpha_4 = 3\pi/4$. Hence, the value of the Leggett-Garg sum for the spin j system is given by $2\sqrt{2}$.

Proposed measurement scheme: We briefly discuss the relevant measurement scheme. Consider single spin j particles emerging from an initial ensemble. These particles are assumed to possess not only a magnetic moment (an interaction energy $\mu B_z J_z$), but also an electric quadrupole moment (an interaction energy proportional to $E_z J_z^2$). There are two major stages of the experiment. In the 1st stage the particles pass through a inhomogeneous electric fields producing beams with $|m_z| = j, j-1...0(1/2)$. Taken separately, each of these beams with given $|m_z|$ are passed through a uniform magnetic field B_z producing energy difference, E, between them. Then an rf pulse generates Rabi oscillations of frequency $\omega = E/\hbar$ among them. This captures the unitary evolution given by (4.6). After evolving for a time t_1, σ_z is measured. Then, each post-measured beam is again evolved, and at a time t_2, σ_z is measured. This same procedure is done many times varying the time of measurements randomly. Correlation statistics are calculated from the measured data of the LG test.

4.1.2 Unsharp measurement and emergence of classicality

The purpose of this section is to show that with sufficiently unsharp measurements, the outcome statistics can be described by classical theory. Using a particular form of coarse graining it was shown in Ref. [42] that when the resolution of the apparatus is much greater than the intrinsic quantum uncertainty, i.e., $\Delta m >> \sqrt{j}$, the outcomes appear to obey classical laws. Under this condition the measurements become fuzzy enough for the non-invasive assumption to become essentially valid and the system dynamics mimics the rotation of a classical spin coherent state for large j. However, there does not exist any sharp cut-off for the value of the apparatus resolution beyond which classicality emerges. Now the question is, can the fuzzy measurement leads to classicality for systems of any dimension, and what is the precise value of the apparatus resolution above which the condition of coarse graining is satisfied ?

Here we follow the theory of unsharp observables [78, 79, 81] which as an element of unsharp reality provides the necessary ingredient in modelling of the emergence of classical behaviour within quantum mechanics in a precise and quantitative manner. Using this formalism we show that below a threshold sharpness of measurement, classicality emerges in case of system of any dimension. Unsharp measurement is described in chapter three. For self consistency of this chapter we briefly recall the formalism of unsharp measurements [78, 79, 81] relevant to our present analysis and how it can be cast as coarse-grained measurement.

In projector valued measurements the observables are self-adjoint operators having projectors as spectra, i.e., $A \equiv \{P_i | \sum P_i = \mathbb{I}, P_i^2 = P_i\}$. The probability of getting the i-th outcome is $tr[\rho P_i]$ for the state ρ. Extending to positive operator valued measures (POVM), the observables are self-adjoint operators but with spectra as positive operators within the interval $[0, \mathbb{I}]$, i.e., $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 < E \leq \mathbb{I}\}$. Similarly, the probability of getting the i-th outcome is $tr[\rho E_i]$. Effects ($E_i s$) represent quantum events that may occur as outcomes of a measurement. Here usharp measurement which is a POVM arises coarse-graining of underlying projectors which have sharply determined properties. Effect operators pertinent to POVM can be decomposed as some linear combinations of projectors. For two outcome measurements this notion is captured by the effect, $E_\lambda = (\mathbb{I} + \lambda n_i \sigma_i)/2, i = 1, 2, 3.$, with $\lambda \in (0, 1]$. Thus, the set of effects can be written as a linear combination of sharp projectors and $E_\lambda \equiv \{E_+^\lambda, E_-^\lambda | E_+^\lambda + E_-^\lambda = \mathbb{I}\}$, given by

$$E_\pm^\lambda = \frac{1+\lambda}{2} P_\pm + \frac{1-\lambda}{2} P_\mp \tag{4.10}$$

Again eq.4.10 can also be written as$= \lambda P_\pm + \frac{1-\lambda}{2}\mathbb{I}$. This can be thought of as projectors becoming noisy reflecting inaccuracy of the experiment. Hence, the sharpness parameter λ can be estimated from the difference between the really observable data and that predicted by sharp observables.

Under this unsharp measurement, the state transformation for the maximally mixed initial state is given by the generalised Lüdder's operation

$$\rho_\pm^{PM}(t_1) = \sqrt{E_{\pm\lambda}}\rho\sqrt{E_{\pm\lambda}}/tr[\sqrt{E_{\pm\lambda}}\rho\sqrt{E_{\pm\lambda}}] = \frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z). \tag{4.11}$$

The probability of getting '$\pm$' outcomes are both $1/2$. In order to formulate the relevant LGI, the $\rho_\pm^{PM}(t_1)$ is evolved for time Δt, giving $\exp^{-i\frac{\omega\Delta t}{2}\sigma_x}\frac{1}{2}(\mathbb{I} \pm \lambda\sigma_z)\exp^{i\frac{\omega\Delta t}{2}\sigma_x} = \frac{\mathbb{I}}{2} \pm \frac{\lambda}{2}(\cos(\omega\Delta t)\sigma_z + \sin(\omega\Delta t)\sigma_y)$. With this post-measurement state we find the conditional probabilities and the two-time correlation function given by $C_{12} = \lambda^2\cos(\omega\Delta t)$. Hence, the LGI with unsharp measurement can be written as $K \equiv \lambda^2\langle LGI\rangle \leq 2$, where $\langle LGI\rangle$ denotes the corresponding expression for sharp measurements. Since $\langle LGI\rangle_{max} = 2\sqrt{2}$, hence it follows that in the case of unsharp measurements the LGI for a spin $1/2$ system is always satisfied when the sharpness parameter upper bounded by $\lambda < 1/2^{1/4}$.

Now, let us consider a system having arbitrary spin. As discussed in our conceptual scheme of measurement in the previous section, in the 1st stage, particles of spin j are sent from the source to an inhomogeneous electric field. After emerging from the field, each beam is effectively described by a two dimensional Hilbert space, and evolves under the same unitary as discussed in previous section. Here we do not consider fuzziness in measurement in the 1st stage but in the 2nd stage where measurements done for LG test. 1st stage of the experiment allows us to get optimal violation for subsequent LG test. it is practically not possible to separate each beams perfectly in 1st stage for higher spin. If beams are taken well distant from the apparatus then each beam can be distinguished in principle [8]. If the 1st stage is omitted and LG test is done with dichotomic measurements (like parity)on spin j then it reduces to problem dealt in [42] with violation lesser than optimal. In this case fuzzy parity measurement can also be treated with unsharp formalism just discussed above. In our case, in the 2nd stage, beams are subjected to non-ideal Stern-Gerlach apparatus [91]. In this scenario the effective spin j observable is given by $Q = \lambda((\Gamma_z + \Pi)/\sqrt{2j+1})$, where $0 < \lambda \leq 1$. Using the lemma it is straightforward to calculate the two-time correlation function. When N is even, we have

$$C_{12} = \lambda^2[\cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2] \tag{4.12}$$

and for N odd, it becomes

$$C_{12} = \lambda^2[2j\cos\alpha_1 \cos\alpha_2 + 2j\sin\alpha_1 \sin\alpha_2 + 1/\sqrt{2}]/(2j+1) \tag{4.13}$$

For both the even and odd cases the optimal value of the sharpness parameter below which no quantum violation of LGI is possible is upper bounded by $1/2^{1/4} \equiv 0.841$. Note in comparison that using the maximal violation of the LGI for large spin obtained in Ref. [42], the required value of this parameter would be 0.8978 in order to ensure satisfaction of the LGI. Note also, that for the case of spatial correlations, a higher value of the sharpness parameter would be required to ensure satisfaction of the relevant local realist inequality, since the maximal bound there drops for the case of integral spin [90]. For the case of spin-1 particles the value of the required sharpness for spatial correlations turns out to be 0.8852 and coincides with our optimal value (0.841) for temporal correlations with infinitely large integral spin.

4.2 Wigner type formulation of LGI

In this section we discuss a new necessary condition of MR following Wigner's argument. This earlier unexplored variant of LGI is formulated by developing an analogy to Wigner's form of local realist inequality [92], parallel to the way the usual form of LGI is obtained analogous to the Bell-CHSH form of local realist inequality [2, 93]. Interesting features of the QM violation of Wigner's form of LGI (WLGI) are studied by considering two state oscillation. Subsequently, the robustness of the *QM violation of WLGI* with respect to the imprecision of measurements involved is investigated by using the formalism of what is known as unsharp measurement [78]. Interestingly, it is found that the QM violation of Wigner's form of LGI is more robust against unsharpness of measurement than the usual forms of LGI.

Furthermore, we consider the other proposed necessary condition for macro-realism known as 'No-signalling in time' (NSIT) [94]. This condition turns out to be most robust against unsharp measurement. We conclude by discussing the significance of such results in the light of analysing the emergence of classicality within quantum theory due to unsharpness of measurements involved.

4.2.1 Derivation of WLGI

We begin by recapitulating Wigner's original argument [92] that derived a local realist testable inequality for a pair of spatially separated spin-1/2 particles in the singlet state. This was based on assuming as a consequence of local realism, the existence of overall joint probabilities of the predetermined definite outcomes of measuring the relevant dichotomic observables of the two particles that would yield the pair-wise marginal joint probabilities which are actually measurable. In the scenario studied by Wigner, the spin components of each of the two spatially separated particles are taken to be measured along three respective directions, say, $\hat{a}, \hat{b}$ and $\hat{c}$. Then consider, for example, the observable joint probability of obtaining both the outcomes +1 if, say, $\vec{\sigma}.\hat{a}$ and $\vec{\sigma}.\hat{b}$ are measured on the first and the second particle respectively, denoted as $P(\hat{a}+, \hat{b}+)$. Using the perfect anticorrelation property of the singlet state in question, $P(\hat{a}+, \hat{b}+)$ can be written as marginal in the form $P(\hat{a}+, \hat{b}+) = \rho(+,-,+;-,+,-) + \rho(+,-,-;-,+,+)$ with $\rho(v_1(\hat{a}), v_1(\hat{b}), v_1(\hat{c}); v_2(\hat{a}), v_2(\hat{b}), v_2(\hat{c}))$ to be the overall joint probability of the predetermined definite outcomes of measurements pertaining to all the relevant observables, where $v_1(\hat{a})$ represents the outcome (± 1) of measurement of the observable $\hat{a}$ for the first particle, and so on. Similarly, considering the expressions for the observable joint probabilities $P(\hat{c}+, \hat{b}+)$ and $P(\hat{a}+, \hat{c}+)$ as marginals, and assuming non-negativity of the overall joint probability distributions, it follows that

$$P(\hat{a}+, \hat{b}+) - P(\hat{a}+, \hat{c}+) - P(\hat{c}+, \hat{b}+) \leq 0 \tag{4.14}$$

which is one of the possible forms of Wigner's version of the local realist inequality.

Next, in order to obtain WLGI by developing an appropriate analogy with the preceding argument, we proceed as follows. Let us focus our attention on an ensemble of systems undergoing temporal evolution involving oscillation between the two states, say, 1 and 2, and let $Q(t)$ be an observable quantity such that, whenever measured, it is found to take a value +1(-1) depending on whether the system is in the state 1(2). Now, consider a collection of sets of experimental runs, each set of runs starting from the same initial state. On the first set of runs, let Q be measured at times t_1 and t_2, on the second Q be measured at t_2 and t_3, and on the third at t_1 and t_3 (here $t_1 < t_2 < t_3$). From such measurements one can then determine the pair-wise joint probabilities like $P(Q_1, Q_2), P(Q_2, Q_3), P(Q_1, Q_3)$ where Q_i is the outcome (± 1) of measuring Q at t_i, $i = 1, 2, 3$. In this context, it is possible to suitably adapt the argument leading to Wigner's inequality (4.14) with the times t_i of measurement playing the role of apparatus settings and by assuming, as a consequence of the assumptions of realism and NIM, the existence of overall joint probabilities $\rho(Q_1, Q_2, Q_3)$ from which by appropriate marginalization the pair-wise joint probabilities can be obtained. For example, the observable joint probability $P(Q_2+, Q_3-)$ of obtaining the outcomes $+1$ and -1 for the sequential measurements of Q at the instants t_2 and t_3 respectively can be written as

$$P(Q_2+, Q_3-) = \sum_{Q_1=\pm} \rho(Q_1, +, -) = \rho(+, +, -) + \rho(-, +, -) \tag{4.15}$$

Writing similar expressions for the other measurable marginal joint probabilities $P(Q_1-, Q_3-)$ and $P(Q_1+, Q_2+)$, we get

$$P(Q_1+, Q_2+) + P(Q_1-, Q_3-) - P(Q_2+, Q_3-) = \rho(+, +, +) + \rho(-, -, -) \tag{4.16}$$

Then, invoking non-negativity of the joint probabilities occurring on the RHS of Eq(4.16), the following form of WLGI is obtained in terms of three pairs of two-time joint probabilities

$$P(Q_2+, Q_3-) - P(Q_1+, Q_2+) - P(Q_1-, Q_3-) \leq 0 \tag{4.17}$$

Similarly, other forms of WLGI involving three pairs of two-time joint probabilities can be derived by using various combinations of the observable joint probabilities, which are as follows

$$P(Q_2+, Q_3+) - P(Q_1-, Q_2+) - P(Q_1+, Q_3+) \leq 0 \tag{4.18a}$$

$$P(Q_2+, Q_3-) - P(Q_1-, Q_2+) - P(Q_1+, Q_3-) \leq 0 \tag{4.18b}$$

$$P(Q_2+, Q_3+) - P(Q_1+, Q_2+) - P(Q_1-, Q_3+) \leq 0 \tag{4.18c}$$

$$P(Q_1+, Q_3-) - P(Q_1+, Q_2-) - P(Q_2+, Q_3-) \leq 0 \tag{4.18d}$$

$$P(Q_1+, Q_3-) - P(Q_1+, Q_2+) - P(Q_2-, Q_3-) \leq 0 \tag{4.18e}$$

$$P(Q_1+, Q_3+) - P(Q_1+, Q_2+) - P(Q_2-, Q_3+) \leq 0 \tag{4.18f}$$

$$P(Q_1+, Q_3+) - P(Q_1+, Q_2-) - P(Q_2+, Q_3+) \leq 0 \tag{4.18g}$$

$$P(Q_1+, Q_2-) - P(Q_1+, Q_3-) - P(Q_2-, Q_3+) \leq 0 \tag{4.18h}$$

$$P(Q_1+, Q_2-) - P(Q_1+, Q_3+) - P(Q_2-, Q_3-) \leq 0 \tag{4.18i}$$

$$P(Q_1+, Q_2+) - P(Q_1+, Q_3+) - P(Q_2+, Q_3-) \leq 0 \tag{4.18j}$$

$$P(Q_1+, Q_2+) - P(Q_1+, Q_3-) - P(Q_2+, Q_3+) \leq 0 \tag{4.18k}$$

Altering the signs in each of the above equations, another set of 12 such 3-term WLGIs can be obtained.

Now, let the observable Q be measured in n different pairs of instants t_i $(i = 1, 2, ..., n)$. From the notion of macro-realism, one can again assume the existence of the overall joint probability distributions $\rho(Q_1, Q_2, ..., Q_n)$. Considering the pair-wise observable joint probabilities as the marginals of the overall joint probability distributions, we then get the following relation

$$\begin{aligned} &P(Q_1+, Q_2-) + P(Q_2+, Q_3-) + ... + P(Q_{n-1}+, Q_n-) \\ &= P(Q_1+, Q_n-) + (n-2)2^{n-2} \text{ non-negative terms} \end{aligned}$$

From the above expression, the form of WLGI in terms of n pairs of two-time joint probabilities can be obtained as follows

$$P(Q_1+, Q_n-) - (\sum_{i=1}^{n-1} P(Q_i+, Q_{i+1}-)) \leq 0 \tag{4.19}$$

Other various forms of the n-term WLGI can be similarly obtained by using different combinations of the joint probabilities for the outcomes (± 1) corresponding to Q_i's. However, for illustrating the basic relevant features concerning the efficacy of WLGI, it suffices for our subsequent treatment to confine our attention to essentially 3-term WLGI involving three pairs of two-time joint probabilities.

4.2.2 Example of two state oscillating system

Next, considering a typical two-state oscillation, let us focus on a system oscillating between the two states $|A\rangle$ and $|B\rangle$ which are degenerate eigenstates of the Hamiltonian H_0 corresponding to energy E_0, with a perturbing Hamiltonian H' inducing oscillatory transition between these two states. Here $\langle A|H'|B\rangle = \langle B|H'|A\rangle = \Delta E$, and $\langle A|H'|A\rangle = \langle B|H'|B\rangle = E'$. At any instant, such a system is found to be either in the state $|A\rangle$ or in the state $|B\rangle$ corresponding to the measurement of the dichotomic observable $Q = |A\rangle\langle A| - |B\rangle\langle B|$. Let the initial state at t_1 be of the general form $\rho_0(t_1) = |\psi_0\rangle\langle\psi_0|$ where

$$|\psi_0\rangle = \cos(\theta)|A\rangle + exp(i\phi)\sin(\theta)|B\rangle \tag{4.20}$$

For the above state, the probability of obtaining the measurement outcome, say, +1 at the instant t_1 is given by $tr(\rho_0(t_1)P_+)$, and after this measurement, the premeasurement state changes to the state given by $\rho_+(t_1) = P_+\rho_0(t_1)P_+^\dagger/tr(\rho_0(t_1)P_+)$ where $P_+ = |A\rangle\langle A| = P_+^\dagger$. Subsequently, the post-measurement state evolves under the Hamiltonian $H = H_0 + H'$ to the state $\rho'_+(t_2) = U_{\Delta t}\rho_+(t_1)U_{\Delta t}^\dagger$ at a later instant t_2 where $U_{\Delta t} = exp(-iH\Delta t)$ taking $\hbar = 1$ and $\Delta t = t_2 - t_1$. Then, considering the subsequent measurement of Q at the instant t_2, the QM value of, say, the joint probability of obtaining both the outcomes +1 at the instants t_1 and t_2 is given by

$$P(Q_1+, Q_2+) = tr(\rho_0(t_1)P_+)tr(\rho'_+(t_2)P_+) = tr(U_{\Delta t}(P_+\rho_0(t_1)P_+)U_{\Delta t}^\dagger P_+) = \cos^2(\theta)\cos^2(\tau) \tag{4.21}$$

where $\tau = \Delta E \Delta t$ (in the units of $\hbar = 1$), and the expression for the unitary matrix $U_{\Delta t} = exp(-iH\Delta t)$ is as follows

$$U_{\Delta t} = e^{-i(E_0+E')\Delta t}[\cos(\tau)\mathbb{I} - i\sin(\tau)(|A\rangle\langle B| + |B\rangle\langle A|)] \tag{4.22}$$

Similarly, one can obtain the QM values of the other relevant joint probabilities occurring on the LHS of the 3-term WLGIs given by Eqs. (4.17)-(4.18) (taking $t_2 - t_1 = t_3 - t_2 = \Delta t$). The QM violations of the inequalities (4.17) and (4.18) can thus be studied by maximizing the QM values of their respective left hand sides with respect to the quantity $\tau = \Delta E \Delta t$.

It has been found that the QM violation of the inequalities (4.17)-(4.18) depend on the the initial state and, among all the 3-term WLGIs (4.17)-(4.18) and the set of other 3-term inequalities obtained from them, the maximum QM violation is obtained of the inequalities (4.17) and (5a) when the LHS = 0.5043. Considering, for example, the inequality (4.17), the maximum QM violation (= 0.5043) occurs for the initial state given by Eq.(4.20) when $\theta \approx 1.07$ rad, $\phi \approx \pi/2$ or $3\pi/2$ corresponding to the choice of $\tau = 1.009$ or 2.135 (in the units of $\hbar = 1$) respectively. In the discussions of the following two sections, we will be specifically using this form of 3-term WLGI given by the inequality (4).

4.2.3 Comparison between WLGI and LGI with respect to unsharp measurement

In the preceding discussions, we have taken the relevant measurements of the observable Q to be essentially 'ideal'. Now, if the 'non-idealness' of actual measurements has to be taken into account, a natural question arises as to what effect this would have on the QM violation of WLGI as compared to that for LGI. In order to address this question, we take recourse to the formalism of what is known as unsharp measurement [78, 81] which can be regarded as a particular case of POVM. We discuss unsharp formalism in earlier section.

With unsharp measurements two time joint probability becomes

$$
\begin{aligned}
&P(Q_1+,Q_2+) = tr(U_{\Delta t}(\sqrt{F_+}\rho_0(t_1)\sqrt{F_+}^{\dagger})U_{\Delta t}^{\dagger}F_+) \\
&= \frac{1}{4}[\lambda^2\cos(2\tau) + 2\lambda\cos(2\theta)\cos^2(\tau) + \lambda\sin(2\theta)\sin(2\tau)\sin(\phi)\sqrt{1-\lambda^2} + 1] \\
&P(Q_2+,Q_3-) = tr(U_{\Delta t}(\sqrt{F_+}(U_{\Delta t}\rho_0(t_1)U_{\Delta t}^{\dagger})\sqrt{F_+}^{\dagger})U_{\Delta t}^{\dagger}F_-) \\
&= \frac{1}{4}[2\lambda\sin^2(\tau)(\cos(2\theta)\cos(2\tau) + \sin(2\theta)\sin(2\tau)\sin(\phi)) - \lambda^2\cos(2\tau) \\
&+ \lambda\sqrt{1-\lambda^2}\sin(2\tau)(\sin(2\tau)\cos(2\theta) - \cos(2\tau)\sin(2\theta)\sin(\phi)) + 1] \\
&P(Q_1-,Q_3-) = tr(U_{2\Delta t}(\sqrt{F_-}\rho_0(t_1)\sqrt{F_-}^{\dagger})U_{2\Delta t}^{\dagger}F_-) \\
&= \frac{1}{4}[\lambda^2\cos(4\tau) - 2\lambda\cos(2\theta)\cos^2(2\tau) - \lambda\sin(2\theta)\sin(4\tau)\sin(\phi)\sqrt{1-\lambda^2} + 1]
\end{aligned}
\tag{4.23}
$$

In the above expressions (4.23), for the parameters characterizing the initial state, we now put $\theta = 1.07$ rad and $\phi = 3\pi/2$ or $\pi/2$. Recall that for these specific choices, as mentioned towards the end of the preceding section, the QM violation of the 3-term WLGI given by (4.17) is maximum using the joint probabilities calculated for ideal measurements. Now with this parameters for unsharp measurement it is found that for $\lambda \leq 0.69$, no violation of WLGI is possible.

The general form of usual LGI involving *n* pairs of two-time correlation functions can be expressed in the following way [44]

$$
\begin{aligned}
&-n \leq K_n \leq n-2 \ \text{ for odd } n \geq 3 \\
&-(n-2) \leq K_n \leq n-2 \ \text{ for even } n \geq 4
\end{aligned}
\tag{4.24}
$$

where $K_n = C_{21} + C_{32} + C_{43} + \ldots.. + C_{n(n-1)} - C_{n1}$, and the correlation function $C_{ij} = \langle Q_i Q_j \rangle$. Considering unsharp measurements, the correlation function for any initial state is obtained in the following form

$$
\begin{aligned}
&\langle Q_i Q_j \rangle_{unsharp} = P(Q_i+,Q_j+) + P(Q_i-,Q_j-) - P(Q_i-,Q_j+) - P(Q_i+,Q_j-) \\
&= tr(U_{\Delta t}(\sqrt{F_+}\rho(t_i)\sqrt{F_+}^{\dagger})U_{\Delta t}^{\dagger}F_+) + tr(U_{\Delta t}(\sqrt{F_-}\rho(t_i)\sqrt{F_-}^{\dagger})U_{\Delta t}^{\dagger}F_-) \\
&- tr(U_{\Delta t}(\sqrt{F_-}\rho(t_i)\sqrt{F_-}^{\dagger})U_{\Delta t}^{\dagger}F_+) - tr(U_{\Delta t}(\sqrt{F_+}\rho(t_i)\sqrt{F_+}^{\dagger})U_{\Delta t}^{\dagger}F_-) \\
&= \lambda^2\cos(2\tau) = \lambda^2\langle Q_i Q_j \rangle_{sharp}
\end{aligned}
\tag{4.25}
$$

where $\langle Q_i Q_j \rangle_{sharp}$ is the correlation function for sharp measurements corresponding to $\lambda = 1$. Using Eq. (4.25) and the result that for any given *n*, the maximum QM value of K_n for sharp measurements has been found to be $n\, cos(\pi/n)$ [44], it follows that for unsharp measurements, if the QM predictions are to satisfy the general form of LGI given by (4.24), the following inequality needs to hold good

$$\lambda^2 n\, cos(\pi/n) \leq n - 2 \qquad \Rightarrow \lambda \leq \sqrt{\frac{n-2}{n\, cos(\pi/n)}} \tag{4.26}$$

Note that as *n* increases, the RHS of (4.26) also increases, thereby implying an increase of the critical value of λ (denoted by, say, λ_c) above which, as measurements become more precise, the QM results can violate the general form of LGI given by (4.24). The minimum value of $\lambda_c(= \sqrt{2/3} = 0.81)$ occurs for $n = 3$. For $n = 4$, λ_c is given by $(1/2)^{1/4} = 0.84$ which is the same as the corresponding λ_c [95] obtained for the Bell-CHSH inequality.

Now, comparing the above mentioned minimum value of $\lambda_c(= 0.81)$ for LGI with the corresponding critical value of $\lambda_c(= 0.69)$ for the 3-term WLGI given by (4.17), it is seen that for the values of λ lying within the range given by $0.69 < \lambda \leq 0.81$, the QM predictions can violate WLGI, but will satisfy LGI. In other words, for the range of values of $\lambda \in (0.69, 0.81]$ corresponding to unsharp or imprecise measurements, the QM violation of macrorealism can be tested using the 3-term WLGI, but not in terms of LGI. This, therefore, underscores the efficacy of WLGI and its non-equivalence with LGI. Such a comparison can be extended for WLGIs involving more than three pairs of two-time joint probabilities. Intuitively, though, it can be argued that, as the number of pairs of measurement increases, the robustness of the QM violation of WLGIs against unsharp measurement is expected to decrease.

4.3 No-Signalling in Time and Unsharp Measurement

As already mentioned in the previous section, an alternative necessary condition for the validity of macrorealism has recently been proposed [94] by assuming that the outcome statistics of a measurement would remain unaffected by any prior measurement. This condition, referred to as 'No-Signalling in Time' (NSIT), is the statistical version of NIM used in deriving LGI and can be viewed as an analogue of the no-signalling condition for the spacelike separated measurements used in the EPR-Bohm scenario, with the difference that while any violation of the latter would violate special relativity, violation of NSIT is not inconsistent with special relativity. Now, in order to express NSIT in a mathematical form, let us again consider a system oscillating in time between two possible states, as discussed earlier. The probability of obtaining a particular outcome, say, +1 for the measurement of a dichotomic observable Q at an instant, say, t_2, without any earlier measurement being performed, be denoted by $P(Q_2 = +1)$. NSIT requires that $P(Q_2 = +1)$ should be the *same* even when an earlier measurement of, say, Q is made at an instant, say, t_1. In other words, if we denote by $P(Q_2 = +1|Q_1 = \pm 1)$ the probability of obtaining an outcome +1 for the measurement of Q at the instant t_2 when an earlier measurement of Q has been performed at t_1 having an outcome ± 1, NSIT can be expressed as the equality condition given by $P(Q_2 = +1) = P(Q_2 = +1|Q_1 = \pm 1)$ which implies that

$$P(Q_2 = +1) = P(Q_1+, Q_2+) + P(Q_1-, Q_2+) \tag{4.27}$$

where the terms on the RHS of Eq. (4.27) are the relevant joint probabilities.

Now, pertaining to the two-state oscillation between the states $|A\rangle$ and $|B\rangle$ with the state $\rho_0(t_1) = |\psi_0\rangle\langle\psi_0|$ at the instant t_1 where $|\psi_0\rangle = \cos\theta|A\rangle + exp(i\phi)\sin\theta|B\rangle$, the QM violation of the condition given by Eq. (4.27) for ideal measurements can be obtained as follows, based on calculations similar to that involving Eq. (4.21) as discussed earlier

$$\begin{aligned}
&P(Q_2 = +1) - [P(Q_1+, Q_2+) + P(Q_1-, Q_2+)] \\
&= tr(U_{\Delta t}\rho_0(t_1)U^{\dagger}_{\Delta t}P_+) - tr(U_{\Delta t}(P_+\rho_0(t_1)P_+)U^{\dagger}_{\Delta t}P_+) \\
&- tr(U_{\Delta t}(P_-\rho_0(t_1)P_-)U^{\dagger}_{\Delta t}P_+) \\
&= \frac{1}{2}\sin(2\tau)\sin(2\theta)\sin(\phi)
\end{aligned} \tag{4.28}$$

It can be seen from Eq. (4.28) that, for sharp or ideal measurements, the maximum QM violation of the NSIT condition as given by Eq. (4.27) is 1/2 corresponding to the choices $\theta = \pi/4, \phi = \pi/2$ and $\tau = \Delta E \Delta t = \pi/4$ (in the units of $\hbar = 1$). Next, taking into account the unsharpness of measurements involved, the QM violation of the NSIT condition of the form (4.27) is obtained as follows on the basis of calculations similar to that leading to

Eqs. (4.23).

$$
\begin{aligned}
&P(Q_2=+1)-[P(Q_1+,Q_2+)+P(Q_1-,Q_2+)]\\
&=tr(U_{\Delta t}\rho_0(t_1)U^{\dagger}_{\Delta t}F_+)-tr(U_{\Delta t}(\sqrt{F_+}\rho_0(t_1)\sqrt{F_+}^{\dagger})U^{\dagger}_{\Delta t}F_+)\\
&-tr(U_{\Delta t}(\sqrt{F_-}\rho_0(t_1)\sqrt{F_-}^{\dagger})U^{\dagger}_{\Delta t}F_+)\\
&=\frac{1}{2}\lambda\sin(2\tau)\sin(2\theta)\sin(\phi)(1-\sqrt{1-\lambda^2}).
\end{aligned}
\tag{4.29}
$$

It is then seen from Eq. (4.29) that, while the magnitude of the QM violation of NSIT depends on the value of the unsharpness parameter λ (this violation is maximum for $\lambda=1$ corresponding to sharp measurement), a particularly noteworthy feature is that unless the state at the instant t_1 is such that either $\sin(2\theta)$ or $\sin(\phi)$ vanishes, the QM violation of NSIT persists for any non-zero value of λ, *i.e.*, for any arbitrarily unsharp measurement. This shows remarkable robustness of the QM violation of NSIT with respect to unsharp or non-ideal measurements.

4.4 Concluding remarks and future perspective

We have shown that optimal violation of the Leggett-Garg inequality [4] is allowed by quantum theory in the context of a suitably adapted measurement scheme for a system possessing arbitrary spin. It may be noted that whereas we obtain maximal violation of macrorealism for an arbitrary spin system, the same Peres-Gisin observable [90] used in the case of spatial correlations does not lead to maximal violation of the corresponding local realist inequality for finite integral spin systems. We have further shown how coarse graining of the measurement process through unsharp observables leads to the satisfaction of LGI. The form of coarse graining used here is quantitative, as different from the coarse-graining employed in a similar context earlier [42].

Three different necessary conditions for the validity of macrorealism are considered, including the two earlier proposed conditions namely LGI, NSIT, and the alternative condition WLGI proposed in this chapter. Comparison between these three conditions in terms of the robustness of their respective QM violations against unsharpness of the measurements is discussed. Emergence of classicality in terms of satisfying the macrorealist inequality WLGI corresponds to greater imprecision or unsharpness of measurements than that for LGI. Next, coming to NSIT, interestingly, we find that its QM violation occurs whatever be the unsharpness of the relevant measurements. For further research it be interesting to make a comprehensive comparison of the results of our present study with that using different characterizations of coarse-grained measurements that are invoked while probing the emergence of classicality within quantum theory under imprecise measurements.

CHAPTER 5

QUANTUM-CLASSICAL TRANSITION

In this chapter we concentrate on the issue of quantum-classical transition in more details basing on two works [96, 97]
i) *Quantum mechanical violation of macrorealism for large spin and its robustness against coarse-grained measurements*, S Mal, D Das, D Home, Phys. rev. A **94**, 062117 (2016).
ii) *Uncovering a Nonclassicality of the Schrdinger Coherent State up to the Macro-Domain*, S Bose, D Home, S Mal, arXiv:1509.00196(2015).
It may be recalled that among the various approaches suggested for addressing the issue of classical limit of QM [7, 98, 99, 100, 101, 102, 103, 104, 105], there are two strands of prevalent wisdom that are relevant to the results obtained in this paper. One is that classical physics emerges from the predictions of QM in the so called 'macroscopic' limit when either the system under consideration is of high dimensionality, for example, large spin system, or if a low dimensional system is of large mass, or if it involves large value of any other relevant parameter such as energy. The other is that classicality arises out of QM under the restriction of coarse grained measurements for which one can empirically resolve only those eigenvalues of a relevant observable that are sufficiently well separated; in other words, this view point stipulates that the limits of observability of quantum effects in an appropriate 'macroscopic limit' determine the way the classicality emerges [42, 106]. As regards the first approach mentioned above, we note that counter-examples questioning it have been pointed out. For instance, in the case of the Bell-EPR scenario, it has been shown that quantum features in the sense of violating local realist inequalities, persist in

the 'macroscopic' limit such as for the large number of constituents of the entangled system [107, 108], or for the large dimensions of the constituents of the entangled system [90, 109]. Further, for the Bell-EPR scenario, the QM violation of the relevant local realist inequalities seems to increase even in the limit of large numbers of particles and large magnitude of spins considered together [110]. On the other hand, in the case of temporal correlations for which the violation of macrorealism (MR) is probed through the violation of LGI, all the relevant studies mentioned earlier [39, 88, 111] reveal that, irrespective of the nature of measurements, the QM violation of LGI persists for arbitrary large value of spin of the system under consideration.

First we discuss on system with arbitrary spin and emergence of classicality with a measurement of varying degree of coarseness in conjunction with fuzziness of measurement. In the second work we consider oscillator system with dichotomic position measurement and investigated quantum-classical transition with increasing mass.

5.1 Violation of MR for large spin and coarse-grained measurements

As regards the second approach mentioned above, it has been shown [42, 106] that for a class of Hamiltonians governing the time evolution, if one goes into the limit of sufficiently large spins, but can experimentally only resolve eigenvalues which are separated by much more than the intrinsic quantum uncertainty, then the measurement outcomes appear to be consistent with that of classical laws. Along this line of research there had been a number of investigations giving more insight into the nature of coarse graining of the measurements and emergence of classicality or persistence of quantumness. In [112] micro-macro nonlocal correlation was established. Quantum violation of local realism has been shown [113] for entangled thermal states with very low detection efficiency, i.e., for extreme coarse grained measurement available. Large amount of violation of Bell inequality has been obtained [114] with human eye as detector in a micro-macro experiment and this violation is robust against photon loss. Precise (non coarse grained) measurements are shown [115] to be essential for demonstrating quantum features at the mesoscopic or macroscopic level and observing nonlocality becoming more difficult with increasing system size.

In the following Section, we explain the relevant features of the system under consideration (an arbitrary spin system in a uniform magnetic field), the specific type of measurement scheme used, its generalisation and the way the fuzziness of the measurements is modelled by unsharp measurement. The key results obtained using LGI and WLGI are discussed, which is followed by Section pertaining to NSIT. Then the key results obtained using LGI, WLGI and NSIT by generalising the scheme by which different measurement outcomes are clubbed together into two different groups are discussed using projective measurement, as well as, using unsharp measurement.

5.1.1 Setting Up Of The Measurement Context

Consider a QM spin j system in a uniform magnetic field of magnitude B_0 along the x direction. The relevant Hamiltonian is $(\hbar = 1)$:$H = \Omega J_x$. where Ω is the angular precession frequency $(\propto B_0)$ and J_x is the x component of spin angular momentum. Consider measurements of the z component of spin (J_z) whose eigenvalues are denoted by m. The measurement scheme used here [39] has the following features:

• The quantity Q is such that $Q = -1$ when $m = -j$ and for any other value of m ranging from $-j+1$ to $+j$, $Q = +1$. We will denote by Q_i and m_i the value of Q and the outcome of J_z measurement respectively at instant t_i. Thus $Q_i = +$ (i.e. $Q_i = +1$) means $m_i = -j + 1$ or, $-j + 2$ or, ... $j - 1$ or, j and $Q_i = -$ (i.e. $Q_i = -1$) means $m_i = -j$.

• We initialize the system so that at t=0, the system is in the state $|-j; j\rangle$ where $|m; j\rangle$ denotes the eigen state of J_z operator with eigenvalue m.

• Consider measurements of Q at times t_1, t_2 and t_3 $(t_1 < t_2 < t_3)$ & set the measurement times as $\Omega t_1 = \Pi$ and $\Omega(t_2 - t_1) = \Omega(t_3 - t_2) = \frac{\Pi}{2}$. For any j, this choice of measurement times may not give the maximum quantum violation of LGI, WLGI or NSIT. However, this choice suffices to give an idea about the nature of QM violations of the relevant inequalities for large j.

• We have also adopted a measurement scheme which is more general than described earlier and more natural in the context of emergence of classicality at the macroscopic limit with coarse grained measurement. This is described bellow:

$Q = -1$ for $m = -j, ..., -j + x$,

$Q = +1$ for $m = -j + x + 1, ..., +j$, where $0 < x \leq$ integer part (j) and x being integer.

The asymmetry in the number of measurement outcomes clubbed together decreases and, hence, the degree of coarse graining of the measurement increases with an increasing value of x. Here for $x = 0$, the aforementioned scheme is reproduced. $x =$ integer part (j) denotes the most macroscopic grouping scheme in the sense of describing the perfect coarse graining of the measurements.

Next, we use the notion of unsharp measurement in the context of treating fuzziness of the measurement. Unsharp measurement, a form of positive operator valued measurement (POVM), is well studied in the quantum formalism. In ideal sharp measurement, the probability of obtaining a particular outcome, say m in case of J_z measurement, and the corresponding post-measurement state are determined by the projector $P_m = |m; j\rangle\langle m; j|$. On the other hand, in the case of unsharp measurement, the probability of an outcome and the corresponding post-measurement state are determined by the effect operator, which is defined as

$$F_m = \lambda P_m + (1 - \lambda)\frac{\mathbb{I}}{d} \tag{5.1}$$

where λ is the sharpness parameter, where $0 \leq \lambda \leq 1$, P_m is the projector onto the state $|m; j\rangle$, $\mathbb{I}$ is the identity operator and d is the dimension of the system (for spin j system, $d = 2j + 1$). Here $(1 - \lambda)$ denotes the amount of white noise present in any unsharp measurement. Given the above specification of the effect operator, the probability of an outcome, say m, is given by $Tr(\rho F_m)$ for which the post-measurement state is given by, $(\sqrt{F_m}\rho\sqrt{F_m}^{\dagger})/Tr(\rho F_m)$, ρ being the state of the system on which measurement is done.

5.1.2 Analysis Using LGI And WLGI

For the purpose of the present section, we shall use the following form of 3-term LGI [44] and WLGI discussed earlier.

$$K_{LGI} = C_{12} + C_{23} - C_{13} \leq 1 \tag{5.2}$$

where $C_{ij} = \langle Q_i Q_j\rangle$ is the correlation function of the variable Q at two times t_i and t_j.

$$K_{WLGI} = P(Q_2+, Q_3+) - P(Q_1-, Q_2+) - P(Q_1+, Q_3+) \leq 0 \tag{5.3}$$

Similarly, other forms of WLGI involving any number of pairs of two-time joint probabilities can be derived by using various combinations of the observable joint probabilities.

Here we consider the specific form of the three term WLGI mentioned above (Eq.(5.3)).

For projective measurement: In order to calculate the expectation values and joint probabilities appearing in the aforementioned forms of LGI and WLGI, we proceed by writing the relevant time evolution operators as, for example, the time evolution operator from the initial time $t = 0$ to the instant of first measurement $t = t_1$, $U(t_1 - 0) = e^{-i\pi J_x} = R^2$ (where $R = e^{-i\frac{\pi}{2}J_x}$), and all the subsequent measurements are equispaced in time. Typically, any joint probability, for example, $P(Q_2+, Q_3+)$ for a spin j system is calculated using the Wigner D-matrix formalism and is of the form given by,

$$P(Q_2+, Q_3+) = 1 - \frac{(4j)!}{4^{2j}[(2j)!]^2} + \frac{1}{2^{4j}} - \frac{1}{2^{2j}} \tag{5.4}$$

Using such expressions, both K_{LGI} and K_{WLGI} can be evaluated. We then obtain in Eq.(5.2) and Eq.(5.3) respectively

$$K_{LGI} = 3 + 4^{1-2j} - 4^{1-j} - \frac{2^{1-4j}(4j)!}{((2j)!)^2} \tag{5.5}$$

$$K_{WLGI} = 1 + 4^{-2j} - 4^{-j} - \frac{4^{-2j}(4j)!}{((2j)!)^2} \tag{5.6}$$

QM violations of LGI and WLGI are quantified by $(K_{LGI} - 1)$ and $(K_{WLGI} - 0)$ respectively. It is found that both these violations increase with increasing values of j. Specific results showing this feature for $j = 1, 10, 100$ are given in Table I. From Eqs. (5.5) and (5.6) it can be seen that for $j \rightarrow \infty$, $(K_{LGI} - 1) \rightarrow 2$ [39] and $(K_{WLGI} - 0) \rightarrow 1$. Thus, in both these cases, the algebraic maxima of both K_{LGI} and K_{WLGI} are attained for infinitely large spin value of the system under consideration.

For unsharp measurement: Next, considering in the context of unsharp measurement, the expression of a typical joint probability distribution is of the form given by,

$$P(Q_1+, Q_2-) = \sum_{k=-j+1}^{j} Tr[F_{-j} U_{\Delta t_2} \sqrt{F_k} U_{\Delta t_1} \rho_i U^{\dagger}_{\Delta t_1} \sqrt{F_k}^{\dagger} U^{\dagger}_{\Delta t_2}] \tag{5.7}$$

where ρ_i = initial state of the system = $|-j; j\rangle\langle -j; j|$, $U_{\Delta t_1} = U(t_1 - 0)$ and $U_{\Delta t_2} = U(t_2 - t_1)$. Now, using the form of the effect operator defined earlier given by Eq.(5.1), Hamiltonian mentioned earlier and using Wigner D Matrix formalism, one can obtain the

j	$(K_{LGI}-1)$	$(K_{WLGI}-0)$
1	0.50	0.44
10	1.75	0.87
100	1.92	0.96

TABLE I: Table showing that the QM violations of LGI and WLGI increase with increasing values of the spin for ideal sharp measurement.

j	*Ranges of λ for which the QM violation*	
	of LGI persists	*of WLGI persists*
1	$(0.85, 1]$	$(0.71, 1]$
10	$(0.35, 1]$	$(0.28, 1]$
100	$(0.12, 1]$	$(0.08, 1]$

TABLE II: Table showing that the ranges of λ for which the QM violations of LGI and WLGI persist for different spin values increase with increasing values of spin.

j	*The magnitude of QM violation*			
	of LGI for		*of WLGI for*	
	$\lambda = 0.7$	$\lambda = 0.5$	$\lambda = 0.7$	$\lambda = 0.5$
10	0.59	0.19	0.31	0.12
50	0.80	0.37	0.40	0.19
100	0.85	0.41	0.43	0.21

TABLE III: Table showing the QM violations of LGI and WLGI for different spin values j and different values of the sharpness parameter λ of the measurement.

j	v_{th} *for LGI*	v_{th} *for WLGI*
1	0.571	0.276
10	0.098	0.052
100	0.010	0.005

TABLE IV: Table showing that the threshold visibilities of LGI and WLGI decrease with increasing values of the spin for ideal sharp measurement.

FIG. 5.1: Four tables showing comparisn between violation of LGI and WLGI considering different spin systems, unsharp measurement and initial mixed state.

joint probability pertaining to our measurement context as the following

$$P(Q_1+, Q_2-) = \frac{x^2\lambda}{2^{2j}} + 2x\lambda\sqrt{\frac{1-\lambda}{2j+1}\frac{1}{2^{2j}}} + \frac{\lambda(1-\lambda)}{2j+1}\frac{2j}{2^{2j}} + \frac{x^2(1-\lambda)}{2j+1} + 2x\left(\frac{1-\lambda}{2j+1}\right)^{\frac{3}{2}} + \left(\frac{1-\lambda}{2j+1}\right)^2 2j, \tag{5.8}$$

where $x = \left(\sqrt{\frac{2j\lambda+1}{2j+1}} - \sqrt{\frac{1-\lambda}{2j+1}}\right)$. Using such joint probabilities, one can obtain

$$\begin{aligned} K_{LGI} = {} & \frac{1}{(1+2j)^2((2j)!)^2} 16^{-j}((16^j + 2(-2+16^j))\lambda^2 + 4j^2(16^j - 4^{1+j}\lambda + 2(2+16^j)\lambda^2) \\ & -4\lambda(-2 + 4^j + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} + 16^j\sqrt{1-\lambda}\sqrt{1+2j\lambda}) - \\ & 4j(16^j + 2\lambda(-2 + 2^{1+2j} - 2^{1+4j} + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} \\ & +16^j\sqrt{1-\lambda}\sqrt{1+2j\lambda})))((2j)!)^2 + 2(1+2j)\lambda(-2+\lambda-2j\lambda+2\sqrt{1-\lambda}\sqrt{1+2j\lambda})(4j)!) \end{aligned} \tag{5.9}$$

and

$$
\begin{aligned}
K_{WLGI} = & \frac{1}{(1+2j)^2((2j)!)^2} 16^{-j}((4j^2\lambda(-4^j + \lambda + 16^j\lambda) - \lambda(-2 + 4^j - 16^j + \lambda \\
& +2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} + 2^{1+4j}\sqrt{1-\lambda}\sqrt{1+2j\lambda}) \\
& -2j(16^j + \lambda(-2 + 2^{1+2j} - 3(16^j) + 2\sqrt{1-\lambda}\sqrt{1+2j\lambda} - 2^{1+2j}\sqrt{1-\lambda}\sqrt{1+2j\lambda} \\
& +2^{1+4j}\sqrt{1-\lambda}\sqrt{1+2j\lambda})))((2j)!)^2 + (1+2j)\lambda(-2+\lambda-2j\lambda+2\sqrt{1-\lambda}\sqrt{1+2j\lambda})(4j)!)
\end{aligned} \tag{5.10}
$$

Now, for a particular value of j, the ranges of λ for which the QM violations of LGI and WLGI persist differ with the range for WLGI being greater than that for LGI. Moreover, the robustness of QM violations of both LGI and WLGI with respect to unsharpness of the measurement increase with increasing values of j. This is illustrated by the results given in Table II, which indicate that the ranges of λ for which the QM violations of LGI and WLGI persist increase with increasing values of j.

Most interestingly, for $j \to \infty$, we get, for the QM violations of LGI and WLGI

$$(K_{LGI} - 1) \to 2\lambda^2 \tag{5.11}$$

and

$$(K_{WLGI} - 0) \to \lambda^2 \tag{5.12}$$

which show that the ranges for which the QM violations of LGI and WLGI persist become equal to $(0, 1]$. On the other hand, for any j, magnitude of the QM violation of LGI (WLGI) decreases for decreasing values of λ. This is illustrated by the results shown in Table III.

Thus it is shown that if one adopts the type of measurement scheme used here, in the macrolimit characterized by infinitely large spin values as well as for any non-zero value of the sharpness parameter (i.e. for an arbitrary degree of fuzziness of the relevant measurement), the QM violation of MR persists for both LGI and WLGI.

Let us investigate whether such kind of behaviour persists when initial state becomes mixed which is the more realistic situation involved in actually testing the macrolimit of quantum mechanics. Here, instead of taking pure initial state $|-j;j\rangle$ at t=0, we initialize the system so that at t=0, the system is in the state ρ given by,

$$\rho = v|-j;j\rangle\langle -j;j| + (1-v)\frac{\mathbb{I}}{d} \tag{5.13}$$

j	Magnitude of the QM violation of LGI for			WLGI for		
	$v = 0.8$	$v = 0.6$	$v = 0.4$	$v = 0.8$	$v = 0.6$	$v = 0.4$
1	0.27	0.03	No violation	0.32	0.20	0.08
10	1.36	0.97	0.58	0.69	0.51	0.32
100	1.53	1.14	0.76	0.77	0.57	0.38

TABLE V: Table showing the QM violations of LGI and WLGI for different spin values and different mixedness incorporated in the initial pure state with ideal sharp measurement.

j	Magnitude of the QM violation of NSIT
1	0.63
10	0.87
100	0.96

TABLE VI: Table showing that the QM violation of NSIT increases with increasing values of the spin for ideal sharp measurement.

FIG. 5.2: Two table showing violation of LGI and WLGI for different mixed initial states and violation of NSIT for a given pure state.

where, v is the visibility parameter which changes the pure state into a mixed state and $(1-v)$ denotes the amount of white noise present in the state $|-j;j\rangle$ $(0 \leq v \leq 1)$, d is the dimension of the system, $\frac{\mathbb{I}}{d}$ is the density matrix of completely mixed state of dimension d. The minimum values of v for which QM violates different necessary conditions of MR signify the maximum amounts of white noise that can be present in the given state for the persistence of the QM violation of the relevant necessary condition of MR, and this value of v is known as the *threshold visibility* (v_{th}) pertaining to the given necessary condition of MR.

We also find for $j \to \infty$, the QM violations of LGI and WLGI are given by

$$(K_{LGI} - 1) \to 2v \tag{5.14}$$

and

$$(K_{WLGI} - 0) \to v \tag{5.15}$$

These results clearly show that for very large j and for any amount of mixedness introduced in the initial state of the system, the QM violation of MR *persists* using LGI or WLGI.

5.1.3 Analysis Using The NSIT Condition

According to the NSIT condition, the measurement outcome statistics for any observable at any instant is independent of whether any prior measurement has been performed. It is discussed in chapter four in detail and the NSIT condition is given by an equality i.e.,

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = 1) + P(Q_2 = -1, Q_3 = -1)] = 0. \tag{5.16}$$

j	Magnitude of the QM violation of NSIT		
	for $\lambda = 0.1$	for $\lambda = 0.5$	for $\lambda = 0.8$
1	0.0004	0.0521	0.2263
10	0.0026	0.1418	0.4502
100	0.0063	0.2085	0.5726
$\to \infty$	0.0100	0.2500	0.6400

TABLE VII: Table showing the QM violations of NSIT for different spin values j and different values of the sharpness parameter λ of the measurement.

j	Magnitude of the QM violation of NSIT		
	for $v = 0.8$	for $v = 0.4$	for $v = 0.2$
1	0.50	0.25	0.13
10	0.70	0.35	0.17
100	0.77	0.38	0.19

TABLE VIII: Table showing the QM violations of NSIT for different spin values and different mixedness incorporated in the initial pure state with ideal sharp measurement.

j	Magnitude of the QM Violation of					
	LGI for		WLGI for		NSIT for	
	$x = 10$	$x = 20$	$x = 10$	$x = 20$	$x = 10$	$x = 20$
40	1.52	1.32	0.76	0.66	0.76	0.66
60	1.61	1.46	0.81	0.73	0.81	0.72
80	1.67	1.53	0.83	0.77	0.83	0.76
100	1.70	1.58	0.85	0.79	0.85	0.79

TABLE IX: Table showing the QM Violations of LGI, WLGI and NSIT for different values of j and x with ideal sharp measurement.

j	The range of sharpness parameter (λ) for which the QM violation of					
	LGI persists for			WLGI persists for		
	$x = 5$	$x = 7$	$x = 9$	$x = 5$	$x = 7$	$x = 9$
10	(0.64, 1]	(0.75, 1]	(0.92, 1]	(0.53, 1]	(0.61, 1]	(0.72, 1]
20	(0.49, 1]	(0.55, 1]	(0.59, 1]	(0.40, 1]	(0.44, 1]	(0.48, 1]
30	(0.42, 1]	(0.47, 1]	(0.51, 1]	(0.33, 1]	(0.37, 1]	(0.40, 1]
40	(0.38, 1]	(0.42, 1]	(0.45, 1]	(0.29, 1]	(0.33, 1]	(0.36, 1]

TABLE X: Table showing the ranges of the sharpness parameter for which QM violations of LGI, WLGI and NSIT persist for different values of j and x.

FIG. 5.3: Four table showing violation of NSIT considering initial mixed state and unsharp measurement and violation of LGI, WLGI for coarse-grained measurements.

QM violation of NSIT is quantified by the non-vanishing value of the LHS of Eq.(4.16).

For projective measurement: For spin j system, for $H = \Omega J_x$, using the measurement scheme discussed in previous section and the choice of measurement times as well as of the initial condition mentioned there, we obtain, using the Wigner D Matrix formalism,

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = -1) + P(Q_2 = -1, Q_3 = -1)] = 1 - \frac{(4j)!}{4^{2j}[(2j)!]^2}. \quad (5.17)$$

It is found that the QM violation of NSIT increases with increasing values of j.

For $j \to \infty$, the QM violation of NSIT $\to 1$, which is the algebraic maximum of the LHS of the NSIT condition.

For unsharp measurement: For 'unsharp measurement' defined in terms of the sharpness parameter λ for spin j system described earlier, the LHS of Eq.(4.16) becomes

$$P(Q_3 = -1) - P(Q_2+, Q_3-) - P(Q_2-, Q_3-) = \frac{2^{-4j}\lambda}{(1+2j)((2j)!)^2}[2 + (-1+2j)\lambda - 2\sqrt{1-\lambda}\sqrt{1+2j\lambda}][16^j((2j)!)^2 - (4j)!]. \quad (5.18)$$

It is then found that for an arbitrary value of spin j, the QM violation of NSIT persists for any non-zero value of the sharpness parameter λ.

For $j \rightarrow \infty$, for any λ, LHS of the NSIT condition is given by

$$P(Q_3 = -1) - [P(Q_2 = +1, Q_3 = -1) + P(Q_2 = -1, Q_3 = -1)] \rightarrow \lambda^2 \tag{5.19}$$

Thus, even in the 'macrolimit' characterized by $j \rightarrow \infty$, for any non-zero value of λ, the QM violation of MR persists using NSIT.

5.1.4 LGI, WLGI, NSIT under generalised coarse-grained measurement

Here we generalise the scheme by which different measurement outcomes are clubbed together into two groups. In this case $Q = -1$ for $m = -j, ..., -j + x$, and $Q = +1$ for $m = -j+x+1, ..., +j$, where $0 < x \leq$ integer part(j) and x being integer. Here the degree of coarse graining of the measurement increases with increase in x. Any fixed value of x denotes a particular grouping scheme.
We initialize the system so that at $t = 0$, the system is in the state $|-j; j\rangle$. We take the aforementioned Hamiltonian and choices of measurement times.
For projective measurement: Here joint probabilities appearing in the aforementioned particular form of LGI, WLGI or NSIT are calculated for ideal sharp measurement using Wigner D Matrix formalism.
From numerical results it is found that for any j (also for arbitrarily large value), QM violation of LGI exists for $x \leq$ integer part($j - 1$) and no violation occurs for $x =$ integer part(j); whereas QM violations of WLGI and NSIT exist for any value of x, where $x \leq$ integer part(j). This indicates that QM violations of different necessary conditions of MR persist for very large degree of coarse graining of the measurement. However, the magnitudes of the violations become *smaller* for increasing values of x, or increasing the degree of coarse graining of the measurement.
For a fixed and finite value of x, magnitudes of QM violations of LGI, WLGI or NSIT become *larger* for increasing values of j. For arbitrarily large values of j ($\frac{j}{x} >> 1$), magnitudes of QM violations of LGI, WLGI or NSIT approach their respective algebraic maxima. However, the QM violations of different necessary conditions of MR approach their respective

algebraic maxima *slowly* as one increases x. These results are shown in Table IX.

For unsharp measurement: Now, instead of projective measurement, let us employ unsharp measurement of spin-z component observable. From numerical results it is observed that, for any j, the ranges of the sharpness parameter for which the QM violations of LGI and WLGI persist become *smaller* for increasing values of x, or increasing the degree of coarse graining of the measurement. And for a fixed and finite value of x, the ranges of the sharpness parameter for which the QM violations of LGI and WLGI persist become *larger* for increasing values of j and for arbitrarily large values of j ($\frac{j}{x} >> 1$), both the ranges approach $(0, 1]$. However, these ranges approach $(0, 1]$ *slowly* as one increases x. This is shown in Table X.

Interestingly, The range of the sharpness parameter for which the QM violation of NSIT persists for arbitrary values of j (including arbitrarily large values of j) and arbitrary values of x is $(0, 1]$. This indicates that, surprisingly, for any particular scheme of branching of the outcomes (i.e. for a fixed value of x), for arbitrarily large values of j ($\frac{j}{x} >> 1$), QM violations of all the necessary conditions of MR persist for almost any nonzero value of the sharpness parameter.

5.2 System with large mass

While the original motivation that led to LGI was to use it for testing the possible limits of QM in the macroscopic regime, e.g., in terms of suitable experiments involving the rf-SQUID device [116], in recent years, a variety of theoretical and experimental studies (reviewed, for example, by Emary et al. [44]) have sought to bring out various fundamental implications of LGI and its variants [42, 117, 118], as well as have probed aspects of LGI pertaining to different types of systems, ranging from, say, solid-state qubits [85, 119], nuclear spins [120], photons [87], elecrons [121], to oscillating kaons and neutrinos [122]. Against this backdrop, in the present section we explores a novel application of LGI using the archetypal example of a linear harmonic oscillator (LHO) which has well defined classical as well as quantum descriptions. Note that systems used so far for probing LGI have been essentially qubits or systems that are isomorphic to qubits. In contrast, the LHO example we consider involves continuous variables. Therefore, to apply LGI in this context, discretization is needed which is ensured by, say, considering coarse-grained measurement of a type that would determine *which* one of the halves of the region of oscillation, the oscillating particle is *in* at any given instant, without any further information about the position of the particle. Then, invoking such dichotomic measurements, it turns out that the LHO example serves to demonstrate the power of LGI in revealing a testable non-classical feature of the Schrödinger coherent state (non-spreading wave packet with minimum position-momentum uncertainty) whose quantum dynamical behaviour is similar to that of a classical oscillator and is regarded as providing the best possible classical-like description of LHO in terms of QM. Using this coherent state, the extent to which for even larger values of mass, the QM violation of LGI persists is investigated.

5.2.1 LGI and the notion of NRM

In the one-dimensional LHO example, the temporal evolution of a particle can be regarded as oscillation between two states, one of which corresponds to the particle being found within, say, the negative half of the region of oscillation ($x = 0$ to $x \to -\infty$) which we call the state 1, while the state 2 pertains to the particle being found within the positive half ($x = 0$ to $x \to +\infty$). Let Q(t) be an observable quantity such that, whenever measured, it is found to take a value $+1(-1)$ depending on whether the system is in the state 1(2). then following LG test discussed earlier one can construct temporal correlations $C_{ij} \equiv \langle Q(t_i)Q(t_j)\rangle$ and LGI

$$C \equiv C_{12} + C_{23} + C_{34} - C_{14} \leq 2 \tag{5.20}$$

C_{ij}s can be written as

$$C_{12} = P_{++}(t_1, t_2) - P_{+-}(t_1, t_2) + P_{--}(t_1, t_2) - P_{-+}(t_1, t_2) \tag{5.21}$$

where $P_{++}(t_1, t_2)$ is the joint probability of finding the particle in the state 1 at both the instants t_1 and t_2; similarly, for $P_{+-}(t_1, t_2), P_{--}(t_1, t_2), P_{-+}(t_1, t_2)$. Note that the derivation of LGI requires essentially the *first measurement* of each such pair to satisfy NIM for macro-observable. This can be ensured through the NRM procedure by arranging the measuring setup so that if, say, the probe is triggered, $Q(t_1) = +1$, while if it is *not*, $Q(t_1) = -1$, thereby ensuring in the latter case that while the untriggered probe provides information about the value of Q, there is no interaction occurring between the probe and the measured particle; in other words, the condition of NIM is then satisfied. Now, if the results of those runs are only used for which $Q(t_1) = -1$, followed by the measurement of Q at t_2, discarding the results of the rest runs, these results can be used for determining the joint probabilities $P_{-+}(t_1, t_2)$ and $P_{--}(t_1, t_2)$. Similarly, for determining the other two joint probabilities $P_{+-}(t_1, t_2)$ and $P_{++}(t_1, t_2)$ occurring in C_{12}, the measuring setup can be inverted so that a value of $Q(t_1) = -1$ triggers the probe, while for $Q(t_1) = +1$, it does *not*. In this way, one can determine C_{12} and, thus, all the 2-time correlation functions occurring in the LGI by ensuring NIM through the use of the NRM procedure for the first measurement of any pair.

5.2.2 LGI USING LHO COHERENT STATE

Having clarified the relevant basics, we now proceed to discuss the details of a specific application of LGI pertaining to the Schrödinger coherent state of LHO. Let us consider the following initial Gaussian wave packet

$$\psi(x, t=0) = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_0}} \exp\left(-\frac{x^2}{4\sigma_0^2} + \frac{ip_0 x}{\hbar}\right) \tag{5.22}$$

which is peaked at $x = 0$ with the initial momentum expectation value p_0 (peak momentum), and has width σ_0 which is taken to be $\sigma_0 = \sqrt{\frac{\hbar}{2m\omega}}$ where ω is the angular frequency of oscillation. Under the LHO potential, the above $\psi(x, 0)$ evolves into $\psi(x, t)$ is obtained in the following way. Propagator for the motion is given by

$$K\left(x', t' = 0; x, t\right) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}} \exp\left[\frac{im\omega}{2\hbar \sin\omega t}\{(x'^2 + x^2)\cos\omega t - 2xx'\}\right] \tag{5.23}$$

Then the time-evolved wave packet at the instant t is given by

$$\psi(x, t) = \int_{-\infty}^{\infty} K(x', t' = 0; x, t)\psi(x', 0)dx' = \sqrt{\frac{1}{\sqrt{2\pi}\sigma_t}} \exp\left(-\sqrt{m\omega}\frac{A(t) + Bx + C(t)x^2}{(2\hbar)^{3/2}\sigma_t}\right) \tag{5.24}$$

where

$$A(t) = \frac{i\hbar p_0^2}{(m\omega)^2}\sin\omega t, \quad B = -\frac{2ip_0\hbar}{m\omega}, C(t) = \hbar\cos\omega t + i\hbar\sin\omega t, \quad \sigma_t = \frac{i\sin\omega t + \cos\omega t}{\sqrt{2m\omega/\hbar.}} \tag{5.25}$$

whence the probability density is given by

$$|\psi(x,t)|^2 = \sqrt{\frac{m\omega}{\hbar\pi}} \exp\left(-m\omega\frac{(x-\frac{p_0}{m\omega}\sin\omega t)^2}{\hbar}\right) \tag{5.26}$$

which oscillates without spreading or changing shape, while its peak follows classical motion, and $\Delta x\Delta p = \hbar/2$ at all instants. Such a wave packet is known as the Schrödinger coherent state [123] - a much-discussed remarkable example of a quasi-classical state in quantum mechanics. Now, in order to apply LGI in this context, we consider coarse-grained measurement of a type, as mentioned earlier, that determines at any instant whether the oscillating particle is in the region between $x \to -\infty$ and $x = 0$ (corresponding to the measurement outcome +1) *or* is in the region between $x = 0$ and $x \to +\infty$ (corresponding to the measurement outcome -1). Such a measurement can be represented by the dichotomic localization operator $\hat{O} = \int_{-\infty}^{0} |x\rangle\langle x|dx - \int_{0}^{\infty} |x\rangle\langle x|dx$ which has two eigenstates $\int_{-\infty}^{0}\langle x|\psi\rangle|x\rangle dx$ and $\int_{0}^{\infty}\langle x|\psi\rangle|x\rangle dx$ corresponding to the eigenvalues $+1, -1$ respectively. Note that the probability of obtaining the outcome +1(-1) for such a measurement at an instant, say, t_1, is given by

$$P_+(t_1) = \int_{-\infty}^{0} |\psi(x,t_1)|^2 dx = \frac{1}{2}\left(1 - Erf(\frac{\langle x(t_1)\rangle}{\sqrt{2}|\sigma_{t_1}|})\right) \tag{5.27}$$

$$P_-(t_1) = \int_{0}^{\infty} |\psi(x,t_1)|^2 dx = \frac{1}{2}\left(1 + Erf(\frac{\langle x(t_1)\rangle}{\sqrt{2}|\sigma_{t_1}|})\right) \tag{5.28}$$

The Error Function $Erf(t_1) = \frac{2}{\sqrt{\Pi}}\int_0^{t_1} exp(-z^2)dz$, $\sigma_{t_1} = (i\hbar\sin\omega t + 2m\omega\sigma_0^2\cos\omega t)/2m\omega\sigma_0$, and $\langle x(t_1)\rangle = (p_0/m\omega)\sin\omega t_1$. Next, given the result of the above measurement at the instant t_1 to be +1(-1), for either of the two outcomes obtained using the NRM procedure, one then considers the post-measurement state $\psi_{\pm}^{PM}(x,t_1)$ that evolves, followed by a measurement at the instant t_2 corresponding to the operator $\hat{O}$. For this latter measurement, the conditional probability of obtaining the outcome +1, contingent upon the outcome +1(-1) obtained for the measurement of $\hat{O}$ at the earlier instant t_1, is given by

$$P_{\pm/+}(t_1,t_2) = \int_{-\infty}^{0} |\psi_{\pm}^{PM}(x,t_2)|^2 dx \tag{5.29}$$

while such a conditional probability for the outcome -1 at the instant t_2 is of the form

$$P_{\pm/-}(t_1,t_2) = \int_{0}^{\infty} |\psi_{\pm}^{PM}(x,t_2)|^2 dx \tag{5.30}$$

where $\psi_{\pm}^{PM}(x,t_2)$ is the time-evolved normalized form of the post-measurement state. This is obtained in the following way. Depending on the outcome $+1(-1)$ of the measurement corresponding to the operator $\widehat{O} = \int_{-\infty}^{0} |x\rangle\langle x|dx - \int_{0}^{\infty} |x\rangle\langle x|dx$ at the instant t_1, the post-measurement state (not normalized) is given by

$$|\psi_{+}^{PM}(t_1)\rangle = \int_{-\infty}^{0} \psi(x',t_1)|x'\rangle dx', \quad |\psi_{-}^{PM}(t_1)\rangle = \int_{0}^{+\infty} \psi(x',t_1)|x'\rangle dx'. \tag{5.31}$$

Subsequently, $\psi_{\pm}^{PM}(t_1)$ evolves up to the instant t_2 by the propagator $K(x',t'=t_1;x,t_2)$ which is of the same form as that given by Eq. (1). The time-evolved normalized form of the post-measurement state at the instant t_2 is then given by

$$\psi_{\pm}^{PM}(x,t_2) = (1/N_{\pm}) \int_{-\infty}^{\infty} K(x',t_1;x,t_2)\psi_{\pm}^{PM}(x',t_1)dx'$$

$$= (1/N_{\pm}) \frac{1}{2\sqrt{\sqrt{2\pi}\sigma_{t_2}}}(1 + Erf[\frac{\chi}{\sqrt{\xi}}]) \exp\left[-\sqrt{m\omega}\frac{A(t_2) + Bx + C(t_2)x^2}{(2\hbar)^{3/2}\sigma_{t_2}}\right] \tag{5.32}$$

where $A(t_2), B, C(t_2)$, and σ_{t_2} are respectively the same as that given by Eqs. (3), (4), (5) and (6), except that t is replaced by t_2, while

$$\chi = -\frac{\sqrt{m\omega}B}{(2\hbar)^{3/2}\sigma_{t_2}} - \frac{im\omega x}{2\hbar \sin\omega(t_2 - t_1)}, \quad \xi = \frac{\sqrt{m\omega}C(t_2)}{(2\hbar)^{3/2}\sigma_{t_2}} - \frac{im\omega\cos\omega(t_2 - t_1)}{2\hbar\sin\omega(t_2 - t_1)} \tag{5.33}$$

and $N_{\pm}$ is the normalisation constant at the instant t_2, given by $N_{\pm} = \int_{-\infty}^{\infty} |\psi_{\pm}^{PM}(x,t_2)|^2 dx$. Using Eqs. (5.27) - (5.30), for suitable choices of the relevant parameters, one can thus compute value of LGI expression. Note that in our setup, the key parameters are m, p_0 and ω. Suitably choosing the values of m, p_0, ω while taking the temporal intervals to be the

TAB. 5.1: LGI violation with increasing mass.

m(amu)	$\sigma_0(m)$	$p_0(kgm/s)$	$v_0(m/s)$	$A_{Cl}(m)$	C
10	3.9×10^{-8}	3.3×10^{-24}	2×10^{2}	10^{-4}	2.62
10^3	3.9×10^{-9}	3.3×10^{-23}	2×10	10^{-5}	2.58
10^6	1.2×10^{-10}	3.3×10^{-21}	2.0	10^{-6}	2.50
10^7	3.8×10^{-11}	3.3×10^{-20}	2.0	10^{-6}	2.34
10^8	1.2×10^{-11}	3.3×10^{-20}	2×10^{-1}	10^{-7}	2.54
10^9	3.8×10^{-12}	3.3×10^{-19}	2×10^{-1}	10^{-7}	2.35
10^{10}	1.2×10^{-12}	3.3×10^{-21}	2×10^{-4}	10^{-10}	2.70
10^{12}	1.2×10^{-12}	3.3×10^{-19}	2×10^{-4}	10^{-10}	2.70
10^{15}	3.9×10^{-15}	3.3×10^{-18}	2×10^{-6}	10^{-12}	2.70
10^{18}	1.2×10^{-16}	3.3×10^{-16}	2×10^{-7}	10^{-13}	2.70
10^{20}	1.2×10^{-17}	3.3×10^{-15}	2×10^{-8}	10^{-14}	2.65

same, i.e., $t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \Delta t$, and by numerically integrating the relevant integrals occurring in Eqs.(5.27) - (5.30), the key results of the quantitative studies are presented in the Tables. Here it needs to be mentioned that for an arbitrarily given m, p_0 and ω, by varying the choices of the time interval Δt and the first instant of measurement t_1, it is found that the maximum value of C on the LHS of the inequality (1) is attained when Δt is chosen within the neighbourhood of T/4 or 3T/4, and t_1 is slightly larger than 0 or is within the neighbourhood of T/2, where T is the time period of oscillation. Note that for computing all the results given in the Tables I - III, we have chosen the *same* values of $\Delta t = 2.4 \times 10^{-6}$s and $t_1 = 1.5 \times 10^{-6}$s where Δt is chosen close to 3T/4 and t_1 is close to T/2 with $T = 3.14 \times 10^{-6}$s which corresponds to $\omega = 2 \times 10^6$Hz taken to be the *same* for evaluating all the results given in the Tables It is found that for $p_0 = 0$, LGI is always satisfied. On the other hand, by appropriately choosing p_0, it is possible to obtain a significant amount of QM violation of LGI. Note that appreciable QM violations of LGI are found by suitable choices of p_0 as given in Table I for, say, masses 10 amu -10^{12} amu, corresponding to which the respective values of A_{Cl} (pertaining to the maximum QM violation of LGI obtained for a given mass) range from 10^{-4}m to 10^{-10}m(for typical parameter of optically levitated system). If the mass is further increased, it is found that in order to obtain significant QM violation of LGI, p_0 needs to be chosen such that the corresponding A_{Cl} becomes much smaller. Also, as m increases, the required value of v_0 (initial peak velocity of the wave packet) for showing the QM violation of LGI becomes increasingly smaller, as can be seen from Table I. Thus, it is evident that in our example, although theoretically one can obtain the QM violation of LGI for any given m and ω by

m(amu)	$\sigma_0(m)$	$v_0(m/s)$	$A_{Cl}(m)$	C
10^2	1.2×10^{-8}	2×10^2	10^{-5}	2.8
10^3	3.8×10^{-9}	2.0	10^{-6}	2.74
10^4	1.2×10^{-9}	2×10^{-1}	10^{-7}	2.65
10^5	3.8×10^{-10}	10^{-2}	10^{-8}	2.54
10^6	1.2×10^{-10}	10^{-3}	10^{-9}	1.56

TAB. 5.2: Taking fixed values of the angular frequency of oscillation $\omega = 2 \times 10^6$Hz and the initial peak momentum (p_0) of the coherent state wave packet to be $p_0 = 3.3 \times 10^{-24}$kgm/s, as the values of mass (m) are increased, gradual decrease of the QM violation of LGI is shown through decreasing values of C.

m(amu)	$p_0(kgm/s)$	$v_0(m/s)$	$\sigma_0(m)$	$A_{Cl}(m)$	C
10^3	3.32×10^{-25}	2×10^{-1}	3.9×10^{-9}	10^{-7}	2.54
	3.32×10^{-24}	2	3.9×10^{-9}	10^{-6}	2.73
	3.32×10^{-23}	2×10	3.9×10^{-9}	10^{-5}	2.6
	1.56×10^{-22}	10^2	3.9×10^{-9}	5×10^{-5}	2.25
	3.32×10^{-22}	2×10^2	3.9×10^{-9}	10^{-4}	1.99

TAB. 5.3: Taking a fixed value of mass $m = 10^3$amu, for increasing values of the intial peak momentum (p_0) of the coherent state wave packet that correspond to increasing values of the classical amplitude (A_{Cl}) of oscillation, the respective computed QM values of the LHS (C) of the LGI inequality (1) are shown which indicate a gradual decrease in the QM violation of LGI as the value of A_{Cl} increases, and eventually LGI is satisfied.

suitably choosing p_0, actual testability of this violation becomes gradually impracticable for sufficiently large mass. The results given in Table II show that if by keeping the parameters p_0, ω fixed, one increases the mass m, the QM violation of LGI gradually diminishes, and eventually for sufficiently large mass, LGI is satisfied; i.e., $C < 2$. (c) For given values of m and ω, if p_0 is increased, the corresponding A_{Cl} is also increased, the QM value of C is found to be gradually decreasing, and eventually $C < 2$ for appropriately large p_0. In other words, given m and ω, for larger value of classical amplitude of oscillation (A_{Cl}), QM satisfies LGI. This is illustrated by the results given in Table III.

The results discussed above, therefore, serve to highlight the efficacy of LGI in not only revealing nonclassicality of the oscillator coherent state, but also in exploring the extent to which such nonclassical feature persists up to macro scale for masses larger than the typical microsopic masses, and the way quantum-classical transition occurs.

5.3 CONCLUDING REMARKS AND FUTURE PERSPECTIVE

For multilevel spin systems, robustness of the quantum mechanical violation of macrorealism (MR) with respect to coarse grained measurements is investigated using three different necessary conditions of MR, namely, the Leggett-Garg inequality (LGI), Wigner's form of the Leggett-Garg inequality (WLGI) and the condition of no-signalling in time (NSIT). It is shown that for dichotomic sharp measurements, in the asymptotic limit of spin, the algebraic maxima of the QM violations of all these three necessary conditions of MR are attained contingent upon a measurement scheme invoked in [111]. These results hold good even when we generalise the grouping scheme. Here the clubbing of the measurement outcomes into two groups makes the measurement coarse grained. However, the boundary between the two groups of outcomes remains precise which is, in general, not true in the realization of the macrolimit. Employing, in conjunction, unsharp measurement makes this boundary also imprecise. Thus, simultaneously clubbing different measurement outcomes together and invoking unsharp measurement enables to describe in a more natural way the coarse graining of the measurements. It is, therefore, emphatically demonstrated that classicality does not emerge for such coarse grained measurement even for arbitrarily large spin value of the system. Considering more general form of coarse graining i. e., invoking biased-ness of measurement and coarsening of time is the further area of study. Then in the other work LGI is applied in the context of a linear harmonic oscillator. Strikingly, it is found that the quantum mechanical (QM) violation of LGI can reveal a testable nonclassical feature associated with the state which is considered the most classical-like of all quantum states, namely the Schrödinger coherent state. In the macrolimit, the extent to which for large values of mass such nonclassicality persists is quantitatively investigated. It is found that while for any given mass and angular frequency of oscillation, by suitably choosing the initial peak momentum of the coherent state wave packet, a significant amount of QM violation of LGI can be obtained, however, as the mass is sufficiently increased, actual observability of this violation becomes increasingly difficult. A potentially feasible experimental setup for studying our example is suggested using optically levitated objects having mass 10^6 amu - 10^9 amu, in which the predicted QM violation of LGI can be tested. For future study it should be mentioned that the effect of decoherence due to coupling with dissipative environment relevant to the setup has to be carefully taken into account in order to examine the extent to which the QM violation of LGI could be observable for the oscillator coherent state.

CHAPTER 6

APPLICATION OF TEMPORAL CORRELATION

In this chapter, we propose an application of violation LGI on certifying randomness by deriving LGI using different set of assumptions: the assumption of no signalling in time (NSIT) and predictability. These assumptions involve only measurement statistics and hence are directly testable in experiments. This derivation of LGI, therefore, allows us to conclude that in a situation where NSIT is satisfied, the violation of LGI will imply violation of predictability i.e., presence of certifiable randomness. Thus LGI finds an important practical implication in a useful information theoretic task.

This chapter based on *Temporal correlations and device-independent randomness*, S Mal, M Banik, S K Choudhury, Quantum Inf Process **15**, pp 2993 3004(2016) [124].

Randomness is a valuable resource for various important tasks ranging from cryptographic applications to numerical simulations such as Monte Carlo method (a useful technique which finds application in computational Physics, Statistical Physics, Physical Chemistry, Computational Biology, Computer Graphics, Finance and many other areas). For various such tasks, the genuineness of the used randomness is of primary concern. Thus, device independent certification and generation of randomness is very important from a practical point of view. Motivated by the work of Pironio and coworkers [125] many interesting results have been obtained, in recent times, in the field of DI certification and generation of randomness. All such methods use nonlocal correlations among spatially separated parties to certify randomness.

From algorithmic information theory it is known that randomness cannot be certified by any mathematical procedure [126]. The generation of randomness, therefore, must be based on unpredictability of some physical phenomena, so that the randomness is guaranteed by the inherent uncertain nature of the physical theory. There is no such thing as true randomness in classical world as any classical phenomenon, can, in principle, be predicted. They appear random to us due to lack of our knowledge and control of all the relevant degrees of freedom. Measurement on a quantum particle, on the other hand, is postulated to give intrinsically random results. The quantum measurements, therefore, can be used to generate true randomness [12, 13]. But, for the reliability of the randomness thus generated, one needs to trust the devices which prepare and measure the quantum states. Can randomness be certified in a Device-Independent way, i.e., can it be certified even without knowing the details of the devices used in its generation– is a topic of current research interest [125, 127, 128, 129].

We derive LGI from NSIT and predictability condition. The assumption of NSIT, described in [94, 130, 131], says that a measurement does not change the outcome statistics of a later measurement, whereas predictability is the assumption that one can predict the outcomes of all possible measurements to be performed on a system [132]. Then we show how much randomness is generated by a certain amount of violation of LGI.

This work provides an important information theoretic application of LGI which can be implemented in laboratory with the present day's technology. From the perspective of experimental implementation the LGI-based DI randomness certification seems more feasible than its spatial analogue as it does not require entanglement.

We begin with a brief review of the ontological framework of an operational theory, as this will later be used in our derivation of LGI.

6.1 Ontological framework of an operational theory and the LGI

In a standard Leggett-Garg test, we consider a macroscopic object which is described by a set of macro variable $\{Q, Q', ...\}$ whose values are considered to be macroscopically distinct by some measure [133]. In a series of runs, the object is prepared in the same initial state, and each preparation defines a new origin of time. Let us consider the case where macro variable $A \in \{Q, Q', ...\}$ is measured at time $t_A (t_A > 0)$ and macro variable $B \in \{Q, Q', ...\}$ at a later time t_B. The correlation function $C_{t_A t_B} \equiv \langle Q_{t_A} Q_{t_B} \rangle$ for measurements at t_A and t_B is obtained from the joint probability $P(A_{t_A} B_{t_B} | Q_{t_A} Q_{t_B})$ of obtaining the results A_{t_A} and B_{t_B} from measurements of Q at time t_A and t_B ($t_B > t_A$) as

$$C_{t_A t_B} = \sum_{A_{t_A} B_{t_B}} A_{t_A} B_{t_B} P(A_{t_A} B_{t_B} | Q_{t_A} Q_{t_B}).$$

In the simplest case, the macro variable may obtain only two different values ± 1. In such cases, *macrorealism* together with *induction* imply the LGI [133] of the Clauser-Horne-Shimony-Holt (CHSH) type [93] ($t_1 < t_2 < t_3 < t_4$):

$$f_4^{LG} = -2 \leq C_{t_1 t_2} + C_{t_2 t_3} + C_{t_3 t_4} - C_{t_1 t_4} \leq 2. \tag{6.1}$$

or of the Wigner type [92]

$$f_3^{LG} = -3 \leq C_{t_1 t_2} + C_{t_2 t_3} - C_{t_1 t_3} \leq 1 \tag{6.2}$$

In the ontological framework, the system's state is described by an ontic variable λ and $P(A_{t_A}, B_{t_B} | Q_{t_A}, Q_{t_B} \lambda, \lambda \rightarrow \lambda')$ denotes the joint probability of obtaining outcome A_{t_A} of measurement Q_{t_A} performed at time t_A and outcome B_{t_B} of measurement Q_{t_B} performed at a later time t_B; $\lambda \rightarrow \lambda'$ denotes the change of the system's ontic state conditioned that A_{t_A} outcome has been obtained in measurement Q_{t_A} at time t_A. The ontological model then predicts for the observed probability as

$$P(A_{t_A} B_{t_B} | Q_{t_A} Q_{t_B}) \quad = \quad \int_\lambda \int_{\lambda'} d\lambda d\lambda' \mu(\lambda) \rho(\lambda' | Q_{t_A}, A_{t_A}, \lambda) P(A_{t_A} B_{t_B} | Q_{t_A} Q_{t_B}, \lambda, \lambda \rightarrow \lambda') \tag{6.3}$$

where $\mu(\lambda)$ and $\rho(\lambda' | Q_{t_A}, A_{t_A}, \lambda)$ respectively denote the distribution of the ontic variables prior to the measurement Q_{t_A} and distribution of the ontic variables after obtaining the result A_{t_A} in the measurement of Q_{t_A}. A crucial step in the derivation of LGI is to establish the following *factorizability* relation which follows from the assumptions of *macrolealism* and *induction* [41, 44] [1]:

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B},\lambda,\lambda\to\lambda') = P(A_{t_A}|Q_{t_A},\lambda)P(B_{t_B}|Q_{t_B},\lambda) \tag{6.4}$$

It is noteworthy that, in contrast to *macrorealism*, quantum mechanics predicts the outcome probability as:

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) = \text{Tr}[\hat{\rho}(t_A)\hat{Q}_A]\text{Tr}[\hat{\rho}_{A_{t_A}}(t_B)\hat{Q}_B], \tag{6.5}$$

where, $\hat{\rho}(t_A)$ is the quantum state of the system at time t_A, $\hat{Q}_A$ and $\hat{Q}_B$ are the measurement operators for outcomes A and B, and $\hat{\rho}_{A_{t_A}}(t_B)$ is the quantum state at time t_B given that at time t_A result A was obtained.

For a two-level system undergoing coherent oscillations between the states with $Q = \pm 1$, the optimal quantum violation of the inequality (6.1) is known to be $2\sqrt{2}$, whereas it is $\frac{3}{2}$ for the inequality (6.2) [117].

6.2 An alternative derivation of the Leggett-Garg Inequality

We, now, state the main result of our paper in the form of the following theorem:
Theorem 1: Any operational theory satisfying the assumptions of predictability and NSIT always satisfies LGIs.
It is known that macrorealism implies both LGI as well as NSIT. But, the assumption of NSIT, alone, does not imply LGI [94]. However, as shown below, it together with the assumption of predictability yield LGI. Before proceeding for the proof of the above theorem, we state the two assumptions in precise mathematical forms:

(a1) NSIT: An operational model is said to satisfy no signalling in time (NSIT) if a measurement does not change the outcome statistics of a later measurement. In the context of the standard Leggett-Garg test, this assumption demands that the probability of obtaining B as measurement outcome for a measurement of an observable Q at time t_B should not depend on any measurement performed at an earlier time

[1] Once the factorizability is achieved, the postulate of induction is further used in calculating the correlation between measurement outcomes at two other different times. It allows one to freely choose the measurement times, independent of the properties of the initially prepared state.

t_A, i.e.,

$$P(B_{t_B}|Q_{t_B}) = P(B_{t_B}|Q_{t_A}Q_{t_B}).$$

(a2) Predictability: A model is said to be predictable if the joint operational statistics $P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) \in \{0,1\}$ for measurements at any time and for all measurement outcomes.

To prove Theorem 1, it is sufficient to prove the following Lemma.
Lemma 1: predictability $\wedge$ NSIT $\Rightarrow$ factorizability relation, i.e. Eq.(6.4).
Proof: The assumption of predictability implies:

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) \in \{0,1\}, \tag{6.6}$$

for measurements at any time and for all measurement outcomes. As

$$P(B_{t_B}|Q_{t_A}Q_{t_B}) = \sum_{A_{t_A}} P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) \tag{6.7}$$

and as all the joint probabilities appearing in the summation above can be either zero or one in a predictable model, hence,

$$P(B_{t_B}|Q_{t_A}Q_{t_B}) \in \{0,1\} \tag{6.8}$$

i.e., measurement outcome at time t_B does not depend on measurement outcome at earlier time t_A. According to Baye's theorem

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) = P(B_{t_B}|A_{t_A}Q_{t_A}Q_{t_B})P(A_{t_A}|Q_{t_A}Q_{t_B}). \tag{6.9}$$

Using the Eq.(6.8) in Eq.(6.9) we get,

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) = P(B_{t_B}|Q_{t_A}Q_{t_B})P(A_{t_A}|Q_{t_A}Q_{t_B}). \tag{6.10}$$

Assuming NSIT, we get,

$$P(B_{t_B}|Q_{t_A}Q_{t_B}) = P(B_{t_B}|Q_{t_B}). \tag{6.11}$$

Due to Induction, which says that measurement statistics at an earlier time should not depend on what would be measured at a later time, we also have.

$$P(A_{t_B}|Q_{t_A}Q_{t_B}) = P(A_{t_B}|Q_{t_A}). \tag{6.12}$$

From Eqs.(6.10), (6.11), and (6.12) finally we get

$$P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) = P(B_{t_B}|Q_{t_A})P(A_{t_A}|Q_{t_A}). \tag{6.13}$$

As the assumption of predictability implies joint operational statistics are deterministic, hence, $P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}, \lambda, \lambda \rightarrow \lambda') = P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B})$. Therefore the factorizability condition (i.e. Eq.(6.4)) follows from conditioning the probabilities in the RHS of the above equation on λ.

The above derivation of LGI implies that either both or at least one of the underlying assumptions is violated whenever LGI is violated. Thus, as a corollary of Theorem 1 we can say,

Corollary: $\neg$ LGI $\wedge$ NSIT $\Rightarrow$ $\neg$ predictability.

Imagine now a situation where LGI is violated but NSIT is satisfied. It would be worth mentioning here that NSIT is experimentally testable. In such situations, we can say that the model cannot be predictable. Using the said situation, in the following, we show that temporal correlations are useful for DI randomness certification.

6.3 LGI and device-independent randomness

In device independent scenario, one does not have detailed knowledge about the experimental apparatuses and hence the experimental setup is like a black-box with inputs and outputs. At the input probe, one can change the parameters of measurement setup and thus can choose different measurements. Measurements are performed at different instants of time on the system and the frequencies $P(A_{t_A}B_{t_B}|Q_{t_A}Q_{t_B}) \in [0, 1]$ of occurrence of a given pair of outcomes for each pair of different time measurements are collected at the output probe either by repeating the experiment many times or by employing an array of many identical systems. The other joint probabilities involved in Leggett-Garg inequalities (6.1) or (6.2) are calculated likewise to observe their violations. These probabilities are also analyzed to see whether NSIT is obeyed. Note that, NSIT is defined operationally

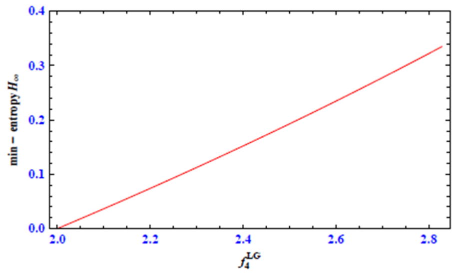

FIG. 6.1: (Color on-line) Certifiable randomness associated with Leggett-Gerg function $f_4^{LG} = f_4^{MR} + \epsilon$. Randomness is achieved for non zero value of ϵ.

and it is the statistical version of non-invasive measurability. Hence, it can be tested from data collected in an input output experiment. In fact, in the appendix of Ref. [94], an explicit scenario involving the Mach-Zehnder interferometer has been presented where LGI is violated but NSIT is satisfied. As we now know that such distribution cannot be predictable and therefore some randomness is associated with it. The associated randomness can be quantified by min-entropy [134] which is a statistical measure of the amount of randomness that a particular distribution contains. For a distribution X, it is defined as

$$H_\infty(X) \equiv \log_2 \frac{1}{\max\limits_{\mathrm{x:Prob(X=x)}} \mathrm{Prob(X=x)}}$$

Thus, to obtain the minimum amount of randomness associated with the violations of LGI (represented by $f_\delta^{LG} = f_\delta^{MR} + \epsilon$, where f_δ^{MR} is the macrorealistic bound of f_δ^{LG}; $\delta = 3$ for inequality (6.2) and for (6.1) $\delta = 4$; $\epsilon > 0$), we need to first solve the following optimization problem:

$$\begin{aligned} P_{NSIT}(Q_{t_\alpha}, Q_{t_\beta}) \quad = \quad & \max_{i,j} P(Q_{t_\alpha} = i, Q_{t_\beta} = j) \\ & \text{subject to } \; f_\delta^{LG} = f_\delta^{MR} + \epsilon \\ & P(Q_{t_\alpha} = i, Q_{t_\beta} = j) \geq 0 \\ & \sum_{i,j} P(Q_{t_\alpha} = i, Q_{t_\beta} = j) = 1 \\ & P(Q_{\mathfrak{T}_\alpha}, Q_{\mathfrak{T}_\beta}) \text{ satisfy NSIT.} \end{aligned} \tag{6.14}$$

Having the optimized solution $P^*_{NSIT}(Q_{t_\alpha}, Q_{t_\beta})$, the minimum randomness is calculated as $H_\infty(Q_{\mathfrak{T}_\alpha}, Q_{\mathfrak{T}_\beta}) = -\log_2 P^*_{NSIT}(Q_{t_\alpha}, Q_{t_\beta})$. We have considered the Leggett-Gerg function f_4^{LG} and f_3^{LG}, one after another, in the optimization problem (6.14) and have numeri-

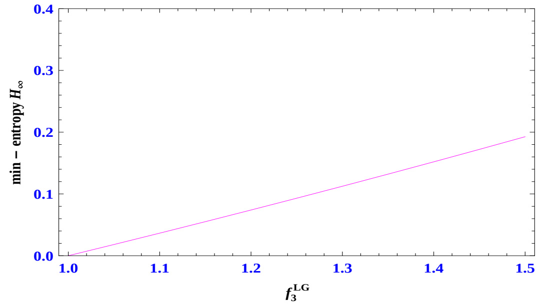

FIG. 6.2: (Color on-line) Certifiable randomness associated with Leggett-Gerg function f_3^{LG}.

cally calculated the minimum amounts of randomness with different values of ϵ. We plot our findings in Fig.6.1 and Fig.6.2 [1]. The curve in Fig.6.1 corresponds to the minimal value of the min-entropy implied by the assumption of NSIT. The function f_4^{LG} is zero at the macrorealistic threshold value $f_4^{LG} = 2$. Temporal correlations that violate the LGI inequality (6.1), on the other hand, have a positive min-entropy. Fig.6.2 represents the minimal value of the min-entropy for the violation of LGI inequality (6.2).

6.4 Concluding remarks and future perspective

In this work, we have shown that temporal correlations which violate Leggett-Gerg inequality, can also be used to certify randomness. It would be worth mentioning here that like Bell's scenario, in the case of temporal correlations too, we need some amount of seed randomness at the input. This is needed for freely choosing the measurement times. From the perspective of experimental implementation the LGI-based DI randomness certification seems more feasible than its spatial analogue as it does not require entanglement.
Bell inequality violation in quantum mechanics is always bounded by Cirelson bound (i.e. $2\sqrt{2}$) [34]. On the other hand the optimal violation of inequality (6.2) is $\frac{3}{2}$ for a qubit system, which remains the same irrespective of the system size for dichotomic measurements. In a recent development, it has been shown that the optimal violation of inequality (6.2) can go beyond $3/2$ for higher dimensional system (for system dimension$\rightarrow \infty$ violation of inequality (6.2) reaches its algebraic maximum) if non degenerate type of projective measurements are considered [39]. The study of 'randomness certification' from temporal correlation in this new scenario is a subject matter of further research.

[1] For the Leggett-Garg function f_4^{LG} , the associated randomness can also be obtained in a closed form as $H_\infty(Q_{\mathcal{T}_\alpha}, Q_{\mathcal{T}_\beta}) \geq -\log_2(\frac{3}{2} - \frac{2+\epsilon}{4})$ (cf. Fig.1). The calculation is similar to Ref.[125], but the context is different here. While in [125], correlations between measurement results from two distantly located physical systems are considered, here the focus is on one and the same physical system to obtain the correlations between measurement outcomes at two different times.

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