Canonical Formulation Of Fluid Dynamics

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To my Grandparents & Bomma....

Canonical Formulation Of Fluid Dynamics

Submitted by: **Arpan Krishna Mitra**

Abstract

This thesis contains a systematic and thorough discussion of different fluid systems in Hamiltonian frame work. A strong physical basis of introducing the Clebsch parameters following Noether's prescription is the foremost topic we have dealt with in the context of non relativistic fluids. The arbitrariness of this parametrisation is discussed in detail. Hamiltonian formulation of an ideal relativistic fluid is presented. Introduction of Clebsch parametrisation reduces the system to first order one demanding a complete constraint analysis. Some subtleties arise on the introduction of a non dynamical interacting gauge field regarding the stress tensor conservation and the equivalence of the Canonical and Symmetric EM tensors. The complimentary role of both the definitions have been discussed, though both the problems are solved subsequently introducing a dynamical gauge field. We have provided a complete description of fluids in light cone coordinates along with developing a new method of non relativistic reduction for non interacting ideal fluid system. A hallmark of a consistent field theory is the Schwinger conditions. Similar conditions in the classical field(not quantum) context particularly in the case of non relativistic fluid is a new finding.

We have developed a generalized fluid model that lives in NC space. How the dynamical equations of fluid, namely the continuity and Euler equations receive NC contributions are discussed. Time evolution of modes of density contrast, in particular the growing modes, dictate the structure formation in Universe. In this thesis we explicitly show how (spatial) Non-Commutativity (NC) can affect the behavior of the modes, that is, we compute NC corrected power law profiles of the density contrast modes.

Arpan Krishna Mitra.

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List of publications

1. *"Hamiltonian analysis of interacting fluids"*
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2. *"Canonical formulation for nonrelativistic Euler fluids and Schwinger type conditions"*
   Rabin Banerjee, **Arpan Krishna Mitra**,
   e-Print: arXiv:1510.00288 [hep-th].

3. *"On the equivalence among stress tensors in a gauge-fluid system"*
   Arpan Krishna Mitra, Rabin Banerjee, Subir Ghosh
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4. *"Noncommutative fluid and Growing Modes of Inhomogeneity in (Newtonian) Cosmology"*
   Arpan Krishna Mitra, Rabin Banerjee, Subir Ghosh
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5. *"Two dimensional hydrodynamics with gauge and gravitational anomalies"*
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Chapter 1

Introduction

The study of fluid dynamics as an applied science has been pursued through ages. Though the description of fluid dynamics as a classical field theory has its origin in the nineteenth century, its generalization as a relativistic field theory is a relatively recent development. The theory of fluid dynamics has continuously evolved through its various ramifications and extensions. Indeed it can and does illuminate several features of particle physics, especially those related to extended structures, and also gravity, that has culminated in the fluid/ gravity correspondence [13]. Thus the study of fluid dynamics, which is interesting in its own right, has relevance and significance in the modern context.

The dynamics of fluids is described by a classical field theory using either the lagrangian or hamiltonian approach. The lagrangian approach deals with the particle description while the hamoltonian approach treats the fluid system as a continuous one. The objective of usual problems of hydrodynamics is to define the Euler variables, namely velocity, density and a thermodynamical variable (pressure or entropy) as the functions of the particle coordinates x_i and time t, since fluid dynamics is most conveniently expressed in terms of them [2–4]. Among these competing approaches, that based on the hamiltonian is quite frequently discussed. In this respect the fluid dynamics differs from the conventional dynamics (classical and quantum). The reasons for that is not only the infinite number of degrees of freedom of fluid. The basic difference between conventional field theory and the fluid is that in the first case we can speak about the dynamics of the field at different points in space, while in the case of the second one, describing the interaction of the neighbouring constituent fluid particles we are not being able to fix its position in the space due to the motion of the fluid itself.

The form of the fluid hamiltonian may be written on general principles and the algebra of variables suitably defined to reproduce the known equations of motion for the fluid [1,2]. Incidentally, this algebra is either posited by inspection [2] or derived by using the Lagrange to Euler map [5]. A self contained derivation within the Eulerian scheme seems to be lacking.

The lagrangian formulation, on the contrary, is quite tricky. For usual canonical brackets the passage from the hamiltonian to the lagrangian is smooth, using an appropriate Legendre transform. However when the brackets are non canonical, as happens in the case of fluids, this transition is far from straightforward. A lagrangian version of fluid dynamics is plagued with obstructions due to the presence of a Casimir operator, which is in three dimension the velocity Chern-Simons term, the vortex helicity (see Jackiw et al. [1, 7] for a modern perspective). Under these circumstances, something different has to be done. A possible way is to introduce Clebsch variables [28] and use certain continuity equations. Clebsch parametrisation is an intrinsically nonlinear vector field decomposition into scalars. This was the original method adopted by Lin and Eckart [29]. This formulation is designed in such a way that the vortex helicity becomes a surface term having no bulk contribution hence does not obstruct the canonical formulation or the construction of the symplectic structure.

However, there were ambiguities since the method was ad-hoc and there was no proper physical basis for the choice of one continuity equation [1]. Another approach based on conservation laws is discussed in [16] but it also suffers from similar criticisms.

Extension of these ideas in a relativistic context has also been dealt with. The velocity-potential version of perfect-fluid dynamics as formulated by Seliger and Whitham, [37] generalized for relativistic fluids by Schutz [17], can be regarded as a nonlinear relativistic field theory for five coupled scalar fields, whose Lagrangian density is simply the pressure of the fluid. But all these studies are concerned with a free (or at best self-interacting) fluid and an in depth hamiltonian analysis of a relativistic fluid with external gauge interactions remains unexplored.

Hydrodynamics is an effective description of nearly equilibrium interacting many body systems. A fluid system is considered to be continuous. The hydrodynamic equations assume that the fluid is in local thermodynamic equilibrium at each point in space and time, inspite of the possible variation of the thermodynamic quantities (the macroscopic or the averaged out quantities) like fluid velocity $v(x,t)$, energy density $\epsilon(x,t)$, pressure $p(x,t)$, fluid density $\rho(x,t)$, etc. The hydrodynamic equations are essentially the local conservation laws supplemented by the constitutive relations that express the stress tensor in terms of the fluid variables. Fluid mechanics can only be applied to the systems where length scales of variation of thermodynamic variables are large compared to the mean free path [1].

Most of the fluids arise out of underlying particle systems. This becomes explicit in the Lagrange description, in which the coordinates of the underlying particle structure are involved. The transition to the Euler (or hamiltonian) description, then, allows us to express the system in terms of continuous fluid degrees of freedom.

The Euler formulation of the fluid system in terms of the density $\rho(x)$, velocity fields $v_i(x)$ (in a non-relativistic framework) and fluid current four vector j^μ ($j^0 = \rho, j^i = \rho v^i$) are found to be suitable for the field theoretic description [3, 4, 6]. This description is consistent with the fluid equations namely Euler (momentum conservation) and continuity (mass conservation) equations.

It may be mentioned that the recent idea of fluid-gravity correspondence [12,13] has brought, to the forefront, the theoretical study of fluid dynamics from a high energy and gravitational physics perspective. The basic premise is that relativistic or non-relativistic fluid dynamics can reproduce the low energy behavior of systems in local thermal equilibrium in a universal way. Indeed, this is an offshoot of the AdS/CFT correspondence [11] that paves the way for studying strongly coupled systems from their weakly coupled analogues in one dimension higher. Generically one exploits AdS/CFT correspondence to study strongly correlated condensed matter systems as boundary conformal theories from results obtained in weakly coupled classical gravity theories in one higher dimension. However, the mutual exchange of ideas can work bothways in fluid-gravity correspondence: fluid systems can yield results relevant in eg. black hole physics, Hawking radiation [14] while gravitational physics can provide new ideas in the context of viscous fluids, turbulence, to name a few. All these considerations require a systematic study of the fluid system as a field theory in the Euler scheme, which is essentially a hamiltonian framework.

From a modern high energy physics perspective, the canonical theory for relativistic perfect isentropic fluids was developed in [1], with special emphasis on symmetry aspects of the theory. Indeed, the classical version of ideal fluid theory is a conformal field theory and this property can be exploited in AdS/CFT correspondence. On the other hand and more interestingly, exploiting the fluid/gravity correspondence there is hope of deriving a theory of non-ideal fluid and even fluid in the presence of turbulence, based on first principles. This is because, the non-ideal fluid, being a strongly coupled one, can be dual to a weakly coupled gravity theory, again thanks to AdS/CFT correspondence. The role of symmetries and their implications in fluid systems is quite crucial in this set up.

Another area of topical interest is the non-relativistic reduction of relativistic fluid systems where light-cone analysis plays a pivotal role. Quite interestingly, it has been demonstrated in [12] that rewriting the conservation relation of relativistic energy-momentum tensor in light-cone variables, and compactifying a spatial light-cone coordinate, one can map the relativistic fluid dynamics to its non-relativistic counterpart in one dimension lower. This also requires a non-trivial map between relativistic and non-relativistic variables that can serve as the constitutive relations. In the present work we have performed a light-cone analysis of the fluid system. This is essential since (it is quite well known [33] that) the constraint structure is altered in a qualitative way as one converts to light-cone variables and one has to recover the dynamical equations directly from the light-cone action, instead of simply expressing the equations of motion (derived from the action in conventional coordinates) in light-cone components, as is done in [12].

The hamiltonian formulation of a nonisentropic fluid system which is interacting with an dynamical gauge field was absent. The most crucial part of this analysis in both the equal time and light cone frame work is to deal with the definitions of the Energy momentum tensor. One of the most important object of a relativistic field theory is the energy momentum tensor. Apart from the physically relevant energy-momentum conservation principles, the tensor components act as spacetime transformation generators that reflect the spacetime symmetries. Standard ways to define this are either by using the Noether's prescription or following the Schwinger definition. It was seen that these two definitions agree in case of non-interacting ideal fluids. But when a background interaction is present they fail to match. Both of these definitions have their own utilities and shortcomings. The components of the e-m tensors derived following Noether's definition generate correct equations of motion but fails to produce the proper conservation equatuion. On the other hand the symmetric definition produces the conservation equation of the stress tensor correctly apart from the presence of the

Lorentz force term, but fails to produce the correct eom for a particular fluid variable.

The main motivation of discussing the fully interacting fluid system is to deal with these discrepancies. Indeed in the presence of a dynamical gauge field the Lorentz force term moves away from the conservation equation of the stress tensors. Moreover the difference between the components of the em tensor following from two different definition become proportional to the Gauss constraint. Hence on the physical subspace these two definitions agree.

Furthermore consistency of the relativistic model depends on the validity of the Schwinger condition [9,10] that is a local property and is stronger than the total energy-momentum conservation principle. The latter appears as an integrated version of the local Schwinger condition. Though it was first discussed in the context of the quantum field theory, the existence of similar conditions had been verified for Chaplygin gas [25]. Whether they hold for classical fluids has been left un explored.

A recent development of Quantum Field Theory is its quantisation to noncommutative space time, that has its origin in low energy limits of String theory [51]. We have investigated the Noncommutative (NC) spacetime effects in ideal fluid dynamics which is turning into an area of recent activity [49]. One way to introduce NC effects in fluid is the introduction of NC algebra in Lagrangian (discrete) fluid degrees of freedom which in turn percolates to the Euler (field) degrees of freedom [52, 54] and NC-extended fluid action.

While discussing Newtonian cosmology we start with the standard Friedman equations. Whether the NC modified fluid equations bring any non trivial change in the Friedman equations is a question we should explore naturally. The other significant changes in the different modes in the density perturbation theory and the changes in the cosmological parameters, if any, remains unexplored.

In this thesis we have presented a systematic and detailed analysis of an ideal relativistic fluid in the hamiltonian framework. Subsequently we have generalised this analysis to include interaction with an external gauge field. Introduction of the Clebsch variables, namely α, β, γ reduces the system to a first order one: a constraint system in the Dirac formalism [32] (see also [33])[1]. We study the systems in hamiltonian framework developed by Dirac. Our analysis reveals that the relativistic Eulerian fluid model poses an intriguing example of a hamiltonian constraint system. This becomes manifest especially when gauge interactions are taken into account.

Next, we proceed towards a generalization of our fluid model [53]. There are two extensions. First, we now consider a non-isentropic fluid, where the entropy of the system is not constant. The fluid potential no longer remain a sole function of density but it becomes function of fluid entropy also. Secondly, and more importantly, it deals with the full interacting theory where the gauge field is also dynamical. This additional input yields new interesting results and puts the interacting fluid model in a clearer perspective. It should be emphasized that our formalism is different from the existing works on fluid in the presence of electromagnetic (U(1)gauge) interactions [6, 17–20].

We have also provided a detailed lightcone analysis has been provided in detail, pri-

[1] A hamiltonian system is said to be constrained if there are some extra conditions imposed on the allowed initial positions and momenta which should remain unchanged during the time evolution of the system. Dirac proposed a self consistent way to handle such systems.

marily because of its role in topical concepts of non-relativistic AdS/CFT and holography [26] and also because of the non-trivial aspects of a relativistic theory in lightcone framework. We have compared and contrasted the results of non-interacting fluid and fluid in presence of gauge interaction in lightcone [15, 53]. This analysis is completely new.

The construction of the stress tensor has been done in some detail. There are two distinct forms of the stress tensor based on two conventional definitions. The canonical $T_{\mu\nu}$ is obtained via Noether prescription and the symmetric $\Theta_{\mu\nu}$ is obtained by metric variation (Schwinger method). Both the forms have their own advantages. For the free theory both definitions agree. However, in the presence of interaction, $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ do not match. We have explained the reason of this mismatch, and have dealt with various aspects of the relations of these two different stress tensors. These are new observations that were not revealed in the literature that dealt with fluid models.

Using the results for the stress tensor, the question of validity of the Schwinger condition [9, 10], a hallmark of a consistent relativistic theory, in the present fluid-gauge model, both in equal time and light-cone coordinates, was investigated. Its role in conservation laws on which the dynamics of fluids is based,is discussed. The fact that the Schwinger condition holds for classical fluids is a new observation.

Finally in this thesis we discuss the extension to NC fluid variable algebra with a discussion on the corresponding Jacobi identities. A study of the generalized continuity and conservation principles was done including comments on spacetime symmetries for NC fluid. Furthermore, we provide an analysis on the effects on cosmological principles induced by NC modified fluid system.

1.1 Outline of the thesis

I will now briefly describe the outline of my thesis

- *Chapter 2* In this chapter we provide a new Lagrangian approach to discuss non-relativistic fluids. We exploit Noether's definition[2] of the stress tensor to obtain the ideal fluid lagrangian which will produce the fluid equations as the equations of motion. The choice of the Clebsh parametrisation is naturally dictated by the analysis. Indeed, the structure of Noether's stress tensor yields this parametrisation and provides the physical basis of the Clebsch variables. The freedom in the choice of Clebsch variables is discussed. Nonisentropic fluids have also been considered. A generalised definition of velocity in terms of Clebsh parameters including entropy is found. A hamiltonian formulation has been given where non canonical brackets are computed directly from the symplectic structure.

[2]We recall the use of Noether's definition for relativistic hydrodynamics [20]. However, since Noethers definition is asymmetric, it has to be used with care when treating relativistic systems where the energy-momentum tensor must be symmetric and suitable improvements have to be done. In the non-relativistic case, of course, this restriction of symmetricity(between space-time) no longer holds.

- *Chapter 3* We have presented a systematic and detailed hamiltonian analysis of an ideal relativistic fluid. Subsequently this analysis is generalised to include interaction with an external gauge field. Introduction of the Clebsch variables reduces the system to a first order one: a constraint system in the Dirac formalism [32] (see also [33]). We study both the systems in Dirac's framework. The relevant constraints are identified and the systems are found to be second class. The modified symplectic structure is the same in both cases. This analysis reveals that the relativistic Eulerian fluid model poses an intriguing example of a Hamiltonian constraint system. This becomes manifest especially when gauge interactions are taken into account.

- *Chapter 4* This chapter deals with the connection between the two distinct forms of the stress tensor based on two conventional definitions. The canonical stress tensor $T_{\mu\nu}$ is obtained via Noether prescription and the symmetric one, $\Theta_{\mu\nu}$, is obtained by metric variation of the action. For the free theory both definitions are found to be in good match. However, in the presence of interaction these two definitions apparently do not agree. When the interaction is with a background gauge field this difference sustains. While the canonical stress tensor yields the dynamical equations for the fluid variables correctly, the symmetric definition fails in this regard. The symmetric stress tensor on the other hnad correctly yields the lorentz force term in the conservation equation the. Here we have been able to demonstrate that the inconsistency regarding this mismatch can be successfully dealt in when the gauge field is dynamical. The point is that the physically relevant quantities are the integrated versions of different components of $T_{\mu\nu}$(or $\Theta_{\mu\nu}$) which define the various space time generators. Interestingly, the integrated versions of $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ agree, modulo terms which are proportional to the Gauss constraint. Thus they are gauge equivalent. Hence, in the physical subspace, the two definitions of the generators agree. Indeed this has been possible only because of the dynamical nature of the gauge field which brings about new constraints in the theory, in particular the Gauss law. This constraint, incidentally, did not appear for non-dynamical gauge fields.

- *Chapter 5* The lightcone analysis has been provided in detail, primarily because of its role in topical concepts of non-relativistic AdS/CFT and holography [26] and also because of the non-trivial theoretical aspects of a relativistic theory itself in lightcone framework. We have compared and contrasted the results with our previous observations in [15] that dealt with non-interacting fluid in lightcone. Lightcone or Infinite Momentum Frame was introduced long ago in the context of formulating bound states of quarks and gluons in relativistic QCD [42] (for a review see [21]). In recent years lightcone quantization has reappeared strongly in the work of Son [26] who has exploited it in non-relativistic generalization of

AdS/CFT and holographic principles [8]. We have done a nonrelativistic reduction for ideal fluids in light cone coordinate.

- *Chapter 6* In this chapter we will concentrate on the commutation relations between the different components of the energy momentum tensor. Our main aim here is to check whether they satisfy the celebrated Schwinger conditions or something of that sort or not. These relations were derived in the relativistic quantum field theory context as another way to show the conservation of the energy momentum tensor. We will show that these relations can be obtained for classical fluids(relativistic) and some equations of similar type is present in the nonrelativistic case as well.

- *Chapter 7* In this chapter we deal with the NC effects on ideal fluids. The model for this purpose as proposed by us rests essentially on the map between the Lagrangian and Eulerian or (Hamiltonian) description of fluid dynamics. The fluid equations of motion are derived from the above as Hamiltons equation of motion. The most relevant result from our perspective is that the Poisson brackets between Euler field variables are explicitly derivable from Poisson brackets between (discrete) Lagrangian d.o.f.. The chain of steps leading from Lagrangian to Eulerian formulation is best suited for our purpose since, as discussed earlier, the NC brackets are given most naturally in point mechanics framework, that is in terms of Lagrangian variables. It is worthwhile to recall here that even the canonical point mechanics (Poisson) brackets lead to a quite involved and non-linear set of operatorial algebra between the Euler variables. Hence it is not entirely surprising that the simplest extension of canonical brackets to NC brackets in Lagrangian setup will lead to an involved and qualitatively distinct NC extended brackets among Euler variables. However, as we will explicitly demonstrate, these NC brackets yield a modified set of continuity equation and Euler force equation. Moreover we clarify issues related to the Jacobi identity of the NC fluid variable algebra.

- *Chapter 8* We introduce cosmological perturbations and explicitly show how the behavior of growing and decaying modes of density contrast are affected by non-commutative (or non-canonical, which is probably more appropriate as pointed out in the paper) corrections. We have explicitly demonstrated that the positive or negative values of the noncommutative parameter σ can decrease or increase the Hubble parameter respectively. The former can be identified with an effective model for dark matter. Similarly positive σ enhances the increasing mode of density contrast which also agrees with the dark matter interpretation mentioned above. We have considered the simplest form of approximation and a more detailed analysis of the model is needed. Specifically one of our future projects is to find solutions of the scale factor directly computed from the noncommutativity extended equations derived here. Finally it would be interesting to exploit the rigorous cosmological averaging principles developed by Buchert and coworkers [71–74] in the present context where the modifications stem from the fact that the evolution and averaging of dynamical variables do not commute.

- *Chapter 9* contains the conclusions and future directions.

Chapter 2

Canonical formulation of nonrelativistic fluids; Physical basis of Clebsch parametrisation

An alternative approach of constructing the fluid lagrangian is provided in this chapter. To do so we will exploit Noether's prescription. Starting from the momentum density and its properties we will unveil the role of Noether's theorem. The Clebsch parametrisation of fluid velocity will come to out to be a natural choice.

This chapter is started with a brief discussion of the existing formalism. Onwards we analyse an irrotational fluid system in our new approach. We use our method to the fluid systems with non zero vorticity. Then a discussion on nonisentrpic fluid where the definition of fluid velocity in terms of Clebsch parameters, including entroty is provided. We end this chapter with a discussion on the arbitrariness in the choice of the Clebsch parameters.

2.1 Review of the standard formalism

In this section we first briefly recapitulate the basic tenets of non-relativistic isentropic Eulerian fluids adopting the standard path. The fluid hamiltonian is given by,

$$H = \int dx(\frac{1}{2}\rho v^2 + V(\rho)) \tag{2.1}$$

where ρ and v_i are the fluid density and velocity, respectively. The fundamental fluid equations, namely the Euler equation and continuity equation, are reproduced by appropriate bracketing with (Eq. (2.1)) by exploiting the following algebra [1, 2],

$$\{\rho(x), \rho(x')\} = 0, \quad \{\rho(x), v_i(x')\} = \partial_i \delta(x - x'), \quad \{v_i(x), v_j(x')\} = -\frac{\omega_{ij}}{\rho}\delta(x - x') \tag{2.2}$$

where,

$$\omega_{ij} = \partial_i v_j - \partial_j v_i \tag{2.3}$$

is the vorticity of the fluid. The continuity equation is reproduced as,

$$\dot{\rho} = \{\rho, H\} = \partial_i(\rho v_i) \tag{2.4}$$

Likewise the Euler equation is obtained as[1],

$$\dot{v_i} = \{v_i, H\} = v_j \partial_j v_i + \partial_i V_\rho(\rho) \tag{2.5}$$

Note that the second term on the right side of (Eq. (2.5)) may be expressed in a familiar form by recalling the definition of pressure P as a Legendre transform of V [1],

$$P(\rho) = \rho V_\rho - V(\rho) \tag{2.6}$$

so that,

$$\frac{1}{\rho}\partial_i P = \partial_i V_\rho \tag{2.7}$$

It is thus the pressure gradient which is consistent with the fact that we are discussing ideal hydrodynamics.

Introducing the current $j_i = \rho v_i$ it is simple to find,

$$\{j_i(x), \rho(x')\} = \rho(x)\partial_i \delta(x - x')$$

$$\{j_i(x), j_k(x')\} = j_k(x)\partial_i \delta(x - x') + j_i(x')\partial_k \delta(x - x') \tag{2.8}$$

This completes a brief summary of the standard formulation. Observe that the brackets (Eq. (2.2)) are posited such that the fluid equations (Eq. (2.4)), (Eq. (2.5)) are reproduced from the hamiltonian (Eq. (2.1)).

2.2 An altrnative approach based on Noether's theorem

An alternative and more economical method based on Noether's prescription is now elaborated. We begin with a simple system and subsequently will deal with more general systems having higher degree of complications.

[1] suffix on V implies a derivative; $V_\rho = \frac{\partial V}{\partial \rho}$

Irrotational fluid

To start with we chose irrotational fluid as our system of concern. Here we recall that the momentum density in a nonrelativistic theory coincides with the current,

$$T^{0i} = j^i = \rho v^i \tag{2.9}$$

The translation generator,

$$P^i = \int d^3x T^{0i} \tag{2.10}$$

acting on the basic variables ρ, v^i should yield the expected transformations,

$$\{\rho(x), P^i\} = \partial^i \rho(x) \tag{2.11}$$

$$\{v^j(x), P^i\} = \partial^i v^j(x) \tag{2.12}$$

This input is sufficient to reproduce the algebra (Eq. (2.2)). Since space is commutative, the fluid density must have a vanishing Poisson bracket. Using (Eq. (2.10)) in (Eq. (2.11)) then yields,

$$\int d^3y\ \rho(y)\{\rho(x), v^i(y)\} = \partial^i \rho(x) \tag{2.13}$$

This immediately reproduces the ρ–v^i algebra given in (Eq. (2.2)). Now inserting (Eq. (2.10)) in (Eq. (2.12)) and exploiting the $\rho - v^i$ algebra leads to,

$$\int d^3y\ \rho(y)\{v^j(x), v^i(y)\} + \partial^j v^i(x) = \partial^i v^j(x) \tag{2.14}$$

The $v^i - v^j$ bracket is now obtained which reproduces (Eq. (2.2)). The crucial role of the relation (Eq. (2.9)) will be further emphasised as we progress.

In order to construct an appropriate Lagrangian let us introduce a variable θ which is canonically conjugate to ρ. Then we may write the lagrangian density as,

$$\mathcal{L} = \rho\dot{\theta} - \mathcal{H} = \rho\dot{\theta} - (\frac{1}{2}\rho v^2(\theta) + V(\rho)) \tag{2.15}$$

Here θ is not an independent variable. It is related to the fluid velocity v_i. The velocity will be expressed as a function of θ so that in the above Lagrangian ρ and θ are the only variables of variation. It is of course necessary to abstract the connection of the velocity with the variable θ before it is possible to show that the correct hydrodynamical equations follow from (Eq. (2.15)). This is shown below.

To understand the meaning of θ in terms of the fluid variables we take recourse to Noether's definition of stress tensor

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu F)}\partial^\nu F - \mathcal{L}g^{\mu\nu} \tag{2.16}$$

where F generically denotes the variables in the lagrangian.

For the lagrangian (Eq. (2.15)) since ρ and θ are the variables of variation, Noether's stress tensor (Eq. (2.16)) reduces to,

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \rho)}\partial^\nu \rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \theta)}\partial^\nu \theta - \mathcal{L}(\rho,\theta) g^{\mu\nu} \tag{2.17}$$

Computing T^{0i} from (Eq. (2.15)) and (Eq. (2.17)), we get,

$$T^{0i} = \rho \partial^i \theta \tag{2.18}$$

which, equated to (Eq. (2.9)), yields the identification,

$$v^i = \partial^i \theta \tag{2.19}$$

Since the vorticity (Eq. (2.3)) vanishes, this corresponds to an irrotational fluid.

The correspondence (Eq. (2.19)) is algebraically consistent with (Eq. (2.2)) because,

$$\{\rho(x), v_i(x')\} = \{\rho(x), \partial_i \theta(x')\} = \partial_i \delta(x - x') \tag{2.20}$$

recalling that (ρ, θ) are a canonical pair, as seen from the first order lagrangian density (Eq. (2.15)),

$$\{\theta(x), \rho(x')\} = \delta(x - x') \tag{2.21}$$

The $v_i - v_j$ bracket following from (Eq. (2.19)) vanishes thereby reproducing the algebra (Eq. (2.2)) for an irrotational fluid.

To complete the demonstration of the validity of (Eq. (2.19)) we show that it reproduces the energy flux T^{i0} and stress tensor T^{ij} from (Eq. (2.16)).

The first step is to express the lagrangian (Eq. (2.15)) in terms of (ρ, θ) variables,

$$\mathcal{L} = \rho\dot{\theta} - (\frac{1}{2}\rho(\partial_i\theta)^2 + V(\rho)) \tag{2.22}$$

Next using (Eq. (2.16)) and (Eq. (2.22)), we obtain,

$$T^{i0} = \rho\partial^i\theta\dot{\theta} \tag{2.23}$$

The $\dot{\theta}$ term is found by bracketing with the hamiltonian, exploiting the algebra (Eq. (2.21)),

$$\begin{aligned}\dot{\theta} = \{\theta, H\} &= \{\theta, \int dy(\frac{1}{2}\rho(\partial_i\theta)^2 + V(\rho))\} \\ &= \frac{1}{2}(\partial_i\theta)^2 + V_\rho(\rho)\end{aligned} \tag{2.24}$$

Thus,

$$T^{i0} = \rho\partial^i\theta(\frac{1}{2}(\partial_i\theta)^2 + V_\rho(\rho)) = \rho v^i(\frac{1}{2}v^2 + V_\rho(\rho)) \tag{2.25}$$

which is the familiar expression for the energy flux.

Likewise the expression for the stress tensor T^{ij}, following from (Eq. (2.16)) and (Eq. (2.22)), is given by[2],

$$T^{ij} = \rho(\partial^i\theta)(\partial^j\theta) - \mathcal{L}g^{ij} = \rho v^i v^j + \mathcal{L}\delta^{ij} \tag{2.26}$$

This may be simplified by writing $\mathcal{L}$ as,

$$\mathcal{L} = \rho\dot{\theta} - (\frac{1}{2}\rho v^2 + V(\rho))$$

Now we use (Eq. (2.24)) to get,

$$\mathcal{L} = \rho(\frac{1}{2}v^2 + V_\rho(\rho)) - (\frac{1}{2}\rho v^2 + V(\rho)) = \rho V_\rho - V \tag{2.27}$$

so that we can express T^{ij} in our familiar form,

$$T^{ij} = \rho v^i v^j + (\rho V_\rho - V)\delta^{ij} \tag{2.28}$$

Thus (Eq. (2.22)) may be regarded as the lagrangian in Euler variables for an irrotational fluid.

We now show that the usual hydrodynamical equations follow from(Eq. (2.22)) which is equivalent to (Eq. (2.15)) with the identification (Eq. (2.19)). Variation with respect to θ gives,

$$\dot{\rho} - \partial_i(\rho(\partial_i\theta)) = 0 \tag{2.29}$$

If we now use(Eq. (2.19)), the above equation reproduces the continuity equation(Eq. (2.4)).

Likewise, variation of ρ in(Eq. (2.22)) yields,

$$\dot{\theta} - \frac{(\partial_i\theta)^2}{2} - V_\rho(\rho) = 0 \tag{2.30}$$

Taking a spatial derivative and again exploiting(Eq. (2.19)) yields,

$$\dot{v_i} - v_j\partial_i v_j - \partial_i V_\rho = 0 \tag{2.31}$$

For an irrotational fluid, $\partial_i v_j = \partial_j v_i$, so that(Eq. (2.31)) just reproduces the Euler equation(Eq. (2.5)).

Relation (Eq. (2.19)) is the Clebsch parametrisation for the velocity of an irrotational fluid. It was obtained on purely physical grounds by equating the T^{0i} component of Noether's stress tensor with the current (Eq. (2.9)). The important role of (Eq. (2.9)) is reinforced.

[2]We use the mostly negative metric, $g^{ij} = -\delta^{ij}$

Extension to a system of fluids with non vanishing vorticity

This idea may be extended to the general (non-vanishing vorticity) case, by an appropriate modification of the velocity parametrisation (Eq. (2.19)). It is not possible to do this by introducing just a single scalar like $\partial_i\beta$ or $\beta\partial_i\beta(=\frac{1}{2}\partial_i\beta^2)$ since they can be absorbed in the original definition (Eq. (2.19)). Thus the simplest nontrivial possibility is to introduce a pair of scalars (α, β) and express the velocity as,

$$v^i = \partial^i\theta + \alpha\partial^i\beta \tag{2.32}$$

It is easy to see that the second term cannot be obtained from the first by a change of variables. For instance $\theta \to \theta + \alpha\beta$ would yield $\partial^i\theta \to \partial^i\theta + \alpha\partial^i\beta + \beta\partial^i\alpha$. Thus the parametrisation (Eq. (2.32)) is intrinsically different from (Eq. (2.19)). The motivation for such a parametrisation will once again be provided by Noether's stress tensor.

The hamiltonian now has the structure,

$$H = \int dx(\frac{1}{2}\rho(\partial_i\theta + \alpha\partial_i\beta)^2 + V(\rho)) \tag{2.33}$$

To see that this yields a meaningful fluid hamiltonian it is useful to construct the lagrangian with a suitable kinetic term. This is done in a way that reproduces the momentum density (Eq. (2.9)) with the velocity given by (Eq. (2.32)), following the Noether prescription (Eq. (2.16)). After a little algebra we get the cherished form of the Lagrangian,

$$\mathcal{L} = \rho(\dot{\theta} + \alpha\dot{\beta}) - (\frac{\rho}{2}(\partial_i\theta + \alpha\partial_i\beta)^2 + V(\rho)) \tag{2.34}$$

To verify our previous statements we compute the momentum density,

$$T^{0i} = \rho(\partial^i\theta + \alpha\partial^i\beta) \tag{2.35}$$

which reproduces (Eq. (2.32)) from (Eq. (2.9)). It is interesting to note that the Clebsch parametrisation (Eq. (2.32)) for a rotational fluid is once again a consequence of (Eq. (2.9)) where T^{0i} is given by Noether's definition (Eq. (2.16)).

Next, the energy flux T^{i0} is computed from (Eq. (2.16)) and(Eq. (2.34)),

$$T^{i0} = \rho(\dot{\theta} + \alpha\dot{\beta})(\partial^i\theta + \alpha\partial^i\beta) \tag{2.36}$$

The time derivatives may be eliminated by considering the equation of motion obtained by a variation of ρ in (Eq. (2.34)),

$$\dot{\theta} + \alpha\dot{\beta} = (\frac{1}{2}(\partial_i\theta + \alpha\partial_i\beta)^2 + V_\rho(\rho)) \tag{2.37}$$

Substituting in (Eq. (2.36)) yields,

$$T^{i0} = \rho v^i[\frac{v^2}{2} + V_\rho(\rho)] \tag{2.38}$$

where we have used (Eq. (2.32)). Thus the desired form of the energy flux is reproduced.

Finally the stress tensorT^{ij} is considered. Again exploiting Noether's definition (Eq. (2.16)) and using (Eq. (2.34)) we find,

$$T^{ij} = \rho(\partial^i\theta + \alpha\partial^i\beta)(\partial^j\theta + \alpha\partial^j\beta) + \mathcal{L}\delta^{ij} = \rho v^i v^j + \mathcal{L}\delta^{ij} \tag{2.39}$$

on recalling the definition (Eq. (2.32)). The Lagrangian density (Eq. (2.34)) simplifies, using (Eq. (2.37)), to the form,

$$\mathcal{L} = \rho V_\rho(\rho) - V(\rho) \tag{2.40}$$

which, together with (Eq. (2.39)), reproduces the expected structure of the stress tensor T^{ij}. Incidentally the above equation (Eq. (2.40)) is expected on general grounds since it expresses the equivalence of the Lagrangian density with the fluid pressure.

Now the hydrodynamical equations are derived from (Eq. (2.34)). Variation with respect to θ yields,

$$\dot{\rho} - \partial_i[\rho(\partial_i\theta + \alpha\partial_i\beta)] = 0 \tag{2.41}$$

Using the form of the velocity (Eq. (2.32)) in the above equation immediately reproduces the continuity equation (Eq. (2.4)).

The demonstration of the Euler equation (Eq. (2.5)) needs some more work. Variation of α yields,

$$\dot{\beta} = v_i\partial_i\beta \tag{2.42}$$

while that of β yields,

$$\dot{\alpha} = v_i\partial_i\alpha \tag{2.43}$$

where, at an intermediate step, the continuity equation (Eq. (2.4)) has been used. Now, time differentiating the relation(Eq. (2.32)), we find,

$$\dot{v_i} = v_j\partial_i v_j + (v_j\partial_j\alpha)\partial_i\beta - (v_j\partial_j\beta)\partial_i\alpha + \partial_i V_\rho \tag{2.44}$$

where all time derivatives appearing on the r.h.s of (Eq. (2.32)) have been eliminated by exploiting (Eq. (2.37)), (Eq. (2.42)) and (Eq. (2.43)). The final point is to calculate the vorticity from (Eq. (2.32)),

$$\omega_{ij} = \partial_i v_j - \partial_j v_i = \partial_i\alpha\partial_j\beta - \partial_j\alpha\partial_i\beta \tag{2.45}$$

and use it to replace $\partial_i v_j$ in favour of $\partial_j v_i$ in (Eq. (2.44)). Immediately this reproduces the Euler equation (Eq. (2.5)).

We have thus shown that (Eq. (2.34)) may be regarded as the lagrangian density for vortical fluids. While the result itself is quite well known, the method of deriving it here is new. Contrary to earlier approaches, Noether's stress tensor plays a pivotal role. Also,it provides a novel way of deriving (Eq. (2.32)) which is the usual Clebsch parametrisation of a vector in terms of three scalars.

Let us now consider the hamiltonian formalism. For this it is necessary to know the algebra of the basic variables. This may be obtained from (Eq. (2.34)), by noting that (ρ, θ) and $(\rho\alpha, \beta)$ are the independent canonical pairs. Then it is possible to to show that, apart from (Eq. (2.21)), the only other nonvanishing algebra among the basic variables is given by,

$$\{\beta(x), \alpha(x')\} = \frac{1}{\rho}\delta(x - x'), \quad \{\alpha(x), \theta(x')\} = \frac{\alpha}{\rho}\delta(x - x') \tag{2.46}$$

with all other brackets being zero.

Now the complete algebra (Eq. (2.2)) for a fluid with vorticity may be verified using (Eq. (2.32)) and the brackets (Eq. (2.21), Eq. (2.46)). Consequently the fluid equations are also reproduced. This completes the hamiltonian analysis the Eulerian fluid model.

It may be observed that the non canonical algebra (Eq. (2.2)) was directly obtained in the hamiltonian formalism either as a postulate or by using generalised coordinates [2,6,34], or by using the map connecting Lagrange to Euler variables. Here they simply follow from the modified symplectic structure.

2.3 Nonisentropic fluids

So far isentropic fluids were considered where entropy has no role. However, for a complete characterisation of a fluid, apart from its density and velocity, one has to include entropy. This section is devoted to a study of nonisentropic fluids along the lines developed in the earlier section.

The fundamental fluid equations are now the continuity equation, the Euler equation and the entropy convection. While the continuity equation (Eq. (2.4)) remains unchanged, the Euler equation (Eq. (2.5)) becomes

$$\dot{v_i} = v_j \partial_j v_i + \partial_i V_\rho(\rho, S) - \frac{\partial_i S}{\rho} V_S(\rho, S) \tag{2.47}$$

where the potential is now a function of both the density and entropy $V(\rho, S)$. Here S is the entropy per unit mass or the specific entropy. The above equation may be expressed in a more conventional form [2] by introducing the variable $U(\rho, S)$ as $V = \rho U$. Then (Eq. (2.47)) reduces to,

$$\dot{v_i} = v_j \partial_j v_i + \frac{1}{\rho}\partial_i(\rho^2 U_\rho) \tag{2.48}$$

which is the general form of the hydrodynamic force balance equation or the Euler equation.

Finally, the entropy convection equation, expressing the fact that heat flow is assumed to vanish, is given by

$$\dot{S} = v_i \partial_i S \tag{2.49}$$

Also the form of the Hamiltonian remains unchanged except that the potential is now a function of both ρ and S,

$$H = \int (\frac{\rho v^2}{2} + V(\rho, S)) \tag{2.50}$$

We now discuss an action principle following our earlier prescription. The idea is to modify the lagrangian so that the momentum density is given by (Eq. (2.9)). This obviously implies that the parametrisation for the velocity (Eq. (2.32)) has to be generalised. Following our earlier logic discussed below (Eq. (2.31)), this may be done in two possible ways,

$$v^i = \partial^i\theta + \alpha\partial^i\beta + S\partial^i\gamma \tag{2.51}$$

or, alternatively,

$$v^i = \partial^i\theta + \alpha\partial^i\beta + \gamma\partial^i S \tag{2.52}$$

It is interesting to note that both these forms lead to a consistent formulation. While the first representation (Eq. (2.51)) was found earlier [37,38] using contact transformations,the second one (Eq. (2.52)) is obtained from the first by changing $\theta \to \theta - S\gamma$ and then replacing $\gamma \to -\gamma$ (or $S \to -S$). The Clebsch decomposition therefore satisfies a duality $S \leftrightarrow \gamma$.

Let us first consider the analysis with (Eq. (2.51)). To construct the Lagrangian corresponding to the Hamiltonian (Eq. (2.50)), the kinetic term has to be defined. As mentioned it is done in a way that reproduces the momentum density (Eq. (2.9)) with the velocity given by (Eq. (2.51)), using Noether's definition (Eq. (2.16)). We find,

$$\mathcal{L} = \rho(\dot\theta + \alpha\dot\beta + S\dot\gamma) - (\frac{\rho}{2}(\partial^i\theta + \alpha\partial^i\beta + S\partial^i\gamma)^2 + V(\rho, S)) \tag{2.53}$$

For a check we compute T^{0i} from (Eq. (2.16)),

$$T^{0i} = \rho v^i = \rho(\partial^i\theta + \alpha\partial^i\beta + S\partial^i\gamma) \tag{2.54}$$

which immediately yields (Eq. (2.51)) from (Eq. (2.9)).

It is now possible to reproduce the expected structure of the flux

$$T^{i0} = \rho v^i[\frac{v^2}{2} + V_\rho(\rho, S)] \tag{2.55}$$

and the stress tensor (Eq. (2.39)) from Noether's definition (Eq. (2.16)). This proves the validity of the Lagrangian (Eq. (2.53)).

The equations of motion for all the variables may be obtained from (Eq. (2.53)) by appropriate variations. Variation with respect to θ just yields the continuity equation(Eq. (2.4)) once the velocity is identified as (Eq. (2.51)). This is exactly as happened in the isentropic theory(Eq. (2.34)). The γ variation gives,

$$\partial_t(\rho S) - \partial_i(\rho S v_i) = 0 \tag{2.56}$$

where we have used (Eq. (2.51)). A simple use of the continuity equation (Eq. (2.4)) now reproduces the entropy convection equation (Eq. (2.49)).

The derivation of the Euler equation (Eq. (2.47)) follows along the earlier isentropic case. First, the equations obtained on varying ρ, S, α, β are found to be, respectively,

$$\begin{aligned}\dot{\theta} + \alpha\dot{\beta} + S\dot{\gamma} &= \frac{v^2}{2} + V_\rho \\ \dot{\gamma} &= v_j\partial_j\gamma + \frac{V_S}{\rho} \\ \dot{\beta} &= v_i\partial_i\beta \\ \dot{\alpha} &= v_i\partial_i\alpha\end{aligned} \tag{2.57}$$

The last two equations are, expectedly, identical to (Eq. (2.42)) and (Eq. (2.43)). Now time differentiating(Eq. (2.51)) and eliminating the time derivatives appearing on its r.h.s by using (Eq. (2.57)), we find,

$$\dot{v}_i = v_j(-\partial_i\alpha\partial_j\beta - (\partial_i S)\partial_j\gamma + (\partial_j\alpha)\partial_i\beta + \partial_j S\partial_i\gamma + \partial_i v_j) + \partial_i V_\rho - \frac{V_S}{\rho}\partial_i S \tag{2.58}$$

Computing vorticity from(Eq. (2.51)),

$$\partial_i v_j - \partial_j v_i = \partial_i\alpha\partial_j\beta + \partial_i S\partial_j\gamma - \partial_j\alpha\partial_i\beta - \partial_j S\partial_i\gamma \tag{2.59}$$

and substituting in (Eq. (2.58)) reproduces the cherished equation(Eq. (2.47)).

For a hamiltonian analysis the noncanonical brackets have to be derived. These are obtained by first noting that an inspection of (Eq. (2.54)) immediately identifies the independent canonical pairs as, (ρ, θ) $(\rho\alpha, \beta)$ and $(\rho S, \gamma)$. Here we compute the brackets involving S since the others have already been found. It is easy to see that the only nonvanishing brackets are given by,

$$\{S(x), \gamma(x')\} = -\frac{1}{\rho}\delta(x - x'), \quad \{S(x), \theta(x')\} = \frac{S}{\rho}\delta(x - x'), \tag{2.60}$$

The bracket of S with v^i is now evaluated using(Eq. (2.60)), and (Eq. (2.51)),

$$\begin{aligned}\{S(x), v^i(x')\} &= \{S(x), (\partial^i\theta + \alpha\partial^i\beta + S\partial^i\gamma)(x')\} \\ &= \partial^i_{x'}(\frac{S}{\rho}\delta(x - x')) + S(x')\partial^i_{x'}(-\frac{1}{\rho}\delta(x - x')) \\ &= \frac{\partial^i S}{\rho}\delta(x - x')\end{aligned} \tag{2.61}$$

Using this algebra the equation (Eq. (2.49)) is reproduced by bracketing S with the hamiltonian (Eq. (2.50)). Likewise the Euler equation (Eq. (2.47)) may also be reproduced.

2.4 Arbitrariness in Clebsch potentials and physically equivalent representations

In this section we will discuss the freedom in choosing the potentials θ, α etc that appear in the definition of the velocity (Eq. (2.51)) or (Eq. (2.52)). The point is that, given a velocity field $\bar{v}$, the potentials are not uniquely determined. Two sets of velocity potentials will be considered physically equivalent if they give the same velocity. We will generate such transformations. Apart from yielding the same velocity, the basic bracket structure (Eq. (2.21)), (Eq. (2.46)), (Eq. (2.60)) and the hamiltonian (Eq. (2.50)) are also preserved. This shows that the equations of motion are invariant under these transformations.

From the equations (Eq. (2.4)), (Eq. (2.49)), and (Eq. (2.57)) we can show that,

$$G = \int d^3x \; \rho g(\alpha, \beta, S) \tag{2.62}$$

is time conserved because,

$$\begin{aligned} \frac{dG}{dt} &= \int d^3x \; (\dot{\rho} g + \rho \dot{g}) \\ &= \int d^3x \; (\partial_i(\rho v_i) g + \rho v_i \partial_i g) \\ &= \int d^3x \; \partial_i(\rho v_i g) = 0 \end{aligned} \tag{2.63}$$

The conserved charge G acts as the generator of infinitesimal transformations. Thus, using the fundamental brackets (Eq. (2.21)), (Eq. (2.46)) and (Eq. (2.60)),

$$\begin{aligned} \delta\alpha = \{\alpha, G\} &= -\frac{\partial g}{\partial \beta} \\ \delta\beta = \{\beta, G\} &= \frac{\partial g}{\partial \alpha} \\ \delta\theta = \{\theta, G\} = g - \alpha\frac{\partial g}{\partial \alpha} &- S\frac{\partial g}{\partial S} \\ \delta\gamma = \{\gamma, G\} &= \frac{\partial g}{\partial S} \end{aligned} \tag{2.64}$$

Also, since G does not involve θ and γ, ρ and S do not change,

$$\delta\rho = \{\rho, G\} = 0; \quad \delta S = \{S, G\} = 0 \tag{2.65}$$

It is thus reassuring to note that only the potentials α, β, θ and γ change but the physical variables, density (ρ) and entropy (S), remain invariant.

Now it can be shown that the above transformations preserve the velocity,

$$\delta v_i = \partial_i(\delta\theta) + \delta\alpha\partial_i\beta + \alpha\partial_i(\delta\beta) + S\partial_i(\delta\gamma) \tag{2.66}$$

$$= \partial_i(g - \alpha\frac{\partial g}{\partial \alpha} - S\frac{\partial g}{\partial S}) - \partial_i\beta\frac{\partial g}{\partial \beta} + \alpha\partial_i\frac{\partial g}{\partial \alpha} + S\partial_i\frac{\partial g}{\partial S} = 0$$

Naturally, the transformations keep the hamiltonian (Eq. (2.50)) invariant since,

$$\delta H = \{H, G\} = -\frac{dG}{dt} = 0 \tag{2.67}$$

as already seen in (Eq. (2.63)). This may also be seen directly by observing that the hamiltonian (Eq. (2.50)) is a function of $\rho, \bar{v}$ and S, each of which is separately invariant. A more nontrivial exercise is to prove that the fundamental bracket structure (Eq. (2.21)), (Eq. (2.46)), (Eq. (2.60)) etc is also preserved. As an illustration, consider the variation of the first equation in (Eq. (2.46)),

$$\{\beta(x), \alpha(x')\} = \frac{1}{\rho}\delta(x - x') \tag{2.68}$$

which yields on the left hand side,

$$\delta\{\beta(x), \alpha(x')\} = \{\delta\beta(x), \alpha(x')\} + \{\beta(x), \delta\alpha(x')\}$$

$$= \{\frac{\partial g}{\partial \alpha(x)}, \alpha(x')\} + \{\beta(x), -\frac{\partial g}{\partial \beta(x')}\}$$

$$= \frac{\partial^2 g}{\partial\beta\partial\alpha}\ \frac{1}{\rho}\delta(x - x') - \frac{\partial^2 g}{\partial\beta\partial\alpha}\ \frac{1}{\rho}\delta(x - x') = 0 \tag{2.69}$$

where, in going from the first to the second line, we have used (Eq. (2.64)) , followed by (Eq. (2.68)). Since ρ is invariant, the variation of the right hand side of (Eq. (2.68)) also vanishes. This proves the invariance of the algebra. Likewise, the complete algebra may be shown to be preserved.

Since both the hamiltonian and the brackets are invariant, it is clear that the equations of motion also remain unchanged under the symmetry transformation. Interestingly, we can also prove the invariance of the lagrangian. Since this is a first order system and the hamiltonian is invariant, we just need to prove the invariance of the kinetic term:

$$\delta[\rho(\dot\theta + \alpha\dot\beta + S\dot\gamma)] = \rho(\delta\dot\theta + \delta\alpha\dot\beta + \alpha\delta\dot\beta + S\dot\gamma)$$

$$= \rho(\dot g - \dot\alpha\frac{\partial g}{\partial \alpha} - \alpha\frac{d}{dt}(\frac{\partial g}{\partial \alpha}) - \dot S\frac{\partial g}{\partial S} - S\frac{d}{dt}(\frac{\partial g}{\partial S}) - \frac{\partial g}{\partial \beta}\dot\beta + \alpha\frac{d}{dt}(\frac{\partial g}{\partial \alpha}) + S\frac{d}{dt}(\frac{\partial g}{\partial S})) = 0 \tag{2.70}$$

Using familiar generating function methods of classical mechanics [23] it is also possible to extract the finite form of the transformations. Let,

$$d\theta' + \alpha' d\beta' + S d\gamma' = d\theta + \alpha d\beta + S d\gamma \tag{2.71}$$

so that,

$$d(\theta' - \theta) = \alpha d\beta - \alpha' d\beta' + S(d\gamma - d\gamma') \tag{2.72}$$

Then there exists a generating function $W(\beta, \beta', \gamma, \gamma')$ such that,

$$\theta' - \theta = W, \quad \frac{\partial W}{\partial \beta} = \alpha, \quad \frac{\partial W}{\partial \beta'} = -\alpha', \quad \frac{\partial W}{\partial \gamma} = -\frac{\partial W}{\partial \gamma'} = S \tag{2.73}$$

Conversely, it is possible to obtain a finite canonical transformation if a generating function is given. In order to establish connection with the infinitesimal transformation considered earlier, let,

$$\beta' = \beta + \dot{\beta}\delta t, \quad \alpha' = \alpha + \dot{\alpha}\delta t, \quad \gamma' = \gamma + \dot{\gamma}\delta t \tag{2.74}$$

where t is some notional time parametrising the change. Then, using (Eq. (2.72)) and the above relations,

$$dW = \alpha d\beta - (\alpha + \dot{\alpha}\delta t)d(\beta + \dot{\beta}\delta t) + S(d\gamma - d(\gamma + \dot{\gamma}\delta t))$$

$$= -\delta t(\alpha d\dot{\beta} + \dot{\alpha}d\beta + Sd\dot{\gamma}) \tag{2.75}$$

Likewise, defining $W = U\delta t$, yields,

$$dU = -(\alpha d\dot{\beta} + \dot{\alpha}d\beta + Sd\dot{\gamma}) \tag{2.76}$$

Making the Legendre transformation,

$$g = U + \alpha\dot{\beta} + S\dot{\gamma} \tag{2.77}$$

gives,

$$dg = \dot{\beta}\delta\alpha - \dot{\alpha}d\beta + \dot{\gamma}dS \tag{2.78}$$

so that,

$$\dot{\beta} = \frac{\partial g}{\partial \alpha}, \quad \dot{\alpha} = -\frac{\partial g}{\partial \beta}, \quad \dot{\gamma} = \frac{\partial g}{\partial S} \tag{2.79}$$

Putting these relations back into (Eq. (2.74)),

$$\begin{gathered} \beta' = \beta + \frac{\partial g}{\partial \alpha}\delta t, \quad \alpha' = \alpha - \frac{\partial g}{\partial \beta}\delta t, \quad \gamma' = \gamma + \frac{\partial g}{\partial S}\delta t \\ \theta' = \theta + U\delta t = \theta + (g - \alpha\frac{\partial g}{\partial \alpha} - S\frac{\partial g}{\partial S})\delta t \end{gathered} \tag{2.80}$$

reproducing the earlier structure (Eq. (2.64)).

The arbitrariness discussed above is consistent with intuitive notions of the number of degrees of freedom in a fluid. This notion implies that it should be possible to describe a fluid completely with four functions at each point, one thermodynamic variable (entropy S) and three independent components of velocity $\bar{v}$. Since we have used five potentials $(\theta, \alpha, \beta, S, \gamma)$ to describe the fluid, only one of these must be completely arbitrary. To visualise this in the above analyis, recall that the symmetry transformations affect θ, α, β and γ. Assume that there is a given set of these potentials. Then a physically equivalent set θ', α', β' and γ' is obtained from (Eq. (2.64)) (infinitesimal version) or (Eq. (2.73)) (finite version). It is clear that only one of these is complete arbitrary (depending on the choice of the generating function g or W). As an illustration consider the simplest example which consists in preserving θ and γ while reshuffling α and β. Then (Eq. (2.64)) implies,

$$\delta\gamma = \frac{\partial g(\alpha, \beta, S)}{\partial S} = 0 \tag{2.81}$$

so that g is a function of only α and β,

$$g = g(\alpha, \beta) \tag{2.82}$$

Moreover,

$$\delta\theta = g - \alpha\frac{\partial g}{\partial \alpha} - S\frac{\partial g}{\partial S} = 0 \tag{2.83}$$

simplifies to,

$$g - \alpha\frac{\partial g}{\partial \alpha} = 0 \tag{2.84}$$

so that,

$$g(\alpha, \beta) = \alpha h(\beta) \tag{2.85}$$

where $h(\beta)$ is some function of β.

Then the transformations in α and β also follow from (Eq. (2.64)),

$$\begin{aligned} \delta\alpha &= -\frac{\partial g}{\partial \beta} = -\alpha\frac{\partial h(\beta)}{\partial \beta} \\ \delta\beta &= \frac{\partial g}{\partial \alpha} = h(\beta) \end{aligned} \tag{2.86}$$

Thus the only arbitrary potential is β whose arbitrariness is characterised by the function $h(\beta)$. Its knowledge uniquely fixes α. It is straightforward to obtain the finite transformations by first expressing (Eq. (2.86)) as,

$$\begin{aligned} \alpha' &= \alpha - \alpha\frac{\partial h}{\partial \beta} \\ \beta' &= \beta + h(\beta) \end{aligned} \tag{2.87}$$

Introducing a new function,

$$f(\beta) = \beta + h(\beta) \tag{2.88}$$

it follows,

$$\beta' = f(\beta)$$

$$\alpha' = \alpha(1 - \frac{\partial h}{\partial \beta}) = \alpha(1 + \frac{\partial h}{\partial \beta})^{-1} = \alpha(\frac{\partial f}{\partial \beta})^{-1} \tag{2.89}$$

Together with,

$$\theta' = \theta; \ \gamma' = \gamma \tag{2.90}$$

this clearly reveals that only one of the potentials (say β) may be completely arbitrary which agrees with the counting of the fluid degrees of freedom as discussed below (Eq. (2.80)).

The finite transformations (Eq. (2.89)), (Eq. (2.90)) preserve the velocity,

$$v_i = \partial_i\theta' + \alpha'\partial_i\beta' + \gamma'\partial_i S' = \partial_i\theta + \alpha\partial_i\beta + \gamma\partial_i S \tag{2.91}$$

where the invariance of S ($S' = S$) has also been used.

Also, the entire algebraic structure of (Eq. (2.60)) (along with (Eq. (2.46))) is preserved. As an example,

$$\{\alpha'(x), \theta'(x')\} = \left\{\alpha\left(\frac{df}{d\beta}\right)^{-1}(x), \theta(x')\right\} = \left(\frac{df}{d\beta}\right)^{-1}\frac{\alpha}{\rho}\,\delta(x-x') = \frac{\alpha'}{\rho}\,\delta(x-x') \tag{2.92}$$

which shows the invariance of the algebra (Eq. (2.46)). Likewise, the invariance of the other relation in (Eq. (2.46)) may be proved,

$$\{\beta'(x), \alpha'(x')\} = \left\{f(\beta)(x), \alpha\left(\frac{df}{d\beta}\right)^{-1}(x')\right\} = \{f(\beta)(x), \alpha(x')\}\left(\frac{df}{d\beta}\right)^{-1}(x') = \frac{1}{\rho}\,\delta(x-x') \tag{2.93}$$

Relations analogous to (Eq. (2.89)), (Eq. (2.90)) were earlier found in [18] for relativistic fluids, directly using finite transformation, where, however, neither the bracket structure nor infinitesimal transformations were considered.

Here we have provided a new set of results in this chapter and fresh insight towards the Clebsch decomposition. Following the Noether's definition which comes out as a natural choice. The generalised velocity for nonisentropic fluids is given. And the subtlety in the choice of the Clebsch parameters is discussed thoroughly.

Chapter 3

Hamiltonian Analysis of Relativistic Fluid

A complete discussion on hamiltonian description of ideal relativistic fluid is given in this chapter. The role of Clebsch parametrisation to bring the fluid system to a first order one is shown explicitly. While developing the hamiltonian structure we found the choice of the auxiliary variable to be quite tricky. The subtlety is discussed in detail. One of the most important part of this discussion is to define the stress tensor. We have shown how crucial the choice of the auxiliary variable is, to show two distinct forms of energy momentum tensors to be identical as expected for a non interacting system. Introduction of a background gauge field however changes the scenario. The canonical and the symmetric definitions of the stress tensor do not agree any more. We have explained the utilities and the shortcomings of these two definitions. Finally we provide a way to modify the canonically defined stress tensor so that it gives the proper conservation equation. The presence of the Lorentz force term in the conservation equation of the energy momentum tensor is also explained.

We start this chapter with the derivation of the Euler Lagrange equation of motion from an existing Lagrangian [1] of an ideal realtivistic fluid. Then we develop the symplectic structure of the fluid. Onwards we establish the equivalence of the canonical and the symmetric stress tensor. In the next section we deal with a fluid system with some background interaction.

3.1 Relativistic fluid mechanics in equal-time coordinates

To start with we will describe the dynamics of an ideal relativistic fluid in this section. The main pillar on which the description of this dynamics is usually based on is the conservation of the stress tensor

$$\partial_\mu \Theta^{\mu\nu} = 0 \tag{3.1}$$

which is further supplemented by the constitutive relation,

$$\Theta_{\mu\nu} = -\eta_{\mu\nu} P_{rel} + (\epsilon_{rel} + P_{rel}) u_\mu u_\nu \tag{3.2}$$

that expresses the Energy momentum tensor in terms of the relativistic fluid variables, the fluid pressure P_{rel}, the energy density ϵ_{rel} and the comoving velocity u_μ. The comoving velocity satisfies the relation $u^\mu u_\mu = 1$.

As we are interested in the Hamiltonian description of the fluid, to serve the purpose we start with a manifestly Lorentz covariant lagrangian density [1].

The appropriate lagrangian density is given by

$$\mathcal{L} = -\eta^{\mu\nu} j_\mu a_\nu - f; \quad \eta^{\mu\nu} = diag(1, -1, -1, -1) \tag{3.3}$$

Here j^μ is the current Lorentz vector $j^\mu = (\rho, \mathbf{j})$ satisfying the continuity equation,

$$\partial_\mu j^\mu = 0 \tag{3.4}$$

so that if necessary one may couple it to background gauge field. We have introduced a generalized scalar potential function $f(\sqrt{j^\mu j_\mu})$ as for instance done by [1] which is a function of the Lorentz covarient $j^\mu j_\mu$, and provides the proper dynamical equations.

Here a_μ is defined following the prescription of Clebsch [28, 29]. For fluids with zero vorticity(irrotational fluid) a_μ is expressed as the divergence of a single scalar, namely, θ

$$a_\mu = \partial_\mu \theta, \tag{3.5}$$

while, for a fluid with non zero vorticity, we need three scalers to express a_μ,

$$a_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta. \tag{3.6}$$

Onwards, We will show that the energy momentum tensor derived from this lagrangian density will satisfy (Eq. (3.1)) and (Eq. (3.2)) while the current entering (Eq. (3.3)) satisfies (Eq. (3.4)).

We take (Eq. (3.3)) as the lagrangian density of an ideal relativistic fluid [1]. It is beneficial to point out a difference between the Lagrangian (point particle) and Euler (field theoretic) frameworks of fluid mechanics. In the former one has constraints so that not all coordinates x_μ are independent whereas no such constraint is present in the latter. Since effectively the (Lagrangian) velocity is replaced by a_μ, a_μ is explicitly written in terms of either *one* degree of freedom (in case of an irrotational fluid) or *three* (and not four) degrees of freedom θ, α, β (for a fluid with non zero vorticity). (For a discussion on this point see [29].) Furthermore, the reason to introduce Clebsch variables has also been discussed in the Introduction. We will go for a short discussion on irrotational fluids where we will get the equations of motion and will check the conservation of the energy momentum tensor and then will move to the rotational fluids.

Irrotational fluids

The equations of motion produced by the lagrangian (Eq. (3.3)), on variation wrt ρ and j_μ are, respectively[1],

$$\dot{\theta} + \frac{\rho}{n} f'(n) = 0. \tag{3.7}$$

$$j_\mu = -\frac{n}{f'(n)} a_\mu = -\frac{n}{f'(n)} \partial_\mu \theta. \tag{3.8}$$

and the θ variation produces the continuity equation (Eq. (3.4)).

The energy momentum tensor we obtain from the lagrangian (Eq. (3.3)) following the Noether's definition is,

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \theta)} \partial_\nu \theta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \rho)} \partial_\nu \rho - \eta_{\mu\nu} \mathcal{L} \\ &= -j_\mu \partial_\nu \theta - \eta_{\mu\nu} \mathcal{L}. \end{aligned} \tag{3.9}$$

Now we will demonstrate the conservation of em tensor,

$$\begin{aligned} \partial^\mu T_{\mu\nu} &= -\partial^\mu (j_\mu \partial_\nu \theta) - \partial_\nu \mathcal{L} \\ &= -j^\mu \partial_\mu \partial_\nu \theta + \partial_\nu (j^\mu \partial_\mu \theta + f(n)) \end{aligned} \tag{3.10}$$

Exploiting (Eq. (3.8)) amd the continuity equation it can be shown that,

$$\partial^\mu T_{\mu\nu} = 0$$

. which completes the demonstration of the conservation of the em tensor.

3.1.1 Rotational fluid

Now, we would like to deal with a more nontrivial system, a fluid with non zero vorticity. The expanded form of the lagrangian (Eq. (3.3)) using (Eq. (3.6)) with $j^\mu j_\mu = n^2$, is

$$\mathcal{L} = -\rho \partial_0 \theta - j^i \partial_i \theta - \rho \alpha \partial_0 \beta - j^i \alpha \partial_i \beta - f(n). \tag{3.11}$$

In the above we have defined $\rho = j^0$. Our prescription is the following: the variables associated with time derivatives like $\rho, \alpha, \beta, \theta$ are treated as dynamical whereas j^i are regarded as auxiliary variables. From the lagrangian (Eq. (3.3)), equations obtained by varying β, α, ρ and j_μ are, respectively,

$$j^\mu \partial_\mu \alpha = 0, \tag{3.12}$$

[1] Prime of a function indicates differentiation, thus $f'(n) = \frac{df(n)}{dn}$.

$$j^{\mu}\partial_{\mu}\beta = 0, \tag{3.13}$$

$$\dot{\theta} + \alpha\dot{\beta} + \frac{\rho}{n}f'(n) = 0. \tag{3.14}$$

$$j_{\mu} = -\frac{n}{f'(n)}a_{\mu} = -\frac{n}{f'(n)}(\partial_{\mu}\theta + \alpha\partial_{\mu}\beta). \tag{3.15}$$

Note that variation of θ reproduces the current conservation law (Eq. (3.4)). We stress that the status of the last equation (Eq. (3.15)) is distinct from the previous ones (Eq. (3.12)-Eq. (3.14)). Its time component is just(Eq. (3.14)). Now, (Eq. (3.12), Eq. (3.13), Eq. (3.14)) represent genuine equations of motion since these involve the velocities[2]. The space component of (Eq. (3.15)), on the contrary, is more like a constraint than an equation of motion since it is bereft of any velocity term. Not surprisingly this equation is obtained by varying j_i which is regarded as an auxiliary variable. It needs to be interpreted carefully and a specific prescription is required (as we will provide later) for its application.

Hamiltonian formulation

Let us now develop a hamiltonian formulation. Being first order in time derivatives the system is a constraint system and has a non-trivial symplectic structure, that can be identified with the Dirac brackets of the variables in a hamiltonian formalism [32]. The first step is to define the conjugate momenta for the dynamical variables, which are

$$\pi_{\theta} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -\rho; \quad \pi_{\alpha} = \frac{\partial\mathcal{L}}{\partial\dot{\alpha}} = 0; \quad \pi_{\beta} = \frac{\partial\mathcal{L}}{\partial\dot{\beta}} = -\rho\alpha, \quad \pi_{\rho} = \frac{\partial\mathcal{L}}{\partial\dot{\rho}} = 0. \tag{3.16}$$

They yield four primary constraints

$$\Omega_1 = \pi_{\theta} + \rho \approx 0; \quad \Omega_2 = \pi_{\alpha} \approx 0; \quad \Omega_3 = \pi_{\beta} + \rho\alpha \approx 0; \quad \Omega_4 = \pi_{\rho} \approx 0. \tag{3.17}$$

Using canonical Poisson brackets of the generic form[3] $\{q(x), \pi_q(y)\} = \delta(\mathbf{x} - \mathbf{y})$, we can easily show that the constraint algebra does not close indicating that they form a set of four second class constraints [32]. In a generic system with n second class constraints Ω_i, $i = 1, 2, ..n$, the modified symplectic structure (or Dirac brackets) are defined in the following way,

$$\{A, B\}^* = \{A, B\} - \{A, \Omega_i\}\{\Omega^i, \Omega^j\}^{-1}\{\Omega_j, B\}, \tag{3.18}$$

[2] For a second order system the true equations of motion involve the accelerations but for a first order system like(Eq. (3.11)), these equations involve the velocities.

[3] Here $\mathbf{x}$ denotes space components x_i.

where $\{\Omega^i, \Omega^j\}$ is the invertible constraint matrix. From now on we will only use Dirac brackets but for notational simplicity we will refer to them as $\{,\}$ instead of $\{,\}^*$. The non-vanishing Dirac brackets are explicitly listed below

$$\{\rho(x), \theta(y)\} = \delta(\mathbf{x}-\mathbf{y});\ \{\alpha(x), \theta(y)\} = -\frac{\alpha}{\rho}\delta(\mathbf{x}-\mathbf{y});\ \{\alpha(x), \beta(y)\} = \frac{\delta(\mathbf{x}-\mathbf{y})}{\rho}. \tag{3.19}$$

Incidentally (Eq. (3.19)) gives rise to two independent canonical pairs (ρ, θ) and $(\alpha, \rho\beta)$. The canonical hamiltonian density for the fluid corresponding to (Eq. (3.11)) is,

$$\begin{aligned}\mathcal{H} &= \pi_\alpha\dot{\alpha} + \pi_\theta\dot{\theta} + \pi_\beta\dot{\beta} + \pi_\rho\dot{\rho} - \mathcal{L} \\ &= j^i\partial_i\theta + j^i\alpha\partial_i\beta + f(n). \end{aligned}\tag{3.20}$$

Using the Dirac brackets (Eq. (3.19)) the hamiltonian equation of motion for ρ is

$$\partial_0\rho = \{\rho, H\}\ , \quad H = \int \mathcal{H} d^3\mathbf{x}, \tag{3.21}$$

and we find

$$\dot{\rho} = -\partial_i j^i, \tag{3.22}$$

yielding the current conservation law (or in fluid dynamics terminology the continuity equation), obtained earlier (Eq. (3.4)). In the same way we can find equations of motion for α, β,

$$\dot{\alpha} = \{\alpha, H\};\quad \dot{\beta} = \{\beta, H\} \tag{3.23}$$

from which we recover

$$\rho\dot{\alpha} = -j^i(\partial_i\alpha) \Rightarrow j^\mu\partial_\mu\alpha = 0, \tag{3.24}$$

and

$$\rho\dot{\beta} = -j^i(\partial_i\beta) \Rightarrow j^\mu\partial_\mu\beta = 0. \tag{3.25}$$

These equations are the same as the Euler-Lagrange equations of motion (Eq. (3.12),Eq. (3.13)). Finally, from $\dot{\theta}$ we find

$$\dot{\theta} = \{\theta, H\} = -\alpha\dot{\beta} - \frac{\rho}{n}f'(n). \tag{3.26}$$

This is same as (Eq. (3.14)) and equivalent to the time component of (Eq. (3.15)). In our case, the space components of (Eq. (3.15)) just correspond to the equation for the nondynamical variable j^i.

At this point let us pause to note the status of the identity (Eq. (3.15)). On one hand the j_i variables are not involved in the symplectic structure (Eq. (3.19)) and so should trivially commute with all degrees of freedom but on the other hand they are directly related to the dynamical variables through (Eq. (3.15)) and infact yield non-zero brackets, *e.g*

$$\{j_i(x), \rho(y)\} = -\frac{n}{f'(n)}\{(\partial_i\theta + \alpha\partial_i\beta)(x), \rho(y)\} = \frac{n}{f'(n)}\partial_i\delta(x-y).$$

It is clear therefore that directly using j_i or replacing it by the identity (Eq. (3.15)) will yield distinct results in the calculation of brackets. This necessitates a specific prescription that will soon be elaborated.

3.1.2 Equivalence of EM tensors

Quite surprisingly, we will find that there are subtleties involved even in the free fluid theory and serious complications in the interacting theory of a fluid with external gauge field, to be treated in a later section. The problem is centered around the implementation of the space component of the relation (Eq. (3.15)) and the construction of the symmetric energy-momentum (or stress) tensor $\Theta_{\mu\nu}$. The stress tensor is obtained from $\mathcal{L}$ in a straightforward way [1]:

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S}{\partial g^{\mu\nu}} = -\mathcal{L}\eta_{\mu\nu} + \frac{j_\mu j_\nu}{\sqrt{j^2}}f'(\sqrt{j^2}). \tag{3.27}$$

From (Eq. (3.3)) and (Eq. (3.15)) the above expression for the stress tensor can be written as,

$$\Theta_{\mu\nu} = -\eta_{\mu\nu}[nf'(n) - f(n)] + \frac{j_\mu j_\nu}{n}f'(n) \tag{3.28}$$

which has the expected structure (Eq. (3.2)). By comparison it is easy to obtain the identifications,

$$P_{rel} = nf'(n) - f(n),\ \epsilon_{rel} + P_{rel} = nf'(n),\ j_\mu = nu_\mu, \tag{3.29}$$

The hamiltonian density from $\Theta_{\mu\nu}$ is given by,

$$\Theta_{00} = \frac{j_i j_i}{n}f'(n) + f(n). \tag{3.30}$$

To rewrite Θ_{00} in terms of Clebsch variables, we use (Eq. (3.15))

$$j_i = -\frac{n}{f'(n)}(\partial_i\theta + \alpha\partial_i\beta), \tag{3.31}$$

and can recover the canonical form of the hamiltonian obtained earlier (Eq. (3.20)), provided we replace only one of the j^i in the quadratic term, leading to

$$\Theta_{00} = j^i(\partial_i\theta + \alpha\partial_i\beta) + f(n). \tag{3.32}$$

We stress that only this prescription will lead to the canonical expression for the hamiltonian computed earlier, (that generated the correct dynamical equations). This is further corroborated by constructing the momentum density,

$$\Theta_{0i} = \frac{j_0 j_i}{n} f'(n) = -\rho(\partial_i\theta + \alpha\partial_i\beta), \tag{3.33}$$

where, once again, the same prescription of replacing j_i is exploited. It is straightforward to show that Θ_{0i} acts as the proper translation generator. Below we explicitly demonstrate this for α:

$$\{\alpha, \int d\bar{x}\Theta_{0i}\} = \{\alpha, \int -\rho(\partial_i\theta + \alpha\partial_i\beta)\} = -(\partial_i\rho)\frac{\alpha}{\rho} + \frac{\partial_i(\rho\alpha)}{\rho} = \partial_i\alpha. \tag{3.34}$$

Likewise one may proceed for other variables.

It is important to note that, like Θ_{00}, Θ_{0i} also agrees with the result obtained from the canonical stress tensor obtained via Noether prescription in (Eq. (3.3)).

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)}\partial_\nu\theta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\beta)}\partial_\nu\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\alpha)}\partial_\nu\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\rho)}\partial_\nu\rho - \eta_{\mu\nu}\mathcal{L} \\ &= -j_\mu\partial_\nu\theta - \alpha j_\mu\partial_\nu\beta - \eta_{\mu\nu}\mathcal{L}. \end{aligned} \tag{3.35}$$

The T_{0i} component is given by,

$$T_{0i} = -\rho\partial_i\theta - \alpha\rho\partial_i\beta \tag{3.36}$$

which reproduces (Eq. (3.33)).

Indeed, following our prescription of replacing j_ν in (Eq. (3.27)) in favour of the Clebsch variables by exploiting(Eq. (3.15)) immediately shows the exact equivalence between $\Theta_{\mu\nu}$(Eq. (3.27)) and $T_{\mu\nu}$(Eq. (3.35)).

As is well known the definition of Noether charges may differ by local counterterms. By appropriate manipulations it is however possible to abstract both $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ from Noether's theorem [27]. However it must be realised that in general $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are not identical. Indeed, by their very definitions (Eq. (3.35)) and (Eq. (3.27)), respectively, it is seen that while $\Theta_{\mu\nu}$ is symmetric, $T_{\mu\nu}$ is not. For gauge theories the difference is proportional to the Gauss constraint so that $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ agree on the physical subspace. The present theory is not a gauge theory as it is bereft of any first class constraint. Nevertheless we find that in the present case $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ are identical provided we interpret j_μ in favour of Clebsch variables(Eq. (3.15)), as already discussed. This interpretation is important and also plays a significant role in the derivation of the Schwinger condition discussed in the next subsection. In the interacting case to be considered in the section 4, however, there is a difference between $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ inspite of this particular interpretation of j_μ. But, by improving $T_{\mu\nu}$ (which is similar to Belinfante's prescription), it becomes identical to $\Theta_{\mu\nu}$.

3.2 Interacting fluid model

The fluid interacting with an non dynamical gauge field is of particular interest. The background gauge field A_μ is introduced in the fluid lagrangian in a conventional way,

$$\mathcal{L} = -\eta^{\mu\nu} j_\mu (a_\nu - A_\nu) - f. \tag{3.37}$$

Here also j^i is regarded as an auxiliary variable. The dynamical equations which are modified by the gauge field are provided below,

$$\dot{\theta} + \alpha\dot{\beta} + \frac{\rho}{n} f'(n) - A_0 = 0. \tag{3.38}$$

$$j_\mu = -\frac{n}{f'(n)}(a_\mu - A_\mu) = -\frac{n}{f'(n)}(\partial_\mu\theta + \alpha\partial_\mu\beta - A_\mu). \tag{3.39}$$

Rest of the equations of motion are same as the free theory, given in (Eq. (3.12), Eq. (3.13)). Notice that the conjugate momenta remain unaffected (Eq. (3.16)) since no new time-derivatives are introduced in the interacting theory and hence the same Dirac bracket structure (as in the free fluid theory) will prevail.

The canonical Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &= \pi_\alpha\dot{\alpha} + \pi_\theta\dot{\theta} + \pi_\beta\dot{\beta} + \pi_\rho\dot{\rho} - \mathcal{L} \\ &= j^i\partial_i\theta + j^i\alpha\partial_i\beta - j^\mu A_\mu + f(n).\end{aligned} \tag{3.40}$$

The θ equation is recovered below,

$$\dot{\theta} = \{\theta, H\} = -\alpha\dot{\beta} - \frac{\rho}{n} f'(n) + A_0. \tag{3.41}$$

Rest of the equations of motion are also derived correctly. Thus the hamiltonian in (Eq. (3.40)) is able to generate the correct dynamics.

Following our free theory analysis we now derive the covariant stress tensor $\Theta_{\mu\nu}$ for the interacting theory,

$$\begin{aligned}\Theta_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\partial S}{\partial g^{\mu\nu}} = -\mathcal{L}\eta_{\mu\nu} + \frac{j_\mu j_\nu}{\sqrt{j^2}} f'(\sqrt{j^2}) \\ &= -(-j^\sigma(a_\sigma - A_\sigma) - f)\eta_{\mu\nu} + \frac{j_\mu j_\nu}{\sqrt{j^2}} f'(\sqrt{j^2}).\end{aligned} \tag{3.42}$$

We express $\Theta_{\mu\nu}$ in terms of Clebsch variables following our earlier prescription of replacing j_ν by exploiting (Eq. (3.39)),

$$\Theta_{\mu\nu} = -(-j^\sigma(a_\sigma - A_\sigma) - f)\eta_{\mu\nu} - j_\mu(\partial_\nu\theta + \alpha\partial_\nu\beta - A_\nu). \tag{3.43}$$

One can directly check that $\Theta_{\mu\nu}$ satisfies the correct conservation law in presence of interactions,

$$\partial^\mu\Theta_{\mu\nu} = -\partial_\nu[-j^\mu(a_\mu - A_\mu) - f] - j_\mu\partial^\mu[\partial_\nu\theta + \alpha\partial_\nu\beta - A_\nu]$$

$$= \partial_\nu j^\mu(\partial_\mu\theta + \alpha\partial_\mu\beta - A_\mu) + j^\mu\partial_\nu(\partial_\mu\theta + \alpha\partial_\mu\beta - A_\mu) + \partial\nu f$$
$$-j_\mu\partial^\mu\partial_\nu\theta - \alpha j_\mu\partial^\mu\partial_\nu\beta + j_\mu\partial^\mu A_\nu \qquad (3.44)$$

$$= j^\mu F_{\mu\nu} + \partial_\nu f + \partial_\nu j^\mu(\partial_\mu\theta + \alpha\partial_\mu\beta - A_\mu) = j^\mu F_{\mu\nu}. \qquad (3.45)$$

where we have exploited the result (Eq. (3.39)). The hamiltonian density obtained from (Eq. (3.43)) is given by,

$$\Theta_{00} = j^i(a_i - A_i) + f = j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) + f. \qquad (3.46)$$

Immediately we are faced with a problem: the expressions for the hamiltonian density given in (Eq. (3.40)) and (Eq. (3.46)) do not match. The mismatch term is j_0A_0 which has nontrivial brackets with θ. Thus the hamiltonian density (Eq. (3.46)) fails to generate the lagrangian equation of motion for the θ variable (Eq. (3.38)). Of course in the absence of interaction the results agree.

The expression for the canonical stress tensor $T_{\mu\nu}$ is straightforward to obtain following the Noether prescription. The result is (Eq. (3.35)) with the lagrangian $\mathcal{L}$ defined in (Eq. (3.37)). Obviously T_{00} agrees with the canonical hamiltonian density (Eq. (3.40)). Also T_{0i} following from (Eq. (3.35)) and (Eq. (3.37)),

$$T_{0i} = \pi_\theta\partial_i\theta + \pi_\beta\partial_i\beta = -\rho(\partial_i\theta + \alpha\partial_i\beta).$$

matches with the non-interacting fluid result (Eq. (3.33)), and behaves like the correct translation generator. In obtaining the final expression we have imposed the constraints (Eq. (3.17)) strongly since Dirac brackets are being ussed. Using (Eq. (3.19)) we obtain,

$$\{\theta, \int d\bar{x}T_{0i}\} = \{\theta, \int -\rho(\partial_i\theta + \alpha\partial_i\beta)\} = \partial_i\theta \qquad (3.47)$$

which is the desired translation law.

However, Θ_{0i} defined from (Eq. (3.43)),

$$\Theta_{0i} = -\rho(\partial_i\theta + \alpha\partial_i\beta - A_i),$$

does not match with T_{0i}, and it does not correctly generate the translation of θ,

$$\{\theta, \int d\bar{x}\Theta_{0i}\} = \{\theta, \int -\rho(\partial_i\theta + \alpha\partial_i\beta - A_i)\} = \partial_i\theta + A_i. \qquad (3.48)$$

Let us next derive the conservation law satisfied by $T_{\mu\nu}$. Taking a four-divergence of (Eq. (3.35)) yields,

$$\partial^\mu T_{\mu\nu} = -\partial^\mu(j_\mu\partial_\nu\theta) - \partial^\mu(\alpha j_\mu\partial_\nu\beta) - \partial_\nu\mathcal{L}.$$

Exploiting the equations of motion we find

$$\partial^\mu T_{\mu\nu} = (\partial_\nu j^\mu)\partial_\mu\theta - j_\mu\alpha\partial^\mu\partial_\nu\beta - (\partial_\nu j^\mu)A_\mu + \partial_\nu f$$

$$= j^\mu F_{\mu\nu} - j^\mu \partial_\mu A_\nu - (\partial_\nu j^\mu) A_\mu + (\partial_\nu j^\mu)\partial_\mu \theta + \alpha(\partial_\nu j^\mu)\partial_\mu \beta + \partial_\nu f$$
$$= j^\mu F_{\mu\nu} - \partial_\mu (j^\mu A_\nu). \tag{3.49}$$

First of all, in the absence of A_μ the stress tensor is conserved. This is compatible with the free fluid theory discussed in section 2. But for the interacting theory the stress tensor does not reproduce the expected conservation law, as computed in (Eq. (3.45)). Apart from the Lorentz force term there is an additional piece. However it is possible to define an 'improved' canonical stress tensor $\tilde{T_{\mu\nu}}$ that yields the desired relation. It is given by,

$$\tilde{T_{\mu\nu}} = T_{\mu\nu} + j_\mu A_\nu \tag{3.50}$$

which satisfies,

$$\partial^\mu \tilde{T_{\mu\nu}} = j^\mu F_{\mu\nu} \tag{3.51}$$

It is now possible to show that this $\tilde{T_{\mu\nu}}$ is exactly identical to $\Theta_{\mu\nu}$ (Eq. (3.42)). From (Eq. (3.35)) and (Eq. (3.50)) we obtain

$$\tilde{T_{\mu\nu}} = -j_\mu(\partial_\nu \theta + \alpha \partial_\nu \beta - A_\nu) - \eta_{\mu\nu}\mathcal{L} \tag{3.52}$$

Exploiting (Eq. (3.39)) we find,

$$\tilde{T_{\mu\nu}} = -\mathcal{L}\eta_{\mu\nu} + \frac{j_\mu j_\nu}{\sqrt{j^2}} f'(\sqrt{j^2}) \tag{3.53}$$

which is the same as $\Theta_{\mu\nu}$ defined in (Eq. (3.42)).

It is worthwhile to observe the complementary roles of the canonical (Noether) stress tensor ($T_{\mu\nu}$) and the symmetric (Schwinger) stress tensor ($\Theta_{\mu\nu}$). While the canonical expression correctly reproduces the equations of motion for all the dynamical variables, the symmetric one fails for the θ variable. On the other hand the symmetric tensor yields the correct Lorentz force term but the canonical tensor fails. Nevertheless, it is possible to redefine the latter from the conservation law such that the expected result is reproduced. Furthermore, this 'improved' canonical tensor matches exactly with the symmetric one.

We have discussed the various aspects regarding relativistic ideal fluid without and with the presence of the gauge field. Developed a hamiltonian structure of the ideal relativistic fluid following Dirac's prescription. Subtlety involved in conservation of the stress tensors of interacting fluids have been dealt with.

Chapter 4

Relativistic, nonisentropic fluid mechanics in equal-time coordinates

In this chapter we will focus on solving the apparent disparity between the definitions of the em tensors that was arouse in the last chapter where we had turned on an interaction which was non dynamical. We will consider a nonisetropic fluid lagrangian which will include the interaction with a dynamical gauge field. Onwards we will show that the difference between the em tensors we obtain by following the Noether's definition and by varying the lagrangian wrt the metric is proportional to the Gauss constraint so they are equivalent to each other on the physical subspace. We will demonstrate this equivalence for the components of the em tensors explicitly.

4.1 Equations of motion and the constraint analysis

We shall start with a quick recapitulation of the free fluid field theory in Eulerian approach. Our fluid system is nonisentropic unlikely the system we had dealt with in the last chapter.

To Construct the fluid Lagrangian we introduce the Clebsch variables [28–31] $\theta, \alpha, \beta, \gamma, S$. The fluid Lagrangian looks like,

$$\mathcal{L} = -\eta^{\mu\nu} j_\mu a_\nu - f; \quad \eta^{\mu\nu} = diag(1, -1, -1, -1) \tag{4.1}$$

in the following combination [35–38],

$$a_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta + \gamma \partial_\mu S. \tag{4.2}$$

We identify S as the entropy. The generalized scalar potential function $f(\sqrt{j^\mu j_\mu})$ dictates the dynamics. In the field theoretic description of relativistic fluid, the dynamical variables are the matter density $j^0 = \rho$ and the currents $j^i = \rho v^i, i = 1,2,3$ that satisfy the continuity equation,

$$\partial_\mu j^\mu = 0. \tag{4.3}$$

From the expanded form of the Lagrangian (Eq. (4.1)), (with $j^\mu j_\mu = n^2$, a relativistic scalar),

$$\mathcal{L} = -\rho\partial_0\theta - j^i\partial_i\theta - \rho\alpha\partial_0\beta - j^i\alpha\partial_i\beta - \rho\gamma\partial_0 S - j^i\gamma\partial_i S - f(n), \tag{4.4}$$

it is now trivial to show that the current conservation law (Eq. (4.3)) follows from the θ-equation of motion.

Let us now posit the relativistic version of a fully interacting model of a fluid and a dynamical $U(1)$ gauge field as,

$$\mathcal{L} = -\eta^{\mu\nu} j_\mu(a_\nu - A_\nu) - f - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{4.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. This ia a natural extension of our previous work [15] discussed in the previous section where we considered an isentropic fluid (with zero entropy) and treated the gauge field as non dynamical.

Variations of the dynamical variables $\beta, \alpha, \gamma, S, \rho(= j^0), j^\mu, A_\mu$ yield the equations of motion,

$$j^\mu\partial_\mu\alpha = 0, \tag{4.6}$$

$$j^\mu\partial_\mu\beta = 0, \tag{4.7}$$

$$j^\mu\partial_\mu S = 0 \tag{4.8}$$

$$j^\mu\partial_\mu\gamma = 0 \tag{4.9}$$

$$\dot{\theta} + \alpha\dot{\beta} + \gamma\dot{S} + \frac{\rho}{n}f'(n) = 0. \tag{4.10}$$

$$j_\mu = -\frac{n}{f'(n)}(a_\mu - A_\mu) = -\frac{n}{f'(n)}(\partial_\mu\theta + \alpha\partial_\mu\beta + \gamma\partial_\mu S - A_\mu). \tag{4.11}$$

$$j_\beta = -\partial^\alpha F_{\alpha\beta} \tag{4.12}$$

It is easy to see that current conservation (Eq. (4.3)) also follows from (Eq. (4.12)). Due to the presence of this conservation, the fluid action corresponding to (Eq. (4.5)) is invariant under the gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \tag{4.13}$$

.

This is exactly what happens in electrodynamics. This similarity persists further if we note that, as in electrodynamics, here we obtain a Gauss constraint which is given by the zeroth component of (Eq. (4.12)) ,

$$\partial_i \pi_i - j_0 = \partial_i \pi_i - \rho = 0, \tag{4.14}$$

where $\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = F_{i0}$ is the conjugate momentum of A^i. The Gauss constraint acts as the generator of the gauge transformation (Eq. (4.13)) and defines the physical subspace as

$$(\partial_i \pi_i - \rho) \mid \Psi >_{Physical} = 0. \tag{4.15}$$

From (Eq. (4.4)) we can identify three independent canonical pairs (ρ, θ), $(\alpha\rho, \beta)$ and $(\rho\gamma, S)$. The fundamental brackets, compatible with the above canonical pairs, follow from the symplectic structure,

$$\{\rho(x), \theta(y)\} = \delta(\mathbf{x}-\mathbf{y}),\ \{\alpha(x), \theta(y)\} = -\frac{\alpha}{\rho}\delta(\mathbf{x}-\mathbf{y}),\ \{\alpha(x), \beta(y)\} = \frac{\delta(\mathbf{x}-\mathbf{y})}{\rho};$$

$$\{\gamma(x), S(y)\} = \frac{\delta(\mathbf{x}-\mathbf{y})}{\rho},\quad \{\gamma(x), \theta(y)\} = -\frac{\gamma}{\rho}\delta(\mathbf{x}-\mathbf{y}). \tag{4.16}$$

All other possible pairs produce vanishing brackets. It is important to note that the apparent singularity in the above symplectic structure for $\rho \to 0$ does not create any problem as this limit is unphysical since the kinetic part of Lagrangian in (Eq. (4.4)) completely disappears for $\rho = 0$.

4.2 Energy momentum tensor

Definition and conservation

We now concentrate on the structure of the energy-momentum tensor. Conventionally there are two parallel definitions. One of these is the symmetric energy-momentum tensor,

$$\Theta_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S}{\partial g^{\mu\nu}}, \tag{4.17}$$

that is obtained by generalizing the lagrangian (Eq. (4.5)) to a curved spacetime which is done by replacing $\eta^{\mu\nu}$ by $g^{\mu\nu}$, varying the action wrt $g^{\mu\nu}$ and finally reverting back to flat spacetime with the replacement of $g^{\mu\nu}$ by $\eta^{\mu\nu}$ in (Eq. (4.17)).

On the other hand, the canonical energy-momentum tensor is obtained following Noether prescription,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \theta)}\partial_\nu \theta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \beta)}\partial_\nu \beta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \alpha)}\partial_\nu \alpha + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \rho)}\partial_\nu \rho + \frac{\partial \mathcal{L}}{\partial(\partial^\mu S)}\partial_\nu S + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\lambda)}\partial_\nu A^\lambda - \eta_{\mu\nu}\mathcal{L} \tag{4.18}$$

Both the definitions are useful in their own ways. $T_{\mu\nu}$ is designed in a way so that it manifestly generates correct space-time transformations of the field variables but it is not symmetric (can be improved following Belinfante prescription) whereas $\Theta_{\mu\nu}$ is manifestly symmetric but its ability to generate appropriate space time transformation is not transparent. In simple cases where the fluid is non interacting, these expressions agree as is natural but situation becomes subtle for the fluid system under consideration. We emphasize that these issues have not been admitted with the due importance so far but become crucial for the consistency of the fluid model.

In our interacting fluid model, the canonical energy-momentum tensor is given by (Eq. (4.18)),

$$T_{\mu\nu} = -j_\mu(\partial_\nu\theta + \alpha\partial_\nu\beta + \gamma\partial_\nu S) - F_{\mu\sigma}\partial_\nu A^\sigma - \eta_{\mu\nu}\mathcal{L}. \tag{4.19}$$

This tensor is conserved. To show this explicitly, we exploit current conservation and other equations of motion to find,

$$\partial^\mu T_{\mu\nu} = \partial^\mu\{-j_\mu(\partial_\nu\theta + \alpha\partial_{nu}\beta + \gamma\partial_\nu S) - F_{\mu\sigma}\partial_\nu A^\sigma - \eta_{\mu\nu}\mathcal{L}\}$$

$$= (\partial_\nu j^\mu)(a_\mu - A_\mu) + \partial_\nu f(n) = 0 \tag{4.20}$$

where the final step is obtained on using (Eq. (4.11)).

On the other hand the symmetric energy momentum tensor is derived from (Eq. (3.27)) as,

$$\Theta_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{j_\mu j_\nu}{n}f' - F^\beta{}_\nu F_{\beta\mu}. \tag{4.21}$$

This is also conserved by applyig the various equations of motion,

$$\partial^\mu\Theta_{\mu\nu} = 0.$$

Here we can see, both symmetric and canonical energy momentum tensors are conserved unlike the case we discussed in the previous section where the interaction was non dynamical.

Now $\Theta_{\mu\nu}$ produces the Hamiltonian

$$\Theta_{00} = -\mathcal{L} + \frac{j_0 j_0}{n}f' - F^j{}_0 F_{j0} \tag{4.22}$$

$$= -\mathcal{L} - \rho(\partial_0\theta + \alpha\partial_0\beta + \gamma\partial_0 S) + \rho A_0 - F^j{}_0 F_{j0}. \tag{4.23}$$

Also $T_{\mu\nu}$ in (Eq. (4.18)) gives rise to canonical Hamiltonian,

$$T_{00} = -\rho(\partial_0\theta + \alpha\partial_0\beta + \gamma\partial_0 S) - F_{0\sigma}\partial_0 A^\sigma - \mathcal{L}. \tag{4.24}$$

Comparison between the two definitions and establishing the equivalence

Let us compute the difference between two Hamiltonian densities,

$$T_{00} - \Theta_{00} = -F_{0i}\partial_0 A^i + F^j{}_0 F_{j0} - \rho A_0 \tag{4.25}$$

$$= -\pi_i \partial_0 A_i - \pi_i^2 - \rho A_0 = -\pi_i(\partial_i A_0 - \pi_i) - \pi_i^2 - \rho A_0 \tag{4.26}$$

$$= -\pi_i \partial_i A_0 - \rho A_0. \tag{4.27}$$

which is clearly nonvanishing. However the physically relevant object is the integrated version of these density terms which correspond to the hamiltonian. Difference in the hamiltonians is found to be,

$$\int d^3x (T_{00} - \Theta_{00}) = -\int d^3x (\pi_i \partial_i A_0 + \rho A_0) \tag{4.28}$$

$$= \int d^3x A_0 (\partial_i \pi_i - \rho). \tag{4.29}$$

which is proportional to the Gauss constraint. Hence, on the physical surface (Eq. (4.15)) these two expressions are equivalent. Comparing this result with the corresponding ones of the last chapter we can conclude, since the gauge field there was non dynamical in nature, there was no Gauss law and the mismatch persisted. While it is clear that the kinetic part of the gauge field, the Maxwell term, rounds off the theory nicely it is still necessary to check whether we can conclude the same for the other important components of the stress tensor.

Let us now consider the momentum density. The relevant expressions are,

$$\begin{aligned} T_{0i} &= -\rho(\partial_i\theta + \alpha\partial_i\beta + \gamma\partial_i S) - F_{0j}\partial_i A^j \\ &= -\rho(\partial_i\theta + \alpha\partial_i\beta + \gamma\partial_i S) - \pi_j \partial_i A^j, \end{aligned} \tag{4.30}$$

and,

$$\begin{aligned} \Theta_{0i} &= \frac{j_0 j_i}{n} f' - F^\beta{}_i F_{\beta 0} \\ &= -\rho(\partial_i\theta + \alpha\partial_i\beta + \gamma\partial_i S) + \rho A_i - F^k{}_i F_{k0}, \end{aligned} \tag{4.31}$$

with the difference

$$T_{0i} - \Theta_{0i} = -\rho A_i - \pi_j \partial_i A_j + \pi_j(\partial_i A_j - \partial_j A_i) = -(\rho A_i + \pi_j \partial_j A_i). \tag{4.32}$$

Once again integration of the above result yields

$$\int d^3x (T_{0i} - \Theta_{0i}) = \int d^3x A_i (\partial_i \pi_i - \rho), \tag{4.33}$$

indicating that the total momenta in the two definitions are equal modulo the first class (Gauss) constraint. Exploiting the covariant notation, the combination of (Eq. (4.28)) and (Eq. (4.33)) is written in a compact form,

$$\int d^3x(T_{0\mu} - \Theta_{0\mu}) = \int d^3x A_\mu(\partial_i \pi_i - \rho), \tag{4.34}$$

which vanishes on the physical subspace.

It is possible to continue this analysis for the angular momentum operator. From Noether's definition, this is given by,

$$M^N_{ij} = \int (x_i T_{0j} - x_j T_{0i} - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \Sigma^{\lambda\sigma}_{ij} A_\sigma) d^3x \tag{4.35}$$

where the spin tensor is defined as,

$$\Sigma^{\lambda\sigma}_{\alpha\beta} = g^\lambda_\alpha \, g^\sigma_\beta - g^\lambda_\beta \, g^\sigma_\alpha. \tag{4.36}$$

We therefore obtain,

$$M^N_{ij} = \int (x_i T_{0j} - x_j T_{0i} - \pi_i A_j + \pi_j A_i) d^3x$$

The angular momentum, following from the symmetric tensor (Eq. (4.17)), is given by

$$M^S_{ij} = \int (x_i \Theta_{0j} - x_j \Theta_{0i}) d^3x$$

Using (Eq. (4.31)) and (Eq. (4.32)) it is seen that the difference between these expressions vanishes, modulo terms proportional to the Gauss constraint,

$$M^N_{ij} - M^S_{ij} = \int d^3x (x_i A_j - x_j A_i)(\partial_k \pi_k - \rho). \tag{4.37}$$

Thus on the physical subspace, the expressions for angular momenta are identical, as happened for the space-time translation generators discussed earlier.

Similarly, the difference in the structures of the boost generators can also be discussed. From Noether's definition, the boost is given by,

$$M^N_{0i} = \int (x_0 T_{0i} - x_i T_{00} - \frac{\partial \mathcal{L}}{\partial \dot{A}^\lambda} \Sigma^{\lambda\sigma}_{0i} . \, A_\sigma) d^3x \tag{4.38}$$

From (Eq. (4.36)) it follows,

$$M^N_{0i} = \int (x_0 T_{0i} - x_i T_{00} - \pi_i A_0) d^3x. \tag{4.39}$$

On the other hand, the definition of boost following from the symmetric tensor (Eq. (3.27)) is,

$$M^N_{0i} = \int (x_0 \Theta_{0i} - x_i \Theta_{00}) d^3x. \tag{4.40}$$

Once again the difference is just proportional to the Gauss constraint,

$$M^N_{0i} - M^S_{0i} = \int d^3x(x_0 A_i - x_i A_0)(\partial_k \pi_k - \rho) \tag{4.41}$$

In fact (Eq. (4.37)) and (Eq. (4.41)) maybe combined to yield a covariant structure,

$$M^N_{\mu i} - M^S_{\mu i} = \int d^3x(x_\mu A_i - x_i A_\mu)(\partial_k \pi_k - \rho). \tag{4.42}$$

Indeed, the above exercise is non-trivial since it reveals the importance of the introduction of the Maxwell gauge field kinetic term and also establishes the spacetime symmetries of the fully interacting relativistic fluid model in a robust way. This explicit demonstration was absent in previous literatures.

In this section we have addressed and solved the problems regarding the conservation of the energy momentum tensors and the mismatch between the components of the same coming from two different definitions.

Chapter 5

Fluid systems in light cone coordinates

In this chapter we will produce a detailed analysis of field theories in light cone coordinate. We will provide a consistent Lagrangian and Hamiltonian formulation entirely in light cone coordinates for both interacting and non interacting ideal fluids. We will derive the symplectic structure of a massless scalar field in this framework. Moreover it will be supplemented with a thorough discussion on the utility of doing field theory in light cone coordinates.

Non relativistic reduction of the fluid equations of motion by compactifying one of the degrees of freedom is given here.

Starting with a discussion on ideal fluids, we will shift our concentration on the non relativistic reduction of the fluid equation following a particular prescription among many. Then the constraint structure of a massless scalar field(as a toy model) is derived and the conservation equations are discussed. Finally we show the equivalence between the definitions of em tensor for an interacting fluid holds good in this framework as well.

5.1 Light cone coordinates

In this chapter we will study various fluid systems in light cone coordinates. Before we commence the actual study we would like to introduce the light cone coordinates.

We define the light-cone coordinates as in [33], $\{x^+, x^-, \bar{x}\}$. They are related to the equal time coordinates in the following way,

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \tag{5.1}$$

and,

$$\bar{x} \equiv x^i, \ where\ i = 1, 2 \tag{5.2}$$

Length element in this coordinate system is presented as,

$$ds^2 = 2dx^- dx^+ + \delta_{ij} dx^i dx^j \tag{5.3}$$

Here, in the length element $x^{\pm}$ appears symmetrically, hence any of the two could have served as time coordinate. We pick up x^+ to play the role of time and $\bar{x}$ are referred as transverse coordinates. Interestingly we can see that time appears in linear order here, unlikely to the equal time frame where it appears in quadratic order.

The nonvanishing metric components, as we can see from (Eq. (5.3)) are $g^{+-} = g^{-+} = 1,\ g^{ii} = -1, i, j = 1, 2$.

5.2 Relativistic fluid mechanics in light-cone (null plane) coordinates:

In this section we study fluid mechanics in light-cone coordinates. Apart from providing a different formulation than the equal time one, there is another motivation which will become clearer in the next section when we discuss the non-relativistic reduction of the fluid model. The fluid lagrangian in this coordinate system is,

$$\mathcal{L} = -j^\mu a_\mu - f(\sqrt{j^\mu j_\mu}) = -(j^+ a_+ + j^- a_- + j^i a_i) - f = -(j_- a_+ + j_+ a_- - j_i a_i) - f$$
$$= -j_+(\partial_- \theta + \alpha \partial_- \beta) - j_-(\partial_+ \theta + \alpha \partial_+ \beta) + j_i a_i - f, \tag{5.4}$$

where, in the last step, we have exploited the definition of a_μ (Eq. (3.6)). Note that x^+ plays the role of time and the dynamical variables are identified following our previous prescription, that is variables involved in x^+-derivatives only are considered as dynamical. In the present setup the degrees of freedom are $j_-, \theta, \alpha, \beta$. The momentum is defined as

$$\pi_\phi = (\partial L)/(\partial(\partial_+ \phi)) \tag{5.5}$$

for a generic ϕ and $\partial_+ \equiv \partial_t$. The first order model (Eq. (5.4)) produces the constraints,

$$\chi_1 = \pi_\theta + \rho \approx 0\ ,\ \chi_2 = \pi_\beta + \rho\alpha \approx 0\ ,\ \chi_3 = \pi_\alpha \approx 0\ ,\ \chi_4 = \pi^- \approx 0. \tag{5.6}$$

where $\pi^-$is the momenta conjugate to j_-. Note that j_- has to be identified with ρ. Constraint analysis once again provides the Dirac brackets

$$\{\rho(x), \theta(y)\} = \delta(\mathbf{x} - \mathbf{y})\ ,\ \{\alpha(x), \theta(y)\} = -(\alpha/\rho)\delta(\mathbf{x} - \mathbf{y})\ ,\ \{\alpha(x), \beta(y)\} = (1/\rho)\delta(\mathbf{x} - \mathbf{y}), \tag{5.7}$$

where $\mathbf{x} = x^-, \bar{x}$ with $\bar{x} = x^1, x^2$ and $\delta(\mathbf{x}-\mathbf{y}) = \delta(x^- - y^-)\delta(\bar{x}-\bar{y})$. It is worthwhile to point out that the above bracket structure in light cone coordinates is same as the one derived earlier in (Eq. (3.19)) in equal time coordinate system. This is simply because the lagrangian (Eq. (3.11)) was also first order. [1] The hamiltonian density is given by

$$\mathcal{H} = \pi_\alpha \dot{\alpha} + \pi_\theta \dot{\theta} + \pi_\beta \dot{\beta} + \pi_- \dot{j_-} - \mathcal{L},$$

from which, using (Eq. (5.4)) and (Eq. (5.6)), the hamiltonian of the fluid is,

$$H = \int dx^- d\bar{x}\; \mathcal{H}(x) = \int dx^- d\bar{x}\; [j_+(\partial_-\theta + \alpha\partial_-\beta) - j_i a_i + f]. \tag{5.8}$$

Before proceeding further we need to check the overall consistency of the light-cone framework mainly because of our specific interpretation of the space component of (Eq. (3.15)) and its subsequent applications.

Let us start by comparing the lagrangian and hamiltonian equations of motion. First comes the continuity equation. From the lagrangian (Eq. (5.4)) by varying θ we obtain,

$$\partial_+ j_- + \partial_- j_+ - \partial_i j_i = \partial_\mu j^\mu = 0 \tag{5.9}$$

which is the continuity equation in light-cone coordinates. On the other hand, in the hamiltonian framework, we have

$$\partial_+ j_-(x) = \{j_-(x), H\} = \{j_-(x), \int dy^- d\bar{y}\; (j_+(\partial_-\theta + \alpha\partial_-\beta) - j_i a_i + f)\}$$
$$= -\partial_- j_+(x) + \partial_i j_i(x), \tag{5.10}$$

which reproduces (Eq. (5.9)). It is interesting to observe that the spatial part is now broken up into two sectors x^- and $\bar{x}$ that are qualitatively somewhat distinct.

Let us rederive the light-cone version of the rest of the lagrangian variational equations (Eq. (3.12)-Eq. (3.14)). The hamiltonian equation,

$$\partial_+\alpha = \{\alpha(x), H\} = -\frac{(\partial_-\alpha)j_+}{j_-} + \frac{(\partial_i\alpha)j_i}{j_-}, \tag{5.11}$$

can be rearranged to yield (Eq. (3.12)) while

$$\partial_+\beta = \{\beta(x), H\} = -\frac{(\partial_-\beta)j_+}{j_-} + \frac{(\partial_i\beta)j_i}{j_-}, \tag{5.12}$$

reproduces (Eq. (3.13)). In a similar way $\partial_+\theta$ obtained below

$$\partial_+\theta = \{\theta(x), H\} = \frac{\alpha j_+(\partial_-\beta)}{j_-} - \frac{\alpha j_i(\partial_i\beta)}{j_-} - \frac{f' j_+}{n} \tag{5.13}$$

is the light-cone version of (Eq. (3.14)).

[1] This can be contrasted with a generic second order system, *e.g.* Klein-Gordon lagrangian, whose light-cone reduction yields a first order system with a drastically altered constraint structure.

5.3 Non-relativistic Light-cone reduction

Non-relativistic conformal field theories have now become an active area of interest due to the possibility of verification of the AdS/CFT correspondence experimentally in physically realizable non-relativistic systems, such as in cold atom [26]. Initially AdS/CFT yielded a mapping between different relativistic theories. Later on it's analogues in non-relativistic cases have also been explored. Indeed, there are several ways to achieve the non-relativistic limit and we here follow the limiting procedure in lightcone coordinates, recommended in [8] and applied in [12]).

We are going to spend some words in order to discuss about the procedure of the non relativistic reduction. The prescription is to reduce the relativistic conformal symmetry to non relativistic Galilean symmetry simply by the compactification of the x^- coordinate. This will induce the selection of a preferred light-cone direction [12]. In practice which implies that we will just omit all x^- dependence that will reduce ∂_- terms to zero.

However, we emphasize that the mapping between relativistic and non-relativistic variables and its application in recovery of the non-relativistic fluid equations from the relativistic dynamics as advocated in [12] is purely algebraic in nature. Moreover the lightcone equations of motion have not been derived from a canonical framework. In some sense the whole process of this mapping and matching is purely at the level of equations of motion [12] and seems ad hoc in nature. In the present work we have aimed at bridging this gap.

Following the above mentioned prescription of dropping x^--dependence, we write the Lagrangian and Hamiltonian density of the fluid in non-relativistic regime as,

$$\mathcal{L} = -j_-(\partial_+\theta + \alpha\partial_+\beta) - j_i a_i - f, \tag{5.14}$$

$$\mathcal{H} = j_i a_i + f. \tag{5.15}$$

We will now write the non-relativistic fluid equations following the same prescription, then, will compare those equations with the usual fluid equations(for non relativistic fluids of course) we shall try to give a map between the non-relativistic and the relativistic fluid variables.

Continuity equation, written in this coordinate system, is

$$\partial_+ j_-(\bar{x}) + \partial_i j_i(\bar{x}) = 0 \tag{5.16}$$

Form of the usual non-relativistic continuity equation is,

$$\dot{\rho} + \bar{\nabla}(\rho\bar{v}) = 0. \tag{5.17}$$

Comparing Eq. (5.16) and Eq. (5.17) we identify $j_- = \rho$ and $j_i = \rho v^i$ Now, to get the non-relativistic Euler equation we start from the relativistic Euler equation, [1]

$$u^\mu(\partial_\nu u_\mu - \partial_\mu u_\nu)f' + (g_{\mu\nu} - u_\mu u_\nu)\partial^\mu n f'' = 0. \tag{5.18}$$

As a particular case, we consider the fluid to be pressureless [1]. Comparing (Eq. (3.28)) with the usual ideal fluid energy momentum tensor tensor

$$T_{\mu\nu} = (\epsilon_{rel} + P_{rel})u_\mu u_\nu - g_{\mu\nu}P_{rel} \tag{5.19}$$

we identify the pressure as

$$P_{rel} = nf' - n. \tag{5.20}$$

The subscript *rel* indicates relativistic variables. Hence, for pressureless condition,

$$P_{rel} = 0 \Rightarrow f(n) = kn, f' = k, f'' = 0 \tag{5.21}$$

with k as a constant. In pressureless condition the relativistic Euler equation (Eq. (5.18)) reduces to,

$$u^\mu(\partial_\nu u_\mu - \partial_\mu u_\nu) = 0. \tag{5.22}$$

We have the non-relativistic Euler equation in terms of the usual fluid variables,as,

$$\dot{\bar{v}} + \bar{v}\bar{\nabla}\bar{v} = 0 \tag{5.23}$$

Finally we look at the Euler equation (with $P = 0$) in the non-relativistic limit, $\nu = -$ gives us

$$u^\mu \partial_\mu u_- = 0 = u^+\partial_+ u_- + u^i\partial_i u_- \tag{5.24}$$

comparing with the (Eq. (5.23)) we get

$$u^+ = -u_- = \alpha,\ u^i = \beta v^i \tag{5.25}$$

where α and β are two arbitrary constants. For $\nu = i$ we have,

$$u^+\partial_+ u_i + u^j\partial_j u_i = 0$$

using ($Eq.$ (5.25)) we get,

$$-\alpha\partial_+(\beta v_i) + \beta^2 v^j\partial_j v_i = 0 \tag{5.26}$$

when compared with (Eq. (5.23)) this equation gives $\alpha = \beta$ hence we have,

$$u^i = \alpha v^i = u^+ v^i \tag{5.27}$$

Now, we go for $\nu = +$, which gives,

$$u^+\partial_+ u_+ + u^i\partial_i u_+ = 0$$

$$\alpha v^+\partial_+ u_+ + \alpha v^i\partial_i u_+ = 0 \tag{5.28}$$

If we define u_+ in the following way,

$$u_+ = \gamma + \frac{\delta}{2}v^2 \tag{5.29}$$

(Eq. (5.28)) produces the usual euler equation. We have two new constants γ and δ

We have another condition in our hand. We will utilise the condition to fix these two constants.

$$u^{\mu}u_{\mu} = -1$$

$$u^{+}u_{+} + u^{-}u_{-} + u^{i}u_{i} = -1$$

$$2\alpha(\gamma + \frac{\delta}{2}v_i^2) + \alpha^2 v_i^2 = -1 \tag{5.30}$$

This equation produces

$$2\alpha\gamma = -1; \quad (\delta + \alpha) = 0$$

$$\gamma = -\frac{1}{2\alpha} = -\frac{1}{2u^{+}}; \quad \delta = -\alpha = -u^{+} \tag{5.31}$$

Now the mapping is clear

$$\begin{aligned} u^{+} &= -u_{-} = \alpha; \quad u^{i} = u^{+}v^{i} \\ u^{-} &= -u_{+} = \frac{1}{2}(\frac{1}{u^{+}} + u^{+}v_i^2) \end{aligned} \tag{5.32}$$

We have successfully given a complete mapping between the relativistic and the nonrelativistic variables as promised.

5.4 Analysis of an interacting fluid in Light cone coordinate system

Light-cone quantization (LCQ) was introduced with two key motivations: as a tool to compute bound state solutions in QCD to represent hadrons as bound states of quarks and gluons in a relativistic framework and also to utilize computers in quantum field theory calculations. (See [21] for an early review.). That LCQ is a convenient alternative to quantization in equal time frame was first pointed out by Dirac [22].

In the context of QCD a related framework, known as Infinite Momentum Frame, was initiated [41, 42] to explain Bjorken scaling in scattering phenomena. The physical meaning of this correspondence is that measurements made by an observer moving at infinite momentum is equivalent to making observations with speed being close to the speed of light and this corresponds to the front form where measurements are made along the front of a light wave.

Coming back to recent times LCQ has generated tremendous interest after the celebrated work of Son [26] on the formulation of a model that represented the experimentally demonstrated trapping of cold atoms at Feshbach resonance, thereby introducing the concept of non-relativistic holographic principle in AdS/CFT correspondence. It is important to note that in light cone variables a second order system, (in terms of time derivative), such as Klein Gordon, can be changed to a first order one, such as Schrodinger. But precisely this algebraic manipulation drastically alters the Hamiltonian structure of the system because the converted first order system turns out to be a constraint system with a non-canonical symplectic structure and reduced number of degrees of freedom. We will explicitly demonstrate that there are subtleties involved in the Hamiltonian analysis since the lightcone coordinate system is qualitatively distinct from the conventional equal time coordinate framework. At this point it is worthwhile to recall our earlier work [15] where, for the first time, a detailed lightcone analysis of the free fluid system was performed. There [15] it was observed that the symplectic structure in lightcone coordinate did not differ from the one in equal time coordinate, the reason being that the free fluid model was a first order system even in equal time coordinate. However, the difference between the two frameworks was manifest in *eg.* Schwinger condition where the spatial coordinates, x_- and transverse ones $\bar{x}$, were clearly separated into different sectors. In the present work, where we consider the fully interacting fluid-Maxwell theory, the situation becomes much more serious since the Maxwell gauge sector is quadratic in nature and upon LCQ leads to complications that puts a question mark on the validity of the Schwinger condition. This is not surprising since, in the hamiltonian framework, LCQ even for a simple massless scalar theory involves subtleties and complications. However, we emphasize, that the total energy of the system remains conserved in LCQ.

5.4.1 A simple model: Massless scalar field

In this section we will see that the intricacies of LCQ can be observed in the simplest of models. We choose massless scalar field for this purpose. It also helps in setting up the notation and introduce some basic formula. The Lagrangian

$$L = \int d^4x \; \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{5.33}$$

generates the equation of motion,

$$\partial_\mu\partial^\mu\phi = 0. \tag{5.34}$$

The same equation is recovered from the Hamiltonian

$$H = \int d^3x \; \mathcal{H}(x) = \int d^3x \; \frac{1}{2}(\pi^2 + \partial_i\phi\partial_i\phi), \tag{5.35}$$

as Hamilton's equation of motion where the equal-time canonical algebra,

$$\{\phi(\bar{x}), \pi(\bar{y})\} = \delta(\bar{x} - \bar{y}), \{\phi(\bar{x}), \phi(\bar{y})\} = \{\pi(\bar{x}), \pi(\bar{y})\} = 0 \tag{5.36}$$

is used.

It is now straightforward to compute the bracket between the energy densities $\{\mathcal{H}(x), \mathcal{H}(y)\}$ to yield,

$$\{\frac{1}{2}(\pi^2 + \partial_i\phi\partial_i\phi)(x), \frac{1}{2}(\pi^2 + \partial_i\phi\partial_i\phi)(y)\} = ((\pi\partial_i\phi)(x) + (\pi\partial_i\phi)(y))\partial_i\delta(\bar{x} - \bar{y}). \tag{5.37}$$

The above equation amounts to,

$$\{\mathcal{H}(x), \mathcal{H}(y)\} = (\mathcal{P}_i(x) + \mathcal{P}_i(y))\partial_i\delta(\bar{x} - \bar{y}), \tag{5.38}$$

thereby yielding the Schwinger condition where $\mathcal{P}_i = \pi\partial_i\phi$ is defined as the momentum density.

The same Lagrangian, now expressed in lightcone coordinates,

$$\mathcal{L} = \partial_+\phi\partial_-\phi - \frac{1}{2}\partial_i\phi\partial_i\phi \tag{5.39}$$

generates the equation of motion,

$$2\partial_+\partial_-\phi = \partial_i\partial_i\phi, \tag{5.40}$$

which is identical to (Eq. (5.34)). However, recovering (Eq. (5.40)) in Hamiltonian formalism is more complicated since the lightcone Lagrangian (Eq. (5.39)) is a constraint system in Dirac's formulation of constraint dynamics. Note that the momentum, defined as $\pi = (\partial\mathcal{L})/(\partial(\partial_+\phi)) = \partial_-\phi$, does not contain a time derivative term and hence (in Dirac's scheme [32]) is interpreted as a primary constraint,

$$\Omega(x) \equiv \pi(x) - \partial_-\phi(x) \approx 0. \tag{5.41}$$

The $\approx$ indicates weak equality which cannot be strongly imposed. This has important implications in the computation of the Poisson algebra of any variable with $\Omega(x)$. Naively, this would vanish. However, due to the weak equality, this is no longer valid and an explicit computation is necessary. Indeed we find,

$$\{\Omega(x), \Omega(y)\} = 2\partial_-\delta(x_- - y_-)\delta(\bar{x} - \bar{y}). \tag{5.42}$$

Canonical brackets (Eq. (5.36)) are used to derive (Eq. (5.42)). Since the constraint algebra (Eq. (5.42)) does not close, the constraint $\Omega(x)$ is said to be second class [32]. This feature is typical of second order systems when expressed in lightcone variables. The next step is to get the Dirac brackets (denoted by a star) which are defined in terms of the Poisson brackets by,

$$\{A(x), B(y)\}^* = \{A(x), B(y)\} - \int \{A(x), \Omega(z_1)\}\{\Omega(z_1), \Omega(z_2)\}^{-1}\{\Omega(z_2), B(y)\}dz_1dz_2 \tag{5.43}$$

where $\{\Omega(z_1), \Omega(z_2)\}^{-1}$ is the inverse of (Eq. (5.42)) defined as,

$$\int dy\{\Omega(x), \Omega(y)\}\{\Omega(y), \Omega(z)\}^{-1} = \delta(x - z) \tag{5.44}$$

By introducing the sign function $\epsilon(x_- - y_-)$ given by,

$$\partial_{x_-}\epsilon(x_- - y_-) = \delta(x_- - y_-) \tag{5.45}$$

it is simple to show that the inverse has the form,

$$\{\Omega(x), \Omega(y)\}^{-1} = \frac{1}{2}\epsilon(x_- - y_-)\delta(\bar{x} - \bar{y}) \tag{5.46}$$

It is now possible to compute the Dirac brackets among the field variables,

$$\begin{aligned}\{\phi(x_+, x_-, \bar{x}), \phi(y_+, y_-, \bar{y})\} &= -\int dz_1\, dz_2\, \{\phi(x), \Omega(z_1)\}[\{\Omega(z_1), \Omega(z_2)\}]^{-1}\{\Omega(z_2), \phi(y)\} \\ &= \frac{1}{2}\epsilon(x^- - y^-)\bar{\delta}(\bar{x} - \bar{y}). \end{aligned}\tag{5.47}$$

The advantage of using Dirac brackets is that the second class constraints can now be strongly imposed. Thus the Dirac brackets of $\Omega(x)$(*Eq.* (5.41)) with any variable vanishes, as may be easily checked.

Note the non-local nature of the lightcone symplectic structure (Eq. (5.47)). Together with the Hamiltonian

$$H = \int dy^- d\bar{y}\ \partial_i\phi\partial_i\phi \tag{5.48}$$

and the algebra (Eq. (5.47)), we compute $\partial_+\phi$,

$$\partial_+\phi = \partial^i_{\bar{x}} \int dy^-\ \partial_i\phi(x_+, \bar{x}, y^-)\frac{1}{2}\epsilon(x^- - y^-). \tag{5.49}$$

Furthermore, on differentiating both sides by ∂_- the non-locality is removed and one recovers the correct equation of motion (Eq. (5.40)). This clearly underlines the fact that lightcone framework, while reproducing the equal time equation of motion (Eq. (5.34)), is qualitatively distinct from the equal time framework with a reduced number of degrees of freedom due to the constraint. The original second order system is converted to first order.

To derive the energy conservation principle, from the covariant form of the symmetric energy momentum tensor for the massless scalar,

$$\Theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \mathcal{L}\eta^{\mu\nu} \tag{5.50}$$

we first write down the different lightcone components as,

$$\Theta^{--} = (\partial_+\phi)^2, \quad \Theta^{i-} = -\partial_i\phi\partial_+\phi, \quad \mathcal{H} \equiv \Theta^{+-} = \partial_i\phi\partial_i\phi \tag{5.51}$$

where Θ^{+-} is identified with the Hamiltonian density $\mathcal{H}$.

Let us now calculate the time derivative of Θ^{+-},

$$\partial_+\Theta^{+-} = \{\mathcal{H}(x), H\} = \{\frac{1}{2}\partial_i\phi(x)\partial_i\phi(x), \int dy^- d\bar{y}\ \frac{1}{2}\partial_i\phi\partial_i\phi\}. \tag{5.52}$$

Using the algebra (Eq. (5.47)) we find

$$\{\frac{1}{2}\partial_i\phi(x)\partial_i\phi(x), \int dy^- d\bar{y}\ \frac{1}{2}\partial_j\phi(y)\partial_j\phi(y)\}$$
$$= \frac{1}{2}(\partial_i\phi)(x)\partial_i^x \int dy^- d\bar{y}(\partial_j\partial_j\phi)(y)\epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y}).$$

Exploiting the lightcone equation of motion (Eq. (5.40)) we obtain,

$$\partial_+\Theta^{+-} = \partial_i\phi\partial_i\partial_+\phi = -\partial_-[\partial_+\phi(x)^2] + \partial_i[\partial_+\phi(x)(\partial_i\phi(x)], \tag{5.53}$$

where the final step is obtained after a simple algebra and reusing (Eq. (5.40)). The factors in parenthesis are identified with Θ^{--} and Θ^{i-}, respectively, as seen from (Eq. (5.51)). Then equation (Eq. (5.53)) is further rewritten as,

$$\partial_\mu\Theta^{\mu-} = \partial_+\Theta^{+-} + \partial_-\Theta^{--} + \partial_i\Theta^{i-} = 0 \tag{5.54}$$

This validates the conservation of energy. Likewise the other components of $\partial_\mu\Theta^{\mu\nu} = 0$ can be shown to hold.

5.4.2 Interacting fluid:

Returning to the interacting fluid model, let us rewrite the Lagrangian (Eq. (4.5)) in lightcone variables,

$$\mathcal{L} = -j^+(\partial_+\theta + \alpha\partial_+\beta - A_+) - j^-(\partial_-\theta + \alpha\partial_-\beta - A_-)$$
$$-j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) - f - \frac{1}{4}(2F^{+-}F_{+-} + 2F^{+i}F_{+i} + 2F^{-i}F_{-i} + f^{ij}F_{ij}) \tag{5.55}$$

with,

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta. \tag{5.56}$$

Interestingly the symplectic structure in the fluid sector remains essentially unaffected since it was already in a first order form (in equal time framework in (Eq. (4.5)). Hence the previous fluid algebra (Eq. (3.19)) suffices. But the constraint structure in the gauge sector is much more involved. The conjugate momenta are,

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_+A^\mu)} = F^{\mu+} \tag{5.57}$$

Only π^- is a true momentum since it involves a time derivative. The other components have to be interpreted as primary constraints,

$$\Omega_1 = \pi^+ \approx 0, \quad \chi_a = \pi^a - F^{a+} \approx 0, \quad a = 1, 2. \tag{5.58}$$

The constraint sector χ_a does not close,

$$\{\chi_a(x), \chi_b(y)\} = \partial^{x-}\delta(x - y)\delta_{ab}$$

and hence is second class. On the other hand Ω_1 closes,

$$\{\Omega_1(x), \Omega_1(y)\} = \{\Omega_1(x), \chi_a(y)\} = 0$$

and hence yields the first class sector. To find the secondary constraints, if any, we have to check the time conservation of $\Omega_1(x)$. To do this the canonical hamiltonian has to be found.

This is obtained following the conventional definition

$$\int \mathcal{H} = \int (\pi^+ \partial_+ A_+ + \pi^- \partial_+ A_- + \pi^a \partial_+ A_a - \mathcal{L})$$
$$= \int [\frac{1}{2}(\pi^-)^2 + \frac{1}{2}F^{12}F_{12} + (\pi^- \partial_- + \pi^i \partial_i - j^+)A^-]. \quad (5.59)$$

Calculating the Poisson bracket of Ω_1 with the Hamiltonian yields a secondary constraint $\Omega_2(x)$,

$$\Omega_2(x) = \{\Omega_1(x), \int \mathcal{H}\} = \{\pi^+(x), \int dy^- d\bar{y}\ \mathcal{H}(y)\} = \partial_i \pi^i(x) + \partial_- \pi^-(x) + j^+(x) \approx 0. \quad (5.60)$$

This is just the time $(+)$ component of the equation of motion (Eq. (4.12)). It is referred as the Gauss constraint since it is the analogue of the Gauss law in pure electrodynamics ($\nabla.\pi = \nabla.E = 0$). No further constraint is generated by $\Omega_2(x)$ since,

$$\{\Omega_2(x), \int \mathcal{H}\} = 0.$$

Now Ω_1, Ω_2 constitute a set of first class constraints indicating a gauge symmetry whereas χ_a, as already stated, turn out to be a second class set of constraints. Using this set of second class constraint and following the previous analysis, the nonvanishing Dirac brackets turn out to be ,

$$\{\pi^-(x), A_i(y)\} = \frac{1}{4}\partial_i^x \epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y}),\ \{\pi^-(x), \pi^-(y)\} = -\frac{1}{4}\nabla^2(x)\epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y}),$$
$$\{A^i(x), A^j(y)\} = \frac{1}{4}\epsilon(x^- - y^-)\delta^2(\bar{x} - \bar{y})\delta_{ij}. \quad (5.61)$$

For computational details the reader is encouraged to consult [33].

First of all we ensure that our earlier observation regarding the equality of the two definitions of the (integrated) energy momentum tensor modulo Gauss constraint remains valid in lightcone. For this we explicitly write down the different components of $T^{\mu\nu}$ and $\Theta^{\mu\nu}$. Following the Noether's prescription we have,

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \theta)}\partial_\nu \theta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \beta)}\partial_\nu \beta + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \alpha)}\partial_\nu \alpha + \frac{\partial \mathcal{L}}{\partial(\partial^\mu j^+)}\partial_\nu j^+ + \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\lambda)}\partial_\nu A^\lambda - \eta_{\mu\nu}\mathcal{L}$$
$$(5.62)$$

Using (Eq. (5.55)) we get after simplification,

$$T^{\mu\nu} = -j^{\mu}(\partial^{\nu}\theta + \alpha\partial^{\nu}\beta) - F^{\mu\sigma}\partial^{\nu}A_{\sigma} - \eta^{\mu\nu}\mathcal{L}. \tag{5.63}$$

Now, it is straightforward to get different components of $T^{\mu\nu}$ explicitly,

$$\begin{aligned} T^{+-} &= \frac{1}{2}(\pi^-)^2 + \frac{1}{2}F_{12}F^{12} + (\pi^-\partial_- + \pi^i\partial_i)A^- + f \\ &\quad + j^-(\partial_-\theta + \alpha\partial_-\beta) + j^i(\partial_i\theta + \alpha\partial_i\beta) - j^{\mu}A_{\mu}, \\ T^{+i} &= -j^+(\partial^i\theta + \alpha\partial^i\beta) + F^{-+}\partial^i A_- - F^{+j}\partial^i A_j, \end{aligned} \tag{5.64}$$

While the the second relation follows trivially from (Eq. (5.63)), some algebra is needed to obtain the first relation.

Now, to give the components of $\Theta^{\mu\nu}$ we start from (Eq. (4.21)). The explicit calculation of Θ^{+i} gives us,

$$\Theta^{+i} = \frac{j^+j^i}{n}f' + F^{\beta +}F_{\beta i} \tag{5.65}$$

which on use of (Eq. (4.11)) produces,

$$\Theta^{+i} = -j^+(\partial^i\theta + \alpha\partial^i\beta - A^i) + F^{-+}F^i_- + F_{ij}\pi_j. \tag{5.66}$$

Similarly we can easily compute,

$$\Theta^{+-} = \frac{1}{2}(\pi^-)^2 + \frac{1}{2}F_{12}F^{12} + j^-(\partial_-\theta + \alpha\partial_-\beta - A_-) + j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) + f, \tag{5.67}$$

It is easy to check that the following relations hold:

$$\int dy^- d\bar{y}\ (T^{+-} - \Theta^{+-}) = -\int dy^- d\bar{y}\ (\partial_i\pi^i + \partial_-\pi^- + j^+)A^-, \tag{5.68}$$

$$\int dy^- d\bar{y}\ (T^{+i} - \Theta^{+i}) = -\int dy^- d\bar{y}\ (\partial_i\pi^i + \partial_-\pi^- + j^+)A^i. \tag{5.69}$$

Integrated forms of the canonical and symmetric structures of the energy momentum tensor differ by a term proportional to the Gauss constraint (Eq. (5.60)) and hence are equal in the physical subspace in lightcone coordinates. Equations (Eq. (5.68), Eq. (5.69)) are the light cone analogues of the equal time relations given in (Eq. (4.34)).

The integrated energy-momentum tensors execute the spacetime translations. This naturally leads to the question as to what happens to rest of the spacetime translations that is rotation. Since the derivation is somewhat tricky we give the details below.

Once again the basic distinction between the spatial coordinates y_- and $\bar{y}$ comes in to play and we need to perform the calculations individually. First of all we provide integrated expressions for the 12-component of angular momentum, M_{12}, in the two definitions, derived respectively from (Eq. (4.35)) and (Eq. (3.27)),

$$M_{12}^N = \int dx^- d\bar{x} \left\{ (x_1 T_{-2} - x_2 T_{-1}) - (\pi_1 A_2 - \pi_2 A_1) \right\}, \tag{5.70}$$

$$M_{12}^S = \int dx^- d\bar{x} \{ (x_1 \Theta_{-2} - x_2 \Theta_{-1}) \}. \tag{5.71}$$

Difference between these two expressions is computed below,

$$M_{12}^N - M_{12}^S = \int dx^- d\bar{x} \; \{ x_1 (T_{-2} - \Theta_{-2}) - x_2 (T_{-1} - \Theta_{-1}) - (\pi_1 A_2 - \pi_2 A_1) \}$$

$$= \int dx^- d\bar{x} \; \{ x_1 (-j^+ A_2 + \pi^- \partial_- A_2 + \pi^i \partial_i A_2) - x_2 (-j^+ A_1 + \pi^- \partial_- A_1 + \pi^i \partial_i A_1) - (\pi_1 A_2 - \pi_2 A_1) \}$$

$$= \int dx^- d\bar{x} \; \{ (x_1 A_2 - x_2 A_1)(\partial_- \pi^- + \partial_i \pi^i + j^+) - (\pi_2 A_1 - \pi_1 A_2) - (\pi_1 A_2 - \pi_2 A_1) \}$$

$$= \int dx^- d\bar{x} \; \{ (x_1 A_2 - x_2 A_1)(\partial_- \pi^- + \partial_i \pi^i + j^+) \}. \tag{5.72}$$

Partial integrations are done to obtain the last step which shows that the expressions are equal modulo Gauss constraint (Eq. (5.60)). Rest of the components of M_{+i}^N and M_{+i}^S are given by,

$$M_{+i}^N = \int dx^- d\bar{x} \left\{ (x_+ T_{-i} - x_i T_{-+}) - (\pi_+ A_i - \pi_i A_+) \right\}, \tag{5.73}$$

$$M_{+i}^S = \int dx^- d\bar{x} \{ (x_+ \Theta_{-i} - x_i \Theta_{-+}) \}. \tag{5.74}$$

Their difference turns out to be,

$$M_{+i}^N - M_{+i}^S = \int dx^- d\bar{x} \; \{ x_+ (T_{-i} - \Theta_{-i}) - x_i (T_{-+} - \Theta_{-+}) - (\pi_+ A_i - \pi_i A_+) \}$$

$$= \int dx^- d\bar{x} \; \{ (x_+ A_i - x_i A_+)(\partial_- \pi^- + \partial_i \pi^i + j^+) - (\pi_i A_+ - \pi_+ A_i) - (\pi_+ A_i - \pi_i A_+) \}$$

$$= \int dx^- d\bar{x} \; \{ (x_+ A_i - x_i A_+)(\partial_- \pi^- + \partial_i \pi^i + j^+) \}.$$

Likewise, the difference among the boosts defined as,

$$M_{-i}^N = \int dx^- \; d\bar{x} \; (x_- T_{-i} - x_i T_{--} - \pi_i A_-) \tag{5.75}$$

and,

$$M^S_{-i} = \int dx^- \ d\bar{x} \ (x_- \Theta_{-i} - x_i \Theta_{--}) \tag{5.76}$$

also turns out to be proportional to the Gauss constraint,

$$M^N_{-i} - M^S_{-i} = \int dx^- d\bar{x} (x_- A_i - x_i A_-)(\partial_- \pi^- + \partial_i \pi^i + j^+) \tag{5.77}$$

Hence we have explicitly demonstrated that, in lightcone coordinates as well, the spacetime symmetry generators, obtained from the Noether and symmetric prescriptions, are equal modulo Gauss constraint which means that they are identical when acting on the physical subspace. It is also straightforward to establish the energy momentum conservation for the fluid gauge model in lightcone framework where the fundamental brackets provided in (Eq. (3.19),Eq. (5.61)) need to be used. We have not given the detailed derivation since it is not very illuminating.

This chapter gains particular importance due to some of the crucial results. Apart from the derivations of the Schwinger conditions and hence the conservation laws for non interacting fluids we have successfully presented the non relstivistic reduction of the fluid equation in a way more consistent manner than it was done before. We have explicitly find out the form of Gauss constraint in this framework. The components of the symmetric and the canonical em tensor differ by this term even in this coordinate is a new finding. Nonetheless the derivation of the sympletic structure of an interacting fluid is quite non trivial.

Chapter 6

Conservation laws in the hamiltonian formulation and Schwinger(type) condition

Schwinger conditions were originally found in the relativistic QFT context. One of the main goal of this formulation was to establish the Poincare algebra using a different route. Here we have obtained similar conditions while dealing with classical fluids in both interacting and non interacting cases. A very similar set of relations are found for the non relativistic fluids as well.

6.1 Relativistic ideal fluids in equal time frame

6.1.1 Without interaction

The analysis of fluids done here strongly rests on the conservation laws (Eq. (3.1)) and (Eq. (4.3)) for the stress tensor and current, respectively. It would be worthwhile to obtain these relations in a hamiltonian approach. That would also clarify the role and utility of the Schwinger condition.

Let us begin by considering the algebra of Θ_{00} with j_0,

$$\{j_0(x), \Theta_{00}(y)\} = \{j_0(x), j^i(\partial_i\theta + \alpha\partial_i\beta)(y) + f(n)(y)\} \tag{6.1}$$

The only nontrivial bracket of j_0(or ρ) is with the θ variable. Using (Eq. (3.19)) we obtain,

$$\{j_0(x), \Theta_{00}(y)\} = j^i(y)\partial_i^y\delta(x-y) \tag{6.2}$$

which reproduces the expected algebra. Its integrated version immediately yields (Eq. (4.3)). To see this consider the above algebra by integrating over y,

$$\{j_0(x), \int d^3y\Theta_{00}(y)\} = \int d^3y j^i(y)\partial_i^y\delta(x-y) \tag{6.3}$$

Recalling that $\int d^3y\Theta_{00}(y)$ is the hamiltonian we obtain, by dropping a surface term,

$$\partial_0 j_0 = -\partial_i j^i \tag{6.4}$$

thereby reproducing (Eq. (4.3)).

We now consider the algebra of Θ_{00} with itself. This algebra is the famous Schwinger condition whose integrated version would yield (Eq. (3.1)), similar to the above derivation of (Eq. (4.3)).

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = \{j^i(\partial_i\theta + \alpha\partial_i\beta)(x) + f(n)(x), j^k(\partial_k\theta + \alpha\partial_k\beta)(y) + f(n)(y)\}. \tag{6.5}$$

Exploiting the basic brackets (Eq. (3.19)) we find,

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = [\frac{j_i(x)f'(x)\rho(x)}{n(x)} + \frac{j_i(y)f'(y)\rho(y)}{n(y)}]\partial_i^x\delta(\mathbf{x}-\mathbf{y}). \tag{6.6}$$

Recalling the identification of j_i in terms of the Clebsch variables (Eq. (3.31)) we obtain,

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = -\big[(\rho(\partial_i\theta + \alpha\partial_i\beta)(x) + \rho(\partial_i\theta + \alpha\partial_i\beta)(y)\big]\partial_i^x\delta(\mathbf{x}-\mathbf{y}). \tag{6.7}$$

The expression on the right side is now expressed in terms of Θ_{0i} by using (Eq. (3.33))

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = (\Theta_{0i}(x) + \Theta_{0i}(y))\partial_i^{(x)}\delta(\mathbf{x}-\mathbf{y}), \tag{6.8}$$

which is the Schwinger condition [9].

Let us now consider its integrated version,

$$\{\Theta_{00}(x), \int d^3y\Theta_{00}(y)\} = \int d^3y(\Theta_{0i}(x) + \Theta_{0i}(y))\partial_i^{(x)}\delta(\mathbf{x}-\mathbf{y}) \tag{6.9}$$

which simplifies, after dropping surface terms, as,

$$\partial_0\Theta_{00} = \partial_i\Theta_{0i} \tag{6.10}$$

which is just the time component of (Eq. (3.1))

$$\partial_\mu\Theta^{\mu 0} = 0 \tag{6.11}$$

Likewise the space component of (Eq. (3.1)) may be obtained from other Schwinger conditions that involve the algebra among $\Theta_{00} - \Theta_{0i}$ and $\Theta_{0i} - \Theta_{0j}$. It is useful to mention that, at an intermediate stage, we have to use the relation,

$$u^\mu(\partial_\nu u_\mu - \partial_\mu u_\nu)f' + (g_{\mu\nu} - u_\mu u_\nu)\partial^\mu n f'' = 0. \tag{6.12}$$

which may also be verified explicitly. This is the relativistic generalization of the Euler equation as noted by [1]. Although in non-relativistic fluid mechanics, Euler equation is frequently used, quite surprisingly the relativistic Euler equation is not very familiar.

It is perhaps pertinent to mention that the Schwinger condition was originally proposed in the context of relativistic QFT. This was an alternative route to establish the conservation of the stress tensor as well as the validity of the Poincare algebra. Nevertheless, it has also found applications in discussing analogous features in the context of classical field theory [24, 25]. The point is that while the validity of the Schwinger condition is not mandatory in the classical context, any deviation must be such that the integrated version leads to the conservation law (Eq. (3.1)). In the present case we find that the Schwinger condition holds exactly. This is a new finding in the context of classical fluids.

It is useful to recall that the Schwinger condition was derived for the symmetric stress tensor $\Theta_{\mu\nu}$ defined in (Eq. (3.27)). Since the proof relies on this symmetricity it does not, in general, hold for $T_{\mu\nu}$ defined in (Eq. (3.35)). The nice point of our analysis is that, subject to the interpretation of j_μ discussed previously, it is possible to recast $T_{\mu\nu}$ in a symmetric form that is identical to $\Theta_{\mu\nu}$. This appears to be a unique characteristic of the theory of classical fluids developed here. There are important physical implications of the Schwinger condition for classical fluids. The first point to note is that the conservation law (Eq. (3.1)) is the fundamental equation on which the dynamics of fluids is based. Establishing Schwinger condition automatically implies (Eq. (3.1)). Next, the role of Clebsch variables gets illuminated. As discussed previously, one of the j_i in Θ_{00}(Eq. (3.30)) has to be eliminated in favour of these variables to get (Eq. (3.32)) which reproduces the equations of motion for the basic variables. It is now found that exploiting precisely this structure of Θ_{00}, the Schwinger condition holds. This serves as an important consistency check on our formalism. As a side remark we find that the same prescription also leads to current conservation(Eq. (4.3)) starting from the algebra (Eq. (6.2)).

6.1.2 Schwinger condition for interacting fluids:

In its simplest form, the Schwinger covariance condition relates the equal-time energy density commutator to the momentum density,

$$[\Theta_{00}(x), \Theta_{00}(x')] = (\Theta_{0i}(x) + \Theta_{0i}(x'))\partial_i\delta(x - x'). \tag{6.13}$$

For some quantum field theoretical applications see [39], where it is referred to as Dirac-Schwinger condition [40].) Validity of this condition in a quantum field theory ensures that the theory is relativistically covariant. However, it can play an important role in field theories even in non-relativistic scenario [15, 35].

Let us now concentrate on the Schwinger condition for the present model. In our previous paper [15] we have demonstrated the validity of the Schwinger condition for the non-interacting fluid model. The situation is more complicated here because the gauge fields being dynamical satisfy a canonical Poisson algebra $\{A_i(\mathbf{x}), \pi^{\mathbf{j}}(\mathbf{y})\} = \delta_{\mathbf{i}}^{\mathbf{j}}\delta(\mathbf{x}-\mathbf{y})$. We need to compute the following bracket,

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = \{j^i(\partial_i\theta + \alpha\partial_i\beta - A_i) + f + \frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}\pi_i^2|_x, j^k(\partial_k\theta + \alpha\partial_k\beta - A_k) + f + \frac{1}{4}F^{lm}F_{lm} + \frac{1}{2}\pi_k^2|_y\}. \quad (6.14)$$

After a long but straightforward calculation, we arrive at the result,

$$\{\Theta_{00}(x), \Theta_{00}(y)\} = [(-\rho(\partial_i\theta+\alpha\partial_i\beta-A_i)+F_{ik}\pi^k|_x)+(-\rho(\partial_i\theta+\alpha\partial_i\beta-A_i)+F_{ik}\pi^k|_y)]\partial_i^x\delta(x-y)$$

$$= (\Theta_{0i}(x) + \Theta_{0i}(y))\partial_i^x\delta(x-y). \quad (6.15)$$

This ensures the validity of the Schwinger condition in the fully interacting fluid-Maxwell theory.

6.1.3 Relativistic non interacting fluids in light cone coordinates

In order to discuss the conservation laws in the light-cone coordinates we have to first identify the appropriate hamiltonian. Consider the Θ_{+-} component of (Eq. (3.28)),

$$\Theta_{+-} = -(nf' - f)g_{+-} + \frac{f'}{n}j_+j_-$$

$$= f - \frac{f'}{n}(j_+j_- - j_ij_i)$$

$$= f + j_+a_- - j_ia_i. \quad (6.16)$$

We identify this with the canonical hamiltonian density ($\mathcal{H}$) defined in (Eq. (5.8)). This may be easily seen by replacing a_- using (Eq. (3.6)).

We are now ready to obtain the various conservation laws. Let us first derive the result (Eq. (5.9)). This will also act as a forerunner for the derivation of the Schwinger condition in light cone coordinates. Consider the algebra,

$$\{j_-(x), \Theta_{+-}(y)\} = \{j_-(x), (j_+a_- - j_ia_i + f)(y)\} \quad (6.17)$$

Replacing a_- and a_i from (Eq. (3.6)) and using the algebra (Eq. (5.7)) yields,

$$\{j_-(x), \Theta_{+-}(y)\} = j_+(y)\partial_-^y\delta(x-y) - j_i(y)\partial_i^y\delta(x-y). \quad (6.18)$$

Taking its integrated version,

$$\{j_-(x), \int d^3y\ \Theta_{+-}(y)\} = \int d^3y\ (j_+(y)\partial_-^y\delta(x-y) - j_i(y)\partial_i^y\delta(x-y)). \quad (6.19)$$

and dropping the surface terms yields,

$$\partial_+ j_-(x) = -\partial_- j_+(x) + \partial_i j_i(x), \tag{6.20}$$

which reproduces (Eq. (5.9)).

We next discuss the Schwinger condition. The relevant algebra is,

$$\{\Theta_{+-}(x), \Theta_{+-}(y)\} = \{[j_+(\partial_-\theta+\alpha\partial_-\beta)-j_i a_i+f](x), [j_+(\partial_-\theta+\alpha\partial_-\beta)-j_i a_i+f](y)\} \tag{6.21}$$

After some algebra we end up with,

$$\{\Theta_{+-}(x), \Theta_{+-}(y)\} = -j_+(x)\partial_-^x\left(\frac{f' j_+ \delta(\mathbf{x}-\mathbf{y})}{n}\right) + j_+(y)\partial_-^y\left(\frac{f' j_+ \delta(\mathbf{x}-\mathbf{y})}{n}\right)$$
$$+ j_i(x)\partial_i^x\left(\frac{f' j_+ \delta(\mathbf{x}-\mathbf{y})}{n}\right) - j_i(y)\partial_i^y\left(\frac{f' j_+ \delta(\mathbf{x}-\mathbf{y})}{n}\right). \tag{6.22}$$

On further simplification we obtain,

$$\{\Theta_{+-}(x), \Theta_{+-}(y)\} = \Big[\frac{f'(j_+)^2}{n}(x)+\frac{f'(j_+)^2}{n}(y)\Big]\partial_-^y\delta(\mathbf{x}-\mathbf{y})+\Big[\frac{f' j_+ j_i}{n}(x)+\frac{f' j_+ j_i}{n}(y)\Big]\partial_i^x\delta(\mathbf{x}-\mathbf{y}). \tag{6.23}$$

From (Eq. (3.15)) and (Eq. (3.28)) we identify the other components of the stress tensor,

$$\frac{f'(j_+)^2}{n} = \Theta_{++}\ , \quad \frac{f' j_+ j_i}{n} = \Theta_{+i}\ ,$$

and thereby recover the cherished form of the Schwinger condition in light-cone coordinates,

$$\{\Theta_{+-}(x), \Theta_{+-}(y)\} = -(\Theta_{++}(x)+\Theta_{++}(y))\partial_-\delta(\mathbf{x}-\mathbf{y})+(\Theta_{+i}(x)+\Theta_{+i}(y))\partial_i\delta(\mathbf{x}-\mathbf{y}). \tag{6.24}$$

We emphasize that this is a completely new result in the context of light-cone formulation of classical fluid.

Integrating over y we recover

$$\partial_+\Theta_{+-} = -\partial_-\Theta_{++} + \partial_i\Theta_{+i} \tag{6.25}$$

or equivalently the energy conservation condition

$$\partial_+\Theta^{+-} + \partial_-\Theta^{--} + \partial_i\Theta^{i-} = 0 \tag{6.26}$$

since this is the $\nu = -$ component of the conservation law(Eq. (3.1)). Note that this computation can be repeated for $\nu = +, i$ but infact that is unnecessary since the covariant conservation law follows directly from the lagrangian (Eq. (3.19)) and we have checked individually that the hamiltonian equations of motion in light-cone coordinates match correctly with their lagrangian counterpart. Finally, as discussed earlier, the light-cone version of the relativistic Euler equation(Eq. (5.18)) will also appear in the present setup.

To the best of our knowledge, in our work, for the first time, the light-cone analysis of relativistic fluid model has been carried through where the specific identification of the physical degrees of freedom with the Clebsch variables has been spelt out.

6.2 Schwinger-type relations for non relativistic fluids

Schwinger conditions are an essential ingredient of relativistic field theory. They were initially proved [9] for the quantum case. However, such conditions are known to exist even for specific examples of classical field theory [24]. These conditions involve the algebra of the components of the energy momentum tensor. The commutator algebra in the quantum case is replaced by the Poisson algebra in the classical case. There are three such relations (involving the $T_{00} - T_{00}$, $T_{00} - T_{0i}$ and $T_{0i} - T_{0j}$ algebra) and they have important applications. In the last section we have veryfied their existence for relativistic classical fluids [15] in both lightcone and equal time frame .

With utter surprise we found that similar conditions appear for nonrelativistic(classical) fluids whose hamiltonian formulation was discussed in the second chapter of the thesis.

The main issue in the nonrelativistic case is to properly define the em tensor. In relativistic theories we usually do by the matric (curved space) variation of the lagrangian an then we come back to flat space. Hence, the stress tensor is symmetric in nature by construction and its expression is used in demonstrating the Schwinger conditions [9]. Since space and time are on the same footing in a relativistic case, there is a practical requirement for a symmetric stress tensor. For nonrelativistic theories time is universal, hence, no such necessity exists in these systems, which implies $T_{0i} \neq T_{i0}$. Rotational symmetry, though, is present, that ensures $T_{ij} = T_{ji}$. These conditions are adequately satisfied by Noether's definition (Eq. (2.16)) and one can explicitly check them from (Eq. (2.18),Eq. (2.25),Eq. (2.28)). Here we show that, using this definition, Schwinger type relations are obtained subject to proper interpretation. Also, some applications of those conditions in this system are discussed.

We first consider the $T^{0i} - T^{0j}$ algebra which is the simplest. Recalling the algebra (Eq. (2.2)) and the identification (Eq. (2.9)), it is possible to obtain,

$$\{T^{0i}(x), T^{0j}(x')\} = T^{0j}(x)\partial^i\delta(x - x') + T^{0i}(x')\partial^j\delta(x - x') \tag{6.27}$$

This is the typical form for one of the Schwinger conditions valid in relativistic field theory. Indeed this structure is inbuilt in the very framework of the Eulerian fluids characterised by the algebra (Eq. (2.8)) and the identification (Eq. (2.9)).

Let us next look at the $T_{00} - T_{00}$ algebra that enters in the fundamental Schwinger condition. After some steps, using (Eq. (2.2)), one obtains,

$$\begin{aligned}\{T_{00}(x), T_{00}(x')\} &= \{(\frac{1}{2}\rho v^2 + V(\rho, S))(x), (\frac{1}{2}\rho v^2 + V(\rho, S))(x')\} \\ &= (T_{i0}(x) + T_{i0}(x'))\partial_i\delta(x - x') \end{aligned} \tag{6.28}$$

where T_{i0} is defined in(Eq. (2.55)). This is the analogue of the famous Schwinger condition. A general proof is given in [9] for its validity in relativistic quantum field theory where the Poisson or Dirac brackets are replaced by appropriate commutators. Interestingly, such a relation was found by us [15] for relativistic classical fluids. Here we show its existence even for non relativistic classical fluids.

Finally, we consider the $T^{00} - T^{0i}$ algebra,

$$\{T^{00}(x), T^{0i}(x')\} = \{(\frac{1}{2}\rho v^2 + V)(x), \rho v^i(x')\} \tag{6.29}$$

Using the basic brackets (Eq. (2.2)), this simplifies to,

$$\{T^{00}(x), T^{0i}(x')\} = T^{ij}(x)\partial^j \delta(x - x') + T^{00}(x')\partial^i \delta(x - x') \tag{6.30}$$

where T^{ij} is defined in (Eq. (2.26)). This is the last Schwinger relation.

There is an important distinction from the standard relativistic relations. Since $T_{0i} \neq T_{i0}$ in the nonrelativistic theory , it is important to note which one occurs in the relations (Eq. (6.27), Eq. (6.28), Eq. (6.30)). In the relativistic theory, the same relations exist but the choice (T^{0i} or T^{i0}) is inconsequential since these are identical. In this sense the Schwinger type relations for nonrelativistic fluids are restrictive. Nevertheless, as we show below, they are consistent with the various symmetries.

To begin with it is easy to see the connection of (Eq. (6.28)) with the conservation of the energy-momentum complex,

$$\partial_\mu T^{\mu\nu} = 0 \tag{6.31}$$

Taking the integral over (x') space on both sides of (Eq. (6.28)), we find,

$$\{T_{00}(x), \int dx' T_{00}(x')\} = \int dx' (T_{i0}(x) + T_{i0}(x'))\partial_i \delta(x - x') \tag{6.32}$$

which simplifies to,

$$\{T_{00}(x), H\} = \partial_i T_{i0}(x) \tag{6.33}$$

on dropping surface terms. Since the l.h.s of the above equation is $\partial_0 T_{00}$, we reproduce the time component of (Eq. (6.31)),

$$\partial_\mu T^{\mu 0} = 0. \tag{6.34}$$

Let us next take the the integral over x on both sides of (Eq. (6.30)). This yields,

$$\{H, T^{0i}(x')\} = -\partial^i T^{ij}(x') \tag{6.35}$$

where a surface term has been dropped. Noting that the l.h.s is $-\partial_0 T^{0i}$ yields,

$$\partial_\mu T^{\mu i} = 0 \tag{6.36}$$

where the symmetry $T^{ij} = T^{ji}$ has been used. The above relation is the space component of the conservation law (Eq. (6.31)).

Observe that in the derivation of (Eq. (6.34)) and (Eq. (6.36)), the desired form (T^{i0} or T^{0i}) appeared, else the conservation law would not emerge. This is an important point of departure from the relativistic case (where $T^{i0} = T^{0i}$) and confirms the viability of the Schwinger type relations for nonrelativistic fluids.

We now provide a demonstration of the Galilean algebra. The closure of Galilean translations,

$$\{P^i, P^j\} = 0; \; P^i = \int dx T^{0i} \tag{6.37}$$

is easily demonstrated by taking the space integrals (over both x and x') on either side of (Eq. (6.27)). Likewise,

$$\{H, H\} = \{H, P^i\} = 0$$

are easily shown by once again taking the space integrals (over x and x') on both sides of (Eq. (6.28)) and (Eq. (6.30)) respectively.

Similarly, the algebra of the rotation generator,

$$M_{ij} = \int (x_i T_{0j} - x_j T_{0i}) dx \tag{6.38}$$

with the translation generator P_k is worked out. An intermediate step is to compute the algebra of P_k with the unintegrated expressions. Taking the integral over x' on both sides of (Eq. (6.27)), we find the expected result,

$$\{T_{0i}(x), P_j\} = \partial_j T_{0i}(x) \tag{6.39}$$

which we obtain by dropping a surface term. Then it follows from (Eq. (6.38)) and (Eq. (6.39)),

$$\{M_{ij}, P_k\} = \int (x_i \partial_k T_{0j} - x_j \partial_k T_{0i}) dx \tag{6.40}$$

Once again, dropping a surface term, yields,

$$\{M_{ij}, P_k\} = -\delta_{ik} P_j + \delta_{kj} P_i \tag{6.41}$$

which is the expected Galelian algebra.

The algebra of the rotation generator is next derived. Using the basic definition,

$$\{M_{ij}, M_{kl}\} = \{\int (x_i T_{0j} - x_j T_{0i}) dx, \int (y_k T_{0l} - y_l T_{0k}) dy, \} \tag{6.42}$$

Taking a particular combination,

$$\{\int (x_i T_{0j}) dx, \int (y_k T_{0l}) dy\} = \int dx dy [x_i y_k (T_{0l}(x) \partial_j \delta(x-y) + T_{0j}(y) \partial_l \delta(x-y))] \tag{6.43}$$

where the relation (Eq. (6.27)) has been exploited .

Dropping the surface term yields,

$$\{\int (x_i T_{0j})dx, \int (y_k T_{0l})dy\} = \int (\delta_{jk} x_i T_{0l} - \delta_{il} x_k T_{0j})dx \tag{6.44}$$

The other combinations may be similarly worked out and the final answer is,

$$\{M_{ij}, M_{kl}\} = \delta_{jk} M_{il} - \delta_{il} M_{kj} - \delta_{ik} M_{jl} + \delta_{jl} M_{ki} \tag{6.45}$$

which reproduces the standard Galilean rotation algebra.

The boost generator,

$$B_i = t \int d^3x T_{0i} - \int d^3x \; x_i \rho \tag{6.46}$$

is not written solely in terms of $T_{\mu\nu}$, due to the presence of the second term. Hence its algebra has to be worked out independently along conventional lines. The Schwinger like relations are not effective in this case.

We have presented a thorough discussion of the bracket structure of the components of the em tensors of various fluid systems and subsequently have derived the conservation laws. That Schwinger type relations can be found in classical systems particularly for fluid systems having unusual symplectic structure is a new finding. This brings out a new facet in the interpretation of Eulerian fluids as a field theory, namely the validity of a closed algebra involving the basic (unintegrated) components of the stress tensor.

Chapter 7

Non-commutative fluids: a hamiltonian description

We introduce NC algebra in Lagrangian (discrete) degrees of freedom which subsequently shows up to the Euler (field) degrees of freedom [52, 54] and NC-extended fluid action [54]. We have followed this approach while introducing the NC effect because, NC generalization can be unambiguously done in this discrete variable set up. We emphasise that our model is constructed following the first principles, entirely based on the map between the Lagrangian (discrete) and Euler formulation of fluid dynamics (check [1] for a detailed discussion on canonical fluid). The NC effect is induced in continuous Euler algebra from the NC extension in discrete Lagrangian variable algebra.

We initiate our discussion with the mapping between the lagrangian to hamiltonian formulation of fluid. Then we introduce non commutativity in the poisson bracket structure of the lagrangian variables and later derive the poisson brackets between the Euler variables and check the algebraic consistency. We then extend the algebra further and check the effects on the fluid equations.

7.1 Hamiltonian formulation of Eulerian fluid: A brief review

Newton's second law for the particle (Lagrangian) coordinate $X_i(t)$ and the velocity $v_i(t) = \dot{X}_i$ is given by,

$$m\ddot{X}_i(t) = m\dot{v}_i(t) = F_i(X(t)), \tag{7.1}$$

where m is the mass of each individual particle and $F_i(X(t))$ is the force on the ith particle . On the other hand, density(ρ), a Eulerian variable, for the single particle is,

$$\rho(t, \mathbf{r}) = m\delta(\mathbf{X}(t) - \mathbf{r}). \tag{7.2}$$

For N number of particles the density field is given by,

$$\rho(t, \mathbf{r}) = m \sum_{n=1}^{N} \delta(\mathbf{X}_n(t) - \mathbf{r}). \tag{7.3}$$

We define the fluid current in a straightforward manner, as,

$$\mathbf{j}(t, \mathbf{r}) = \mathbf{v}(t, \mathbf{r})\rho(t, \mathbf{r}) = m \sum_{n=1}^{N} \dot{\mathbf{X}}_n(t)\delta(\mathbf{X}_n(t) - \mathbf{r}). \tag{7.4}$$

Finally we replace the discrete particle labels by continuous spatial arguments (omitting time t) which leads to,

$$\rho(\mathbf{r}) = \rho_0 \int \delta(X(x) - r)dx, v_i(\mathbf{r}) = \frac{\int dx \dot{X}_i(x)\delta(X(x) - r)}{\int dx \delta(X(x) - r)}. \tag{7.5}$$

The integration is over the entire relevant volume. (The dimensionality of the measure will be specified only for the formulas which are dimension specific.) ρ_0 is a background mass density, so that the volume integral of density ρ is the total mass.

In a Hamiltonian formulation the canonical Poisson bracket structure is given by

$$\{\dot{X}^i, X^j\} = (i/m)\delta^{ij}, \quad \{X^i, X^j\} = 0, \quad \{\dot{X}^i, \dot{X}^j\} = 0. \tag{7.6}$$

For the Lagrangian fluid this is generalized to [1],

$$\{\dot{X}^i(\mathbf{x}), X^j(\mathbf{x}')\} = \frac{1}{\rho_0}\delta^{ij}\delta(\mathbf{x} - \mathbf{x}'); \quad \{X^i(\mathbf{x}), X^j(\mathbf{x}')\} = \{\dot{X}^i(\mathbf{x}), \dot{X}^j(\mathbf{x}')\} = 0. \tag{7.7}$$

It is straightforward to show that the above bracket structure satisfies the Jacobi identity.

Using the definitions of density (ρ) and current ($\mathbf{j}$) in terms of $\mathbf{X}$ and $\dot{\mathbf{X}}$ given above (Eq. (7.5)), a straightforward computation leads to the Poisson algebra between the Euler variables ρ and $\mathbf{j}$ [1] (details of the computation are provided in the appendix):

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = 0 \tag{7.8}$$

$$\{j^i(\mathbf{r}), \rho(\mathbf{r}')\} = \rho(\mathbf{r})\partial_i\delta(\mathbf{r} - \mathbf{r}') \tag{7.9}$$

$$\{j^i(\mathbf{r}), j^j(\mathbf{r}')\} = j^j(\mathbf{r})\partial_i\delta(\mathbf{r} - \mathbf{r}') + j^i(\mathbf{r}')\partial_j\delta(\mathbf{r} - \mathbf{r}'). \tag{7.10}$$

Since $\mathbf{j} = \rho\mathbf{v}$ an equivalent set of brackets follows [1]:

$$\{v^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i\delta(\mathbf{r} - \mathbf{r}'), \tag{7.11}$$

$$\{v^i(\mathbf{r}), v^j(\mathbf{r}')\} = -\frac{\omega_{ij}(\mathbf{r})}{\rho(\mathbf{r})}\delta(\mathbf{r} - \mathbf{r}'), \tag{7.12}$$

where

$$\omega_{ij}(\mathbf{r}) = \partial_i \ v_j(\mathbf{r}) - \partial_j v_i(\mathbf{r}) \tag{7.13}$$

is the fluid vorticity, $\omega = 0$ for an irrotational fluid.

The Hamiltonian for a generic barotropic fluid (where pressure depends only on density) is taken as,

$$H = \int dV \ \mathcal{H} = \int dV \ (\frac{1}{2}\rho v^2 + V(\rho)). \tag{7.14}$$

The pressure P is related to V via $P(\rho) = \rho V' - V$, where $V' = \frac{\partial V}{\partial \rho}$. Fluid dynamical equations follow from the Hamiltonian equations of motion. As an example,

$$\dot{\rho} = \{\rho, H\} = \{\rho, \int dV \ (\frac{1}{2}\rho v^2 + V(\rho))\} \tag{7.15}$$

Then using (Eq. (7.8)) and (Eq. (7.11)) we obtain,

$$\dot{\rho} = -\partial_i(\rho v_i) \tag{7.16}$$

which is the continuity equation $\partial_\mu j^\mu = 0$, with $j^\mu = (\rho, \mathbf{j})$. Similarly, using (Eq. (7.8)) and (Eq. (7.12)) we obtain Euler equation,

$$\dot{v_k} = \{v_k, H\} = -v_i \partial_i v_k - \partial_k V'(\rho). \tag{7.17}$$

7.2 Non Commutative generalization

Let us now generalize the above to NC space. We start with the usual minimal (and most popular) form of extended NC Poisson brackets between the Lagrangian variables, [60],

$$\{X_i(\mathbf{x}), X_j(\mathbf{y})\} = \frac{\theta_{ij}}{\rho_0}\delta(\mathbf{x}-\mathbf{y}), \ \{\dot{X}_i(\mathbf{x}), X_j(\mathbf{y})\} = \frac{1}{\rho_0}\delta_{ij}\delta(\mathbf{x}-\mathbf{y}), \ \{\dot{X_i(\mathbf{x})}, \dot{X}_j(\mathbf{y})\} = 0, \tag{7.18}$$

where the NC parameter tensor θ_{ij} is constant and antisymmetric ($\theta_{ij} = -\theta_{ji}$). This is the simplest extension of the canonical algebra (and qualitatively equivalent to the NC proposed by Seiberg and Witten in [60] to NC space.

In Eulerian description in NC space we define the fluid variables in the same way as in (Eq. (7.4)) and (Eq. (7.5)) and the induced NC field algebra appears below (computational details are given in Appendix A1),

$$\{\rho(\mathbf{r}), \rho(\mathbf{r}')\} = - \ \theta_{ij}\partial_i \rho \partial_j \delta(\mathbf{r} - \mathbf{r}'), \tag{7.19}$$

$$\{\rho(\mathbf{r}), j^i(\mathbf{r})\} = \rho(\mathbf{r}')\partial_i \delta(\mathbf{r} - \mathbf{r}') - \theta^{jk}\partial_k \delta(\mathbf{r} - \mathbf{r}')\partial_j j^i(\mathbf{r}), \tag{7.20}$$

$$\{j_i(\mathbf{r}), j_j(\mathbf{r})\} = j_i(\mathbf{r}')\partial_k\delta(\mathbf{r}-\mathbf{r}') + j_k(\mathbf{r})\partial_i\delta(\mathbf{r}-\mathbf{r}') - \theta_{lm}\partial_m\delta(\mathbf{r}-\mathbf{r}')\partial_l(\frac{j_i(\mathbf{r})j_k(\mathbf{r})}{\rho(\mathbf{r})}). \tag{7.21}$$

Again we have a similar set of equations between the density (ρ) and the fluid velocity (v^i).

$$\{v_i(\mathbf{r}), \rho(\mathbf{r})\} = -\theta_{jk}\partial_k\delta(\mathbf{r}-\mathbf{r})\partial_j v_i(\mathbf{r}') + \partial_i\delta(\mathbf{r}-\mathbf{r}'), \tag{7.22}$$

$$\{v_i(\mathbf{r}), v_j(\mathbf{r}')\} = \frac{\partial_j v_i - \partial_i v_j}{\rho}\delta(\mathbf{r}-\mathbf{r}') + \theta^{lm}\frac{\partial_l v_i \partial_m v_j}{\rho}\delta(\mathbf{r}-\mathbf{r}'). \tag{7.23}$$

This is the complete NC algebra between the Eulerian fluid variables which reduces to the usual canonical form for $\theta_{ij} = 0$.

7.2.1 Algebraic consistency and Jacobi identity

Consistency of a generalized Poisson bracket structure in Hamiltonian framework demands validity of the Jacobi identity. In the present theory since we have posited a hitherto unknown algebra we must ensure that it satisfies the Jacobi identity. Because of the non-linear nature of the algebra, explicit demonstration of Jacobi identity is indeed a non trivial exercise.

The Jacobi identity for a generic set of variables a, b, c, is given by,

$$J(a,b,c) = \{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0.$$

It proves to be convenient to work in momentum space via Fourier transforms. We write down the density and current in momentum space as,

$$\tilde{\rho}(\mathbf{p}) = \int \mathbf{dr}\ \mathbf{e}^{\mathbf{ip.r}}\rho(\mathbf{r}), \quad \tilde{\mathbf{j^i}(\mathbf{p})} = \int \mathbf{dr}\mathbf{e}^{\mathbf{ipr}}\mathbf{j^i}(\mathbf{r}). \tag{7.24}$$

We recalculate the brackets in momentum space,

$$\{\tilde{\rho}(\mathbf{p}), \tilde{\rho}(\mathbf{q})\} = \{\int dr e^{i\mathbf{p}.\mathbf{r}}\rho(\mathbf{r}), \int dr' e^{i\mathbf{q}.\mathbf{r}'}\rho(\mathbf{r}')\}$$

$$= -\theta^{ij}\int dr p_j(p_i+q_i)e^{i(p+q)r}\rho(\mathbf{r}); \ = i\hbar\theta^{ij}p_i q_j\tilde{\rho}(\mathbf{p}+\mathbf{q}) \tag{7.25}$$

$$\{\tilde{\rho}(\mathbf{p}), \tilde{j}_i(\mathbf{q})\} = ip_i\tilde{\rho}(\mathbf{p}+\mathbf{q}) - \theta_{jk}q_j p_k\tilde{j}_i(\mathbf{p}+\mathbf{q}) \tag{7.26}$$

To begin with, we take the $J(\rho,\rho,\rho)$ which in momentum space reads,

$$J(\rho(\mathbf{p}), \rho(\mathbf{q}), \rho(\mathbf{r})) = [\theta^{ij}\theta^{lm}p^i q^j(p^l+q^l)r^m] + cyclic\ terms.$$

After some algebra(details of the explicit demonstration of (Eq. (7.27)) are provided in the Appendix A2.) we recover

$$J(\rho(\mathbf{p}),\rho(\mathbf{q}),\rho(\mathbf{r})) = \theta_k\theta_n\epsilon_{ijk}\epsilon_{lmn}[(p^iq^j(p^l+q^l)r^m+q^ir^j(q^l+r^l)p^m+r^ip^j(r^l+p^l)q^m] = 0. \quad (7.27)$$

with,

$$\theta_{ij} = \epsilon_{ijk}\theta_k.$$

To prove the next nontrivial identity (in momentum space) we need to check

$$J(\rho(\mathbf{p}),\rho(\mathbf{q}),v_k(\mathbf{r})) = \{\{\rho(\mathbf{p}),\rho(\mathbf{q})\},v_k(\mathbf{r})\}+\{\{\rho(\mathbf{q}),v_k(\mathbf{r})\},\rho(\mathbf{p})\}+\{\{v_k(\mathbf{r}),\rho(\mathbf{p})\},\rho(\mathbf{q})\} = 0. \quad (7.28)$$

Again a detailed but reasonably straightforward computation reveals that

$$\{\{\rho(\mathbf{p}),\rho(\mathbf{q})\},v_k(\mathbf{r})\} = \epsilon^{ijk}\theta_kp^iq^j[i(p_k+q_k)\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})+\epsilon_{lmn}\theta_n(p_l+q_l)r_mv_k(\mathbf{p}+\mathbf{q}+\mathbf{r})] = 0.$$

which implies the validity of $J(\rho(\mathbf{p}),\rho(\mathbf{q}),v_k(\mathbf{r})) = 0$.

In a similar way, validity of rest of the non-trivial Jacobi identities can be checked as well.

7.2.2 Modified Non Commutative algebra

It is natural to consider further extensions of the NC structure that we have already considered by introducing a new set of NC parameters σ_{ij} in the $\{\dot{X}_i(\mathbf{x}),X_j(\mathbf{y})\}$ bracket in (Eq. (7.18)),

$$\{X_i(\mathbf{x}),X_j(\mathbf{y})\} = \frac{\theta_{ij}}{\rho_0}\delta(\mathbf{x}-\mathbf{y}),\ \{\dot{X}_i(\mathbf{x}),X_j(\mathbf{y})\} = \frac{1}{\rho_0}(\delta_{ij}+\sigma_{ij})\delta(\mathbf{x}-\mathbf{y}),\ \{\dot{X}_i(\mathbf{x}),\dot{X}_j(\mathbf{y})\} = 0. \quad (7.29)$$

Indeed, we stress that this new extension is not for purely academic purpose. Onwards we will see that it has important consequence in cosmology. Adopting the same procedure the new NC Euler algebra is found as,

$$\{\rho(\mathbf{r}),\rho(\mathbf{r}')\} = \rho_0^2\{\int dx\delta(X(\mathbf{x})-\mathbf{r}),\int dy\delta(X(\mathbf{y})-\mathbf{r}'\}$$

$$= -\ \partial_i\rho\theta_{ij}\partial_j\delta(\mathbf{r}-\mathbf{r}'), \quad (7.30)$$

$$\{\rho(r),j^i(r')\} = \rho_0^2\{\int dx\delta(X(\mathbf{x})-\mathbf{r}),\int dy\ \dot{X}^i(y)\delta(X(\mathbf{y})-\mathbf{r}'\}$$

$$= \rho(\mathbf{r}')\partial_i\delta(\mathbf{r}-\mathbf{r}') - \theta^{jk}\partial_k\delta(\mathbf{r}-\mathbf{r}')\partial_jj^i(\mathbf{r}) + \sigma_{ij}\rho(\mathbf{r}')\partial_j\delta(\mathbf{r}-\mathbf{r}') \quad (7.31)$$

$$\{j^i(\mathbf{r}),j^k(\mathbf{r})\} = j_k(\mathbf{r})\partial_i\delta(\mathbf{r}-\mathbf{r}') + j_i(\mathbf{r}')\partial_k\delta(\mathbf{r}-\mathbf{r}') + \sigma_{ij}j_k(\mathbf{r})\partial_j\delta(\mathbf{r}-\mathbf{r}')$$
$$+\sigma_{kj}j_i(\mathbf{r}')\partial_j\delta(\mathbf{r}-\mathbf{r}') + \theta_{lm}\partial_m\delta(\mathbf{r}-\mathbf{r}')\partial_l\frac{j_i(\mathbf{r})j_k(\mathbf{r})}{\rho(\mathbf{r})} \quad (7.32)$$

Again we provide the algebra between the velocity and the density,

$$\{v^i(\mathbf{r}), \rho(\mathbf{r}')\} = \partial_i \delta(\mathbf{r} - \mathbf{r}') + \sigma_{ij}\partial_j \delta(\mathbf{r} - \mathbf{r}') + \theta_{kj}\partial_k \delta(\mathbf{r} - \mathbf{r}')\partial_j v_i(\mathbf{r}') \tag{7.33}$$

$$\{v_i(\mathbf{r}), v_j(\mathbf{r}')\} = \frac{\partial_j v_i - \partial_i v_j}{\rho}\delta(\mathbf{r}-\mathbf{r}')+\theta^{lm}\frac{\partial_l v_i \partial_m v_j}{\rho}\delta(\mathbf{r}-\mathbf{r}')+\frac{1}{\rho}(\sigma_{kj}\partial_k v_i - \sigma_{ik}\partial_k v_j)\delta(\mathbf{r}-\mathbf{r}'). \tag{7.34}$$

Note that the density algebra remains unaltered but the rest receive σ_{ij}-contribution. Keeping the form of Hamiltonian unaltered,

$$H = \int dV \ \mathcal{H} = \int (\frac{1}{2}\rho v^2 + V(\rho))$$

the NC-generalized Euler dynamics follows,

$$\dot{\rho} = \{\rho, H\} = -\partial_i(\rho v_i) - \sigma_{ij}\partial_j(\rho v_i) = -\partial_i(\rho v_i + \sigma_{ji}\rho v_j), \tag{7.35}$$

$$\dot{v_k} = \{v_k, H\} = -v_i \partial_i v_k - \sigma_{ij} v_i \partial_j v_k - \partial_k V'(\rho) - \sigma_{kj}\partial_j V'(\rho) + \theta_{ji}\partial_i V'(\rho)\partial_j v_k. \tag{7.36}$$

From (Eq. (7.35)) we can certainly conclude that the total mass remains unaltered as the NC modification showed up in the equation as a total derivative term. The flux though, changes, and both the NC terms have their contributions in it.

In this chapter we have discussed how the introduction of noncommutativity at the lagrangian level remoulds the Poisson bracket between the eulerian variables. Then we have shown that these extended brackets satisfies Jacobi identity . We then have extended the algebra further and observe the modifications it brings in the dynamical equations of fluid. The purpose of this extension will become clear in the following chapter where we are going to discuss the effects of non commutatative modifications in fluids in the cosmological context.

Chapter 8

Noncommutative effects of fluid in cosmology

The structure formation of this universe is dictated by the time evolution modes, the growing modes in particular, of the density contrast. In this chapter we explicitly show how (spatial) Non-Commutativity (NC) can affect the temporal dependence of the modes, that is , here, we compute NC corrected power law profiles of the density contrast modes. We develop a generalized fluid model that lives in NC space. The dynamical equations of fluid, namely the continuity and Euler equations receive NC contributions. When mapped to comoving coordinates these generate the NC extended versions of continuity and Friedmann equations for cosmology. Introducing cosmological perturbations finally yield the NC corrected evolution of density contrast modes.

8.1 Basic Discussion

Finally we discuss the implications of NC fluid dynamics in cosmological context. To start with we would like to have a short discussion on some basic equations which are contextual. The present model lives in flat space. (For introductory reference in cosmological perturbation see for example [50].)

The canonical continuity and Euler equations in Friedmann-Robertson-Walker (FRW) cosmology are given by.

$$\dot{\rho} = -3H(\rho + P) = -3\frac{\dot{a}}{a}(\rho + P), \tag{8.1}$$

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M^2} + \frac{\Lambda}{3}, \tag{8.2}$$

with pressure P and cosmological constant Λ and $M = (8\pi G)^{-1/2}$ with G the Newton's constant. $H(t) = \dot{a}/a$ is the Hubble parameter, indicative to the rate of expansion of the universe. The Friedmann equation follows:

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{\rho}{3M^2} + \frac{\Lambda}{3} - \frac{k}{a^2}. \tag{8.3}$$

We know that the canonical fluid equations in lab frame, when expressed in comoving frame, gets mapped on to the FRW equations (Eq. (8.1), Eq. (8.2)). Following same route we will map the NC fluid equations (Eq. (7.35), Eq. (7.36)) in comoving frame and will interpret those equations as the NC FRW equations.

As is customary in cosmology we now work in a comoving frame $(a(t), \mathbf{x})$ where the map between laboratory and comoving coordinates ($\mathbf{r}$ and $a(t), \mathbf{x}$ respectively) is given by,

$$\mathbf{r}(t) = a(t)\mathbf{x} \tag{8.4}$$

with $a(t)$ being the scale factor and $\mathbf{x}$, the time independent comoving distance.

The relations between space and time derivatives between the laboratory and comoving frames are obtained following (Eq. (8.4)). They are,

$$\frac{\partial}{\partial r} = \frac{1}{a}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t}|_r = \frac{\partial}{\partial t}|_x - \frac{\dot{a}}{a}(x.\partial_x). \tag{8.5}$$

8.1.1 Noncommutative FRW from noncommutative fluid

This section is dedicated to the studies of the consequences of noncommutative modified fluid from cosmological perspective. As observed in the last chapter the sole contribution to the non commutative continuity equation comes from the σ_{ij} term. The θ_{ij} term due to its anti symmetric nature does not appear in the modified continuity equation. Hence the crucial role played by σ_{ij}, appearing in our extended form of noncommutativity in fluid, will come in front.

Our first step will be to write the NC modified fluid equations (Eq. (7.35)), (Eq. (7.36)),namely the continuity and the Euler equation, in comoving frame, exploiting (Eq. (8.5)),

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho + \partial_i(\rho v_i) + \frac{\sigma_{ij}}{a}\partial_j(\rho\dot{a}x_i + \rho v_i) = 0 \tag{8.6}$$

and,

$$\begin{aligned}
&\ddot{a}x_k + \frac{\partial v_k}{\partial t} + \frac{\dot{a}}{a}v_k + \frac{1}{a}v_i\partial_i v_k + \frac{\dot{a}}{a}\sigma_{ik}(\dot{a}x_i + v_i) + \frac{1}{a}\sigma_{ij}(\dot{a}x_i + v_i)\partial_j v_k \\
&= -\frac{1}{a}[\frac{\partial_k P}{\rho} + \sigma_{kj}\frac{\partial_j P}{\rho} + \frac{\dot{a}}{a\rho}\theta_{ik}\partial_i P + \frac{1}{a\rho}\theta_{ij}\partial_i P\partial_j v_k + \frac{4\pi}{3}aG\rho x_k + \partial_k\phi].
\end{aligned} \tag{8.7}$$

Note that $\partial_k P = \frac{\partial P}{\partial\rho}\partial_k\rho = c_s^2\partial_k\rho$ where c_s is the adiabatic sound speed. Thus the above equation reads

$$\begin{aligned}
&\ddot{a}x_k + \frac{\partial v_k}{\partial t} + \frac{\dot{a}}{a}v_k + \frac{1}{a}v_i\partial_i v_k + \frac{\dot{a}}{a}\sigma_{ik}(\dot{a}x_i + v_i) + \frac{1}{a}\sigma_{ij}(\dot{a}x_i + v_i)\partial_j v_k \\
&= -\frac{1}{a}[c_s^2\frac{\partial_k\rho}{\rho} + \sigma_{kj}\frac{\partial_j P}{\rho} + \frac{\dot{a}}{a\rho}\theta_{ik}\partial_i P + \frac{1}{a\rho}\theta_{ij}\partial_i P\partial_j v_k + \frac{4\pi}{3}aG\rho x_k + \partial_k\phi].
\end{aligned} \tag{8.8}$$

Continuity equation: Let us focus our attention on the Continuity equation (Eq. (8.6)). If we expand (Eq. (8.6)) order by order, using (Eq. (8.23)), the background part satisfies

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 + \frac{1}{a}\sigma_{ij}\partial_j(\rho_0\dot{a}x_i) = 0 \quad \rightarrow \dot{\rho}_0 + \frac{\dot{a}}{a}\rho_0(3+\sigma) = 0 \tag{8.9}$$

where $Tr(\sigma_{ij}) = \sigma$. Clearly the NC effect modifies the background continuity equation. If we set the NC contribution zero ($\sigma_{ij} = 0$) we will get the continuity equation (Eq. (8.1)) back (with zero pressure). [1] We can make a further simplification by dropping the peculiar velocity contributions, namely $\mathbf{v} = \mathbf{0}$ in the full equation (Eq. (8.6)) leading to

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\sigma_{ij}\partial_j(\rho\dot{a}x_i) = 0. \tag{8.10}$$

This is the usual way to generate the conventional ($\sigma_{ij} = \theta_{ij} = 0$) FRW equation from fluid dynamics. In the present case we have derived the NC corrected continuity equation, even with vanishing peculiar velocity.

Euler equation: Let us now concentrate on the Euler equation (Eq. (8.7)). We follow the conventional procedure of isolating structurally similar terms in (Eq. (8.7)) and requiring that the combinations vanish separately. In the present case the x_i-dependent terms read (with ρ replaced by its homogeneous background value ρ_0):

$$[(\ddot{a} + \frac{4\pi}{3}G\rho_0)\delta_{ik} + \dot{a}H\sigma_{ik}]x_i = 0. \tag{8.11}$$

To satisfy the above for arbitrary x_i we require determinant of the coefficient matrix of x_i to vanish,

$$\begin{vmatrix} (\lambda + \dot{a}H\sigma_{11}) & \dot{a}H\sigma_{12} & \dot{a}H\sigma_{13} \\ \dot{a}H\sigma_{21} & (\lambda + \dot{a}H\sigma_{22}) & \sigma_{23}\dot{a}H \\ \dot{a}H\sigma_{31} & \dot{a}H\sigma_{32} & (\lambda + \dot{a}H\sigma_{33}) \end{vmatrix} = 0, \tag{8.12}$$

where,

$$\lambda = \ddot{a} + \frac{4\pi}{3}G\rho_0.$$

Expanding the determinant yields,

$$(\lambda + \dot{a}H\sigma_{11})[(\lambda + \dot{a}H\sigma_{22})(\lambda + \dot{a}H\sigma_{33}) - (\dot{a}H)^2\sigma_{23}\sigma_{32}]$$
$$+(\dot{a}H)\sigma_{12}[(\dot{a}H)^2\sigma_{23}\sigma_{31} - \dot{a}H\sigma_{21}(\lambda + \dot{a}H\sigma_{33})] \tag{8.13}$$
$$+\dot{a}H\sigma_{13}[(\dot{a}H)^2\sigma_{21}\sigma_{32} - \dot{a}H\sigma_{31}(\lambda + \dot{a}H\sigma_{22})] = 0. \tag{8.14}$$

Since we are interested in $O(\sigma)$ contributions, the above equation reduces to,

$$(\lambda)^3 + \lambda^2\dot{a}H(\sigma_{11} + \sigma_{22} + \sigma_{33}) \approx 0, \tag{8.15}$$

[1] However, in an interesting variant of our model in [49] it is shown the θ_{ij} can also modify the continuity equation.

leading to

$$\lambda + \dot{a}H\sigma = 0 \tag{8.16}$$

which is a modified Euler equation in cosmology,

$$\ddot{a} + \frac{4\pi}{3}G\rho_0 + \dot{a}H\sigma = 0, \tag{8.17}$$

augmented by the σ_{ij} contribution.

After a little more algebra we find that (Eq. (8.10)) and (Eq. (8.17)) together yield,

$$\frac{1}{2}\frac{d}{dt}(\dot{a}^2) = \frac{4\pi G\rho}{3}[\frac{1}{\rho}(\frac{d}{dt}(\rho a^2) + \frac{a}{\rho}\sigma\partial_j(\rho\dot{a}))] - \dot{a}^2 H\sigma \tag{8.18}$$

Finally the cherished Friedmann equation with NC correction is recovered:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{8\pi G}{3a^2}\int dt\ a\partial_j(\rho\dot{a})\sigma - \frac{2}{a^2}\int dt\ \dot{a}^2 H\sigma.$$

$$= \frac{8\pi G\rho}{3} - \frac{k_{eff}}{a^2}, \tag{8.19}$$

where

$$k_{eff} = k - \sigma(\frac{8\pi G}{3}\int dt\ a\dot{a}\rho - 2\int dt\ \dot{a}^2 H). \tag{8.20}$$

The original (curvature) constant k is scaled to $0, \ \pm 1$ signifying flat, closed or open universe respectively. But in NC space this feature will be dictated by the effective curvature k_{eff}. For instance for a flat universe in NC cosmology $k_{eff} = 0$ will lead to a relation,

$$k = \sigma(\frac{8\pi G}{3}\int dt\ a\dot{a}\rho - 2\int dt\ \dot{a}^2 H) \tag{8.21}$$

that can provide a bound on the value of σ_{ij}.

8.1.2 Cosmological perturbation

Let us introduce the cosmological perturbation scheme. As usual we are assuming that at sufficiently large distance scales, (may be beyond the galaxy clusters), the space inhomogeneities average out leaving behind an isotropic and homogeneous background.

To that end velocities in the laboratory and comoving frames are related by,

$$\dot{\mathbf{r}} = \mathbf{u} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}}. \tag{8.22}$$

Note that, here $\mathbf{x}$ is time-dependent and generates the second term, known as peculiar velocity $a\dot{\mathbf{x}} = \mathbf{v}$ which appears as a perturbation, (such that $|\mathbf{v}| \ll |\dot{a}\mathbf{x}|$). Conventionally we do not consider $\mathbf{v}$ in the canonical set of FRW equations.

In constructing the perturbation theory we split the fields into a flat FRW background part and a perturbation part that can be analyzed order by order. We introduce perturbations in the following way,

$$\begin{aligned} \rho(\mathbf{x},t) &= \rho_0(t) + \delta\rho((\mathbf{x},t)) = \rho_0 + \rho_1 + \rho_2 + ... \\ P(\mathbf{x},t) &= P_0(t) + \delta P(\mathbf{x},t) = P_0 + P_1 + P_2 + ... \\ H(\mathbf{x},t) &= H_0(t) + \delta H(\mathbf{x},t) = H_0 + H_1 + H_2 ... \\ \phi(\mathbf{x},t) &= \phi_0(t) + \delta\phi(\mathbf{x},t) = \phi_0 + \phi_1 + \phi_2 + .. \\ \mathbf{u} &= \dot{a}\mathbf{x} + \mathbf{v} = \dot{a}\mathbf{x} + \mathbf{v_1} + \mathbf{v_2} .... \end{aligned} \tag{8.23}$$

It needs to be emphasised that this scheme of introducing inhomogeneity through perturbation about a homogeneous background is the conventional one. Keeping the background variables spatially invariant is valid since we are introducing the NC effect perturbatively and are considering NC corrections only up to first order. The novelty of our scheme lies in the fact that noncommutativity provides a natural seed for generating inhomogeneity.

The peculiar velocity $\mathbf{v}$ in (Eq. (8.23)) is considered to be the perturbation in the velocity field. We here define a quantity called density contrast(of order n) as,

$$\delta_n = \frac{\rho_n}{\rho_0}. \tag{8.24}$$

ϕ is the gravitational potential which satisfies,

$$\nabla_x^2 \phi = 4\pi G a^2 \rho.$$

Hence the zero'th order background equation is,

$$\nabla_x^2 \phi_0 = 4\pi G a^2 \rho_0 \tag{8.25}$$

with the solution for the background potential

$$\phi_0 = \frac{2\pi}{3} G (ax)^2 \rho_0 \tag{8.26}$$

using Newtonian model for gravitational potential due to a sphere of uniform density ρ_0. Furthermore, equations for the perturbations appear as

$$\nabla_x^2 \phi_n = 4\pi G a^2 \rho_n$$

where n is the order of perturbation.

Perturbed FRW equations :

The aim of of introducing perturbations in the FRW "Standard Model" of cosmology is to explain how large scale structures were formed in the expanding Universe. In particular, this means that starting from an isotropic and homogeneous universe with an average background density ρ_0, how does the fluctuation $\delta\rho = \rho - \rho_0$ grow so that the density contrast $\delta = \delta\rho/\rho_0$ can reach unity. Once δ reaches values of the order of unity, their growth becomes non-linear. From then onwards, they rapidly evolve towards bound structures such as star formation and other astrophysical process, eventually leading to formation of galaxies and clusters of galaxies.

Now, we would like to write the perturbation equation corresponding to the Euler equation (Eq. (8.7)) (without the terms in (Eq. (8.17)) that has already been taken in to account). The perturbed equation is,

$$\frac{\partial v_k}{\partial t} + (H_0 + \delta H)v_k + \frac{1}{a}v_i\partial_i v_k + (H_0 + \delta H)\sigma_{ik}v_i + \frac{1}{a}v_i\partial_i v_k$$
$$= -\frac{1}{a}[c_s^2\frac{\partial_k(\rho_0 + \delta\rho)}{\rho_0 + \delta\rho} + \sigma_{kj}\frac{\partial_j(P_0 + \delta P)}{\rho_0 + \delta\rho} + (H_0 + \delta H)\theta_{ik}\frac{\partial_i(P_0 + \delta P)}{\rho_0 + \delta\rho}$$
$$+\frac{1}{a}\theta_{ij}\frac{\partial_i(P_0 + \delta P)}{\rho_0 + \delta\rho}\partial_j v_k + \partial_k\delta\phi]. \quad (8.27)$$

Here we will confine ourselves upto 1st order in perturbation so that terms of the form $\frac{\partial_k(\rho_0+\delta\rho)}{\rho_0+\delta\rho} \approx \frac{\partial_k\delta\rho}{\rho_0}$. Thus we find

$$\dot{v_k}^1 + H_0(v_k^1 + \sigma_{ik}v_i^1) = -[\frac{1}{a}c_s^2\frac{\partial_k\delta\rho}{\rho_0} + \partial_k\phi_1 + \frac{1}{a\rho_0}H_0\theta_{ik}\partial_i P_1 + \frac{1}{a\rho_0}H_1\theta_{ik}\partial_i P_0$$
$$+\frac{1}{a\rho_0}\sigma_{kj}\partial_j P_1 + \frac{1}{a^2\rho_0}\theta_{ij}\partial_i P_0\partial_j v_k] \quad (8.28)$$

It is straightforward to see from (Eq. (8.9)) that the linear equations satisfied by the first order perturbations [50] are,

$$H^1 = \frac{1}{3}\partial_i v_i^1, \qquad \partial_k^2\Phi^1 = 4\pi G\delta\rho^1,$$

$$\dot{(\rho^1)} = -\rho^0 H^1(3+\sigma) - H^0\rho^1(3+\sigma). \quad (8.29)$$

Evidently the last relation is modified due to the non commutative modifications in (Eq. (7.29)). Here we recall that $\rho_0 \propto a^{-3}$ which leads to a further simplification [50] in the last relation in (Eq. (8.29)),

$$\dot{(\delta^1)} = -H^1(3+\sigma). \quad (8.30)$$

We are interested in finding out the changes brought in by the non commutative considerations in the density perturbation equation. For that we would like to work with the density contrast (Eq. (8.24)) over $\delta\rho$ and derive the density perturbation equation. Taking divergence of the perturbation equation (Eq. (8.27)) results in,

$$\partial_k \dot{v_k}^1 + H_0 \partial_k (v_k^1 + \sigma_{ik} v_i^1) = -\frac{1}{a}[c_s^2 \frac{\partial_k^2 \rho^1}{\rho_0} + \partial_k^2 \phi^1 + \sigma_{kj} \frac{\partial_k \partial_j P^1}{\rho_0} + \frac{1}{a^2 \rho_0} \theta_{ij} \partial_i P_0 \partial_j v_k]. \tag{8.31}$$

Now, the relation connecting the divergence of the peculiar velocity and the Hubble parameter (Eq. (8.29)), is used. Some more algebra yields,

$$\dot{H}^1 = -2H^0 H^1 - \frac{1}{3a}[c_s^2 \frac{\partial_k^2 \rho^1}{a\rho_0} + \frac{\partial_k^2 \phi^1}{a} + H_0 \sigma_{ik} \partial_k v_i^1 + \sigma_{kj} \frac{\partial_k \partial_j P^1}{a\rho_0} + \frac{1}{a^2 \rho_0} \theta_{ij} \partial_i P_0 \partial_j v_k]. \tag{8.32}$$

8.1.3 Wave Equation for Growth of Small Density Perturbations

Eventually using (Eq. (8.30)) we derive the cherished form density perturbation equation:

$$\ddot{\delta^1} = -2H_0 \dot{\delta^1} + \frac{(3+\sigma)}{3a}[H_0 \sigma_{ik} \partial_k v_i^1 + c_s^2 \frac{\partial_k^2 \delta^1}{a} + \sigma_{kj} \frac{\partial_k \partial_j P^1}{a\rho_0} + \frac{\partial_k^2 \phi^1}{a} + \frac{1}{a^2 \rho_0} \theta_{ij} \partial_i P_0 \partial_j v_k]. \tag{8.33}$$

The noncommutative parameter σ_{ij} being small, we can ignore terms quadratic in σ_{ij}. The term containing θ_{ij} is ignored compared to the other terms since it varies as $\frac{1}{a^3}$. Furthermore, seeking solutions of the form $\delta^1 \sim exp\ i(\mathbf{k_c}.\mathbf{x} - \omega \mathbf{t})$ we note that $c_s^2 \partial_k^2 \delta^1 = -c_s^2 k_c^2 \delta^1 = -c_s^2 k^2 a^2 \delta^1$ where $\mathbf{k_c}$ and $\mathbf{k}$ are respectively the comoving and proper wave vector,

$$\begin{aligned} \ddot{\delta^1} &= -2H_0 \dot{\delta^1} + \frac{\partial_k^2 \phi^1}{a^2} + c_s^2 \frac{\partial_k^2 \delta^1}{a^2} + \frac{\sigma}{3} \frac{\partial_k^2 \phi^1}{a^2} + \frac{1}{a} H_0 \sigma_{ik} \partial_k v_i^1 + \sigma_{kj} \frac{\partial_k \partial_j P^1}{a^2 \rho_0} \\ &= -2H \dot{\delta^1} + (4\pi G \rho_0 - c_s^2 k^2) \delta^1 + \frac{4\pi G \rho_0}{3} \sigma \delta^1 + \frac{1}{a} \sigma_{ik} (H_0 \partial_k v_i^1 + \frac{\partial_i \partial_k P^1}{a\rho_0}). \end{aligned} \tag{8.34}$$

Finally we have reached our goal of obtaining the density perturbation equation. This equation governs the dynamics of small density fluctuations in a noncommutative fluid for an expanding background cosmology without cosmological constant.

We rewrite the above equation in the convenient form,

$$\ddot{\delta^1} = -2H \dot{\delta^1} + 4\pi G \rho_0 (1 + \frac{\sigma}{3}) \delta^1 - c_s^2 k^2 \delta^1 + \Sigma, \tag{8.35}$$

where $\Sigma = \sigma_{ik} \frac{\partial_i \partial_k P^1}{a^2 \rho_0}$ where we have dropped the term $\frac{1}{a} \sigma_{ik} H_0 \partial_k v_i^1$ from Σ since it is $O(\sigma v^1)$.

Thus σ and Σ are both NC contributions.

Jeans' instability in expanding medium: The pressure terms are negligible except on small scales just before the matter radiation equality [50]. Hence in the long wavelength limit[2] we can drop the terms generated by pressure and consider a reduced form of (Eq. (8.35)),

$$\ddot{\delta}^1 = -2H_0\dot{\delta}^1 + 4\pi G\rho_0(1+\frac{\sigma}{3})\delta^1. \tag{8.36}$$

In the linear regime, density fluctuations on different scales evolve independently. Thus it is useful to write the equations (Eq. (8.30)), (Eq. (8.35)) in the Fourier space as,

$$H^1_k = -\frac{\dot{\delta}_k}{3+\sigma},$$

$$\ddot{\delta}^1_k + 2H_0\dot{\delta}^1_k = 4\pi G\rho_0(1+\frac{\sigma}{3})\delta^1_k + \Sigma_k \tag{8.37}$$

where $\Sigma_k = -\sigma_{ik}\frac{k^2}{a^2\rho_0}P^1_k$ is the Σ written in the Fourier space. We will drop this term since we are neglecting pressure as explained earlier. The modified (Eq. (8.37)) can be written as,

$$\ddot{\delta}^1_k + 2H_0\dot{\delta}^1_k = 4\pi G\rho_0(1+\frac{\sigma}{3})\delta^1_k. \tag{8.38}$$

We will try to find solution of the equation (Eq. (8.38)) in a flat space which implies at critical density ($\rho = \rho_c$). Under these conditions we have to find out the dependence of a and ρ_0 on time and subsequently we would like to solve (Eq. (8.38)).

Before proceeding further to derive explicit form of δ^1_k it is important to stress that the background, (about which the fluctuations are being studied), is no longer the conventional one since it has already received a NC correction, as is seen from (Eq. (8.9)). So the first task is to ascertain the NC modified background density ρ_0 for which we consider the modified background continuity equation (Eq. (8.9)). The solution is given by,

$$\rho_0 = \bar{\rho}a^{-(3+\sigma)}. \tag{8.39}$$

As we are confining ourselves upto first order in σ we are allowed to use the canonical time dependence of $a(= A_0 t^{\frac{2}{3}})$ [50] and the solution of the modified continuity equation (Eq. (8.9)) to get the time dependence of k under flat space condition from (Eq. (8.21)). A straightforward computation yields[3],

$$k(t) = \frac{8}{3}\sigma t^{-2/3}(-\pi G\bar{\rho}A_0^{-(1+\sigma)}t^{-2\sigma/3} + \frac{A_0^2}{3}). \tag{8.40}$$

[2]Long wavelength limit refers to $\lambda\rangle\rangle\lambda_J = c_s\sqrt{\frac{\pi}{G\rho_0}}$, λ_j is the Jeans' wavelength in conventional cosmology.

[3]$k = \sigma(\frac{8\pi G}{3}\bar{\rho}\int dt\ a\dot{a}a^{-(3+\sigma)} - 2\int dt\ \dot{a}^2 H)$

Quite obviously this $k(t)$ is proportional to the NC parameter σ and vanishes in the conventional (flat space) case. On using this k in the Friedmann equation (Eq. (8.3)) (with $\Lambda = 0$, no cosmological constant) we get,

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{\rho_0}{3M^2} - \frac{\frac{8}{3}\sigma t^{-2/3}(-\pi G\bar{\rho}A_0^{-(1+\sigma)}t^{-2\sigma/3} + \frac{A_0^2}{3})}{a^2}. \tag{8.41}$$

We want to obtain the solution of a as a polynomial in t restricting ourselves to the first non-trivial σ-correction. In the RHS of (Eq. (8.41)) we substitute

$$\rho_0 = \bar{\rho}a^{-(3+\sigma)}, \quad a = A_0 t^{2/3}, \tag{8.42}$$

that amounts to taking account of the σ-corrected background and conventional form of $a(t)$ so that (Eq. (8.41)) will yield the $O(\sigma)$ corrected $a(t)$. Here A_0 and $\bar{\rho}$ are simply two constants that take care of the dimensions. It is straightforward to get a solution of the form,

$$t = Aa^{\frac{3+\sigma}{2}} + Ba^{3(\frac{1+\sigma}{2})} \tag{8.43}$$

where A and B are constants,

$$A = \frac{2(1-\sigma)}{3+\sigma}\sqrt{\frac{3}{8\pi G\bar{\rho}}}, \quad B = \frac{8\sigma A_0^3}{27(1+\sigma)}(\frac{3}{8\pi G\bar{\rho}})^{\frac{3}{2}}.$$

We need to invert (Eq. (8.43)) to express a as a function of t in the familiar form,

$$a = (\frac{t}{A})^{\frac{2}{3+\sigma}}[1 - \frac{BA_0^{2\sigma/3}}{A}t^{\frac{2\sigma}{3}}]^{\frac{2}{3+\sigma}} \tag{8.44}$$

where, $\frac{B}{A} = \frac{2A_0^3\sigma}{3\pi G\bar{\rho}}$. First of all it is reassuring to note that for $\sigma = 0$ the familiar form, $a(t) \sim t^{2/3}$ is recovered. For convenience we further approximate $a(t) \sim t^{2/(3+\sigma)}$ in subsequent analysis. Putting everything together in (Eq. (8.38)) provides the cherished evolution equation of δ_k^1:

$$\ddot{\delta}_k^1 + \frac{4(1-\sigma/3)}{3t}\dot{\delta}_k^1 - \frac{2}{3t^2}(1+\frac{\sigma}{6})\delta_k^1 = 0. \tag{8.45}$$

By inspection a power law solution $\delta_k^1 \sim t^n$ yields

$$n = \frac{1}{6}[-1 + \frac{4\sigma}{3} \pm 5\sqrt{1 - \frac{11}{75}\sigma}] \approx \frac{1}{6}[-1 + \frac{4\sigma}{3} \pm 5(1 - \frac{11}{150}\sigma)]. \tag{8.46}$$

The NC corrected values of n are

$$n = \frac{2}{3} + \frac{29}{180}\sigma, \quad n = -1 + \frac{51}{180}\sigma. \tag{8.47}$$

Note that σ can be either positive or negative. Positive and negative values of n signify growing or decaying modes. Obviously allowed values of σ have to be such that the original nature of the mode (growing or decaying) is not altered. This constitutes the other significant result of our paper. In the next section we discuss some of the consequences of NC fluid model in cosmology.

8.2 Noncommutative effect on Hubble parameter and structure formation

Indeed it is pertinent to ask to what extent can NC can affect the curvature and related evolutionary history of the universe in quantitative way. Generically numerical upper bounds of NC parameters, obtained from areas in quantum mechanics or particle physics are in fact extremely small. From a theoretical perspective NC effects are expected to become relevant at approximately around Planck scale when the spacetime continuum tends to get replaced by discreteness with noncommutativity manifesting itself by inducing an inherent length scale. However, we should emphasize the distinction between the above scenario and the present context because, strictly speaking, in the latter, we are dealing with a *non-canonical* Poisson bracket structure in classical physics, rather than a noncommutative structure in the quantum commutators. Even though the non-canonical structure carries the legacy of the NC-extended (Heisenberg) quantum commutation relations or vice-versa and both affect classical and quantum physics respectively in similar fashion, there are important differences between NC generalizations in Poisson brackets in classical mechanics and commutation relations in quantum mechanics, notable among them being that dimensionally the NC parameters in the two scenarios are different.

One of the most important observables in cosmology is the Hubble parameter $H(t)$. Let us concentrate on the NC effect on H. Using the explicit form of NC-modified scale factor $a(t)$ we compute $H(t)$ and plot it against t for two values of $\sigma = \pm 0.1$ and $\sigma = \pm 0.5$ (since σ can take positive or negative values). This is depicted in **Figure 8.1** where profiles for $H(t)$ for $\sigma = \pm 0.1$ and $\sigma = \pm 0.5$ are plotted. These can be compared with the conventional case, $\sigma = 0$, the middle black line. In our simplified scheme we have

$$H(t) = \frac{2}{(3+\sigma)t}. \tag{8.48}$$

Thus larger negative values of σ tend to stay more and more above the $\sigma = 0$ line whereas larger positive values of σ stay below the $\sigma = 0$ line. Comparing with a conventional matter dominated universe $H \sim 2/(3t)$, one might conclude that the NC correction for positive σ reduces H indicating that the rate of expansion of universe slows down, thereby simulating a dark matter like behavior whereas values of negative σ seem to behave in a way that opposes the conventional matter contribution. Furthermore Hubble parameter also indicates the physical distance at which objects are receding at the speed of light, which is referred to as the Hubble distance given by $R_H = c/H$. Thus the Hubble distance increases (decreases) for negative (positive) values of σ.

The other object of interest related to structure formation is the NC-correction in the evolution of the density contrast modes $\delta_k^1 \sim t^n$ where NC-modified n is provided in (Eq. (8.46)). Once again for $\sigma = 0$ the conventional values $n = -1$ and $n = +2/3 \approx 0.66$ are recovered out of which the latter increasing mode is of

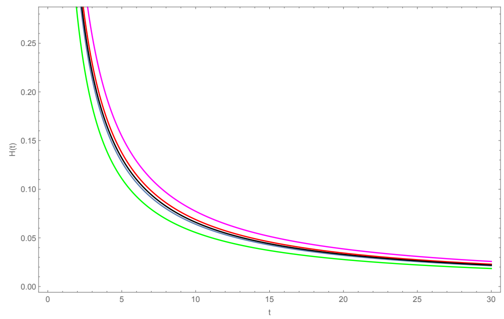

Figure 8.1: $H(t)$ (ordinate) is plotted against t (abscissa) for $\sigma = 0$ black line (conventional case), $\sigma = \mp 0.1$ for blue and green lines respectively $\sigma = \mp 0.5$ for pink and red lines respectively.

interest. From (Eq. (8.47)) we get for $\sigma = \pm 0.1$, n changes to $0.68,\ 0.63$ respectively and for $\sigma = \pm 0.5$, n changes to $0.74,\ 0.58$ respectively for the increasing mode. In **Figure 8.2** we have plotted δ_k^1 against t for the above four values of n along with $n = +2/3$ (for $\sigma = 0$) for comparison. The nature of the profiles presented in **Figure 8.2** reveal that positive values of σ enhances the growing modes so that structure formation is favored. In this sense our model of generalized fluid dynamics in the cosmological perspective becomes interesting since it might lead to a dark matter model, (that is essential for explaining the observed large-scale structure in the Universe), remaining rooted in classical physics.

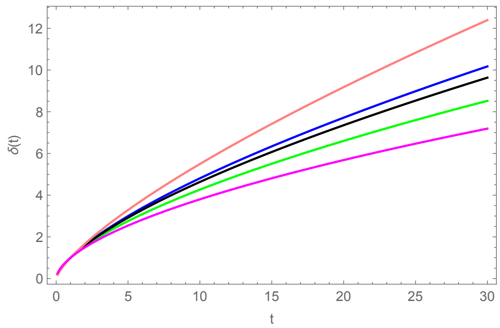

Figure 8.2: $\delta_k^1(t)$ (ordinate) is plotted against t (abscissa) for $n = 2/3$, $\sigma = 0$ black line (conventional case), $n = 0.68$, $n = 0.63$ for $\sigma = \pm 0.1$ for blue and green lines respectively and $n = 0.74$, $n = 0.58$ for $\sigma = \pm 0.5$ for orange and magenta lines respectively.

In the present chapter we have considered cosmological implications of a generalized fluid model in Newtonian framework. Our model is an extension of noncommutative fluid model we have proposed in the previous chapter.

In the second part we have introduced cosmological perturbations and rigorously derive how the behaviour of decaying and growing modes of density contrast are altered by noncommutative (or non-canonical, which is probably more appropriate) corrections. We have demonstrated that the positive and negative values of the noncommutative parameter σ can respectively decrease or increase the Hubble parameter. The former can be identified with an effective model for dark matter. The positive σ boosts the increasing mode of density contrast which also agrees with the dark matter interpretation mentioned above.

Chapter 9

Conclusions

Despite the fact that the origins of fluid dynamics lie in nineteenth century physics, its relevance has not abated even now. It has stood the test of time like Maxwell's electrodynamics. The theory of fluid dynamics has continuously evolved through its various ramifications and extensions. The dynamics of fluid is usually described through classical field theory. Infact fluid dynamics is one of the earliest known examples of a field theory. It is thus possible to provide a canonical formulation for fluid dynamics, either at the lagrangian or hamiltonian levels. Here we have presented the canonical formulation for the fluid dynamics to study various aspect of this system and also provided some formal applications in cosmological context.

In this section we would like to provide a chapter wise summary of our work.

In *chapter 1* we have discussed about the motivations and the outline of this thesis.

In *chapter 2* we came up with a fresh approach for the introduction of the Clebsch variables in the fluid action. Usually this is done in somewhat ad hoc manner. We have provided the physical basis of this parametrisation using Noether's definition of fluid current which is identified with momentum density. Starting with a simple irrotational fluid we have extended the analysis for the rotational fluid as well.

An elaborate analysis of the arbitrariness in the Clebsch decomposition was presented afterwards. The generator of infinitesimal transformations was given. Finite transformations and their connection with infinitesimal ones were discussed. It was shown that only one of the velocity potentials is completely arbitrary. This agrees with the usual counting of degrees of freedom, as discussed below (Eq. (2.80)). Such a detailed and systematic analysis of the arbitrariness in the Clebsch variables of nonisentropic nonrelativistic fluids is a new feature.

In *chapter 3* we have discussed several kinematic and dynamic aspects in detail both for free and interacting fluids, the latter in the presence of non dynamical gauge fields. We have used a hamiltonian formalism as this framework is most suitable for studying symmetry properties. The Clebsch parametrization plays an essential role in our framework where the fluid turns out to be a second class constraint system.

Another major drive of the work is in the study of fluids interacting with a external gauge field. We have demonstrated that the canonical (Noether) and symmetric forms of energy momentum tensors do not match although both have essential properties pertaining to it such as generating proper dynamics (in case of the canonical one) and satisfying correct conservation principle (in case of the symmetric one). In this sense the two definitions of the stress tensor complement each other. However, there still remains a lack of a single stress tensor that shows both these properties. We have also shown how an elegant modification of the canonical stress tensor leads to the symmetric one. In this analysis we have once again used the same interpretation of the auxiliary variable in terms of the physical ones as done for the free theory. This establishes the robustness of the aforesaid interpretation.

In *chapter 4* we have dealt with a way more complicated system compared to our previous work [15]. We have added the entropy term to the fluid sector, where we have exploited the freedom of adding scalar variables according to the prescription of Clebsch parametrisation and have included the Maxwell term in the gauge sector. The latter ensures that the gauge field is dynamical so we are now considering a fully interacting gauge-fluid system. We have concentrated mainly on the relativistic aspect of the theory and have studied in detail the structures of energy momentum tensor, derived from two definitions, *ie.* the symmetric one and the canonical (Noether) one. In the equal-time framework, we have shown that all the space time symmetry generators obtained from these two definitions agree modulo the Gauss constraint. This equivalence in the physical sector has been achieved only because of the kinetic term of the gauge fields. We consider this finding to be an extremely important one since, in the absence of this term, this equivalence, as we found in the last chapter, cannot be shown.

In *chapter 5* we have explored the fluid model in light-cone coordinate system.

An interesting and non-trivial form of mapping between relativistic and non-relativistic variables was suggested in [12] in a purely algebraic framework without paying attention to a dynamic framework. We have explicitly constructed a lightcone Lagrangian and Hamiltonian model with a symplectic structure to show that correct fluid dynamics is reproduced under the mapping [12]. Hence our work lends credence and consistency to the work of [12].

Moreover a we provide a detailed analysis of the gauge-fluid model in the lightcone formalism. This lightcone analysis though has numerous non-trivial features but a rigorous analysis of it, particularly, in the context of fluids was lacking. We have carried out a detailed study since the lightcone framework has become quite a sensational topic in recent times. We have shown that the validity of the conservation principles. Furthermore we have explicitly demonstrated that the space time symmetry generators differ by the lightcone version of Gauss law as we found in the equal time case.

In *chapter 6* we have derived the validity of the Schwinger condition, a hallmark of any relativistic field theory in both, equal-time and light-cone coordinates. Although these conditions were initially given for relativistic QFT, instances are there [24, 25] where they hold for the classical cases also. We find here that it is valid for relativistic classical fluids.

Proceeding one step further, we have explicitly demonstrated the validity of the Schwinger condition in the fully interacting gauge-fluid theory which ensures that the unconventional symplectic structure of fluid (with the presence of auxiliary fluid variables) does not hamper the relativistic covariance of the model.

A completely new element of this paper is the demonstration of the closure of the algebra involving the components of Noether's stress tensor (Eq. (2.16)) for non relativistic fluid. This algebra, given in (Eq. (6.27)), (Eq. (6.28)) and (Eq. (6.29)) is a new finding for such systems. It has a remarkable resemblance with the Schwinger conditions [9] found in relativistic field theory where the stress tensor is obtained as a response to metric variations. This brings out a new facet in the interpretation of Eulerian fluids as a field theory, namely the validity of a closed algebra involving the basic (unintegrated) components of the stress tensor.

In *chapter 7* we consider a non commutative fluid model. In the first part we concentrate on rigorous derivation of some formal aspects of the noncommutative fluid model in hamiltonian feamework. We clarify issues related to the Jacobi identity of the NC fluid variable algebra. We then include some non trivial extension to our model. This extended NC algebra leads to a modified form of fluid equations.

In *chapter 8* we introduce cosmological perturbations and explicitly show how the behavior of growing and decaying modes of density contrast are affected by non-commutative (or non-canonical, which is probably more appropriate as pointed out in the paper) corrections. The modification that appears in Hubble parameter has also been shown.

As a future goal to extend this thesis we are planning to discuss Quantum hydrodynamics and its applications in Newtonian cosmology. A description of cosmological dark matter fluid in Schrodinger formalism is already available. We would like to observe how the NC modifications moulds the fluid equations for a quantum fluid system and its subsequent consequences in the cosmological parameters. In the analysis of NC fluid we have considered the simplest form of approximation and a more detailed analysis of the model is perhaps possible. One of our specific future projects is to find solutions of the scale factors directly from the noncommutativity extended equations derived here. Moreover we would like to exploit the cosmological averaging principles [71–75] contextually where the modifications stem from the fact that the evolution and averaging of dynamical variables do not commute.

Appendix

A. Calculation of the brackets between ρ and j

$$\{\rho(r), j^i(r')\} = \rho_0^2\{\int dx\delta(X(x)-r), \int dy\ \dot{X}^i(y)\delta(X(y)-r')\}$$

$$= \rho_0^2[\int dxdy\{\delta(X(x)-r), \dot{X}^i(y)\}\delta(X(y)-r') + \{\delta(X(x)-r), \delta(X(y)-r')\}\dot{X}^i(y)]$$

$$= \rho(r')\partial_i\delta(r-r') - \theta^{jk}\partial_k\delta(r-r')\partial_j j^i(r) \tag{1}$$

B. Explicit calculation of one of the Jacobi identities,

$$J(\rho(p), \rho(q), \rho(r)) = \{\{\rho(p), \rho(q)\}, \rho(r)\} + \text{cyclic terms}$$

$$= \theta_k\theta_n\epsilon_{ijk}\epsilon_{lmn}[(p^iq^j(p^l+q^l)r^m + q^ir^j(q^l+r^l)p^m + r^ip^j(r^l+p^l)q^m]$$

$$\begin{aligned}= \theta_k\theta_n[\delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})][(p^iq^j(p^l+q^l)r^m + \\ q^ir^j(q^l+r^l)p^m + r^ip^j(r^l+p^l)q^m]\end{aligned} \tag{2}$$

$$\begin{aligned}= \theta^2(p^2(q.r) + (p.q)(q.r) - (p.r)(p.q) - q^2(p.r)) - \theta_m\theta_n(p^2q^nr^m + (p.q)q^nr^m - (p.q)p^nr^m \\ -q^2p^nr^m) + \theta_n\theta_l((p.r)q^np^l + (p.r)q^nq^l - p^np^l(q.r) - (q.r)p^nq^l) \\ +\theta^2(q^2(r.p) + (q.r)(r.p) - (q.p)(q.r) - r^2(q.p)) - \theta_m\theta_n(q^2r^np^m + (q.r)r^np^m - (q.r)q^np^m \\ -r^2q^np^m) + \theta_n\theta_l((q.p)r^nq^l + (q.p)r^nr^l - q^nq^l(r.p) - (r.p)q^nr^l) \\ +\theta^2(r^2(p.q) + (r.p)(p.q) - (r.q)(r.p) - p^2(r.q)) - \theta_m\theta_n(r^2p^nq^m + (r.p)p^nq^m - (r.p)r^nq^m \\ -p^2r^nq^m) + \theta_n\theta_l((r.q)p^nr^l + (r.q)p^np^l - r^nr^l(p.q) - (p.q)r^np^l)\end{aligned} \tag{3}$$

$$\begin{aligned}= -\theta_m\theta_n(p^2q^nr^m + (p.q)q^nr^m - (p.q)p^nr^m - q^2p^nr^m) + \theta_n\theta_l((p.r)q^np^l - (q.r)p^nq^l) \\ -\theta_m\theta_n(q^2r^np^m + (q.r)r^np^m - (q.r)q^np^m - r^2q^np^m) + \theta_n\theta_l((q.p)r^nq^l - (r.p)q^nr^l) \\ -\theta_m\theta_n(r^2p^nq^m + (r.p)p^nq^m - (r.p)r^nq^m - p^2r^nq^m) + \theta_n\theta_l((r.q)p^nr^l - (p.q)r^np^l) \\ = 0\end{aligned} \tag{4}$$

C. Explicit calculation of the ρ, j with the modified NC algebra

$$\{\rho(r), j^i(r')\} = \rho_0^2 \int dxdx'\{\delta(X(x)-r), \dot{X}^i\delta(X(x')-r')\}$$

$$= \rho_0^2 \int dxdx' \int [\partial_j^{X(x)}\delta((X(x)-r))\partial_k^{X(x')}\delta((X(x')-r'))\{X_j, X_k\}\dot{X}_i(x')$$

$$+\partial_j^{X(x)}\delta((X(x)-r))\delta(X(x')-r')\{X_j, \dot{X_i}\}]$$

$$= \rho_0 \int dxdx' \int [\partial_j^{X(x)}\delta((X(x)-r))\partial_k^{X(x')}\delta((X(x')-r'))\theta_{jk}\delta(x-x')\dot{X}_i(x')$$

$$-\partial_j^{X(x)}\delta((X(x)-r))\delta(X(x')-r')(\delta_{ij}+\sigma_{ji})\delta(x-x')]$$

$$= -\rho_0\theta_{jk}\partial_k^r\delta(r-r')\partial_j^r[\int dx\dot{X}_i\delta(X(x)-r)] - \partial_i^{r'}\delta(r-r')\rho(r') - \sigma_{ji}\partial_j^{r'}\delta(r-r')\rho(r')$$

$$= \rho(r')\partial_i\delta(r-r') - \theta^{jk}\partial_k\delta(r-r')\partial_j j^i(r) + \sigma_{ji}\rho(r')\partial_j\delta(r-r') \quad (5)$$

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